MATHEMATICAL THEORY OF EXCHANGE-DRIVEN GROWTH

EMRE ESENTURK

Abstract. Exchange-driven growth is a process in which pairs of clusters interact and exchange a single unit of mass. The rate of exchange is given by an interaction kernel $K(j,k)$ which depends on the masses of the two interacting clusters. In this paper we establish the fundamental mathematical properties of the mean field kinetic equations of this process for the first time. We find two different classes of behaviour depending on whether $K(j,k)$ is symmetric or not. For the non-symmetric case, we prove global existence and uniqueness of solutions for kernels satisfying $K(j,k) \leq Cjk$. This result is optimal in the sense that we show for a large class of initial conditions with kernels satisfying $K(j,k) \geq Cj^{\beta}$ ($\beta > 1$) the solutions cannot exist. On the other hand, for symmetric kernels, we prove global existence of solutions for $K(j,k) \leq C(j^{\mu}k^{\nu} + j^{\nu}k^{\mu})$ ($\mu, \nu \leq 2$, $\mu + \nu \leq 3$), while existence is lost for $K(j,k) \geq Cj^{\beta}$ ($\beta > 2$). In the intermediate regime $3 < \mu + \nu \leq 4$, we can only show local existence. We conjecture that the intermediate regime exhibits finite-time gelation in accordance with the heuristic results obtained for particular kernels.

1. Introduction

Growth processes are ubiquitous in nature. Surprisingly diverse phenomena at contrasting scales (from microscopic level polymerization processes to cloud formation to galaxy formation mechanisms at huge scales) have similar driving mechanisms [1], [2]. One of the commonly occuring mechanisms is the cluster growth by coagulation for which Smoluchoswki and Becker-Doring models are classical examples. For these models, an extensive mathematical theory has been established [3], [20] relating the properties of the cluster size distribution to the structure of the interaction kernel, $K(j,k)$, encoding the rate of coagulation of clusters of sizes j and k. Exchange-driven growth (EDG) is a much less studied natural mechanism for non-equilibrium cluster growth where pairs of clusters interact by exchanging a single unit of mass (monomer) [4]. It has also been considered as a model of social phenomena like migration [5], population dynamics [6] and wealth exchange [7]. However, no rigorous mathematical results on the corresponding mean-field kinetic equations have been obtained to date. There has recently been increased mathematical interest in the properties of EDG since the corresponding kinetic equations can be obtained as scaling limits of a class of interacting particle systems, including zero-range processes [10], [11], [12], [13], and more general misanthrope processes [14], [15], [16], that have been intensively studied for a range of condensation phenomena that they exhibit. The purpose of this paper is to provide the mathematical theory on the main properties of solutions of the kinetic equations for EDG.

The main mathematical object in our version of the kinetic formulation of the EDG problem is $c_j(t)$, describing the volume fraction of the system which is occupied by clusters of size j, including $j = 0$ corresponding to the empty (available) volume fraction not occupied by clusters. As we show later, inclusion of empty volume introduces another conserved quantity in addition to total mass, and the $c_j(t)$ sum to a constant (or to 1 when normalized with rescaled time) for all times $t > 0$. This interpretation is motivated by studies on coarsening dynamics in condensing particle systems. We note that, the formulation of the EDG problem including empty volume or clusters of 'size' 0 is based on a fundamentally different motivation than the approach of physicists which does not include volume. The two approaches are related and our results directly translate to this classical interpretation, as we will discuss in detail in the conclusion.

Date: October 19, 2017.
Key words and phrases. Exchange-driven growth, Aggregation.
Corresponding author email: E.esenturk.1@warwick.ac.uk.

Symbolically, the exchange process can be described in the following way. If $< j >, < k >$ denote the non-zero clusters of sizes $j, k > 0$, then the rule of interaction is

$$< j > \oplus < k > \to < j \pm 1 > \oplus < k \mp 1 > .$$

If, one of the clusters is a zero-cluster (0-cluster), then the rule is given by

$$< j > \oplus < 0 > \to < j - 1 > \oplus < 1 > .$$

If all the clusters interact uniformly, $K(j,k)c_j c_k$ denotes the rate of any cluster of size "j" exporting a single particle to a cluster of size "k". The details of such microscopic processes are coded in the function $K(j,k)$, known as the interaction kernel. In general, the processes of export and import need not be symmetric. So, generally $K(j,k) \neq K(k,j)$. But, in most natural systems the rate of these reactions are equal, and it is common to take K as a symmetric function of its arguments. Mathematically, these coupled exchange reactions can be represented by an infinite set of nonlinear ordinary differential equations (ODEs) with given initial conditions as below

$$\dot{c}_0 = c_1 \sum_{k=0}^{\infty} K(1,k) c_k - c_0 \sum_{k=1}^{\infty} K(k,0) c_k, \tag{1.1}$$

$$\dot{c}_j = c_{j+1} \sum_{k=0}^{\infty} K(j+1,k) c_k - c_j \sum_{k=0}^{\infty} K(j,k) c_k \tag{1.2}$$

$$- c_j \sum_{k=1}^{\infty} K(k,j) c_k + c_{j-1} \sum_{k=1}^{\infty} K(k,j-1) c_k, \tag{1.3}$$

$$c_j(0) = c_{j,0} \quad \{j = 0, 1, 2, ...\}. \tag{1.4}$$

In this article our main goal is to prove the fundamental properties of this infinite system of equations such as existence of global solutions, uniqueness, positivity and possible classes of cases leading to non-existence.

In order to put our work into context, we give a brief summary of other growth systems which have been extensively studied. Basic aggregation models are quite old and date back to the works of Smoluchowski [17] (1917) and Becker-Doring [18] (1935) (see [3] for other related works). Over the decades, systematic mathematical analysis of the resulting equations have been carried out [19], [20] and mathematical questions concerning existence and uniqueness of these systems have been investigated in fair generality for kernels satisfying bounds, $K(j,k) \leq C$ [21], $K(j,k) \leq C(j+k)$ [22], $K(j,k) \leq Ca(j)a(k)$ ($a(j) = o(j)$) [23].

One of the striking results of these studies was that when the interaction kernel grows fast enough, drastic changes take place in the dynamics of the problem. For instance, when the kernel is super-linear the solutions ceases to exist [20] for the Becker-Doring model, while in the Smoluchowski model, the system undergoes a phase transition and begins behaving very differently. The latter case, known as gelation [24], [26], [27] is a counter-intuitive phenomenon where some of the mass in the system "escapes" to infinity. At the same time the uniqueness of the solution is lost along with a change in scaling behavior. So, it is physically and mathematically very important to identify the regions where such strange behaviors may happen. For the exchange-driven growth problem, heuristic studies suggest [4] that for symmetric kernels of the form $K(j,k) = (jk)^{\mu}$, no gelation occurs if $\mu \leq 3/2$ (regular case). When $2 \geq \mu > 3/2$ however, gelation takes place at some finite time T_g. For, $\mu > 2$, even more strangely, gelation takes place right at the beginning at $t = 0$, known as instantaneous gelation. This behavior is significantly different from the Smoluchowski model in which ordinary gelation occurs for $1 \geq \mu > 1/2$ and post gel solutions continue to exist for $t > T_g$, while instantaneous gelation takes place for $\mu > 1$ [28], [29].

In this article, we investigate the "regular case" for the EDG problem in the sense described above. In particular, we prove rigorously, for a system with general non-symmetric kernel satisfying the bound $K(j,k) \leq Cjk$ that the solution exists globally and is unique and conserves the mass. However, if the growth of the kernel is faster, i.e., $K(j,k) \geq Cj^{\beta}$ ($\beta > 1$) then under some assumptions on the initial conditions, the solutions can be shown to be non-existent. So, in this sense the growth rate on the kernel for global existence is optimal. For symmetric kernels, the results can be extended considerably. We prove that, if $K(j,k) \leq C(j^\mu k^\nu + j^\nu k^\mu)$ $(\mu,\nu \leq 2,\ \mu+\nu \leq 3)$ then the solutions are global and mass-conserving. We also identify an intermediate regime $(\mu,\nu \leq 2,\ \mu+\nu \leq 4)$ where the solutions exist locally. We conjecture that this is the gelation regime where there is a loss of mass after a finite time (the gelation time). Beyond this regime, i.e., if $K(j,k) \geq Cj^\beta$ $(\beta > 2)$, under fair assumptions, we show that the solutions cease to exist.

To prove the existence we employ a truncation method (due to McLeod) [19], [30] which suits well to the discrete structure of the equations. The truncated finite ODE system is useful in providing basic estimates on the total mass allowing one to pass to the limit which we will prove to solve the original (infinite) ODE system. The main assumption is that initial cluster distributions decay sufficiently fast (some higher moments exist). For the symmetric kernels, we show that one can actually obtain better estimates than just bounding the total mass (which is intuitively obvious). The arguments follow by fortunate cancellations due to symmetry and use of some fundamental inequalities. For the uniqueness of solutions we provide two results for the non-symmetric and symmetric kernels. The ideas are based on controlling the difference of (supposedly distinct) solutions. Again, one needs to produce different routes of steps for the two cases (non-symmetric and symmetric kernels). The non-existence, on the other hand, is based on the idea of obtaining lower bounds to the tails of the distributions and arguing that these lead to contradictions. To prove the non-existence for the non-symmetric kernel we need to make additional assumption that the kernel selectively favors growth. For the symmetric kernel, we do not need such selectivity (and it is clearly disallowed by the symmetry). However, in that case, non-existence will take place only for fast growing kernels (faster than quadratic) as expected.

