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The Cauchy–Kowalevski theorem applied for counting connections with a prescribed Ricci tensor

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Abstract: How many linear connections are there with a prescribed Ricci tensor? The question is answered in the analytic case by using the Cauchy–Kowalevski theorem.

Key words: Linear connection, Ricci tensor, Cauchy–Kowalevski theorem

1. Introduction

Our study is inspired by the recent paper by Dušek and Kowalski [3]. Roughly speaking, the question is how many structures of a prescribed type exist. By a satisfactory answer we mean a theorem saying that the set of such structures is parametrized by some families (finite) of arbitrarily chosen functions. We consider the local setting of the question. It turns out that the theorem of Cauchy–Kowalevski can be used as a tool in answering it. Of course, using this tool implies that we must restrict ourselves to analytic structures. However, the advantage is that the tool belongs to the fundamentals of mathematics and a procedure of getting structures is explicit modulo solving a Cauchy–Kowalevski system of differential equations. On the other hand, it seems that the method fits only very special situations.

More precisely, the main goal of the present paper is to determine the number of analytic functions that define an analytic connection with a prescribed Ricci tensor. One can say that this is an 'inverse type problem': given a Ricci tensor, find a connection.

The question of existence of connections with a prescribed Ricci tensor was studied, for instance, in [1, 2, 4, 5]. In particular, it was proved in [5] that if $n \geq 2$ then any analytic symmetric tensor of type $(0, 2)$ on an n-manifold can be locally realized as the symmetric part of the Ricci tensor of some torsion-free connection. We extend this result to not necessarily symmetric prescribed tensors and the whole Ricci tensors. Namely, we observe that a necessary condition for a tensor of type $(0, 2)$ to be (locally) the Ricci tensor of some torsion-free connection is that its antisymmetric part is a closed form. If $n \geq 2$ then for an analytic tensor field on an n-manifold the closedness of the antisymmetric part is also a sufficient condition for a local realization as the Ricci tensor of a torsion-free connection. Moreover, we show that if $n \geq 2$ then the set of germs at a point in $\mathbf{R}^n$ of all analytic torsion-free connections ∇ with a prescribed Ricci tensor (whose antisymmetric part is closed) depends bijectively on $\frac{n^3-3n}{2} + 1$ analytic functions of n variables and $\frac{n^2+n}{2}$ analytic functions of $(n-1)$ variables. In particular, the functions of n variables are some Christoffel symbols of ∇. Choosing them

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in special ways one can produce structures with additional properties. In the case of connections with arbitrary torsion we prove, modifying the proof of the main theorem from [3], that if $n \geq 2$ then the set of germs of all analytic connections with a prescribed Ricci tensor depends bijectively on $n^3 - n^2$ analytic functions of n variables and n^2 functions on $n-1$ variables. In particular, if $n \geq 2$ we get that any analytic tensor field of type $(0,2)$ around $0 \in \mathbb{R}^n$ can be realized locally around 0 as the Ricci tensor of an analytic connection. We also consider the case where the trace of the torsion vanishes.

A remarkable paper concerning the topic under consideration is [2]. In [2] more advanced techniques (than the Cauchy–Kowalevski theorem) are used and the analyticity assumption can be dropped. Moreover, the global setting of the problem is considered. On the other hand, the paper deals only with the Riemannian case and manifolds of dimension at least 3. In the present paper we put emphasis on the generality of the situation (arbitrary affine connections with Ricci tensor of any algebraic type) and the simplicity of the method. The method also allows us to give an answer to the considered problem in the Riemannian case for 2-dimensional manifolds. More precisely, an answer is provided in the 2-dimensional case for nondegenerate Ricci tensors. Here the Cauchy–Kowalevski theorem of the second order is used.

2. Preliminaries

Recall the theorem of Cauchy–Kowalevski in the version we need for our considerations. We adopt the notation $(f)_i = \frac{\partial f}{\partial x^j}$, $(f)_{jk} = \frac{\partial^2 f}{\partial x^j \partial x^k}$ for a function on a domain endowed with a coordinate system $(x^1, ..., x^n)$. All coordinate systems used in this paper are analytic.

