Cronfa - Swansea University Open Access Repository

Paper:
Fathizadeh, F. (2015). On the scalar curvature for the noncommutative four torus. *Journal of Mathematical Physics, 56*(6), 062303
http://dx.doi.org/10.1063/1.4922815

ON THE SCALAR CURVATURE FOR THE NONCOMMUTATIVE FOUR TORUS

FARZAD FATHIZADEH

Abstract. The scalar curvature for the noncommutative four torus $\mathbb{T}^4_\Theta$, where its flat geometry is conformally perturbed by a Weyl factor, is computed by making the use of a noncommutative residue that involves integration over the 3-sphere. This method is more convenient since it does not require the rearrangement lemma and it is advantageous as it explains the simplicity of the final functions of one and two variables, which describe the curvature with the help of a modular automorphism. In particular, it readily allows to write the function of two variables as the sum of a finite difference and a finite product of the one variable function. The curvature formula is simplified for dilatons of the form sp, where s is a real parameter and $p \in C^\infty(\mathbb{T}^4_\Theta)$ is an arbitrary projection, and it is observed that, in contrast to the two dimensional case studied by A. Connes and H. Moscovici, unbounded functions of the parameter s appear in the final formula. An explicit formula for the gradient of the analog of the Einstein-Hilbert action is also calculated.

Contents

1. Introduction

The computation of scalar curvature [9, 16] for noncommutative two tori $\mathbb{T}^2_\theta$ was stimulated by the seminal work [10] of A. Connes and P. Tretkoff on the Gauss-Bonnet theorem for these C^*-algebras, and its extension carried out in [13] to general translation-invariant conformal structures. Flat geometries of $\mathbb{T}^2_\theta$ [2, 3], whose conformal classes are represented by positive Hochschild cocycles [4], are conformally perturbed by means of a positive invertible element e^{-h}, where $h = h^* \in C^\infty(\mathbb{T}^2_\theta)$ is a dilaton [10]. Local geometric invariants, such as scalar curvature, can then be computed by considering small time asymptotic expansions, which depend on the action of the algebra on a Hilbert space and the distribution at infinity of the eigenvalues of a relevant geometric operator, namely the Laplacian of the conformally perturbed metric.

Following these works, the local differential geometry of noncommutative tori equipped with curved metrics has received considerable attention in recent years [1, 15, 11, 14, 12]. See also [28]. It should be mentioned that conformal geometry in the noncommutative setting is intimately related to twisted spectral triples and we refer to [8, 10, 9, 25, 20] for detailed discussions. Also it is closely related to the spectral action computations in the presence of a dilaton [5]. For noncommutative four tori $\mathbb{T}^4_\Theta$, the scalar curvature is computed in [14] and it is shown that flat metrics are the critical points of the analog of the Einstein-Hilbert action. Also noncommutative residues for noncommutative tori were studied in [17, 24, 14] (see also [29]). We refer to [30, 21] and [23] for detailed discussion on noncommutative residues for classical manifolds.

A crucial tool for the local computations on noncommutative tori has been Connes' pseudodifferential calculus, developed for C^*-dynamical systems in [2], which can be employed to work in the heat kernel scheme of elliptic differential operators and index theory (cf. [19]). An obstruction in these calculations, which is a purely noncommutative feature, is the appearance of integrals of functions over the positive real line that are C^*-algebra valued. This is overcome by the rearrangement lemma [10, 9] (cf. [1, 16, 22]), which uses the modular automorphism of the state, which implements the conformal perturbation, and delicate Fourier analysis to reorder the integrands and computes the integrals explicitly. The integrals are then expressed as somewhat complicated functions of the modular automorphism acting on relevant elements of the C^*-algebra. This lemma has been generalized in [22], and the work in [12] is an instance where the generalization is used.

A striking fact about the final formulas for the curvature of noncommutative tori is their simplicity and their fruitful properties such as being entire. Considering the numerous functions from the rearrangement lemma that get involved in hundreds of terms in the computations, the final simplicity indicates an enormous amount of cancellations, which are carried out by computer assistance. One of the aims of this paper is to explain this simplicity by computing the scalar curvature for $\mathbb{T}^4_\Theta$ without using the rearrangement lemma. We then study the curvature formula for the dilatons that are associated with projections in $C^\infty(\mathbb{T}^4_\Theta)$. The gradient of the Einstein-Hilbert action is also calculated, which prepares the ground for studying its associated flow in future works.

This article is organized as follows. In §2 we recall the formalism and notions used in [14] concerning the conformally perturbed Laplacian on $\mathbb{T}^4_\Theta$. In §3 we use a noncommutative residue that involves integrations on the 3-sphere $\mathbb{S}^3$ to compute the scalar curvature. This method is quite convenient as it does not require any help from the rearrangement lemma and it is advantageous as it explains the simplicity of the final formula (3). Also it readily allows to write the function of two variables in (3) as the sum of a finite difference and a finite product of the one variable function. In §4 the curvature formula is simplified for dilatons of the form sp, where s is a real parameter and $p \in C^\infty(\mathbb{T}^4_\Theta)$ is an arbitrary projection. It is observed that, in contrast to the two dimensional case studied in [9], unbounded functions of the parameter s appear in the final formula. In §5, we compute an explicit formula for the gradient of the analog of the Einstein-Hilbert action in terms of finite differences (cf. [9, 22]) of a one variable function that describes this action [14].

2. Preliminaries

The noncommutative four torus $C(\mathbb{T}^4_\Theta)$ is the universal C^*-algebra generated by four unitaries U_1, U_2, U_3, U_4, which satisfy the commutation relations

$$U_k U_j = e^{2\pi i \theta_{kj} U_j U_k},$$

where $\Theta = (\theta_{kj}) \in M_4(\mathbb{R})$ is an antisymmetric matrix. For simplicity elements of the form $U_1^{\ell_1} U_2^{\ell_2} U_3^{\ell_3} U_4^{\ell_4} \in C(\mathbb{T}^4_\Theta)$ shall be denote by U^ℓ for any 4-tuple of integers $\ell = (\ell_1, \ell_2, \ell_3, \ell_4)$.

There is a natural action of $\mathbb{R}^4$ on this C^*-algebra, which is defined by

$$\alpha_s(U^\ell) = e^{is\cdot\ell} U^\ell, \qquad s \in \mathbb{R}^4, \qquad \ell \in \mathbb{Z}^4,$$

and is extended to a 4-parameter family of C^*-algebra automorphisms $\alpha : \mathbb{R}^4 \to \mathrm{Aut}\big(C(\mathbb{T}^4_\Theta)\big)$. The infinitesimal generators of this action, denoted by $\delta_1, \delta_2, \delta_3, \delta_4$, are defined on the smooth subalgebra

$$C^\infty(\mathbb{T}^4_\Theta) = \{a \in C(\mathbb{T}^4_\Theta);\ \text{the map } \mathbb{R}^4 \ni s \mapsto \alpha_s(a) \text{ is smooth}\},$$

which is a dense subalgebra of $C(\mathbb{T}^4_\Theta)$ and can alternatively be defined as the space of elements of the form $\sum_{\ell\in\mathbb{Z}^4} a_\ell U^\ell$ with rapidly decaying complex coefficients $(a_\ell) \in \mathcal{S}(\mathbb{Z}^4)$. These derivations are determined by the relations $\delta_j(U_j) = U_j$ and $\delta_j(U_k) = 0$, if $j \neq k$.

