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Chapter 13

New properties of the families of convergent and divergent permutations - Part I

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13.1 Introduction

In this paper, the permutations p of $\mathbb{N}$ are divided into two types, namely, the permutations preserving convergence of rearranged real series (i.e. the ones for which the series $\sum a_{p(n)}$ is convergent for any convergent real series $\sum a_n$) and the other permutations, without this property. The first ones will be called the convergent permutations, the second ones – the divergent permutations (so for any divergent permutation p of $\mathbb{N}$ there exists a convergent real series $\sum a_n$ such that the $p-$rearranged series $\sum a_{p(n)}$ is divergent). Convergent permutations have been characterized by many authors and in many ways. The following, very illustrative description is given by Agnew [1]:

Theorem 13.1. *A permutation p of $\mathbb{N}$ is the convergent permutation if and only if there exists a positive integer n such that for any interval I of $\mathbb{N}$ (i.e. a subset of $\mathbb{N}$ having the form $\{k,k+1,\dots,k+m-1\}$ for some $k,m\in\mathbb{N}$) the set $p(I)$ is a union of at most n mutually separated intervals (abbrev.: n **MSI**).*

We have noticed that, on the ground of the above theorem, one can formulate the following dual characterization of the divergent permutations:

Theorem 13.2. *A permutation p of $\mathbb{N}$ is the divergent permutation if and only if for every $n \in \mathbb{N}$ there exists an interval I of $\mathbb{N}$ such that the set $p(I)$ is a union of at least n* ***MSI****.*

Wituła [17] has generalized the description given in Theorem 13.1 onto the functions $f\colon \mathbb{N} \to \mathbb{N}$ preserving convergence of the real series (i.e. the functions for which the series $\sum a_{f(n)}$ is convergent whenever the series $\sum a_n$ is convergent).

For any subset A of $\mathbb{N}$ the notation: "the set A is a union of n (or at most n or at least n, respectively) **MSI**" means that there exists a family $\mathfrak{I}$ of n (or at most n or at least n, respectively) mutually separated intervals of positive integers forming a partition of A. We say that the set $\mathfrak{I}$ is a family of mutually separated intervals of positive integers if each element of $\mathfrak{I}$ is an interval of $\mathbb{N}$ and if for any two distinct members K and L of $\mathfrak{I}$ the following inequality $\min\{|k-l|\colon k \in K \wedge l \in L\} \geq 2$ holds.

We will denote by $\mathfrak{P}$ the family of all permutations of $\mathbb{N}$.

It is easy to check that Theorem 13.1 describes also all permutations $p \in \mathfrak{P}$ rearranging any convergent series $\sum a_n$ with real terms into the convergent series $\sum a_{p(n)}$ and preserving the sum of rearranged series, i.e. such that $\sum a_{p(n)} = \sum a_n$.

The family of all convergent permutations will be denoted by $\mathfrak{C}$. Notice that $\mathfrak{C}$ is closed with respect to the composition of functions i.e. $\mathfrak{C}$ is a semigroup (with the unity of course). The family of all divergent permutations will be denoted by $\mathfrak{D}$. Obviously we have $\mathfrak{D} = \mathfrak{P} \setminus \mathfrak{C}$.

Moreover, we introduce the following notation

$$\mathfrak{CC},\ \mathfrak{CD},\ \mathfrak{DC} \quad \text{and} \quad \mathfrak{DD}$$

for the nonempty subsets of $\mathfrak{P}$ defined by the relation

$$p \in AB \quad \text{if and only if} \quad p \in A \text{ and } p^{-1} \in B$$

for any $A, B \in \{\mathfrak{C}, \mathfrak{D}\}$ and $p \in \mathfrak{P}$.

We present now the construction of some permutation $p \in \mathfrak{DC}$.