Theorem 2.1 *Consider a system of differential equations for unknown functions $U^1, ..., U^N$ in a neighborhood of $0 \in \mathbf{R}^n$ and of the form*

$$
\begin{aligned}
&(U^1)_1 = H^1(x^1, ..., x^n, U^1, ..., U^N, (U^1)_2, ..., (U^1)_n, ..., (U^N)_2, ..., (U^N)_n),\\
&(U^2)_1 = H^2(x^1, ..., x^n, U^1, ..., U^N, (U^1)_2, ..., (U^1)_n, ..., (U^N)_2, ..., (U^N)_n),\\
&\;\;...\\
&(U^N)_1 = H^N(x^1, ..., x^n, U^1, ..., U^N, (U^1)_2, ..., (U^1)_n, ..., (U^N)_2, ..., (U^N)_n),
\end{aligned}
$$

where H^i, $i = 1, ..., N$, are analytic functions of all variables in a neighborhood of $(0, ..., 0, \varphi^1(0), ..., \varphi^N(0), (\varphi^1)_2(0), ..., (\varphi^1)_n(0), ..., (\varphi^N)_2(0), ..., (\varphi^N)_n(0)) \in \mathbf{R}^{(N+1)n}$ for analytic functions $\varphi^1, ..., \varphi^N$ given in a neighborhood of $0 \in \mathbf{R}^{n-1}$.

Then the system has a unique solution $(U^1(x^1, ..., x^n), ..., U^N(x^1, ..., x^n))$ that is analytic around $0 \in \mathbf{R}^n$ and satisfies the initial conditions

$$
U^i(0, x^2, ..., x^n) = \varphi^i(x^2, ..., x^n) \quad for \quad i = 1, ..., N.
$$

In the second-order Cauchy–Kowalewski theorem we additionally prescribe analytic functions $\psi^1, ..., \psi^N$ defined in a neighborhood of $0 \in \mathbf{R}^{n-1}$. We have $(U^1)_{11}, ..., (U^N)_{11}$ on the left-hand sides and we add to the set of arguments of $H^1, ..., H^N$ on the right-hand sides the first derivatives $(U^1)_1, ..., (U^N)_1$ and the second derivatives $(U^i)_{jk}$ for $i = 1, ..., N$, $j = 1, ..., n$ and $k = 2, ..., n$. To the initial conditions we add the conditions

$$(U^i)_1(0, x^2, ..., x^n) = \psi^i(x^2, ..., x^n)$$

for the prescribed functions ψ^i, $i = 1, ..., N$.

Since the problems we study are of local nature, we shall locate geometric structures in open neighborhoods of $0 \in \mathbf{R}^n$. For the beginning a neighborhood can be equipped with any analytic coordinate system, for instance, the canonical one.

In the following theorems, when we write about objects in a neighborhood of $0 \in \mathbf{R}^n$, for instance connections, tensor fields, functions, we mean, in fact, their germs at 0.

3. How many connections are there with a prescribed Ricci tensor?

For a fixed coordinate system $(x^1, ..., x^n)$ the Ricci tensor Ric of a linear connection ∇ with Christoffel symbols Γ^i_{jk} is expressed by the formula

$$\mathrm{Ric}\,(\partial_i, \partial_j) = \sum_{k=1}^{n}[(\Gamma^k_{ij})_k - (\Gamma^k_{kj})_i] + \sum_{k,l=1}^{n}[\Gamma^l_{ij}\Gamma^k_{kl} - \Gamma^l_{kj}\Gamma^k_{il}]. \tag{1}$$

Let r be an analytic tensor field of type $(0, 2)$ around $0 \in \mathbf{R}^n$. Set $r_{ij} = r(\partial_i, \partial_j)$. Modifying arguments from [3] we will prove how many real analytic linear connections ∇ exist such that Ric $= r$.

The condition Ric $= r$ is equivalent to the system of equations

$$\sum_{k=1}^{n}[(\Gamma^k_{ij})_k - (\Gamma^k_{kj})_i] = \sum_{k,l=1}^{n}[\Gamma^l_{kj}\Gamma^k_{il} - \Gamma^l_{ij}\Gamma^k_{kl}] + r_{ij}\ ,\ i, j = 1, ..., n. \tag{2}$$