One can consider a complex structure on $C(\mathbb{T}^4_\Theta)$ (cf. [14]) by introducing the analog of the Dolbeault operators

$$\partial_1 = \delta_1 - i\delta_3, \qquad \partial_2 = \delta_2 - i\delta_4, \qquad \bar\partial_1 = \delta_1 + i\delta_3, \qquad \bar\partial_2 = \delta_2 + i\delta_4,$$

and by setting

$$\partial = \partial_1 \oplus \partial_2, \qquad \bar\partial = \bar\partial_1 \oplus \bar\partial_2,$$

which are maps from $C^\infty(\mathbb{T}^4_\Theta)$ to $C^\infty(\mathbb{T}^4_\Theta) \oplus C^\infty(\mathbb{T}^4_\Theta)$.

There is a canonical positive faithful trace $\varphi_0 : C(\mathbb{T}^4_\Theta) \to \mathbb{C}$, which is defined on the smooth algebra by

$$\varphi_0\Big(\sum_{\ell\in\mathbb{Z}^4} a_\ell U^\ell\Big) = a_0.$$

Following the method introduced in [10], φ_0 is viewed as the volume form, and conformal perturbation of the metric is implemented in [14] by choosing a dilaton $h = h^* \in C^\infty(\mathbb{T}^4_\Theta)$ and by considering the linear functional $\varphi : C(\mathbb{T}^4_\Theta) \to \mathbb{C}$ given by

$$\varphi(a) = \varphi_0(ae^{-2h}), \qquad a \in C(\mathbb{T}^4_\Theta).$$

This is a KMS state and we consider the associated 1-parameter group $\{\sigma_t\}_{t\in\mathbb{R}}$ of inner automorphisms given by

$$\sigma_t(a) = e^{ith} a e^{-ith}, \qquad a \in C(\mathbb{T}^4_\Theta),$$

and will use the following operators substantially

$$\Delta(a) = \sigma_i(a) = e^{-h} a e^h, \qquad \nabla(a) = \log\Delta(a) = [-h, a], \qquad a \in C(\mathbb{T}^4_\Theta).$$

Denoting the inner product associated with the state φ by

$$(a, b)_\varphi = \varphi(b^* a), \qquad a, b \in C(\mathbb{T}^4_\Theta),$$

the Hilbert space completion of $C(\mathbb{T}^4_\Theta)$ with respect to this inner product is denoted by $\mathcal{H}_\varphi$, and the analog of the de Rham differential is defined in [14] by

$$d = \partial \oplus \bar\partial : \mathcal{H}_\varphi \to \mathcal{H}_\varphi^{(1,0)} \oplus \mathcal{H}_\varphi^{(0,1)}.$$

The Hilbert spaces $\mathcal{H}_\varphi^{(1,0)}$ and $\mathcal{H}_\varphi^{(0,1)}$ are respectively the completions of the analogs of $(1,0)$-forms and $(0,1)$-forms, namely the spaces $\{\sum_{i=1}^n a_i \partial b_i; a_i, b_i \in C^\infty(\mathbb{T}^4_\Theta), n \in \mathbb{N}\}$ and $\{\sum_{i=1}^n a_i \bar\partial b_i; a_i, b_i \in C^\infty(\mathbb{T}^4_\Theta), n \in \mathbb{N}\}$, with the appropriate inner product related to the conformal factor.

The Laplacian $d^*d : \mathcal{H}_\varphi \to \mathcal{H}_\varphi$ is then computed and shown to be anti-unitarily equivalent to the operator

$$\triangle_\varphi = e^h \bar\partial_1 e^{-h} \partial_1 e^h + e^h \partial_1 e^{-h} \bar\partial_1 e^h + e^h \bar\partial_2 e^{-h} \partial_2 e^h + e^h \partial_2 e^{-h} \bar\partial_2 e^h.$$

3. Scalar Curvature and its Functional Relations

The scalar curvature of the conformally perturbed metric on $\mathbb{T}^4_\Theta$ is the unique element $R \in C^\infty(\mathbb{T}^4_\Theta)$ such that

$$\mathrm{res}_{s=1} \mathrm{Trace}(a \triangle_\varphi^{-s}) = \varphi_0(aR), \qquad \forall a \in C^\infty(\mathbb{T}^4_\Theta).$$

Since the linear functional

$$⨍ P = \mathrm{res}_{s=0} \mathrm{Trace}(P \triangle_\varphi^{-s})$$

defines a trace on the algebra of pseudodifferential operators [7, 18], it follows from the uniqueness of traces on the algebra of pseudodifferential operators [17, 14, 24] that it coincides with the noncommutative residue defined in [14]. Therefore there exists a constant c such that for any P

$$⨍ P = c \int_{\mathbb{S}^3} \varphi_0(\rho_{-4}(\xi))\, d\Omega,$$

where ρ_{-4} is the homogeneous term of order -4 in the expansion of the symbol of P, and $d\Omega$ is the invariant measure on the sphere $\mathbb{S}^3$. Therefore, in order to compute the curvature, we can write

$$\mathrm{res}_{s=1} \mathrm{Trace}(a \triangle_\varphi^{-s}) = \mathrm{res}_{s=0} \mathrm{Trace}(a \triangle_\varphi^{-s-1}) = ⨍ a \triangle_\varphi^{-1} = c\, \varphi_0\Big(\int_{\mathbb{S}^3} a\, b_2(\xi)\, d\Omega \Big),$$

where b_j is the homogeneous term of order $-2-j$ in the asymptotic expansion of the symbol of the parametrix of $\triangle_\varphi$. Hence

$$R = c \int_{\mathbb{S}^3} b_2(\xi)\, d\Omega.$$

We compute b_2 by applying Connes' pseudodifferential calculus [2] to the symbol of $\triangle_\varphi$, which is the sum of the following homogeneous components:

$$a_2(\xi) = e^h \sum_{i=1}^4 \xi_i^2, \quad a_1(\xi) = \sum_{i=1}^4 \delta_i(e^h) \xi_i, \quad a_0(\xi) = \sum_{i=1}^4 \big(\delta_i^2(e^h) - \delta_i(e^h) e^{-h} \delta_i(e^h) \big).$$

That is, we solve the following equation explicitly up to b_2

$$(b_0 + b_1 + b_2 + \cdots) \circ (a_0 + a_1 + a_2) \sim 1,$$

which in general yields

$$b_0 = a_2^{-1} = \big(e^h \sum_{i=1}^4 \xi_i^2 \big)^{-1}, \qquad b_n = - \sum_{\substack{2+j+|\ell|-k=n, \\ 0 \le j < n,\, 0 \le k \le 2}} \frac{1}{\ell!} \partial^\ell(b_j) \delta^\ell(a_k) b_0 \qquad (n > 1).$$

Here, for any $\ell = (\ell_1, \ell_2, \ell_3, \ell_4) \in \mathbb{Z}^4_{\geq 0}$, ∂^ℓ denotes $\partial^{\ell_1}_{\xi_1}\partial^{\ell_2}_{\xi_2}\partial^{\ell_3}_{\xi_3}\partial^{\ell_4}_{\xi_4}$ and δ^ℓ denotes $\delta_1^{\ell_1}\delta_2^{\ell_2}\delta_3^{\ell_3}\delta_4^{\ell_4}$. Note that the composition rule for pseudodifferential symbols [2],

$$\rho \circ \rho' = \sum_{\ell \in \mathbb{Z}^4_{\geq 0}} \frac{1}{\ell!} \partial^\ell \rho(\xi)\, \delta^\ell(\rho'(\xi)),$$

is used in the derivation of the above recursive formula for b_n.