Example 13.1. Let us put $n_k = k^2 + 3k$, for $k \in \mathbb{N}_0$. Then $n_{k+1} - n_k = 2(k+2)$, for $k \in \mathbb{N}$. Next, let us set $p(n_k + i) = n_k + 2i$ and $p(n_k + k + 2 + i) = n_k + 2i - 1$, for $i = 1, 2, \ldots, k+2$ and $k \in \mathbb{N}_0$. Hence the set

$$p([n_k + 1, n_k + k + 2])$$

is a union of $(k+2)$ MSI for every $k \in \mathbb{N}$. Notice that for any interval I the set $p^{-1}(I)$ is a union of at most 3 MSI. Therefore, by Theorems 13.1 and 13.2, we get $p \in \mathfrak{DC}$.

Consequently, the families $\mathfrak{CD}$ and $\mathfrak{DC}$ are nonempty, as it was claimed above.

In part II of this paper the following fundamental properties of all four "two-sided" families of permutations, defined above, will be proven:

$$\mathfrak{U} \circ \mathfrak{U} = \mathfrak{U} \tag{13.1}$$

and

$$\mathfrak{DD} \circ \mathfrak{U} = \mathfrak{U} \circ \mathfrak{DD} = \mathfrak{DD} \cup \mathfrak{U} = \begin{cases} \mathfrak{D} & \text{if} \quad \mathfrak{U} \subset \mathfrak{D}, \\ \mathfrak{D}^{-1} & \text{otherwise}, \end{cases} \tag{13.2}$$

where $\mathfrak{B}^{-1} := \{p^{-1} \colon p \in \mathfrak{B}\}$ for every $\mathfrak{U} = \mathfrak{CD}$ or $\mathfrak{DC}$ and $\mathfrak{B} \subset \mathfrak{P}$, $\mathfrak{B} \neq \emptyset$ (in other words, both $\mathfrak{CD}$ and $\mathfrak{DC}$ are the subsemigroups of $\mathfrak{P}$)[1], and

$$\mathfrak{CC} \circ \mathfrak{U} = \mathfrak{U} \circ \mathfrak{CC} = \mathfrak{U} \tag{13.3}$$

for every $\mathfrak{U} = \mathfrak{CC}$, $\mathfrak{CD}$, $\mathfrak{DC}$ or $\mathfrak{DD}$ (in fact, family $\mathfrak{CC}$ is a proper subgroup of $\mathfrak{P}$). Family $\mathfrak{CC}$ only seems to be small with respect to the composition of permutations. For example, it is proven in paper [16] that for any $p \in \mathfrak{D}^{-1}$ and $q \in \mathfrak{D}$ there exists a permutation $\rho \in \mathfrak{CC}$ such that $\rho^2 = \mathrm{id}_{\mathbb{N}}$ and $p\rho q \in \mathfrak{DD}$.

The symbol $\circ$ denotes here the composition of nonempty subsets of $\mathfrak{P}$ defined by

$$A \circ B = \{p \circ q = pq \colon p \in A \text{ and } q \in B\}$$

for every $A, B \in \mathfrak{P}$, $A \neq \emptyset$, $B \neq \emptyset$ and where $pq(n) \colon = p(q(n))$ for every $n \in \mathbb{N}$.

For brevity, we will write $K < L$ for any two nonempty subsets K and L of $\mathbb{N}$, whenever $k < l$ for any $k \in K$ and $l \in L$. In the sequel, we will write $k < L$ ($k > L$, respectively) instead of $\{k\} < L$ ($\{k\} > L$, respectively) for any $k \in \mathbb{N}$ and $L \subset \mathbb{N}$, $L \neq \emptyset$. The symbol $\subset$ denotes here the relation of being the proper subset, whereas the symbol $\subseteq$ denotes the relation of being a subset.

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[1] It seems that, considering the subject-matter discussed here, within the framework of $\mathfrak{P} = \mathrm{Sym}(\mathbb{N})$ there appear some completely new problems of algebraic nature! Some classical results in this subject are connected with the results obtained by J. Schreier and S. Ulam, recalled, among others, in paper [7].