Set

$$\Lambda_{ij} = \sum_{k,l=1}^{n}[\Gamma^l_{kj}\Gamma^k_{il} - \Gamma^l_{ij}\Gamma^k_{kl}] \tag{3}$$

and rewrite the system (2) in the form

$$[(\Gamma^1_{ij})_1 + ... + (\Gamma^n_{ij})_n] - [(\Gamma^1_{1j})_i + ... + (\Gamma^n_{nj})_i] = \Lambda_{ij} + r_{ij}\ ,\ i, j = 1, ..., n. \tag{4}$$

For $i = 1$ and $j = 1, ..., n$, we keep each derivative $(\Gamma^n_{nj})_1$ on the left-hand side of the corresponding equation. We denote the sum of all remaining terms on the left-hand side of the corresponding equation by Λ'_{1j} and move it to the right-hand side. For $i > 1$ and $j = 1, ..., n$, we keep each derivative $(\Gamma^1_{ij})_1$ on the left-hand side of the corresponding equation. We denote the sum of all remaining terms on the left-hand side of the corresponding equation by Λ'_{ij} and move it to the right-hand side. Then we obtain the (equivalent) system

$$\begin{aligned} &(\Gamma^n_{nj})_1 = -\Lambda_{1j} - r_{1j} + \Lambda'_{1j}\ ,\ j = 1, ..., n, \\ &(\Gamma^1_{ij})_1 = \Lambda_{ij} + r_{ij} - \Lambda'_{ij}\ ,\ i = 2, ..., n\ ,\ j = 1, ..., n\ . \end{aligned} \tag{5}$$

We see that the first derivatives on the left-hand sides of this system are not present in any terms on the right-hand sides.

Theorem 3.1 *Let $n \geq 2$ and r be an analytic tensor field of type $(0,2)$ around $0 \in \mathbf{R}^n$. The family of all analytic linear connections ∇ defined around 0 with the Ricci tensor* $\mathrm{Ric} = r$ *depends bijectively on $n^3 - n^2$ analytic functions of n variables and n^2 analytic functions of $n-1$ variables.*

Proof We can choose $n^3 - n^2$ Christoffel symbols Γ_{ij}^k not present on the left-hand side of (5) as arbitrary analytic functions. Then n^2 analytic functions of $n-1$ variables appear by solving the system (5) using the Cauchy–Kowalevski theorem. □

For a linear connection ∇ with torsion $\mathrm{T}(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y]$, we have the 1-form τ given by

$$\tau(Y) = \mathrm{tr}\,(X \to \mathrm{T}(X,Y))\ . \tag{6}$$

Using a similar method as above, given an analytic tensor field r around $0 \in \mathbf{R}^n$, we will describe all analytic linear connections Γ such that $\tau = 0$ and $\mathrm{Ric} = r$.

Clearly, this problem is equivalent to finding all solutions of the system consisting of the system (5) and

$$\sum_{i=1}^{n} (\Gamma_{ik}^i - \Gamma_{ki}^i) = 0\ ,\ \ k = 1, ..., n\ . \tag{7}$$

Theorem 3.2 *Let $n \geq 3$ and r be an analytic tensor field of type $(0,2)$ around $0 \in \mathbf{R}^n$. The family of all analytic linear connections ∇ with $\tau = 0$ and* $\mathrm{Ric} = r$ *depends bijectively on $n^3 - n^2 - n$ analytic functions of n variables and n^2 analytic functions of $n-1$ variables.*

Proof From (7) we have

$$\begin{aligned} \Gamma_{k,k+1}^{k+1} &= -\sum_{i=1}^{k-1} \Gamma_{ki}^i - \sum_{i=k+2}^{n} \Gamma_{ki}^i \\ &\quad + \sum_{i=1}^{k-1} \Gamma_{ik}^i + \sum_{i=k+1}^{n} \Gamma_{ik}^i, \qquad k = 1, ..., n-1, \\ \Gamma_{n,n-1}^{n-1} &= -\sum_{i=1}^{n-2} \Gamma_{ni}^i + \sum_{i=1}^{n-1} \Gamma_{in}^i. \end{aligned} \tag{8}$$