Computing b_2 and restricting it to $\mathbb{S}^3$ by the substitutions

$$\begin{aligned} \xi_1 &= \cos(\psi), & \xi_2 &= \cos(\theta)\sin(\psi), \\ \xi_3 &= \sin(\theta)\cos(\phi)\sin(\psi), & \xi_4 &= \sin(\theta)\sin(\phi)\sin(\psi), \end{aligned}$$

with $0 \leq \psi < \pi$, $0 \leq \theta < \pi$, $0 \leq \phi < 2\pi$, we perform its integral over the sphere and find that

$$\int_{\mathbb{S}^3} b_2(\xi)\, d\Omega = \int_0^{2\pi}\int_0^\pi\int_0^\pi b_2(\xi)\sin(\theta)\sin^2(\psi)\, d\psi\, d\theta\, d\phi$$

$$\begin{aligned} = \sum_{i=1}^4 \Big(&\big(-2\pi^2\big)\, b_0\delta_i\delta_i(e^h)b_0 + \big(2\pi^2\big)\, b_0\delta_i(e^h)\frac{1}{e^h}\delta_i(e^h)b_0 + \frac{5\pi^2}{2} b_0\delta_i(e^h)b_0\delta_i(e^h)b_0 \\ &+ \big(3\pi^2\big)\, b_0 e^h b_0\delta_i\delta_i(e^h)b_0 + \big(-8\pi^2\big)\, b_0e^hb_0\delta_i(e^h)b_0\delta_i(e^h)b_0 \\ &+ \big(-2\pi^2\big)\, b_0e^hb_0e^hb_0\delta_i\delta_i(e^h)b_0 + \big(-\pi^2\big)\, b_0\delta_i(e^h)b_0e^hb_0\delta_i(e^h)b_0 \\ &+ \big(2\pi^2\big)\, b_0e^hb_0\delta_i(e^h)b_0e^hb_0\delta_i(e^h)b_0 + \big(4\pi^2\big)\, b_0e^hb_0e^hb_0\delta_i(e^h)b_0\delta_i(e^h)b_0\Big) \end{aligned}$$

$$= \pi^2\sum_{i=1}^4 \Big(-e^{-h}\delta_i^2(e^h)e^{-h} + \frac{3}{2}e^{-h}\delta_i(e^h)e^{-h}\delta_i(e^h)e^{-h}\Big). \tag{1}$$

The fact that, over $\mathbb{S}^3$, b_0 reduces to e^{-h} is crucial in the last equation, which leads to such a simple final formula.

We then use the following identities [10, 9, 16] to write the expression (1) in terms of $\nabla = \log\Delta = -\mathrm{ad}_h$ and $\delta_i(h)$:

$$e^{-h}\delta_i(e^h) = g_1(\Delta)(\delta_i(h)), \qquad e^{-h}\delta_i^2(e^h) = g_1(\Delta)(\delta_i^2(h)) + 2g_2(\Delta,\Delta)(\delta_i(h)\delta_i(h)),$$

where

$$g_1(u) = \frac{u-1}{\log u}, \qquad g_2(u,v) = \frac{u(v-1)\log(u) - (u-1)\log(v)}{\log(u)\log(v)(\log(u)+\log(v))}. \tag{2}$$

This yields

$$\begin{aligned} \int_{\mathbb{S}^3} b_2(\xi)\, d\Omega &= \frac{1}{c}R \\ &= \pi^2\sum_{i=1}^4 \Big(-e^{-h}\Delta^{-1}g_1(\Delta)(\delta_i(h)) - 2e^{-h}\Delta^{-1}\big(g_2(\Delta,\Delta)(\delta_i(h)^2)\big) \\ &\qquad + \frac{3}{2}e^{-h}\Delta^{-1}\big(g_1(\Delta)(\delta_i(h))g_1(\Delta)(\delta_i(h))\big)\Big) \\ &= \pi^2 e^{-h}k(\nabla)\Big(\sum_{i=1}^4 \delta_i^2(h)\Big) + \pi^2 e^{-h}H(\nabla,\nabla)\Big(\sum_{i=1}^4\delta_i(h)^2\Big), \end{aligned} \tag{3}$$

where

$$k(s) = -e^{-s} g_1(e^s) = \frac{e^{-s} - 1}{s},$$

$$\begin{aligned} H(s,t) &= -2e^{-s-t} g_2(e^s, e^t) + \frac{3}{2} e^{-s-t} g_1(e^s) g_1(e^t) \\ &= \frac{e^{-s-t}\left(\left(e^s - 1\right)\left(3e^t + 1\right)t - \left(e^s + 3\right)s\left(e^t - 1\right)\right)}{2st(s+t)}. \end{aligned} \tag{4}$$

This formula matches with the one obtained in [14] (up to the multiplicative factor $1/c = 2\pi^2$).

Theorem 3.1. *Let*

$$\tilde{k}(s) = e^s k(s), \qquad \tilde{H}(s,t) = e^{s+t} H(s,t),$$

where k and H are the functions in the final formula for the scalar curvature. We have

$$\tilde{H}(s,t) = 2\frac{\tilde{k}(s+t) - \tilde{k}(s)}{t} + \frac{3}{2}\tilde{k}(s)\tilde{k}(t). \tag{5}$$

Proof. It follows from (4) and the following relation between the functions introduced in (2):

$$\begin{aligned} g_2(u,v) &= \int_0^1 s u^s g_1(v^s)\, ds = \frac{1}{\log(v)}\Big(\frac{uv-1}{\log(uv)} - \frac{u-1}{\log(u)}\Big) \\ &= \frac{1}{\log(v)}\big(g_1(uv) - g_1(u)\big). \end{aligned}$$

□

4. Projections and the Scalar Curvature

Similar to the illustration in [9] of the scalar curvature of $\mathbb{T}_\theta^2$ for dilatons associated with projections, we consider dilatons of the $h = sp$, where $s \in \mathbb{R}$ and $p = p^* = p^2 \in C^\infty(\mathbb{T}_\Theta^4)$ is an arbitrary projection and simplify the expression (3) for these cases. We shall also study the behaviour of the functions of the parameter s that appear in the final formula.