Only the intervals of positive integers will be discussed here. For any $n, m \in \mathbb{N}$, such that $n \leq m$, we shall use the following notations:

$$[n,m] := \{n, n+1, \ldots, m\},$$
$$[n,\infty) := \{n, n+1, \ldots\}, \quad (n,\infty) := \{n+1, n+2, \ldots\},$$
$$etc.$$

13.2 Group generated by the convergent permutations

Algebraic and combinatoric properties of convergent and divergent permutations are discussed in many papers (see for example [2, 3, 11, 12, 13, 14]). Especially important is the paper by Pleasants [10] in which it is proven that the group $\mathfrak{G}$ generated by $\mathfrak{C}$ is not equal to $\mathfrak{P}$. We want to note here that the following description of the group $\mathfrak{G}$ (see [22]) could be easily deduced:

$$\mathfrak{G} = \mathfrak{CD} \cup \mathfrak{CD} \circ \mathfrak{DC} \cup \mathfrak{CD} \circ \mathfrak{DC} \circ \mathfrak{CD} \cup \ldots =$$
$$= \mathfrak{DC} \cup \mathfrak{DC} \circ \mathfrak{CD} \cup \mathfrak{DC} \circ \mathfrak{CD} \circ \mathfrak{DC} \cup \ldots$$

Let us set

$$\mathfrak{C}_1 := \mathfrak{CD}, \quad \mathfrak{D}_1 := \mathfrak{DC},$$
$$\mathfrak{C}_2 := \mathfrak{DC} \circ \mathfrak{CD}, \quad \mathfrak{C}_3 := \mathfrak{CD} \circ \mathfrak{C}_2, \quad \mathfrak{C}_4 := \mathfrak{DC} \circ \mathfrak{C}_3, \quad \ldots$$
$$\mathfrak{D}_2 := \mathfrak{CD} \circ \mathfrak{DC}, \quad \mathfrak{D}_3 := \mathfrak{DC} \circ \mathfrak{D}_2, \quad \mathfrak{D}_4 := \mathfrak{CD} \circ \mathfrak{D}_3, \quad \ldots$$

Then $\mathfrak{G} = \bigcup\limits_{n\in\mathbb{N}} \mathfrak{C}_n = \bigcup\limits_{n\in\mathbb{N}} \mathfrak{D}_n$ and even $\mathfrak{G} = \bigcup\limits_{n\in\mathbb{N}} \mathfrak{C}_{f(n)} = \bigcup\limits_{n\in\mathbb{N}} \mathfrak{D}_{f(n)}$ for any increasing function $f\colon \mathbb{N} \to \mathbb{N}$.

We note that by (13.1) we get

$$\mathfrak{DC} \circ \mathfrak{C}_{2n} = \mathfrak{C}_{2n}, \quad \mathfrak{CD} \circ \mathfrak{C}_{2n-1} = \mathfrak{C}_{2n-1},$$
$$\mathfrak{DC} \circ \mathfrak{D}_{2n-1} = \mathfrak{D}_{2n-1}, \quad \mathfrak{CD} \circ \mathfrak{D}_{2n} = \mathfrak{D}_{2n},$$

simultaneously we obtain

$$\mathfrak{C}_n \circ \mathfrak{DC} = \mathfrak{D}_{n+1}, \quad \mathfrak{D}_n \circ \mathfrak{CD} = \mathfrak{C}_{n+1},$$
$$\mathfrak{C}_n \circ \mathfrak{CD} = \mathfrak{C}_n, \quad \mathfrak{D}_n \circ \mathfrak{DC} = \mathfrak{D}_n.$$

With the distinguished families $\mathfrak{C}_n$ and $\mathfrak{D}_n$, $n \in \mathbb{N}$, there is connected a number of interesting properties as well as several unsolved problems.