Since $n \geq 3$, the Christoffel symbols on the left-hand sides of (8) are not present on the left-hand sides of the n^2 equalities of (5). We substitute the above n equalities (8) into the n^2 equalities of (5). We obtain

$$\begin{aligned} (\Gamma_{nj}^n)_1 &= -\tilde{\Lambda}_{1j} - r_{1j} + \tilde{\Lambda}'_{1j}\ ,\ \ j = 1, ..., n\ , \\ (\Gamma_{ij}^1)_1 &= \tilde{\Lambda}_{ij} + r_{ij} - \tilde{\Lambda}'_{ij}\ , i = 2, ..., n\ , j = 1, ..., n\ , \end{aligned} \tag{9}$$

where $\tilde{\Lambda}_{1j}$, $\tilde{\Lambda}'_{1j}$, $\tilde{\Lambda}_{ij}$, $\tilde{\Lambda}'_{ij}$ are Λ_{1j}, Λ'_{1j}, Λ_{ij}, Λ'_{ij} respectively, after the substitutions. It is easy to see that the first derivatives on the left-hand sides of the system (9) are not present on the right-hand sides. We can now choose $n^3 - n^2 - n$ Christoffel symbols Γ_{jk}^i not present on the left-hand sides of (9) and of (8) as arbitrary analytic functions. Then n^2 analytic functions of $n-1$ variables appear by solving (9) by means of the Cauchy–Kowalevski theorem. □

If $n = 2$ then the condition $\tau = 0$ yields $\mathrm{T} = 0$. Hence the connection is torsion-free. We shall now study this case for any dimension. Set

$$D_j = div^{\nabla}\partial_j = \mathrm{tr}\,(X \to \nabla_X \partial_j) = \sum_{k=1}^{n} \Gamma^k_{kj}. \tag{10}$$

Then the formula for the Ricci tensor can be written as follows:

$$\mathrm{Ric}\,(\partial_i, \partial_j) = \sum_{k=1}^{n} (\Gamma^k_{ij})_k - (D_j)_i + \Lambda_{ij}. \tag{11}$$

We decompose the Ricci tensor into its symmetric and antisymmetric parts, that is, Ric $= s + a$, where

$$s(X,Y) = \frac{\mathrm{Ric}\,(X,Y) + \mathrm{Ric}\,(Y,X)}{2}, \qquad a(X,Y) = \frac{\mathrm{Ric}\,(X,Y) - \mathrm{Ric}\,(Y,X)}{2}. \tag{12}$$

For a torsion-free connection the portions $\sum_{k=1}^{n}(\Gamma^k_{ij})_k$ and Λ_{ij} are symmetric for i and j. Hence for a torsion-free connection we have

$$a_{ij} = a(\partial_i, \partial_j) = \frac{(D_i)_j - (D_j)_i}{2}, \tag{13}$$

$$s_{ij} = s(\partial_i, \partial_j) = \sum_{k=1}^{n} (\Gamma^k_{ij})_k - \frac{(D_j)_i + (D_i)_j}{2} + \Lambda_{ij}. \tag{14}$$

In [6] the following proposition was proved. Since its proof is short, we cite it here.

Proposition 3.3 *For a torsion-free connection on a paracompact manifold M the antisymmetric part of its Ricci tensor is exact.*

Proof By the first Bianchi identity we have

$$\mathrm{tr}\, R(X,Y) = \mathrm{Ric}\,(Y,X) - \mathrm{Ric}\,(X,Y)$$

for a torsion-free connection ∇, where R is its curvature tensor. Let ∇' be any torsion-free connection whose Ricci tensor Ric$'$ is symmetric. It can be the Levi-Civita connection of some metric. Denote by Q the difference tensor between ∇ and ∇', that is, $Q(X,Y) = Q_X Y = \nabla_X Y - \nabla'_X Y$. Define the 1-form δ on M by

$$\delta(X) = \mathrm{tr}\, Q_X.$$

Then

$$d\delta(X,Y) = \frac{1}{2}\{\mathrm{tr}\,\nabla' Q(X,Y,\cdot) - \mathrm{tr}\,\nabla' Q(Y,X,\cdot)\}.$$

The curvature tensors R and R' for ∇ and ∇' are related by the formula

$$R(X,Y)Z = R'(X,Y)Z + \nabla' Q(X,Y,Z) - \nabla' Q(Y,X,Z) + Q_X Q_Y Z - Q_Y Q_X Z.$$

It follows that $\mathrm{tr}\, R(X,Y) = \mathrm{tr}\, R'(X,Y) + 2d\delta(X,Y) = 2d\delta(X,Y)$. □

Since we study problems of local nature, we replace the exactness of the form in the above theorem by its closedness. We shall prove