Proposition 4.1. *Let $p = p^* = p^2 \in C^\infty(\mathbb{T}_\Theta^4)$ be a projection. For the dilaton $h = sp$, $s \in \mathbb{R}$, the formula for the scalar curvature reduces to*

$$R = e^{-sp}\big(f_1(s)\triangle(p) + f_2(s)\triangle(p)p + f_3(s)p\triangle(p) + f_4(s)p\triangle(p)p\big),$$

where

$$f_1(s) = \frac{1}{4}(-2\sinh(s) + \cosh(s) - 1), \qquad f_2(s) = \frac{1}{2}\sinh^2\left(\frac{s}{2}\right),$$

$$f_3(s) = \frac{-s + \sinh(s)}{2} - \frac{\cosh(s) - 1}{4}, \qquad f_4(s) = s - \sinh(s).$$

Proof. Our method is quite similar to the one used in [9]. That is, we first use the identity

$$\triangle(p) = p\triangle(p)p + p\triangle(p)(1-p) + (1-p)\triangle(p)p + (1-p)\triangle(p)(1-p),$$

to decompose $\triangle(p)$ to the sum of eigenvectors of $\nabla = -\mathrm{ad}_{sp}$ with eigenvalues $0, -s, s, 0$. Therefore

$$\begin{aligned} k(\nabla)(\triangle(h)) &= sk(\nabla)(\triangle(p)) \\ &= sk(0)\big(p\triangle(p)p + (1-p)\triangle(p)(1-p)\big) \\ &\quad +sk(-s)\big(p\triangle(p)(1-p)\big) + sk(s)\big((1-p)\triangle(p)p\big) \\ &= -s\big(p\triangle(p)p + (1-p)\triangle(p)(1-p)\big) \\ &\quad +(1-e^s)\big(p\triangle(p)(1-p)\big) + (e^{-s}-1)\big((1-p)\triangle(p)p\big). \end{aligned}$$

Then, using the identity $\delta_i(p) = \delta_i(p)p + p\delta_i(p)$, one can see that

$$\begin{aligned} &H(\nabla,\nabla)(\delta_i(h)\delta_i(h)) \\ &= \frac{s^2}{2}\Big(\big(H(s,-s)+H(-s,s)\big) + \big(H(s,-s)-H(-s,s)\big)(1-2p)\Big)\big(\delta_i(p)\delta_i(p)\big) \\ &= \frac{s^2}{2}\Big(\frac{2(\cosh(s)-1)}{s^2} + \frac{4(s-\sinh(s))}{s^2}(1-2p)\Big)\big(\delta_i(p)\delta_i(p)\big) \\ &= \Big((\cosh(s)-1) + 2(s-\sinh(s))(1-2p)\Big)\big(\delta_i(p)\delta_i(p)\big) \\ &= \big(2s - 2\sinh(s) + \cosh(s) - 1 - 4(s-\sinh(s))p\big)\big(\delta_i(p)\delta_i(p)\big). \end{aligned}$$

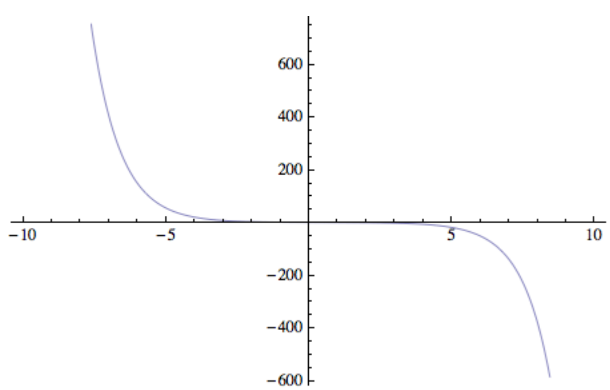
Using the identity $2\sum \delta_i(p)^2 = (1-p)\triangle(p) - \triangle(p)p$, we sum the above expressions and find that the formula (3), for the dilaton $h = sp$, reduces to

$$\begin{aligned} &\frac{1}{2}(-2\sinh(s) + \cosh(s) - 1)\triangle(p) + (-s + \sinh(s) - \frac{\cosh(s)}{2} + \frac{1}{2})p\triangle(p) \\ &+ \sinh^2\Big(\frac{s}{2}\Big)\triangle(p)p + 2(s-\sinh(s))p\triangle(p)p, \end{aligned}$$

up to multiplication from left by $e^{-h} = e^{-sp}$.

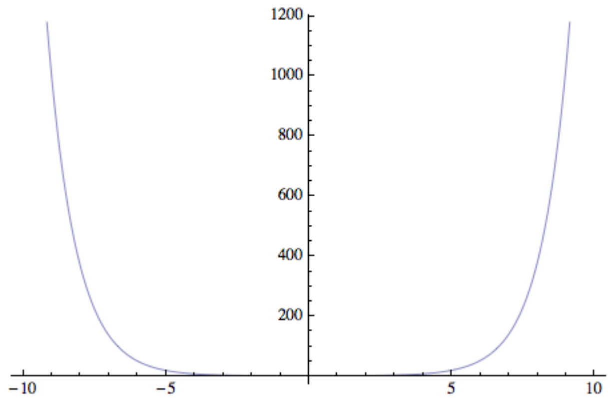
□

In contrast to the two dimensional case (cf. [9]), the functions of the variable $s \in \mathbb{R}$ that appear in Proposition 4.1 are not bounded as they tend to $\pm\infty$ as $|s| \to \infty$. The graphs of these functions are given below and some relations between these functions are investigated. First we graph f_1.



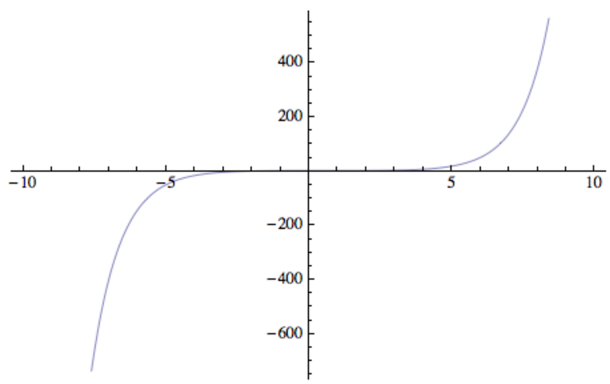
Graph of the function $f_1(s) = \frac{1}{4}(-2\sinh(s) + \cosh(s) - 1)$.

Among these functions, f_2 is the only one that is bounded below, whereas the other functions are neither bounded above nor bounded below. In fact f_2 is a non-negative even function.



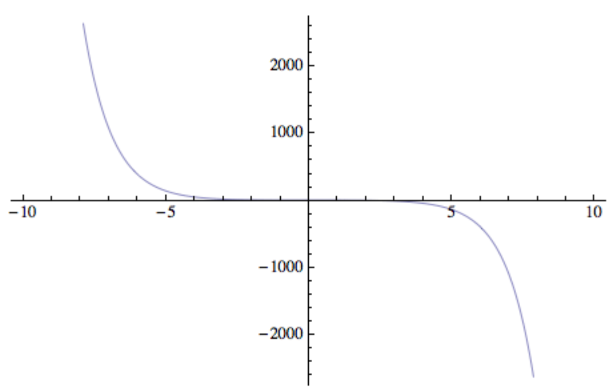
Graph of the function $f_2(s) = \frac{1}{2}\sinh^2\left(\frac{s}{2}\right)$.