The structure of this article is as follows. In Section 2, we detail the truncation method and show some of its basic properties which hold true uniformly for arbitrarily large finite systems. We then use these preliminary results to prove, after a number of technical steps, global existence of solutions for the non-symmetric and symmetric kernels. In Section 3, we show the other important results related to the same EDG system: uniqueness, positivity and non-existence of solutions. In Section 4, we conclude the paper by discussing the relationship between our formulation of the problem and existing physics literature. We also point out possible extensions of the current work and suggest some other future research directions.

2. Existence of Solutions

We start by giving the setting of the problem and some definitions. Let $X_\mu = \{x = (x_j),\ x_j \in \mathbb{R}; \|x\|_\mu < \infty\}$ be the space of sequences equipped with the norm $\|x\|_\mu = \sum_{j=1}^\infty j^\mu x_j$ where $\mu \geq 0$. Also, let $K(\cdot,\cdot) : \mathbb{R}\times\mathbb{R} \to [0,\infty)$ be the cluster interaction kernel which we assume to be non-negative throughout. We set $K(0,j) \equiv 0$ identically.

Definition 1: We say the system has a solutions iff for all $j \geq 0$

(i) $c_j(t) : [0,\infty) \to [0,\infty)$ is continuous and $\sup_{t\in[0,\infty)} c_j(t) < \infty$

(ii) $\int_0^t \sum_{k=0}^\infty K(j,k)c_k ds < \infty$, $\int_0^t \sum_{k=1}^\infty K(k,j)c_k ds < \infty$ for all $t \in [0,T)$ $(T \leq \infty)$

(iii) $$c_j(t) = c_j(0) + \int_0^t \left(c_{j+1}\sum_{k=0}^\infty K(j+1,k)c_k - c_j \sum_{k=0}^\infty K(j,k)c_k\right) ds + \int_0^t \left(-c_j \sum_{k=1}^\infty K(k,j)c_k + c_{j-1}\sum_{k=1}^\infty K(k,j-1)c_k\right) ds \quad \{j > 0\}$$

$$\dot{c}_0(t) = c_0(0) + \int_0^t c_1 \sum_{k=0}^\infty K(1,k)c_k - c_0 \sum_{k=1}^\infty K(k,0)c_k.$$

Definition 2: For a sequence $(c_j)_{j=1}^N$, we call the quantity $M_p^N(t) = \sum_{j=0}^N j^p c_j(t)$ as the $p^{th}-$moment of the sequence. If the sequence is infinite, then we denote the $p^{th}-$moment with $M_p(t) = \sum_{j=0}^\infty j^p c_j(t)$.

Definition 3: We say that the kernel $K(j,k)$ is nearly symmetric iff $K(j,k) = K(k,j)$ for all $j,k \geq 1$.

To prove the existence, we first consider a truncated system which respects, even at the finite dimensional level, the key features of the original infinite dimensional ODE system. Then, we obtain, for the truncated system, some uniform bounds. With the help of these bounds the limit of the truncated system is shown to be well defined and is actually a solution of the original problem.

Now, consider the truncated EDG system where we cut off the equations at a finite order N (that is, setting $c_j \equiv 0$ identically for $j > N$)

$$\dot{c}_0^N = c_1^N \sum_{k=0}^{N-1} K(1,k)c_k^N - c_0^N \sum_{k=1}^{N} K(k,0)c_k^N, \tag{2.5}$$

$$\begin{aligned}\dot{c}_j^N &= c_{j+1}^N \sum_{k=0}^{N-1} K(j+1,k)c_k^N - c_j^N \sum_{k=0}^{N-1} K(j,k)c_k^N \\ &- c_j^N \sum_{k=1}^{N} K(k,j)c_k^N + c_{j-1}^N \sum_{k=1}^{N} K(k,j-1)c_k^N, \ \{1 \le j \le N-1\}\end{aligned} \tag{2.6}$$

$$\dot{c}_N^N = -c_N^N \sum_{k=0}^{N-1} K(N,k)c_k^N + c_{N-1}^N \sum_{k=1}^{N} K(k,N-1)c_k^N, \tag{2.7}$$

with the initial conditions given by

$$c_j^N(0) = c_{j,0} \ge 0, \quad \{0 \le j \le N\}. \tag{2.8}$$

The existence and uniqueness of this system comes from the standard ODE theory. It is also known that the solutions are continuously differentiable.

Next, some preliminary lemmas are in order. The first lemma below demonstrates (as a corollary) that the truncated system has two conserved quantities. The significance of this result will shortly be clear when getting the uniform estimates (in N) for the growth of cluster size distributions.

Lemma 1. *Let g_j be a sequence of non-negative real numbers. Then,*

$$\sum_{j=0}^{N} g_j \frac{dc_j}{dt} = \sum_{j=1}^{N} (g_{j-1} - g_j)c_j^N \sum_{k=0}^{N-1} K(j,k)c_k^N + \sum_{j=0}^{N-1} (-g_j + g_{j+1})c_j^N \sum_{k=1}^{N} K(k,j)c_k^N. \tag{2.9}$$

If $K(\cdot,\cdot)$ is nearly symmetric, then one has

$$\begin{aligned}\sum_{j=0}^{N} g_j \frac{dc_j}{dt} &= \sum_{j=1}^{N-1} (g_{j-1} - 2g_j + g_{j-1})c_j^N \sum_{k=1}^{N-1} K(j,k)c_k^N \\ &+ \sum_{j=1}^{N-1} \left((g_{j-1} - g_j) + (g_1 - g_0)\right) K(j,0)c_j^N c_0^N \\ &+ \sum_{j=1}^{N-1} ((g_{j+1} - g_j) + (g_{N-1} - g_N))c_j^N K(N,j)c_N^N \\ &+ ((g_{N-1} - g_N) + (g_1 - g_0))c_j^N K(N,0)c_0^N.\end{aligned} \tag{2.10}$$

Proof. Writing $\dot{c}_j^N(t)$ from (2.5)-(2.7) and taking the summation for the $g(j)\dot{c}_j^N$ and shifting the indices on the terms having c_{j+1}, c_{j-1}, we get

$$\sum_{j=0}^{N} g_j \frac{dc_j}{dt} = \sum_{j=1}^{N} g_{j-1}c_j^N \sum_{k=0}^{N-1} K(j,k)c_k^N - \sum_{j=1}^{N} g_j c_j^N \sum_{k=0}^{N-1} K(j,k)c_k^N \tag{2.11}$$

$$- \sum_{j=0}^{N-1} g_j c_j^N \sum_{k=1}^{N} K(k,j)c_k^N + \sum_{j=0}^{N-1} g_{j+1}c_j^N \sum_{k=1}^{N} K(k,j)c_k^N. \tag{2.12}$$

Collecting the 1^{st},2^{nd} and 3^{rd}, 4^{th} terms in (2.11),(2.12) together yields the first identity

$$\sum_{j=0}^{N} g_j \frac{dc_j}{dt} = \sum_{j=1}^{N} (g_{j-1} - g_j)c_j^N \sum_{k=0}^{N-1} K(j,k)c_k^N \tag{2.13}$$

$$+ \sum_{j=0}^{N-1} (g_{j+1} - g_j)c_j^N \sum_{k=1}^{N} K(k,j)c_k^N. \tag{2.14}$$

For the second identity we first split the sums in (2.13), (2.14) and recombine the terms that are alike, while accounting for the "boundary terms". Let A, B denote the sums on the right hand side of (2.13) and (2.14). Then, one has

$$\begin{aligned} A = &\sum_{j=1}^{N-1}(g_{j-1}-g_j)c_j^N\sum_{k=1}^{N-1}K(j,k)c_k^N + \sum_{j=1}^{N-1}(g_{j-1}-g_j)c_jK(j,0)c_0^N \\ &+ (g_{N-1}-g_N)c_N^N\sum_{k=1}^{N-1}K(N,k)c_k^N + (g_{N-1}-g_N)c_N^NK(N,0)c_0^N, \end{aligned}$$

$$\begin{aligned} B = &(g_1-g_0)c_0^N\sum_{k=1}^{N-1}K(k,0)c_k^N + (g_1-g_0)c_j^NK(N,0)c_0^N \\ &+ \sum_{j=1}^{N-1}(g_{j+1}-g_j)c_j^N\sum_{k=1}^{N-1}K(k,j)c_k^N + \sum_{j=1}^{N-1}(g_{j+1}-g_j)c_j^NK(N,j)c_N^N. \end{aligned}$$

Taking the sum $A+B$, rearranging the terms and using the symmetry of K yield result. □

Corollary 1. *For a general kernel K, the zeroth moment and the first moment of the truncated system (2.5)-(2.8) are conserved in time.*

Proof. By setting $g_j = 1$, we see that all the terms in the first identity of Lemma 1 cancels each other

$$\sum_{j=0}^{N}\dot{c}_j(t) = 0,$$

and hence the zeroth moment is conserved. To see that the first moment is also conserved we set $g_j = j$. Then again, by the first identity of Lemma 1 we get

$$\sum_{j=0}^{N}j\dot{c}_j(t) = \sum_{j=1}^{N}(-1)c_j^N\sum_{k=0}^{N-1}K(j,k)c_k^N + \sum_{j=0}^{N-1}(1)c_j^N\sum_{k=1}^{N}K(k,j)c_k^N = 0,$$

which gives conservation of the first moment. □

For the proofs of existence theorems, we will need bounds for the (truncated) solutions from above and below. The following lemma shows the non-negativity of solutions of the truncated system provided that the initial cluster distributions are non-negative (hence giving a lower bound).