Theorem 3.4 *Let $n \geq 2$. An analytic tensor field r around $0 \in \mathbf{R}^n$ of type $(0,2)$ can be locally realized as the Ricci tensor of a torsion-free connection if and only if its antisymmetric part a, that is, $a(X,Y) = \frac{r(X,Y)-r(Y,X)}{2}$, is closed. For a given tensor field r in a neighborhood of $0 \in \mathbf{R}^n$ satisfying the above conditions the set of all analytic torsion-free connections defined around $0 \in \mathbf{R}^n$ and whose Ricci tensor is r depends bijectively on $\frac{n^3-3n}{2}+1$ arbitrarily chosen analytic functions of n variables and $\frac{n^2+n}{2}$ arbitrarily chosen analytic functions of $n-1$ variables.*

Proof Let s stand for the symmetric part of r. The functions $a_{ij} = a(\partial_i, \partial_j)$, $s_{ij} = s(\partial_i, \partial_j)$ are given. Since the form a is closed and r is analytic, around the fixed point 0 there is an analytic 1-form α such that $a = -d\alpha$. The 1-form α is chosen up to one function, that is, α can be replaced by $\alpha + d\phi$ for any analytic function ϕ. Let $\alpha = \alpha_1 dx^1 + ... + \alpha_n dx^n$. We have $2a_{ij} = -2d\alpha(\partial_i, \partial_j) = (\alpha_i)_j - (\alpha_j)_i$. Suppose that r is the Ricci tensor of some torsion-free connection whose Christoffel symbols Γ^k_{ij} are unknown. Then

$$\frac{(D_i)_j + (D_j)_i}{2} = a_{ij} + (D_j)_i \tag{15}$$

for $i, j = 1, ..., n$. Set $D_i = \alpha_i$ for $i = 1,,,.,n$. We have already used (13) and from now on the functions $D_1, ..., D_n$ are given.

All the conditions from (14) must be satisfied. We have

$$s_{11} = \sum_{k=1}^{n} (\Gamma^k_{11})_k - (D_1)_1 + \Lambda_{11},$$

and hence

$$(\Gamma^1_{11} + \Gamma^2_{21} + ... + \Gamma^n_{n1})_1 = (\Gamma^1_{11})_1 + (\Gamma^2_{11})_2 + ... + (\Gamma^n_{11})_n + \Lambda_{11} - s_{11}.$$

We can write it equivalently as

$$(\Gamma^2_{12})_1 = \sum_{k=2}^{n} (\Gamma^k_{11})_k - \sum_{k=3}^{n} (\Gamma^k_{k1})_1 + \Lambda_{11} - r_{11}. \tag{16}$$

For $i > 1$ we have

$$s_{1i} = \sum_{k=1}^{n} (\Gamma^k_{1i})_k - \frac{(D_i)_1 + (D_1)_i}{2} + \Lambda_{1i}.$$

By using (15) we get

$$(\Gamma^1_{1i})_1 = -(\Gamma^2_{1i})_2 - ... - (\Gamma^n_{1i})_n - \Lambda_{1i} + a_{i1} + (D_1)_i + s_{i1}.$$

We can write it as follows:

$$(\Gamma^1_{1i})_1 = -(\Gamma^2_{1i})_2 - ... - (\Gamma^n_{1i})_n - \Lambda_{1i} + (D_1)_i + r_{i1}. \tag{17}$$

For i, j, where $1 < i \leq j \leq n$, we have

$$s_{ij} = \sum_{k=1}^{n} (\Gamma^k_{ij})_k - \frac{(D_j)_i + (D_i)_j}{2} + \Lambda_{ij},$$

that is,

$$s_{ij} = (\Gamma^1_{ij})_1 + (\Gamma^2_{ij})_2 + ... + (\Gamma^n_{ij})_n - a_{ij} - (D_j)_i + \Lambda_{ij}.$$

We shall write it as follows:

$$(\Gamma^1_{ij})_1 = -(\Gamma^2_{ij})_2 - ... - (\Gamma^n_{ij})_n - \Lambda_{ij} + (D_j)_i + r_{ij}. \tag{18}$$