The function f_3, similar to f_1, does not satisfy any symmetry properties.



Graph of the function $f_3(s) = \frac{1}{2}(\sinh(s) - s) + \frac{1}{4}(1 - \cosh(s))$.

The last function f_4 is obviously an odd function whose graph is given here:



Graph of the function $f_4(s) = 2(s - \sinh(s))$.

It is interesting to observe that these functions, which describe the scalar curvature for the dilaton $h = sp$, where p is an arbitrary projection, satisfy the following relations:

$$f_1(s) + f_1(-s) = f_2(s) + f_2(-s) = -\big(f_3(s) + f_3(-s)\big) = -\sinh^2\left(\frac{s}{2}\right),$$
$$f_3(s) - f_3(-s) = -\frac{1}{2}f_4(s) = \sinh(s) - s.$$

5. Gradient of the Einstein-Hilbert Action

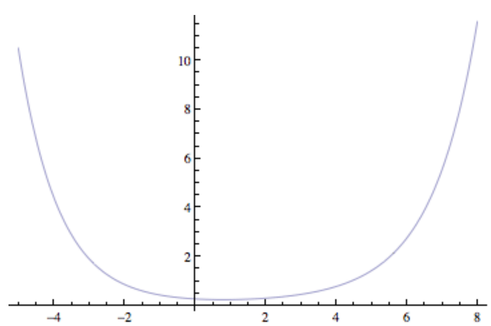
Denoting the Einstein-Hilbert action associated with the dilaton $h = h^* \in C^\infty(\mathbb{T}^4_\Theta)$ by $\Omega(h)$, we compute its gradient, namely an explicit formula for an element $\mathrm{Grad}_h\Omega$ that represents the derivative at $\varepsilon = 0$ of $\Omega(h + \varepsilon a)$, where h, a are selfadjoint smooth elements. The final formula for the gradient is expressed in terms of finite differences of the function T, obtained in the proof of Theorem 5.3 of [14]. We recall from the proof of this theorem that

$$\Omega(h) = \varphi_0(R) = \sum_{i=1}^{4} \varphi_0\big(e^{-h}T(\nabla)(\delta_i(h))\delta_i(h)\big),$$

where

$$T(s) = \frac{-2s + e^s - e^{-s}(2s+3) + 2}{4s^2}$$

with the following graph.



Graph of the function T.

The fact that this function is non-negative played a crucial role in identifying the extrema of the Einstein-Hilbert action in [14].

In order to compute the gradient of Ω, first we need the following lemmas. The proofs follow closely the techniques given in [9] for the computation of the gradient such functionals (see also [22]).

Theorem 5.1. *For any selfadjoint* $h \in C^\infty(\mathbb{T}^4_\Theta)$*, we have*

$$\mathrm{Grad}_h\Omega = \sum_{i=1}^{4} \Big(e^{-h}\omega_1(\nabla)(\delta_i^2(h)) + e^{-h}\omega_2(\nabla,\nabla)(\delta_i(h)^2)\Big),$$

where

$$\omega_1(s) = \frac{-2\sinh(s) + \sinh(2s) - \cosh(2s) + 1}{4s^2},$$

$$\omega_2(s,t) =$$
$$\frac{1}{4s^2t^2(s+t)^2}(-s(-(s^2+st+3t^2)\sinh(s)+(s^2+5st+t^2)\cosh(s)-t^2)(\sinh(3(s+t))+\cosh(3(s+t)))+s((s^2(8t+5)+st(12t+17)+t^2(4t+5))(\sinh(s)+\cosh(s))-4st^2-4t^3-5t^2)(\sinh(s+t)+\cosh(s+t))+(-(5s^3+s^2t(4t+15)+2st^2(2t+5)+5t^3)(\sinh(s)+\cosh(s))+t^2(-(s+t))-t^2(4s(s+t-2)-5t)(\sinh(2s)+\cosh(2s))+t(s+t)(2s+t)(\sinh(3s)+\cosh(3s)))(\sinh(s+2t)+\cosh(s+2t))+s(s\sinh(s)+s\cosh(s)+t)((s+t)(\sinh(s)+\cosh(s))-t))(\cosh(3s+2t)-\sinh(3s+2t)).$$

Proof. We have

$$\begin{aligned}
&\Omega(h+\varepsilon a)\\
&=\sum_{i=1}^{4}\varphi_0\big(e^{-h-\varepsilon a}\,T(\nabla_{h+\varepsilon a})(\delta_i(h+\varepsilon a))\delta_i(h+\varepsilon a)\big)\\
&=\sum_{i=1}^{4}\Big(\varphi_0\big(e^{-h-\varepsilon a}\,T(\nabla_{h+\varepsilon a})(\delta_i(h))\delta_i(h)\big)+\varepsilon\varphi_0\big(e^{-h-\varepsilon a}\,T(\nabla_{h+\varepsilon a})(\delta_i(a))\delta_i(h)\big)\\
&\quad+\varepsilon\varphi_0\big(e^{-h-\varepsilon a}\,T(\nabla_{h+\varepsilon a})(\delta_i(h))\delta_i(a)\big)+\varepsilon^2\varphi_0\big(e^{-h-\varepsilon a}\,T(\nabla_{h+\varepsilon a})(\delta_i(a))\delta_i(a)\big)\Big).
\end{aligned}$$

Therefore

$$\begin{aligned}
&\frac{d}{d\varepsilon}|_{\varepsilon=0}\Omega(h+\varepsilon a)\\
&=\varphi_0\big((\frac{d}{d\varepsilon}|_{\varepsilon=0}e^{-h-\varepsilon a})\,T(\nabla)(\delta(h))\delta(h)\big)+\varphi_0\big(e^{-h}\,\frac{d}{d\varepsilon}|_{\varepsilon=0}T(\nabla_{h+\varepsilon a})(\delta(h))\delta(h)\big)\\
&\quad+\varphi_0\big(e^{-h}T(\nabla)(\delta(h))\delta(a)\big)+\varphi_0\big(e^{-h}T(\nabla)(\delta(a))\delta(h)\big).
\end{aligned}$$

Hence, using the following lemmas we obtain the explicit formula,

$$\mathrm{Grad}_h\Omega=\sum_{i=1}^{4}\Big(e^{-h}\omega_1(\nabla)(\delta_i^2(h))+e^{-h}\omega_2(\nabla,\nabla)(\delta_i(h)\delta_i(h))\Big),$$

where

$$\omega_1(s)=-T(s)-T(-s)e^{-s},$$

$$\begin{aligned}
\omega_2(s,t)&=E(s,t)+L(s,t)-P(s,t)-Q(s,t)\\
&=\frac{e^{-s-t}-1}{s+t}T(s)+e^{-s-t}\Big(\frac{T(-t)-T(s)}{s+t}+\frac{T(t)-T(-s)}{s+t}e^{t}\Big)\\
&\quad-\Big(T(s+t)\frac{e^{-s}-1}{s}+\frac{T(t)-T(s+t)}{s}e^{-s}+\frac{T(s+t)-T(s)}{t}\Big)\\
&\quad-\Big(T(-s-t)e^{-s-t}\frac{e^{-s}-1}{s}+\frac{T(t)-T(s+t)}{s}e^{-s}+\frac{T(s+t)-T(s)}{t}\Big).
\end{aligned}$$

Then one can find the above explicit functions in the statement of the theorem by direct computer assisted computations. ∎

For simplicity in the notation, in the following lemmas, δ can be taken to be any of the canonical derivations δ_i introduced in §2.