Lemma 2. *Let $c_j^N(t)$ be a solution of the truncated system (2.5)-(2.8) where $K(j,k) \geq 0$. If $c_j^N(0) \geq 0$ for all $j \geq 0$, then $c_j^N(t) \geq 0$.*

Proof. Let $S(j,c^N) = \sum_{k=0}^{N-1}K(j,k)c_k^N$ and $\bar{S}(j,c^N) = \sum_{k=1}^{N}K(k,j)c_k^N$. Then the system (2.5)-(2.8) can be written as

$$\frac{dc_0^N}{dt} + \bar{S}(0,c^N)c_0^N = S(1,c^N)c_1^N,$$

$$\frac{dc_j^N}{dt} + (S(j,c^N)+\bar{S}(j,c^N))c_j = c_{j+1}^N(t)S(j+1,c^N) + c_{j-1}^N(t)\bar{S}(j-1,c^N) \quad \{j \geq 1\}. \tag{2.15}$$

Now, if the assertion in the theorem were not true, then there would be a very first time $t_0 \in [0,\tau)$ and some $i \in \mathbb{N}$, such that $c_i^N(t_0) = 0$ and $(c_i^N)'(t_0) < 0$. Suppose $i > 0$ (similar argument can be repeated if $i = 0$). Then for the left hand side of (2.15) we have

$$\frac{dc_i^N(t_0)}{dt} + (S(i,c^N(t_0)) + \bar{S}(i,c^N(t_0)))c_i^N(t_0) < 0. \tag{2.16}$$

However, the right hand side of (2.15) gives

$$c_{i+1}^N(t_0)S(i+1,c^N(t_0)) + c_{i-1}^N(t_0)\bar{S}(i-1,c^N(t_0)) \geq 0 \tag{2.17}$$

since $c_j(t_0) \geq 0$. But this contradicts with (2.16). Hence we have $c_j(t) \geq 0$ for all j and t. □

Now, we state and prove the main theorems of this section. We provide two different versions of the existence theorems for each of the non-symmetric and nearly symmetric kernel cases. As the assumptions of the theorems are different, the results do not imply each other. In the first version, we demand more on the moments of the initial cluster distribution. This was the approach taken in [22] for the Smoluchowski equation. In the second version we demand more on the growth of the kernel.

In the sequel, we denote, by $C \geq 0$, a dummy constant which may take different values at different steps.

Theorem 1. *Consider the EDG system given by (1.2)-(1.4). Let $K(j,k)$ be a general kernel satisfying $K(j,k) \leq Cjk$ for large enough j,k. Assume further that $M_p(0) = \sum_{k=0}^{\infty} j^p c_j(0) < \infty$ for some $p > 1$. Then the infinite system (1.2)-(1.4) has a global solution $(c_j) \in X_1$.*

Proof. The key ingredient of the proof is the constancy of the zeroth and first moment of the truncated system $M_1^N(t)$. This then will imply that $c_j^N(t)$ and $\dot{c}_j^N(t)$ and are bounded uniformly. Indeed, since $c_j^N(t)$ are non-negative, the bound on the zeroth moment

$$\sum_{j=0}^{N} c_j^N(t) = \sum_{j=0}^{N} c_j^N(0) \leq \sum_{j=0}^{\infty} c_j(0) = M_0(0)$$

yields $c_j^N(t) \leq M_0(0)$ for all N and $j \geq 0$. Similarly, for the derivatives, we have (when $j \geq 1$)

$$\begin{aligned}
\left|\dot{c}_j^N(t)\right| &\leq \sum_{k=0}^{N-1} c_{j+1}^N K(j+1,k) c_k^N + c_j^N \sum_{k=0}^{N-1} K(j,k) c_k^N \\
&\quad + \sum_{k=1}^{N} K(k,j) c_k^N c_j^N + \sum_{k=1}^{N} K(k,j-1) c_k^N c_{j-1}^N \\
&\leq C \sum_{k=0}^{N} jk c_j^N c_k^N \leq CM_1(0)^2.
\end{aligned}$$

where, to get to the third line, we simply shifted the "j" indices and used the bound on $K(j,k)$. Similarly we can show $\left|\dot{c}_0^N(t)\right| \leq CM_1(0)$. Hence the sequence (c_j^N) is uniformly bounded and equicontinuous. Then by Arzela-Ascoli theorem there is a subsequence $\{c_j^{N(i)}\}$ which converges uniformly to a continuous function, say $c_j(t)$. Let us denote the subsequence $N(i)$ also with N for brevity. To show that $c_j(t)$ is a solution to the original problem we need to show the series $\sum_{j=1}^{N} K(j,k) c_k^N$ converges uniformly on bounded intervals of time $[0,T]$. To prove this, we need the boundedness of a higher moment. Let $g(s) = s^p$ for some $1 < p \leq 2$ without loss of generality. By the mean value theorem $j^p - (j-1)^p = p(j-\theta_1)^{p-1}$ and $(j+1)^p - j^p = p(j+\theta_2)^{p-1}$ for some $0 < \theta_1, \theta_2 < 1$. Then, from the first identity in Lemma 1

$$\begin{aligned}
\dot{M}_p^N(t) &= \sum_{j=0}^{N} j^p \dot{c}_j(t) = \sum_{j=1}^{N} p(j-\theta_1)^{p-1} c_j^N \sum_{k=0}^{N-1} K(j,k) c_k^N + \sum_{j=0}^{N-1} p(j+\theta_2)^{p-1} c_j^N \sum_{k=1}^{N} K(k,j) c_k^N \\
&\leq C \sum_{j=1}^{N} p j^{p-1} j c_j^N \sum_{k=0}^{N-1} k c_k^N + C \sum_{j=1}^{N-1} p j (j+1)^{p-1} c_j^N \sum_{k=1}^{N} k c_k^N \leq CM_p^N(t) M_1(0).
\end{aligned}$$

Hence one has $M_p^N(t) \leq M_p^N(0) e^{Ct} \leq M_p(0) e^{Ct}$ by Gronwall inequality. Now, $\sum_{j=1}^{N-1} K(j,k) c_k^N$ converges uniformly to $\sum_{j=1}^{\infty} K(j,k) c_k$. To see this we observe

$$\left|\sum_{k=1}^{\infty} K(j,k) c_k^N - \sum_{k=1}^{\infty} K(j,k) c_k\right| \leq \sum_{k=1}^{N_2} K(j,k) \left|c_k^N - c_k\right| + \left|\sum_{k=N_2+1}^{\infty} K(j,k)(c_k + c_k^N)\right|. \tag{2.18}$$

In the limit, the second term on the right hand side of (2.18) can be made arbitrarily small for N_2 large enough since

$$\left|\sum_{k=N_2+1}^{\infty} K(j,k)(c_k + c_k^N)\right| \leq 2Cj \sum_{k=N_2+1}^{\infty} k k^{-p} k^p (c_k + c_k^N) \leq CjN_2^{1-p} M_p^N(t).$$

The first term on the right hand side of (2.18) can be made arbitrarily small be letting N become large. Hence $\sum_{k=1}^{\infty} K(j,k)c_k^N$ converges uniformly. Similarly, $\sum_{k=1}^{N} K(k,j)c_k^N$ also converges uniformly. Now, if we write the truncated system in the integral form

$$c_j^N(t) = c_j^N(0) + \int_0^t c_{j+1}^N(s) \sum_{k=0}^{N-1} K(j+1,k)c_k^N(s) - \int_0^t c_j^N \sum_{k=0}^{N-1} K(j,k)c_k^N(s)ds \tag{2.19}$$
$$- \int_0^t c_j^N(s) \sum_{k=1}^{N} K(k,j)c_k^N(s) + \int_0^t c_{j-1}^N(s) \sum_{k=1}^{N} K(k,j-1)c_k^N(s)ds$$

we see that we can pass to the limit $N \to \infty$, on the right hand side, under the integral sign since the functions $c_j^N(t)$ and $\sum_{k=1}^{N-1} K(j,k)c_k^N$ converge uniformly. This shows that c_j as the limit, is a solution of the system (1.2)-(1.4). □

From the construction in the above theorem, considering the integral form of the equations, it is immediate that the limit solution $c_j(t)$ is differentiable due to the uniform convergence of c_j^N and the sums involved. We also note that, under the conditions of Theorem 1, with the boundedness of the higher moments, i.e., $M_p^N(t) < C(c(0),t) < \infty$ for $p > 1$, the approximate (truncated) solutions converge strongly to the limit function, i.e., $\lim_{i\to\infty} \left\| c_j^{N(i)}(t) - c_j(t) \right\|_\mu \to 0$ for $\mu < p$. In particular, we have the following corollary as a consequence.

Corollary 2. *Let c_j be the solution of the (1.2)-(1.4) under the conditions of Theorem 1 for some $p > 1$. Then $c_j(t)$ is continuously differentiable. Moreover $M_p(t) < \infty$ and*

$$\sum_0^\infty c_j(t) = \sum_0^\infty c_j(0),$$
$$\sum_0^\infty jc_j(t) = \sum_0^\infty jc_j(0).$$

If the kernel K is nearly symmetric, by some further cancellations and use of a simple induction argument together with a fundamental inequality, we can prove a stronger result for exponents satisfying $\mu + \nu \leq 3$.