Collecting the equations from (16)–(18) we get the following Cauchy–Kowalevski system of $\frac{n(n+1)}{2}$ equations (equivalent to (14)):

$$\begin{aligned} &(\Gamma^2_{12})_1 = \sum_{k=2}^n (\Gamma^k_{11})_k - \sum_{k=3}^n (\Gamma^k_{1k})_1 + \Lambda_{11} - r_{11}, \\ &(\Gamma^1_{1i})_1 = -(\Gamma^2_{1i})_2 - ... - (\Gamma^n_{1i})_n - \Lambda_{1i} + (D_1)_i + r_{i1}, \qquad i > 1, \\ &(\Gamma^1_{ij})_1 = -(\Gamma^2_{ij})_2 - ... - (\Gamma^n_{ij})_n - \Lambda_{ij} + (D_j)_i + r_{ij}, \qquad 1 < i \le j \le n. \end{aligned} \tag{19}$$

The quantities r_{11}, $(D_1)_i + r_{i1}$, $(D_j)_i + r_{ij}$ are given.

Except for the dependence given by (19) the Christoffel symbols are related by the following system of equations:

$$\begin{aligned} &D_1 = \Gamma^1_{11} + [\Gamma^2_{21}] + ... + \Gamma^n_{n1}, \\ &D_2 = [\Gamma^1_{12}] + \Gamma^2_{22} + ... + \Gamma^n_{n2}, \\ &\vdots \\ &D_n = [\Gamma^1_{1n}] + \Gamma^2_{2n} + ... + \Gamma^n_{nn}, \end{aligned} \tag{20}$$

where, by using brackets, we marked the Christoffel symbols from the right-hand side of (20) that appear on the left-hand side of (19). Observe also that on the right-hand sides of (20) there are no Christoffel symbols that repeat because of the symmetry of Γ^k_{ij} in lower indices.

From each of the equations in (20) we want to determine one Christoffel symbol and then insert it into (19) by the expression obtained from (20). Of course, we should not determine and substitute any marked symbol. Moreover, we have to do it in such a way that, after the substitution into (19), the derivatives from the left-hand side of (19) will not appear on the right-hand side of (19). Therefore, from the first equation of (20) we can only take $\Gamma^1_{11} = D_1 - \Gamma^2_{21} - ... - \Gamma^n_{n1}$. From the next equations we can take Γ^k_{kk} (but here it is not necessary to do it in this way).

For the modified system (19) (after the substitutions) we can apply the Cauchy–Kowalevski theorem.

We shall now count how many Christoffel symbols can be chosen arbitrarily. Note that all Christoffel symbols for which the upper index is equal to one or two of lower indices are on the right-hand side of (20). We see that from (20) we can choose $n(n-2)$ symbols arbitrarily. Consider now the Christoffel symbols for which the upper index is different than each of the lower indices. Consider first the symbols whose upper index is 1. All of them appear on the left-hand side of (19) and so we cannot choose them. Finally consider those Christoffel symbols whose upper index is k, where $1 < k \le n$, and k is different than any of the lower indices. They do not appear either on the left-hand side of (19) or on the right-hand side of (20). All of them can be chosen arbitrarily. There are $(n-1)\frac{(n-1)n}{2} = \frac{(n-1)^2 n}{2}$ such symbols. Therefore, we can choose $n(n-2) + \frac{(n-1)^2 n}{2} = \frac{n^3 - 3n}{2}$ Christoffel symbols arbitrarily. The function ϕ from the beginning of the proof is also an arbitrarily chosen function of n variables. □

Remark 3.5 For $n = 2$ we have

$$\begin{aligned} D_1 &= \Gamma^1_{11} + [\Gamma^2_{12}], \\ D_2 &= [\Gamma^1_{21}] + \Gamma^2_{22}. \end{aligned} \tag{21}$$

None of the Christoffel symbols from the right-hand side of (21) can be chosen arbitrarily (in the above procedure). We have $\frac{n^3-3n}{2} = 1$. The only Christoffel symbol that can be arbitrarily chosen in this case is Γ^2_{11}. In particular, we can choose it as 0 and then the vector field $\nabla_{\partial_1}\partial_1$ is parallel to ∂_1 (but we cannot assume that $\nabla_{\partial_1}\partial_1$ vanishes). For any dimension the functions Γ^k_{11} for $k = 2, ..., n$ are up to choice. In particular, one can choose them as 0, which means that $\nabla_{\partial_1}\partial_1$ is parallel to ∂_1. However, we cannot assume that $\Gamma^1_{11} = 0$. From the last equation of (19) it is clear that we cannot assume that for some $i > 1$ we have $\Gamma^k_{ii} = 0$ for all indices k, because we cannot choose Γ^1_{ii} arbitrarily.