Lemma 5.1. *We have*

$$\varphi_0\big(\big(\frac{d}{d\varepsilon}|_{\varepsilon=0}e^{-h-\varepsilon a}\big)G(\nabla)(x)x\big)=\varphi_0\big(ae^{-h}E(\nabla,\nabla)(xx)\big),$$

where

$$E(s,t)=\frac{e^{-s-t}-1}{s+t}G(s).$$

Proof. Using

$$\frac{d}{d\varepsilon}\Big|_{\varepsilon=0}e^{-h-\varepsilon a}=\frac{1-e^{\nabla}}{\nabla}(a)e^{-h},$$

we have

$$\begin{aligned}\varphi_0\big(\big(\frac{d}{d\varepsilon}|_{\varepsilon=0}e^{-h-\varepsilon a}\big)G(\nabla)(x)x\big) &= \varphi_0\big(\frac{1-e^{\nabla}}{\nabla}(a)e^{-h}G(\nabla)(x)x\big)\\ &= \varphi_0\big(ae^{-h}\,\frac{e^{-\nabla}-1}{\nabla}(G(\nabla)(x)x)\big)\\ &= \varphi_0\big(ae^{-h}E(\nabla,\nabla)(xx)\big).\end{aligned}$$

□

Lemma 5.2. *For any $x\in C^\infty(\mathbb{T}^4_\Theta)$, we have*

$$\varphi_0\Big(e^{-h}\frac{d}{d\varepsilon}|_{\varepsilon=0}G(\nabla_{h+\varepsilon a})(x)x\Big)=\varphi_0(ae^{-h}\,L(\nabla,\nabla)(xx)),$$

where

$$L(s,t)=e^{-s-t}\Big(\frac{G(-t)-G(s)}{s+t}+\frac{G(t)-G(-s)}{s+t}e^{t}\Big).$$

Proof. Writing $G(v)=\int e^{-itv}g(t)\,dt$ and using the following identity [9])

$$\frac{d}{d\varepsilon}|_{\varepsilon=0}G(\nabla_{h+\varepsilon a})=\int_0^1\int it\,\sigma_{ut}\,\mathrm{ad}_a\,\sigma_{(1-u)t}\,g(t)\,dt\,du,$$

we find that

$$\varphi_0\Big(e^{-h}\frac{d}{d\varepsilon}|_{\varepsilon=0}G(\nabla_{h+\varepsilon a})(x)x\Big)=\varphi_0(e^{-h}a\,L_0(\nabla,\nabla)(xx)),$$

where

$$L_0(s,t)=\frac{G(-t)-G(s)}{s+t}+\frac{G(t)-G(-s)}{s+t}e^{t}.$$

Therefore

$$\varphi_0\Big(e^{-h}\frac{d}{d\varepsilon}|_{\varepsilon=0}G(\nabla_{h+\varepsilon a})(x)x\Big)=\varphi_0(ae^{-h}\,L(\nabla,\nabla)(xx)),$$

where

$$L(s,t)=e^{-s-t}L_0(s,t).$$

□

Lemma 5.3. *For any $x\in C^\infty(\mathbb{T}^4_\Theta)$ one has*

$$\delta\big(G(\nabla)(x)\big)=G(\nabla)(\delta(x))+M_1(\nabla,\nabla)(\delta(h)x)+M_2(\nabla,\nabla)(x\delta(h)),$$

where

$$M_1(s,t)=\frac{G(t)-G(s+t)}{s},\qquad M_2(s,t)=\frac{G(s+t)-G(s)}{t}.$$

Proof. It can be seen by writing $G(v)=\int e^{-itv}g(t)\,dt$ and using the identity [9]

$$\delta_i\sigma_t=\sigma_t\delta_i+it\int_0^1\sigma_{ut}\,\mathrm{ad}_{\delta_i(h)}\sigma_{(1-u)t}\,du.$$

□

Lemma 5.4. *We have*

$$\varphi_0\big(e^{-h}G(\nabla)(\delta(h))\delta(a)\big)=-\varphi_0\big(ae^{-h}G(\nabla)(\delta^2(h))\big)-\varphi_0\big(ae^{-h}P(\nabla,\nabla)(\delta(h)\delta(h))\big),$$

where

$$P(s,t)=G(s+t)\frac{e^{-s}-1}{s}+M_1(s,t)e^{-s}+M_2(s,t).$$

Proof. We start by writing

$$\begin{aligned}&\varphi_0\big(e^{-h}G(\nabla)(\delta(h))\delta(a)\big)=-\varphi_0\big(a\,\delta(G(\nabla)(e^{-h}\delta(h)))\big)\\&=-\varphi_0\big(a\,G(\nabla)(\delta(e^{-h}\delta(h)))\big)-\varphi_0\big(aM_1(\nabla,\nabla)(\delta(h)e^{-h}\delta(h))\big)\\&\quad-\varphi_0\big(aM_2(\nabla,\nabla)(e^{-h}\delta(h)\delta(h))\big)\\&=-\varphi_0\big(ae^{-h}\,G(\nabla)(e^{h}\delta(e^{-h})\delta(h))\big)-\varphi_0\big(a\,G(\nabla)(e^{-h}\delta^2(h))\big)\\&\quad-\varphi_0\big(ae^{-h}M_1(\nabla,\nabla)(e^{-\nabla}(\delta(h))\delta(h))\big)-\varphi_0\big(ae^{-h}M_2(\nabla,\nabla)(\delta(h)\delta(h))\big).\end{aligned}$$

Then, using the fact that $e^h\delta(e^{-h})=\frac{e^{-\nabla}-1}{\nabla}(\delta(h))$, one can find the above expression for $\varphi_0\big(e^{-h}G(\nabla)(\delta(h))\delta(a)\big)$ in the statement of the lemma.