Theorem 2. *Consider the infinite EDG system (1.2)-(1.4). Let $K(j,k)$ be nearly symmetric and satisfy $K(j,k) \leq C(j^\mu k^\nu + j^\nu k^\mu)$ for j,k large and $\mu+\nu \leq 3$, $\mu,\nu \leq 2$. Assume also that $M_p(0) < \infty$ for some $p > 2$. Then the system (1.2)-(1.4) has a global solution $(c_j) \in X_2$.*

Proof. The general idea of the proof is similar to the previous one. However, we now allow faster growth on K and therefore, boundedness of $M_1(t)$ is not sufficient. We need estimates on the higher moments which will be done by bounding uniformly the higher moments of the truncated system. To see this, we use the second identity in the Lemma 1.

Let us first show that the second moment of the truncated system is uniformly bounded. We first observe that, in Lemma 1, the second line of (2.10) is non-positive. Indeed, choosing $g_j = j^2$ we have, for $1 \leq j \leq N-1$,

$$(g_{j-1} - g_j) + (g_1 - g_0) = -2j + 2 \leq 0$$

Similarly, the third and fourth lines of (2.10) are also non-positive since $j \leq N-1$, giving

$$(g_{j+1} - g_j) + (g_{N-1} - g_N) = 2j + 1 - 2N + 1 \leq 0,$$
$$(g_1 - g_0) - (g_N - g_{N-1}) = 1 - (2N-1) \leq 0.$$

Then we have the following inequality for $M_2^N(t)$

$$\sum_0^N j^2 \dot{c}_j^N(t) \leq \sum_{j=0}^{N-1} ((j+1)^2 - 2j^2 + (j-1)^2)c_j^N \sum_{k=0}^{N-1} K(j,k)c_k^N \tag{2.20}$$

$$\leq 2C \sum_{j=0}^{N-1} \sum_{k=0}^{N-1} (j^\mu k^\nu + j^\nu k^\mu)c_j^N c_k^N. \tag{2.21}$$

Now since $\mu+\nu\le 3$ for the exponents in (2.21), there exists $\bar\mu\ge\mu$ and $\bar\nu\ge\nu$ such that $\bar\mu+\bar\nu=3$. Now, by Young's inequality we have

$$j^{\bar\mu}k^{\bar\nu}\le\left(\frac{2\bar\mu-\bar\nu}{3}j^2k+\frac{2\bar\nu-\bar\mu}{3}jk^2\right). \tag{2.22}$$

Then (2.22) and inequality (2.21) together give

$$\begin{aligned}\dot M_2^N(t)&\le C\sum_{j=0}^{N-1}\sum_{k=0}^{N-1}(j^{\bar\mu}k^{\bar\nu}+j^{\bar\nu}k^{\bar\mu})c_j^Nc_k^N\le C\sum_{j=0}^{N-1}\sum_{k=0}^{N-1}(j^2k+jk^2)c_j^Nc_k^N\\&\le CM_2^N(t)M_1^N(t)\le CM_2^N(t),\end{aligned}$$

from which we deduce, by Gronwall's inequality,

$$M_2^N(t)\le M_2(0)e^{Ct},$$

which is a uniform bound for all N. Then arguing as in Theorem 1 we find a subsequence $c_j^{N(i)}(t)$ which converges uniformly to $c_j(t)$. However, to prove that c_j is a solution in the sense of Definition 1 we need boundedness of higher moments, i.e., $M_p(t)<\infty$ for some $p>2$. But, this now can be achieved using the boundedness of $M_2(t)$ which we just have proved. Indeed, let $g_j=j^p$ and take, without loss of generality, $2<p<3$ where $M_p(0)<\infty$. Then, by the mean value theorem we see

$$(g_{j+1}-g_j)-(g_N-g_{N-1})=\frac{3}{2}(j+\theta_1)^{p-1}-(N-1+\theta_2)^{p-1}\le 0.$$

Similarly $(g_{j-1}-g_j)+(g_1-g_0)\le 0$ and $(g_1-g_0)-(g_N-g_{N-1})\le 0$, Hence, by Lemma 1, we have

$$\sum_0^N j^p\dot c_j^N(t)\le\sum_{j=0}^{N-1}((j+1)^p-2j^p+(j-1)^p)c_j^N\sum_{k=0}^{N-1}K(j,k)c_k^N.$$

Expanding the function $g(s)=s^p$ around $s=j$ in Taylor series up to second order gives $|(j+1)^p-2j^p+(j-1)^p|\le Cj^{p-2}$ and hence

$$\sum_0^N j^p\dot c_j^N(t)\le C\sum_{j=0}^{N-1}\sum_{k=0}^{N-1}j^{p-2}(j^\mu k^\nu+j^\nu k^\mu)c_j^N(t)c_k^N(t)\le C\sum_{j=0}^{N-1}\sum_{k=0}^{N-1}j^{p-2}(j^2k+jk^2)c_j^N(t)c_k^N(t)$$

In the second step above we again used Young's inequality. Taking the sums on the furthest right yields

$$\sum_0^N j^p\dot c_j^N(t)\le C(M_p^N(t)M_1+M_{p-1}^N(t)M_2(t))\le C(M_1+M_2(t))M_p^N(t)$$

which, by another use of Gronwall inequality, gives the bound $M_p^N(t)\le M_p(0)e^{\int_0^t C(M_1+M_2(s))ds}$. Repeating the arguments in Theorem 1 proves that $c_j(t)$ is indeed a solution. □

Remark 1. *The growth assumption $K(j,k)\le C(j^\mu k^\nu+j^\nu k^\mu)$ ($\mu+\nu\le 3$, $\mu,\nu\le 2$) in the theorem was crucial to get the global existence. This is in accordance with the physical studies which derived regular growth for the same regime assuming very specific type of kernels. For general symmetric kernels growing faster than the aforementioned rates we can only prove local existence of solutions as shown in the following corollary.*

Corollary 3. *Consider the infinite EDG system (1.2)-(1.4). Let $K(j,k)$ be nearly symmetric and satisfy $K(j,k)\le Cj^2k^2$ (for j,k large) and $M_p(0)<\infty$ for some $p>2$. Then the system (1.2)-(1.4) has a local solution $(c_j)\in X_2$.*

Proof. The proof takes similar steps to Theorem 2. Indeed, under the assumption $K(j,k)\le j^2k^2$, we again consider $\dot M_2^N(t)$

$$\dot M_2^N(t)=\sum_0^N j^2\dot c_j^N(t)\le 2C\sum_{j=0}^{N-1}\sum_{k=0}^{N-1}j^2k^2c_j^Nc_k^N\le 2C\left(M_2^N(t)\right)^2. \tag{2.23}$$

Hence we obtain

$$M_2^N(t) \leq \frac{1}{\frac{1}{M_2^N(0)} - 2Ct} \leq \frac{1}{\frac{1}{M_2(0)} - 2Ct} \text{ for } t < 1/(2M_2(0)C),$$

a uniform bound which is valid up to some certain finite time. This nevertheless allows us to construct a subsequence $c_j^{N(i)}$, as before, which converges uniformly to a limit function $c_j(t)$. After that, using the arguments in Theorem 2, we can get a bound for $M_p(t)$ (valid up to a finite time T) and show that the partial sums in the truncated system converges uniformly up to time T which proves the existence of local solutions. □

Theorem 2 and Corollary 3 give us signs of an intermediate regime where the solutions behave differently. Previous heuristic studies with special kernels of the form $K(j,k) = j^\mu k^\nu + j^\nu k^\mu$ suggest that $\mu + \nu = 3$ is the critical line for the onset of finite time gelation. So, in light of the previous theorems we can make the following conjecture.

Conjecture: *Consider the infinite EDG system (1.2)-(1.4). Let the nearly symmetric kernel satisfy $K(j,k) \geq Cj^\mu k^\nu$, $\mu + \nu > 3$. Then gelation occurs in finite time.*

The previous two theorems crucially made use of the boundedness of the initial moments. We can relax this assumption by sacrificing on the growth rate of K. This was the approach taken by [23] for the Smoluchowski equation. More precisely, if we assume, for the non-symmetric kernel, the growth rate

$$(I) : K(j,k) \leq a(j)b(k) \quad \text{with } a(j), b(j) = o(j)$$

then we have the following.