We shall now give an easy proof to the question of how many Levi-Civita connections on a 2-dimensional domain are those whose Ricci tensor is a prescribed symmetric tensor r of type $(0, 2)$.

For a metric tensor field g (not necessarily positive definite) the Christoffel symbols of its Levi-Civita connection are given by

$$\Gamma^s_{ij} = \frac{1}{2}\sum_{k=1}^{n} g^{sk}\left((g_{ki})_j + (g_{jk})_i - (g_{ji})_k\right),$$

where $g_{ij} = g(\partial_i, \partial_j)$ and (g^{sk}) is the inverse matrix of the matrix (g_{ij}). If $n = 2$ and the matrix (g_{ij}) has a diagonal form in the coordinate system then

$$\begin{aligned} &\Gamma^1_{11} = \tfrac{1}{2}g^{11}(g_{11})_1,\ \Gamma^2_{11} = -\tfrac{1}{2}g^{22}(g_{11})_2,\ \Gamma^1_{21} = \Gamma^1_{12} = \tfrac{1}{2}g^{11}(g_{11})_2, \\ &\Gamma^2_{21} = \Gamma^2_{12} = \tfrac{1}{2}g^{22}(g_{22})_1,\ \Gamma^1_{22} = -\tfrac{1}{2}g^{11}(g_{22})_1,\ \ \Gamma^2_{22} = \tfrac{1}{2}g^{22}(g_{22})_2, \end{aligned} \tag{22}$$

where $g^{11} = \frac{1}{g_{11}}$ and $g^{22} = \frac{1}{g_{22}}$. The Ricci tensor Ric of the Levi-Civita connection for g satisfies the equality $\mathrm{Ric} = fg$, where f is the Gaussian curvature of g. Using (22), by a straightforward computation one gets

$$\begin{aligned} f = &-\tfrac{1}{2}g^{11}g^{22}[(g_{11})_{22} + (g_{22})_{11}] \\ &+\tfrac{1}{4}g^{11}(g^{22})^2[(g_{22})_2(g_{11})_2 + ((g_{22})_1)^2] \\ &+\tfrac{1}{4}(g^{11})^2g^{22}[(g_{11})_1(g_{22})_1 + ((g_{11})_2)^2]. \end{aligned} \tag{23}$$

Note that for an analytic metric tensor field on a 2-dimensional manifold there is an analytic orthogonal coordinate system around each point of the domain of the metric tensor field.

Theorem 3.6 *Let r be an analytic nondegenerate tensor field of type $(0, 2)$ such that its matrix is diagonal in an analytic coordinate system (x^1, x^2) on a neighborhood of $0 \in \mathbf{R}^2$. The set of all analytic metric tensor fields around 0 such that their Ricci tensors are equal to r depends bijectively on arbitrarily chosen pairs (φ, ψ) of analytic functions of one variable with $\varphi(0) \neq 0$.*

Proof Suppose that g is an analytic metric tensor field around $0 \in \mathbf{R}^2$ such that its Ricci tensor Ric is equal to r. Then $g = hr$ for some analytic map h around $0 \in \mathbf{R}^2$ with $h(0) \neq 0$. By (23) the equality $\mathrm{Ric} = r$ is equivalent to the partial differential equation

$$\begin{aligned} &-\frac{1}{2hr_{22}}[(hr_{11})_{22}+(hr_{22})_{11}] \\ &+\frac{1}{4(hr_{22})^2}[(hr_{22})_2(hr_{11})_2+((hr_{22})_1)^2] \\ &+\frac{1}{4h^2r_{11}r_{22}}[(hr_{11})_1(hr_{22})_1+((hr_{11})_2)^2]=r_{11}. \end{aligned} \tag{24}$$

Applying the Leibniz rule one sees that this equation can be transformed equivalently into the one of the forms

$$(h)_{11}=F(x,h,(h)_1,(h)_2,(h)_{12},(h)_{22})$$

for some analytic map F. Our theorem now follows from the Cauchy–Kowalevski theorem of order 2, where two analytic functions φ, ψ of one variable are prescribed and the initial conditions are: $h(0,x^2)=\varphi$, $(h)_1(0,x^2)=\psi$. □

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