□

Lemma 5.5. *We have*

$$\varphi_0\big(e^{-h}G(\nabla)(\delta(a))\delta(h)\big)=-\varphi_0\big(ae^{-h}\bar G(\nabla)(\delta^2(h))\big)-\varphi_0\big(ae^{-h}Q(\nabla,\nabla)(\delta(h)\delta(h))\big),$$

where

$$\begin{aligned}\bar G(s)&=G(-s)e^{-s},\\Q(s,t)&=\bar G(s+t)\frac{e^{-s}-1}{s}+M_1(s,t)e^{-s}+M_2(s,t).\end{aligned}$$

Proof. It follows from the previous lemma after writing

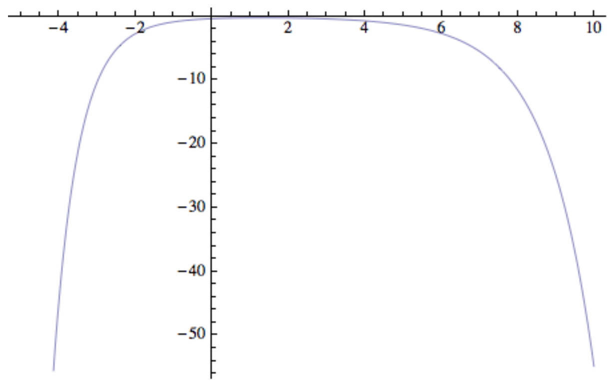
$$\varphi_0\big(e^{-h}G(\nabla)(\delta(a))\delta(h)\big)=\varphi_0\big(e^{-h}G(-\nabla)e^{-\nabla}(\delta(h))\delta(a)\big).$$

□

The Taylor series at $s=0$ of the function of one variable ω_1 appearing in the formula for the above gradient is given by

$$\omega_1(s)=-\frac{1}{2}+\frac{s}{4}-\frac{s^2}{6}+\frac{s^3}{16}-\frac{s^4}{45}+\frac{s^5}{160}+O\left(s^6\right),$$

and the following is its graph.

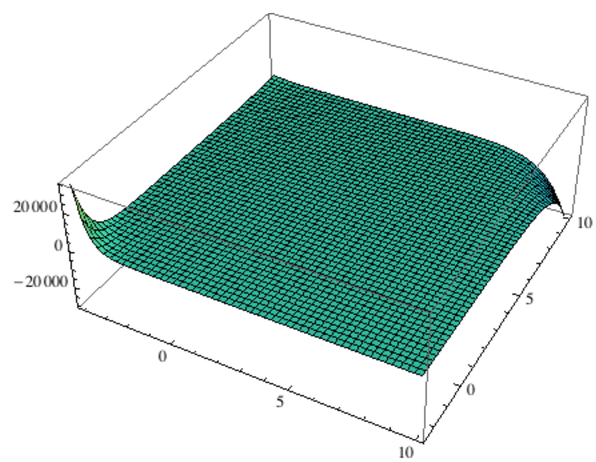


Graph of the function ω_1.

The function of two variables ω_2 appearing in the formula for the above gradient has the following Taylor expansion at the origin

$$\begin{aligned}\omega_2(s,t) \quad = \quad & \left(\frac{1}{4}+\frac{t}{24}+\frac{13t^2}{240}-\frac{7t^3}{360}+O\left(t^4\right)\right)+s\left(-\frac{3}{8}+\frac{5t}{48}-\frac{17t^2}{120}+\frac{t^3}{14}+O\left(t^4\right)\right)\\ & +s^2\left(\frac{47}{240}-\frac{t}{6}+\frac{77t^2}{480}-\frac{1159t^3}{13440}+O\left(t^4\right)\right)\\ & +s^3\left(-\frac{83}{720}+\frac{169t}{1260}-\frac{697t^2}{5760}+\frac{151t^3}{2304}+O\left(t^4\right)\right)+O\left(s^4\right),\end{aligned}$$

and here is its graph:



Graph of the function ω_2.

We now look at the behavior of the function ω_2 on the diagonals. We have

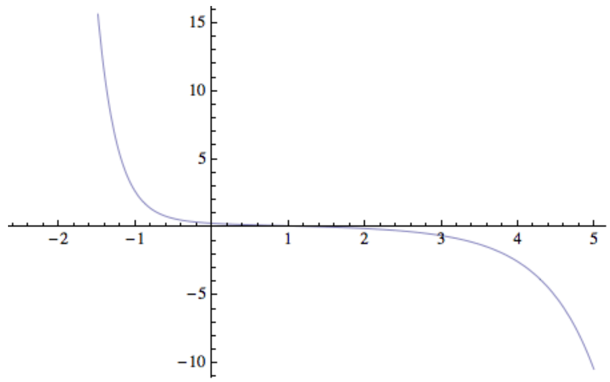
$$\omega_2(s,s) =$$

$$-\frac{e^{-3s/2}\sinh\left(\frac{s}{2}\right)\left(8s+(8s-5)\sinh(s)-3\sinh(2s)+\sinh(3s)-8\cosh(s)-3\cosh(2s)+11\right)}{4s^3},$$

with the Taylor expansion

$$\omega_2(s,s) = \frac{1}{4} - \frac{s}{3} + \frac{17s^2}{48} - \frac{319s^3}{720} + \frac{623s^4}{1440} - \frac{155s^5}{448} + O\left(s^6\right),$$

and the following graph.

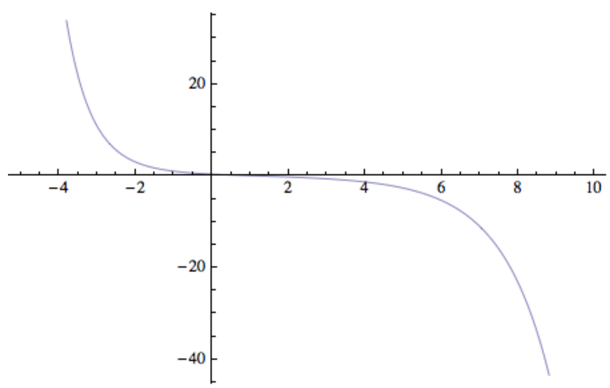


Graph of the function $s \mapsto \omega_2(s,s)$.

On the other diagonal we have

$$\omega_2(s,-s) = \frac{4s + e^{-2s} - 2e^s + 1}{4s^2} = \frac{1}{4} - \frac{5s}{12} + \frac{7s^2}{48} - \frac{17s^3}{240} + \frac{31s^4}{1440} - \frac{13s^5}{2016} + O\left(s^6\right),$$

whose graph is the following.



Graph of the function $s \mapsto \omega_2(s,-s)$.

6. Discussion

Considering the variety of geometric spaces that fit into the paradigm of noncommutative geometry [3, 4], it is of great importance to develop different methods for computing their local geometric invariants such as scalar curvature. Different methods of computation can also help with conceptual understanding of such invariants for specific examples.

The appearance of functions of a modular automorphism in the final formulas for the scalar curvature of noncommutative tori is a purely noncommutative feature

[9, 16, 14], which accompanies the following striking facts. First, the final one and two variable functions in the curvature formulas are significantly simple, which indicates an enormous amount of cancellations in the algebraically lengthy formula that involves hundreds of terms with numerous functions from the rearrangement lemma [10, 9, 1, 16] involved. Second, the function of two variables for the curvature of $\mathbb{T}_\theta^2$ can be recovered from the one variable function [9] by finite differences.