Theorem 3. *Consider the EDG system given by (1.2)-(1.4). Let $K(j,k)$ be a general kernel satisfying the growth condition (I) above. Suppose that the system has finite initial total mass. Then the infinite system (1.2)-(1.4) has a global solution $(c_j) \in X_1$.*

Proof. As in Theorem 1, we can use the boundedness of the zeroth and first moments of the truncated system (for any N) to construct a sequence of solutions that converge uniformly to a continuous function on bounded time intervals $[0,T]$. However, to show that this is the desired solution, we also need prove that $\sum_{k=0}^{N-1} K(j,k)c_k^N \rightrightarrows \sum_{k=0}^{\infty} K(j,k)c_k$ as $N \to \infty$. This can be shown using the growth rate of the kernel assumed in the theorem

$$\left| \sum_{j=1}^{N-1} K(j,k)c_k^N - \sum_{j=1}^{\infty} K(j,k)c_k \right| = \left| \sum_{j=1}^{N_2} K(j,k)c_k^N - \sum_{j=1}^{N_2} K(j,k)c_k \right| + \left| \sum_{j=N_2+1}^{\infty} K(j,k)(c_k + c_k^N) \right|.$$

Now, the second term can be made arbitrarily small since the growth rate of K is slower than the decay of c_k, i.e., for large enough N

$$\left| \sum_{j=N_2+1}^{\infty} K(j,k)(c_k + c_k^N) \right| \leq 2a(j)\frac{b(N_2)}{N_2}M_1(0) \to 0.$$

The first term can be made as small as desired by letting N grow (since $c_k^N \to c_k$). Repeating the arguments of Theorem 1 completes the proof. □

We can prove a similar version of the above theorem for the symmetric kernels assuming

$$(II) : K(j,k) \leq a(j^\mu)a(k^\nu) + a(j^\nu)a(k^\mu),\ a(j) = o(j) \tag{2.24}$$

Theorem 4. *Consider the EDG system given by (1.2)-(1.4). Let $K(j,k)$ be a nearly symmetric kernel satisfying the growth condition (II) with $\mu, \nu \leq 2$ and $\mu + \nu \leq 3$. Suppose for the initial distribution that $M_2(0) < \infty$. Then the infinite system (1.2)-(1.4) has a global solution.*

Proof. The proof follows steps similar to Theorem 2. The difference is that, now, we only have $M_2(0) < \infty$ for the initial distribution. Since condition (II) holds, by Young's inequality one can show, as in Theorem 2, $M_2(t) < \infty$ on any interval $(T < \infty)$ which allows us to construct sequences of functions $c_j^N(t)$ which converge uniformly to some function $c_j(t)$ which is continuous. To prove that $c_j(t)$ have the desired properties as a solution, it is sufficient to show, arguing as in the Theorem 3, that $\sum_{j=N}^{\infty} K(j,k)(c_k + c_k^N)$ vanishes as $N \to \infty$.

Indeed, it is clear that, for any $j \in \mathbb{N}$, we have the bound $a(j) \leq Cj$. Now consider the case $\mu, \nu \geq 1$ (the other cases can be done similarly). Since $a(j) = o(j)$, for given $1 > \varepsilon > 0$ arbitrarily small, we can choose $N_2 \in \mathbb{N}$ large enough that, on the interval $[0, T]$, one has $\frac{a(k)}{k} < \varepsilon$ for $k > N_2$. Then, we find

$$\begin{aligned}\left|\sum_{j=N_2}^{\infty} K(j,k)(c_k^N + c_k)\right| &\leq C \left|\sum_{j=N_2}^{\infty} \left(a(j^\mu)a(k^\nu) + a(j^\nu)a(k^\mu)\right)(c_k^N + c_k)\right| \\ &\leq C \left|\sum_{j=N_2}^{\infty} (\varepsilon j^2 k + \varepsilon j k^2)(c_k^N + c_k)\right| \\ &\leq 2C\varepsilon \sup_{t\in[0,T]} M_2(t) M_1.\end{aligned}$$

where again we used Young's inequality in the second line. Following the steps of the previous theorems finishes the proof. ☐

3. UNIQUENESS, POSITIVITY AND NON-EXISTENCE

Although the truncated system (2.5)-(2.8) has a unique solution by the general ODE theory, the method of proof of existence we used in the previous section does not guarantee uniqueness of the infinite system as there may be many subsequences of c_j^N which converges to different limit functions. Hence, uniqueness has to be analyzed separately.

We provide two uniqueness results. Our first uniqueness result is for systems with non-symmetric kernel. The idea is to control the "absolute" value of the differences of two solutions, say c_j and d_j, and show that $c_j(t) = d_j(t)$ identically. The tricky part is the non-linear terms which are of different signs.

Theorem 5. *Consider the infinite ODE system (1.2)-(1.4). Let the conditions of Theorem 1 (or Theorem 3) be satisfied with $M_2(0) < \infty$. Then there is exactly one solution.*

Proof. We show the proof under the conditions of Theorem 1. The case for Theorem 3 can be done in a similar way. Let $c_j(t)$ and $d_j(t)$ be two different solutions with the same initial conditions. Let $e_j(t) = c_j(t) - d_j(t)$. Define $M_e(t) := \frac{1}{2}\sum_{j=1}^{\infty} e_j(t)^2$. Consider the partial sum $T_N(t) = \frac{1}{2}\sum_{j=1}^{N} c_j(t)^2$. Clearly, $\lim_{N\to\infty}\sum_{j=1}^{N} c_j(t)^2$ is finite. This is because $\sum_{j=0}^{\infty} c_j(t)$ being finite implies, for N large enough, that $c_j < 1$ when $j > N$ and hence $c_j^2 < c_j$. Also, the derivative, $\dot{T}_N(t)$ converges uniformly. To show this, take arbitrary $\varepsilon > 0$. We can choose a number N_2 large enough that $\left|\sum_{j=N_2}^{\infty} e_j \dot{e}_j(t)\right| < \varepsilon$. Indeed, writing $\dot{e}_j(t)$ from the rate equations and noting $c_j c_k - d_j d_k = c_j e_k + e_j d_k$ we have

$$\begin{aligned}\left|\sum_{j=N_2}^{\infty} e_j \dot{e}_j(t)\right| \leq& \sum_{j=N_2}^{\infty} |e_j| \sum_{k=0}^{\infty} K(j,k)\, |c_{j+1}e_k + e_{j+1}d_k| + \sum_{j=N_2}^{\infty} |e_j| \sum_{k=0}^{\infty} K(j,k)\, |c_j e_k + e_j d_k| \\ &+ \sum_{j=N_2}^{\infty} |e_j| \sum_{k=1}^{\infty} K(k,j)\, |c_j e_k + e_j d_k| + \sum_{j=N_2}^{\infty} |e_j| \sum_{k=1}^{\infty} K(k,j-1)\, |c_{j-1}e_k + e_{j-1}d_k|\,.\end{aligned} \tag{3.25}$$

The four double-sums on the right hand side of (3.25) can each be made arbitrarily small. We show this for the first double-sum. The others are similar. Let $A(N) = \sum_{j=N}^{\infty} |e_j| \sum_{k=0}^{\infty} K(j+1,k)\, |c_{j+1}e_k + e_{j+1}d_k|$. Splitting the sum for the ˜$c_j e_k$ and ˜$e_j d_k$ terms and noting that $|e_j| \leq c_j + d_j \leq C$ and $K(j,k) \leq Cjk$ we get

$$A(N_2) = \sum_{j=N_2}^{\infty} |e_j| \sum_{k=0}^{\infty} K(j+1,k)\, |c_{j+1}e_k| + \sum_{j=N_2}^{\infty} |e_j| \sum_{k=0}^{\infty} K(j+1,k)\, |e_{j+1}d_k|$$

$$\leq 2C \sum_{j=N_2}^{\infty} |e_j|\,(j+1)c_{j+1}M_1 + C \sum_{j=N_2}^{\infty} |e_j|\,(j+1)\,|e_{j+1}|\,M_1$$
$$\leq C \sum_{j=N_2}^{\infty} (j+1)c_{j+1} + C \sum_{j=N_2}^{\infty} (j+1)\,|e_{j+1}| \leq C \sum_{j=N_2+1}^{\infty} j(c_j+d_j),$$

where we used $\sum_{k=1}^{\infty} |e_k|\, k \leq \sum_{k=0}^{\infty} (c_k+d_k)k \leq CM_1(t)$ in the second line and $|e_j| \leq c_j + d_j \leq C$ in the third line. Hence, by choosing N_2 large enough we get $\left|\sum_{N_2}^{\infty} e_j\dot{e}_j(t)\right| < \varepsilon$.

Now, we differentiate $M_e(t)$ and obtain four terms as in equation (3.25). It is again sufficient to show the algebra for the first one since the other terms are similar. By the bound on K one gets

$$A(1) \leq C\sum_{j=0}^{\infty} |e_j|\,(j+1)c_{j+1}M_1 + C\sum_{j=0}^{\infty} |e_j|\,(j+1)(c_{j+1}+d_{j+1})M_1$$
$$\leq C\sum_{j=1}^{\infty} e_j(j+1)(c_{j+1}+d_{j+1})M_1$$

Using the Cauchy-Schwarz inequality we obtain

$$A \leq C\left(\sum_{j=1}^{\infty} e_j^2\right)^{1/2}\left(\sum_{j=1}^{\infty} j^2(c_j+d_j)^2\right)^{1/2}$$

The other double-summed terms in (3.27) can be estimated in a similar way. Then, since $\sum_{j=1}^{\infty} j^2(c_j+d_j)^2 \leq CM_2(t)$, altogether we get

$$\dot{M}_{e_1}(t) \leq C\,(M_{e_1}(t))^{1/2}\,(M_2(t))^{1/2}\,.$$

By the assumption in the theorem and arguments in Theorem 1 $M_2(t)$ is bounded on any finite interval. Then, solving the differential inequality yields

$$M_{e_2}(t) \leq C\,(M_{e_2}(0))^2\,.$$

Hence $e_j(t) = 0$ for $j \geq 1$. To complete the proof we also need to show $c_0(t) = d_0(t)$. But this follows immediately from the corollary of Theorem 1. Indeed, by $c_j(0) = d_j(0)$ and the conservation of zeroth moment we have

$$\sum_{k=1}^{\infty} c_j(t) = \sum_{k=1}^{\infty} c_j(0) = \sum_{k=1}^{\infty} d_j(t).$$

Then, since $c_j(t) = d_j(t)$ for $j \geq 1$ as shown just above we necessarily have $c_0(t) = d_0(t)$ proving uniqueness. ☐

Our second result in this section addresses the uniqueness of solutions for kernels with faster growth. The proof is based on a similar idea. However, we will need the following lemma which is important in its own right.