Using the noncommutative residue that involves integration of the sphere [17, 14], we computed the scalar curvature of $\mathbb{T}_\Theta^4$ in this paper without using the rearrangement lemma. This method avoids the complexity stemming from the functions coming from this lemma and also explains the simplicity of the final functions in the curvature formula. It should be emphasized that there is another rather technical simplifying factor in this method, compared to the use of parametric pseudodifferential calculus, which is due to the fact that the first term b_0 of the parametrix of the Laplacian reduces on the sphere to a power of the Weyl factor. It can be seen in the derivation of (1) that this softens out some further complexities and leads to the final expression, whose summand consists of only a few terms.

Similar to the concrete illustration in [9] of the scalar curvature of $\mathbb{T}_\theta^2$ for dilatons associated with projections, which exist in abundance for noncommutative tori [27], we have worked out a concrete formula in the case of $\mathbb{T}_\Theta^4$. For a dilaton of the form $h = sp$, where $s \in \mathbb{R}$ and $p \in C^\infty(\mathbb{T}_\Theta^4)$ is an arbitrary projection, the final concrete formula for the curvature involves unbounded functions of the parameter s, which is in contrast to the striking fact about the boundedness of the functions obtained in the two dimensional case [9]. The question that arises is whether there is a conceptual meaning behind this contrast. Also, the explicit computation of the gradient of the analog of the Einstein-Hilbert action for $\mathbb{T}_\Theta^4$ prepares the ground for further studies of the natural associated geometric flows in this context, cf. [1, 9].

Acknowledgments

The author thanks the Max-Planck Institute for Mathematics, the Banach Center at the Institute for Mathematics Polish Academy of Sciences, and the Hausdorff Research Institute for Mathematics for their support and hospitality.

References

[1] T. A. Bhuyain, M. Marcolli, *The Ricci flow on noncommutative two-tori,* Lett. Math. Phys. 101 (2012), no. 2, 173-194..

[2] A. Connes, *C^*-algèbres et géométrie différentielle*, C.R. Acad. Sc. Paris, t. 290, Série A, 599-604, 1980.

[3] A. Connes, *Noncommutative differential geometry,* Inst. Hautes Études Sci. Publ. Math. No. 62, 257-360, 1985.

[4] A. Connes, *Noncommutative geometry*, Academic Press, 1994.

[5] A. H. Chamseddine, A. Connes, *Scale invariance in the spectral action,* (English summary) J. Math. Phys. 47 (2006), no. 6, 063504, 19 pp.

[6] A. Connes, M. Marcolli, *Noncommutative Geometry, Quantum Fields and Motives*, American Mathematical Society Colloquium Publications, 55, 2008.

[7] A. Connes, H. Moscovici, *The local index formula in noncommutative geometry,* Geom. Funct. Anal. 5, no. 2, 174-243, 1995.

[8] A. Connes, H. Moscovici, *Type III and spectral triples,* Traces in number theory, geometry and quantum fields, 57–71, Aspects Math., E38, Friedr. Vieweg, Wiesbaden, 2008.

[9] A. Connes, H. Moscovici, *Modular curvature for noncommutative two-tori,* J. Amer. Math. Soc. 27 (2014), no. 3, 639–684.

[10] A. Connes, P. Tretkoff, *The Gauss-Bonnet theorem for the noncommutative two torus*, Noncommutative geometry, arithmetic, and related topics, 141-158, Johns Hopkins Univ. Press, Baltimore, MD, 2011.

[11] L. Dabrowski, A. Sitarz, *Curved noncommutative torus and Gauss–Bonnet,* J. Math. Phys. 54, 013518 (2013).

[12] L. Dabrowski, A. Sitarz, *Asymmetric noncommutative torus,* arXiv:1406.4645.

[13] F. Fathizadeh, M. Khalkhali, *The Gauss-Bonnet theorem for noncommutative two tori with a general conformal structure*, J. Noncommut. Geom. 6 (2012), no. 3, 457–480.

[14] F. Fathizadeh, M. Khalkhali, *Scalar curvature for noncommutative four-tori,* To appear in the Journal of Noncommutative Geometry, arXiv:1301.6135.

[15] F. Fathizadeh, M. Khalkhali, *Weyl's Law and Connes' Trace Theorem for Noncommutative Two Tori,* Lett. Math. Phys. 103 (2013), no. 1, 1-18.

[16] F. Fathizadeh, M. Khalkhali, *Scalar curvature for the noncommutative two torus.* J. Noncommut. Geom. 7 (2013), no. 4, 1145–1183.

[17] F. Fathizadeh, M. W. Wong, *Noncommutative residues for pseudo-differential operators on the noncommutative two-torus*, Journal of Pseudo-Differential Operators and Applications, 2(3) (2011), 289-302.

[18] N. Higson, *The local index formula in noncommutative geometry,* (English summary) Contemporary developments in algebraic K-theory, 443–536 (electronic), ICTP Lect. Notes, XV, Abdus Salam Int. Cent. Theoret. Phys., Trieste, 2004.

[19] P. Gilkey, *Invariance theory, the heat equation, and the Atiyah-Singer index theorem*, Mathematics Lecture Series, 11. Publish or Perish, Inc., Wilmington, DE, 1984.

[20] M. Greenfield, M. Marcolli, K. Teh, *Twisted spectral triples and quantum statistical mechanical systems,* p-Adic Numbers Ultrametric Anal. Appl. 6 (2014), no. 2, 81–104.

[21] V. Guillemin, *A new proof of Weyl's formula on the asymptotic distribution of eigenvalues,* Adv. in Math. 55 (1985), no. 2, 131-160.

[22] M. Lesch, *Divided differences in noncommutative geometry: rearrangement lemma, functional calculus and Magnus expansion,* arXiv:1405.0863.

[23] M. Lesch, C. Neira Jiménez, *Classification of traces and hypertraces on spaces of classical pseudodifferential operators,* J. Noncommut. Geom. 7 (2013), no. 2, 457–498.

[24] C. Levy, C. Neira Jiménez, S. Paycha, *The canonical trace and the noncommutative residue on the noncommutative torus,* arXiv:1303.0241.

[25] H. Moscovici, *Local index formula and twisted spectral triples,* (English summary) Quanta of maths, 465–500, Clay Math. Proc., 11, Amer. Math. Soc., Providence, RI, 2010.

[26] R. Ponge, H. Wang, *Noncommutative geometry, conformal geometry, and the local equivariant index theorem,* arXiv:1210.2032.

[27] M. A. Rieffel, *C*-algebras associated with irrational rotations,* Pacific J. Math. 93 (1981), no. 2, 415–429.

[28] J. Rosenberg, *Levi-Civita's theorem for noncommutative tori,* (English summary) SIGMA Symmetry Integrability Geom. Methods Appl. 9 (2013), Paper 071, 9 pp.

[29] A. Sitarz, *Wodzicki residue and minimal operators on a noncommutative 4-dimensional torus,* J. Pseudo-Differ. Oper. Appl. 5 (2014), no. 3, 305–317.

[30] M. Wodzicki, *Noncommutative residue. I. Fundamentals,* in K-theory, Arithmetic and Geometry (Moscow, 1984–1986), Lecture Notes in Mathematics 1289, Springer, 320-399, 1987.