Lemma 3. *Consider the EDG system (1.2)-(1.4). Let $K(j,k)$ satisfy the conditions of Theorem 2. Let $p > 1$ be an arbitrary integer. and assume that $M_p(0) < \infty$. Then $M_p(t) < \infty$ on all finite intervals $[0,T]$ ($T < \infty$).*

Proof. The proof is by induction. By Theorem 2, we have $M_2(t) < \infty$. Let the assertion be true for $p-1$, i.e., $M_{p-1}(t) = \sum_{j=1}^{\infty} j^{p-1}c_j(t) < \infty$. Consider the derivative of the finite sum

$$\sum_{j=1}^{N} j^p\dot{c}_j(t) = \sum_{j=1}^{N} j^p \sum_{k=0}^{\infty} K(j+1,k)c_{j+1}c_k - \sum_{j=1}^{N} j^p \sum_{k=0}^{\infty} K(j,k)c_jc_k$$
$$\sum_{j=1}^{N} j^p \sum_{k=1}^{\infty} K(k,j)c_jc_k + \sum_{j=1}^{N} j^p \sum_{k=1}^{\infty} K(k,j-1)c_{j-1}c_k.$$

Note that this is different from the truncated system that we discussed in Section 2. By shifting the indices of the first and fourth terms on the right hand side we get

$$\sum_{j=1}^{N} j^p \dot{c}_j(t) = \sum_{j=2}^{N}(j-1)^p \sum_{k=0}^{\infty} K(j,k)c_jc_k - \sum_{j=1}^{N} j^p \sum_{k=0}^{\infty} K(j,k)c_jc_k$$
$$- \sum_{j=1}^{N} j^p \sum_{k=1}^{\infty} K(k,j)c_jc_k + \sum_{j=0}^{N}(j+1)^p \sum_{k=1}^{\infty} K(k,j)c_jc_k.$$

Rearranging the terms and using that $K(j,k)$ is nearly symmetric yields

$$\sum_{j=1}^{N} j^p \dot{c}_j(t) = \sum_{j=1}^{N} ((j-1)^p - 2j^p + (j+1)^p) \sum_{k=1}^{\infty} K(j,k)c_jc_k \tag{3.26}$$
$$+ \sum_{j=1}^{\infty}((j-1)^p - j^p)K(j,0)c_jc_0 + \sum_{k=1}^{\infty} K(k,0)c_0c_k$$

Since $(j-1)^p - j^p + 1 \leq 0$, the second line of (1.2) is non-positive and we get

$$\sum_{j=1}^{N} j^p \dot{c}_j(t) \leq \sum_{j=1}^{N} ((j-1)^p - 2j^p + (j+1)^p) \sum_{k=1}^{\infty} K(j,k)c_jc_k$$
$$\leq C \sum_{j=1}^{N} j^{p-2} \sum_{k=1}^{\infty} (j^\mu k^\nu + j^\nu k^\mu)c_jc_k$$
$$\leq C \sum_{j=1}^{N} j^{p-2} \sum_{k=1}^{\infty} (jk^2 + j^2k)c_jc_k,$$

where we used Taylor expansion in the second line and Young's inequality in the third line. Now, taking the limit $N \to \infty$ we arrive at

$$\dot{M}_p(t) \leq C(M_{p-1}(t)M_2(t) + M_p(t)M_1).$$

Since $M_{p-1}(t) < \infty$ by the induction hypothesis, then applying a Gronwall type lemma we conclude $M_p(t) < \infty$, completing the proof. □

Theorem 6. *Let $c_j(t)$ be the solution of (1.2)-(1.4) as in Theorem 2 with the additional assumption $M_4(0) < \infty$. Then there is exactly one solution to this system.*

Proof. Let $c_j(t)$ and $d_j(t)$ be two different solutions with the same initial conditions and let $e_j(t) = c_j(t) - d_j(t)$. Consider again the series $M_e(t) = \frac{1}{2}\sum_{j=1}^{\infty} e_j(t)^2 < \infty$. By the differentiability of $c_j(t)$ and uniform convergence of partial sums the series can be differentiated term by term. Indeed, let $\varepsilon > 0$ be given and the conditions of Theorem 2 be satisfied ($K(j,k) \leq C(j^\mu k^\nu + j^\nu k^\mu)$ in particular). Then, by use of Young's inequality and doing some algebra we find

$$\left|\sum_{j=N}^{\infty} e_j \dot{e}_j(t)\right| \leq C \sum_{j=N_2}^{\infty} |e_j| \left(|c_{j+1}| + |e_{j+1}|\right)((j+1)^2 M_1 + (j+1)M_2(t)) \tag{3.27}$$
$$+ 2C \sum_{j=N_2}^{\infty} |e_j| \left(|c_j| + |e_j|\right)(j^2 M_1 + jM_2(t))(|c_j| + |e_j|)$$
$$+ C \sum_{j=N_2}^{\infty} |e_j| \, |e_j| \left(|c_{j-1}| + |e_{j-1}|\right)((j-1)^2 M_1 + (j-1)M_2(t))$$
$$\leq C(M_1(t) + M_2(t)) \sum_{j=N_2-1}^{\infty} (c_j + d_j)(j^2 + j),$$

where again we used $|e_j| \leq c_j + d_j \leq C$ and $\sum_{k=1}^{\infty} |e_k| k^2 \leq \sum_{k=1}^{\infty} (c_k + d_k)k^2 \leq CM_2(t)$. Since $M_2(t)$ is bounded on any finite interval $[0,T]$, it is clear that $\sum_{j=N_2-1}^{\infty}(c_j + d_j)(j^2 + j)$

can be made less than ε by choosing N_2 large. Hence, the $\sum_{j=1}^{N} e_j\dot{e}_j(t)$ converge uniformly and $\sum_{j=0}^{\infty} e_j(t)^2$ can be differentiated term by term.

Now, we take the derivative of $M_e(t)$ and estimate it in a similar way to the right hand side of (3.27). It is enough to estimate $A = \sum_{j=1}^{\infty} |e_j|\,(j^2+j)(|c_j|+|d_j|)(M_1(t)+M_2(t))$ as the other terms are similar. Using the Cauchy-Schwarz inequality we obtain

$$A \le C\left(\sum_{j=1}^{\infty} e_j^2\right)^{1/2}\left(\sum_{j=1}^{\infty}(j^2+j)^2(|c_j|+|d_j|)^2\right)^{1/2}(M_1+M_2(t)).$$

Since $\sum_{j=1}^{\infty}(j^2+j)^2(|c_j|+|d_j|)^2 \le CM_4(t)$ we can write

$$A \le C\left(\sum_{j=1}^{\infty} j^2 e_j^2\right)^{1/2}(M_1+M_2(t))M_4(t).$$

The other double-summed terms can be estimated in a similar way. Then, altogether we get

$$\dot{M}_{e_2}(t) \le C\left(M_{e_2}(t)\right)^{1/2} M_4(t)(M_1+M_2(t)).$$

Since, by the previous lemma, $M_4(t)$ is bounded on all bounded intervals, solving the differential inequality yields

$$M_{e_2}(t) \le C\left(M_{e_2}(0)\right)^2.$$

Hence $e_j(t)=0$ implying $c_j(t)=d_j(t)$ for $j\ge 1$. Arguing as in the previous theorem, by the conservation of $M_0(t)$, we also have $c_0(t)=d_0(t)$ completing the proof. □

Next we address another important property of the solutions: positivity, which is not apparent from the equations as $\dot{c}_j$ terms have both, positively and negatively signed terms. The next result guarantees this.

Theorem 7. *Let $c_j(t)$ be a solution of (1.2)-(1.4) as in Theorem 1 (or Theorem 2). Suppose that $c_j(0)>0$. Then, $c_j(t)>0$ for all $t>0$.*

Proof. Let $S(j,c)=\sum_{k=0}^{\infty}K(j,k)c_k$, $\bar{S}(j,c)=\sum_{k=1}^{\infty}K(k,j)c_k$. Arguing as in Lemma 2, since $c_j(0)\ge 0$, we can easily show that $c_j(t)\ge 0$. To strengthen the result, we rearrange the rate equations and multiply the terms by the appropriate integrating factor to get

$$\frac{d}{dt}\left[c_0(t)e^{\int_0^t \bar{S}(0,c(s))ds}\right] = c_1(t)S(1,c)e^{\int_0^t \bar{S}(0,c(s))ds},$$

$$\frac{d}{dt}\left[c_j(t)e^{\int_0^t (S(j,c(s))+\bar{S}(j,c(s)))ds}\right] = (c_{j+1}(t)S(j+1,c)+c_{j-1}(t)\bar{S}(j-1,c))e^{\int_0^t (S(j,c(s))+\bar{S}(j,c(s)))ds}.$$

The operations on the left hand side are allowed since $S(j,c)$ and $\bar{S}(j,c)$ are continuous by uniform convergence. Integrating this equation we get

$$\begin{aligned} c_j(t)e^{\int_0^t (S(j,c(s))+\bar{S}(j,c(s)))ds} = c_j(0) + \int_0^t & c_{j+1}(\tau)S(j+1,c)e^{\int_0^\tau (S(j,c(s))+\bar{S}(j,c(s)))ds}d\tau \\ & + c_{j-1}(\tau)\bar{S}(j-1,c)e^{\int_0^\tau (S(j,c(s))+\bar{S}(j,c(s)))ds}d\tau. \end{aligned}$$

from which it follows that if $c_j(t)>0$ for all j since the integrals on the right hand side are non-negative. □

Our final results concern the non-existence of solutions. It has been known [28] and in some cases has been rigorously shown, that, for the Smoluchowski and Becker-Doring type models, super-linearly growing kernels may lead to non-existence [20], [29].

In EDG systems, we showed in the previous section that global solutions exist for non-symmetric kernels satisfying $K(j,k)\le Cjk$ and local solutions persist for nearly symmetric kernels satisfying $K(j,k)\le Cj^2k^2$. For specific kernels of the form $K(j,k)=j^\mu k^\nu + j^\nu k^\mu$ ($\mu,\nu>2$) physical studies [4] suggest that gelation takes place instantaneously which is a sign of a pathological behavior. Below, taking the approach of [20], we show, under some technical conditions on the initial data, that the solutions cannot exist.

To prove the result one needs to understand how the tail of the distribution behaves with fast growing kernels. For this purpose, it will be useful to write the infinite system as a system of mass-flow equations, i.e.,

$$\dot{c}_j(t) = I_{j-1}(c) - I_j(c),$$

where

$$I_j(c) = c_j \sum_{k=1}^{\infty} K(k,j)c_k - c_{j+1}\sum_{k=0}^{\infty} K(j+1,k)c_k. \tag{3.28}$$

Again we provide two different results for the non-symmetric kernel and symmetric kernel. For both of the results we will need the following lemma which is a straightforward computation.

Lemma 4. *Let $c_j(t)$ be a solution of the EDG system (1.2)-(1.4). Then one has the following identities*

$$\sum_{j=m}^{\infty} c_j(t) - \sum_{j=m}^{\infty} c_j(0) = \int_0^t I_{m-1}(c(s))ds,$$
$$\sum_{j=m}^{\infty} jc_j(t) - \sum_{j=m}^{\infty} jc_j(0) = \int_0^t \sum_{j=m}^{\infty} I_j(c(s))ds + m\int_0^t I_{m-1}(c(s))ds,$$
$$\sum_{j=m}^{\infty} j^2c_j(t) - \sum_{j=m}^{\infty} j^2c_j(0) = \int_0^t \sum_{j=m}^{\infty} (2j+1)I_j(c(s))ds + m^2\int_0^t I_{m-1}(c(s))ds.$$

For the non-symmetric kernel we make the extra assumption that cluster interaction kernels are biased, i.e., $K(k,j) > K(j,k)$ for $j > k$. This is reasonable assumption for systems that prefers exchanges towards bigger clusters (e.g. migration towards bigger cities). If the exchange rate grows faster than linearly this may cause non-existence as we see in the next theorem.

Theorem 8. *Consider the infinite EDG system (1.2)-(1.4) with $c_0(0) > 0$. Let $K(j,k) \geq Cj^{\beta}$ hold for some $\beta > 1$. Assume that $K(j,0) = 0$, $K(k,j) \geq (1+\varepsilon)K(j,k)$ for $j > k \geq 1$ and some $\varepsilon > 0$. Assume further that $\lim_{m\to\infty} e^{\delta m^{\beta-1}} \sum_{j=m}^{\infty}(j-m)c_j(0) \nrightarrow 0$ for all $\delta > 0$. Then there exists no solution $c_j(t) \in X_1$ of (1.2)-(1.4) on any interval $[0,T)$ $(T > 0)$.*

Proof. We will prove the result by contradiction. Suppose that there is a solution. From the first and second identity of Lemma 4 above one has

$$\sum_{j=m}^{\infty}(j-m)c_j(t) - \sum_{j=m}^{\infty}(j-m)c_j(0) = \int_0^t \sum_{j=m}^{\infty} I_j(c(s))ds. \tag{3.29}$$

Writing in the expression for $I_j(c(s))$ from (3.28) on the right hand side of (3.29) reads

$$\int_0^t \sum_{j=m}^{\infty} I_j(c(s))ds = \int_0^t \sum_{j=m}^{\infty} c_j(s)\sum_{k=1}^{\infty} K(k,j)c_k(s)ds \tag{3.30}$$
$$- \int_0^t \sum_{j=m}^{\infty} c_{j+1}(s)\sum_{k=0}^{\infty} K(j+1,k)c_k(s)ds. \tag{3.31}$$

Shifting the index on the second term of (3.31), using $K(j,0) = 0$ to remove the $k = 0$ terms and matching the lower bounds of the sums we have

$$\int_0^t \sum_{j=m}^{\infty} I_j(c(s))ds =$$
$$\int_0^t \sum_{j=m}^{\infty}\sum_{k=1}^{\infty} c_j(s)K(k,j)c_k(s)ds - \int_0^t \sum_{j=m}^{\infty}\sum_{k=1}^{\infty} c_j(s)K(j,k)c_k(s)ds + \int_0^t c_m(s)\sum_{k=1}^{\infty} K(m,k)c_k(s)ds.$$

Splitting the sums as $\sum_{j=m}^{\infty}\sum_{k=1}^{\infty}(...) = \sum_{j=m}^{\infty}\sum_{k=1}^{m-1}(...) + \sum_{j=m}^{\infty}\sum_{k=m}^{\infty}(...)$ and using the non-negativity of $\sum_{k=1}^{\infty} K(m,k)c_k(s)$ sum yields

$$\int_0^t \sum_{j=m}^{\infty} I_j(c(s))ds \geq \int_0^t \sum_{j=m}^{\infty}\sum_{k=1}^{m-1} c_j(s)(K(k,j)-K(j,k))c_k(s)ds$$
$$+\int_0^t \sum_{j=m}^{\infty}\sum_{k=m}^{\infty} c_j(s)(K(k,j)-K(j,k))c_k(s)ds.$$

Note that the second double-sum on the right hand side is zero by the symmetry of the sum and hence by (3.29) we are left with

$$\sum_{j=m}^{\infty}(j-m)c_j(t) - \sum_{j=m}^{\infty}(j-m)c_j(0) \geq \int_0^t \sum_{j=m}^{\infty}\sum_{k=1}^{m-1} c_j(s)(K(k,j)-K(j,k))c_k(s)ds \tag{3.32}$$

$$\geq \int_0^t \sum_{j=m}^{\infty}\sum_{k=1}^{m-1} \varepsilon c_j(s)K(j,k)c_k(s)ds. \tag{3.33}$$

Using the lower bound $K(j,k) \geq Cj^{\beta}$ one has

$$\sum_{j=m}^{\infty}(j-m)c_j(t) \geq \sum_{j=m}^{\infty}(j-m)c_j(0) + \int_0^t \varepsilon C \sum_{j=m}^{\infty} j^{\beta}c_j(s)\sum_{k=1}^{m-1} c_k(s)ds \tag{3.34}$$

$$\geq \sum_{j=m}^{\infty}(j-m)c_j(0) + \varepsilon C m^{\beta-1}\int_0^t \sum_{j=m}^{\infty} jc_j(s)ds. \tag{3.35}$$

For the second line we used $\sum_{k=1}^{m-1} c_k(s) \geq C > 0$ which is a consequence of the fact $c_0(t)$ is strictly increasing under the assumption $K(j,0) = 0$ as a result of the rate equation (1.1). Since $\sum_{j=m}^{\infty} jc_j(t) > \sum_{j=m}^{\infty}(j-m)c_j(t)$, one gets the differential inequality below

$$\sum_{j=m}^{\infty} jc_j(t) \geq \sum_{j=m}^{\infty}(j-m)c_j(0) + \varepsilon C m^{\beta-1}\int_0^t \sum_{j=m}^{\infty} jc_j(s)ds,$$

from which we get the inequality

$$\sum_{j=m}^{\infty} jc_j(t) \geq e^{\varepsilon C m^{\beta-1}t}\sum_{j=m}^{\infty}(j-m)c_j(0).$$

Since $\lim_{m\to\infty} e^{\varepsilon C m^{\beta-1}t}\sum_{j=m}^{\infty}(j-m)c_j(0) \nrightarrow 0$ for any $t>0$ by our assumption we arrive at $\lim_{m\to\infty}\sum_{j=m}^{\infty} jc_j(t) > 0$ which is a contradiction. □

Example: The condition $\lim_{m\to\infty} e^{\delta m^{\beta-1}}\sum_{j=m}^{\infty}(j-m)c_j(0) \nrightarrow 0$ in the above theorem can be achieved by many kinds of initial distributions $c_j(0)$ with algebraically decaying tails. Consider, for instance, $c_j(0) = \frac{1}{j^q}$ with any $q > 2$. Then,

$$\sum_{j=m}^{\infty}(j-m)c_j(0) = \sum_{j=m}^{2m}(j-m)c_j(0) + \sum_{j=2m+1}^{\infty}(j-m)c_j(0) \geq \left(\frac{s(m)}{m^q} + \sum_{j=2m+1}^{\infty}\frac{j}{2j^q}\right)$$

where, by some algebra, $s(m) = (m^2-m)/2$. Then comparing the sum $\sum_{j=2m+1}^{\infty}\frac{j}{2j^q}$ with the integral $\int_{2m+1}^{\infty}\frac{dy}{2y^{q-1}}$, we obtain

$$\sum_{j=m}^{\infty}(j-m)c_j(0) \geq \frac{m^2-m}{2m^q} + \frac{1}{2(q-2)(2m+1)^{q-2}}$$

Then in the limit $m\to\infty$ the condition of the theorem is satisfied for any $q>2$ since

$$\lim_{m\to\infty} e^{\delta m^{\beta-1}}\left(\frac{m^2-m}{2m^q} + \frac{1}{2(q-2)(2m+1)^{q-2}}\right) > 0.$$

If we assume faster growth such as $\beta > 2$, then the condition in the theorem is satisfied even by distributions with light tails. Indeed, let $c_j(0) = \kappa^j$, $\kappa < 1$. Then, $\sum_{j=m}^{\infty}(j-m)c_j(0) = \frac{\kappa^{m+1}}{(1-\kappa)^2}$. Hence one has, for any δ,

$$\lim_{m\to\infty} e^{\delta m^{\beta-1}} \sum_{j=m}^{\infty}(j-m)c_j(0) = \lim_{m\to\infty} e^{\delta m^{\beta-1}-C-m\ln(\kappa/(1-\kappa)^2)} > 0$$

satisfying the condition of the theorem.

The previous theorem relied on the assumption that pairwise interactions favored bigger sizes. For symmetric kernels, there is no such favoring and non-existence cannot take place unless $K(j,k)$ grows faster (agreeing with the existence results of the previous section). However, we have the following result.

Theorem 9. *Consider the infinite EDG system (1.2)-(1.4). Suppose that the symmetric kernel satisfies $K(j,k) \geq Cj^{\beta}$ for some $\beta > 2$. Suppose that $K(j,0) = 0$. Assume also that $\lim_{m\to\infty} e^{\delta m^{\beta-2}} \sum_{j=m}^{\infty}(j^2-m^2)c_j(0) \nrightarrow 0$ for all $\delta > 0$. Then there exists no solution $c_j(t) \in X_2$ of (1.2)-(1.4) on any interval $[0,T)$ $(T > 0)$.*

Proof. We go by contradiction as in Theorem 8. Let $c_j(t) \in X_2$ be a solution on $[0,T)$. Then $M_2(t) < \infty$ for $t < T$. Using the first and third identities of Lemma 4 we have

$$\sum_{j=m}^{\infty}(j^2-m^2)c_j(t) - \sum_{j=m}^{\infty}(j^2-m^2)c_j(0) = \int_0^t \sum_{j=m}^{\infty}(2j+1)I_j(c(s))ds. \tag{3.36}$$

Pulling $I_j(c(s))$ from (3.28) and placing it on the right hand side of (3.36) and shifting the index for the c_{j+1} term reads

$$\int_0^t \sum_{j=m}^{\infty}(2j+1)I_j(c(s))ds = \int_0^t \sum_{j=m}^{\infty}(2j+1)c_j(s)\sum_{k=1}^{\infty}K(k,j)c_k(s)ds \tag{3.37}$$

$$- \int_0^t \sum_{j=m+1}^{\infty}(2j-1)c_j(s)\sum_{k=0}^{\infty}K(j,k)c_k(s)ds. \tag{3.38}$$

Matching the lower indices in (3.37), (3.38) for the j sums one gets the inequality

$$\int_0^t \sum_{j=m}^{\infty}(2j+1)I_j(c(s))ds \geq$$

$$2\int_0^t \sum_{j=m}^{\infty}c_j(s)\sum_{k=1}^{\infty}K(k,j)c_k(s) + \int_0^t \sum_{j=m}^{\infty}(2j-1)c_j(s)\sum_{k=1}^{\infty}(K(k,j)-K(j,k))c_k(s)ds.$$

where we used the non-negativity of $\int_0^t(2m-1)c_j(s)\sum_{k=0}^{\infty}K(m,k)c_k(s)ds$. Notice, by symmetry, the second term in the second line is zero. Then, placing the remaining inequality in equation (3.36) we see

$$\sum_{j=m}^{\infty}(j^2-m^2)c_j(t) - \sum_{j=m}^{\infty}(j^2-m^2)c_j(0) \geq 2\int_0^t \sum_{j=m}^{\infty}c_j(s)\sum_{k=1}^{\infty}K(k,j)c_k(s)ds. \tag{3.39}$$

Now, by the bounds for K assumed in the theorem, we can write, from (3.39), the following

$$\begin{aligned}\sum_{j=m}^{\infty}j^2c_j(t) &\geq \sum_{j=m}^{\infty}(j^2-m^2)c_j(0) + 2C\int_0^t \sum_{j=m}^{\infty}j^{\beta}c_j(s)\sum_{k=1}^{\infty}c_k(s)ds \\ &\geq \sum_{j=m}^{\infty}(j^2-m^2)c_j(0) + 2Cm^{\beta-2}\int_0^t \sum_{j=m}^{\infty}j^2c_j(s)ds.\end{aligned}$$

where in the second line we used $\sum_{k=1}^{\infty}c_k(s) \geq C > 0$ as in Theorem 8. Solving the differential inequality yields the inequality

$$\sum_{j=m}^{\infty}j^2c_j(t) \geq e^{2Cm^{\beta-2}t}\sum_{j=m}^{\infty}(j^2-m^2)c_j(0)$$

which contradicts, in the limit $m \to \infty$, with the boundedness of $M_2(t)$ (i.e., $\lim_{m\to\infty} \sum_{j=m}^{\infty} j^2 c_j(t) = 0$) on $[0, T)$. □

4. CONCLUSION

In this article, as an initial mathematical investigation of the subject, we studied fundamental properties of the EDG systems. For the last two decades, these dynamic processes have attracted considerable attention in the interdisciplinary communities as such models have found applications in physics, migration dynamics, socioeconomic behavior etc. These processes (e.g. zero-range processes) are also of significant interest for probabilists as the rate equations involved can be obtained as scaling limits of underlying stochastic dynamics. Our article is motivated by the latter approach.

The connection between the two approaches is simple but subtle. For a physicist the exchange processes are meaningfully defined only between clusters that have non-zero mass and growth is unidirectional. So, when a monomer is absorbed into another cluster there remains nothing behind. In the course of the time the total mass $\sum_{j=1}^{\infty} jc_j(t)$ is the only conserved quantity and total number of clusters $\sum_{j=1}^{\infty} c_j(t)$ decreases in time. This picture is no contradiction with the probabilists' view where the particles sit on lattice sites (or on a complete graph) each of which can accommodate arbitrary number of particles. In this view, lattice sites interact in pretty much the same way clusters interact in the mind of a physicist, that is, by exchanging single particles among each other at a time. There is one significant difference however, namely the "empty sites" or "empty (available) volume". In our formulation, which is the more general one, particles are allowed to hop from a massive cluster to an empty (available) volume creating a single monomer which can continue to interact with the rest of the system in the usual way. And when a monomer is taken by another cluster the remaining space is still available to be occupied. In this regard, the "total volume" or total number of clusters including the zero-cluster (or the available volume), i.e., $\sum_{j=0}^{\infty} c_j(t)$ is conserved. These two views are compatible with each other and in fact one can be "obtained" from the other. By setting $K(j, 0) = 0$ in our general formulation, we disallow hopping to the available volume and system grows indefinitely creating more and more available volume in time. Indeed, looking at the rate equations (1.2)-(1.4), if $K(j, 0) = 0$, we observe that $c_0(t)$ monotonically increases which means that $\sum_{j=1}^{\infty} c_j(t)$ must decrease due to conservation of total volume just as a physicist would reason. We also observe that the rate equations for $j \geq 1$ is completely decoupled from the $c_0(t)$ and evolve independently again agreeing with physicists' picture of the process. However, the main theorems on the existence and uniqueness that are proven in this article remain intact and give us all the existence and uniqueness results for the classical EDG system (after choosing a "free" initial condition for c_0).

To recapitulate our results, we showed that growth assumptions on the kernel determine whether the solutions exist globally, locally or do not exist at all. In particular, for general non-symmetric kernels whose growth is bounded as $K(j, k) \leq Cjk$, unique classical solutions exist globally. For symmetric kernels however, we showed that the existence result can be generalized to kernels whose growth rate is lying in the range $K(j, k) \leq C(j^{\mu}k^{v} + j^{\nu}k^{\mu})$, with $\mu, \nu \leq 2$, $\mu + v \leq 3$. This fact was first discovered by physicists based on scaling arguments [4]. On the other hand, for non-symmetric kernels which grow fast enough (i.e., $K(j, k) \geq Cj^{\beta}$) ($\beta > 1$) we showed that the solutions may not exist at all. Similarly, for symmetric kernels, we proved an analogous results stating that, for kernels which grow with the rate $K(j, k) \geq Cj^{\beta}$) ($\beta > 2$) solutions may cease to exist if some assumptions on the initial conditions are satisfied.

A number of questions remain still open for investigation. First of all, the intriguing question of existence of gelling solutions (solutions that do not conserve mass) is not addressed in this article. Physical studies suggest that $\mu + \nu = 3$ is the critical line beyond which gelation takes place. A separate but related question in this matter is whether the gelling solutions (if they exist) can be extended beyond the gelation time. Also, physical studies suggest that for kernels that grow super-quadratically, gelation takes place instantaneously for general initial conditions. Although, our non-existence result is a step in that direction, it is by no means a complete resolution of the problem as we restricted ourselves to specific initial conditions.

Another whole area which deserves detailed analysis and which we have made no attempt to analyze is the existence of self-similar solutions and large time behavior of general solutions. In recent years there has been revived interest on the subject and several seminal results has been obtained for Smoluchowski type models concerning self-similarity [31], [32], [33], [34] and the long time behaviour [35]. Similar results are likely to be true for the case of the EDG systems and have been considered by physicists for kernels with special form [25].

Acknowledgements. I thank Colm Connaughton and Stefan Grosskinsky for fruitful discussions. The author is supported by the Marie Curie Fellowship of European Commission, grant agreement number: 705033.

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University of Warwick, Mathematics Institute, UK
E-mail address: E.esenturk.1@warwick.ac.uk