Theorem 13.3. *The following inclusions hold*

$$\mathfrak{P} \setminus \mathfrak{D}\mathfrak{D} = \mathfrak{C} \cup \mathfrak{C}^{-1} \subset \mathfrak{C}_2 \cap \mathfrak{D}_2, \tag{13.4}$$

$$\mathfrak{C}_n \subseteq \mathfrak{C}_{n+1}, \quad \mathfrak{D}_n \subseteq \mathfrak{D}_{n+1}, \tag{13.5}$$

$$\mathfrak{C}_n \cup \mathfrak{D}_n \subseteq \mathfrak{C}_{n+1} \cap \mathfrak{D}_{n+1}. \tag{13.6}$$

Proof. **(13.4)**: Let $p \in \mathfrak{C}\mathfrak{D}$ and

$$p \circ p := q \in \mathfrak{C}\mathfrak{D}.$$

Then we have

$$p = q \circ p^{-1} \in \mathfrak{D}_2,$$
$$p = p^{-1} \circ q \in \mathfrak{C}_2,$$

i.e.

$$p \in \mathfrak{C}_2 \cap \mathfrak{D}_2.$$

Similar reasoning can be executed for permutation $p \in \mathfrak{D}\mathfrak{C}$. Thus $\mathfrak{C}\mathfrak{D} \cup \mathfrak{D}\mathfrak{C} \subseteq \mathfrak{C}_2 \cap \mathfrak{D}_2$.

Next, from equality (13.3) we get

$$\mathfrak{C}\mathfrak{C} \subset \mathfrak{C}_2 \cap \mathfrak{D}_2.$$

Inclusion (13.4) is sharp and the respective example is given in paper [16].

(13.5) and (13.6): We conduct the inductive proof.

The case $n = 1$ obeys the inequality (13.4). Suppose that relations (13.5) and (13.6) hold for some $n \in \mathbb{N}$. We have either $\mathfrak{C}_{n+2} = \mathfrak{C}_2 \circ \mathfrak{C}_n$, if n is even, or $\mathfrak{C}_{n+1} = \mathfrak{C}_2 \circ \mathfrak{C}_n$, if n is odd, i.e. $\mathfrak{D}_2 \circ \mathfrak{C}_n = \mathfrak{C}_{n+2}$, which by (13.4) implies either

$$\mathfrak{C}_n \cup \mathfrak{C}_{n+1} = (\mathfrak{C}\mathfrak{D} \cup \mathfrak{D}\mathfrak{C}) \circ \mathfrak{C}_n \subseteq \mathfrak{C}_2 \circ \mathfrak{C}_n = \mathfrak{C}_{n+2}$$

or

$$\mathfrak{C}_n \cup \mathfrak{C}_{n+1} = (\mathfrak{C}\mathfrak{D} \cup \mathfrak{D}\mathfrak{C}) \circ \mathfrak{C}_n \subseteq \mathfrak{D}_2 \circ \mathfrak{C}_n = \mathfrak{C}_{n+2},$$

i.e. $\mathfrak{C}_{n+1} \subseteq \mathfrak{C}_{n+2}$. Thus by (13.4) and by the inductive assumption we have

$$\mathfrak{D}_{n+2} = \mathfrak{D}_n \circ \mathfrak{D}_2 \supseteq \mathfrak{D}_n \circ (\mathfrak{C}\mathfrak{D} \cup \mathfrak{D}\mathfrak{C}) = \mathfrak{D}_n \cup \mathfrak{C}_{n+1} = \mathfrak{C}_{n+1},$$

i.e.

$$\mathfrak{C}_{n+2} \cap \mathfrak{D}_{n+2} \supseteq \mathfrak{C}_{n+1}.$$

Similarly we obtain the inclusions

$$\mathfrak{D}_{n+1} \subseteq \mathfrak{D}_{n+2} \quad \text{and} \quad \mathfrak{C}_{n+2} \cap \mathfrak{D}_{n+2} \supseteq \mathfrak{D}_{n+1},$$

i.e.

$$\mathfrak{C}_{n+2} \cap \mathfrak{D}_{n+2} \supseteq \mathfrak{C}_{n+1} \cup \mathfrak{D}_{n+1}.$$

By virtue of the principle of mathematical induction it means that the inequalities (13.5) and (13.6) hold true for every $n \in \mathbb{N}$. □

Problem 13.1. Are the inclusions (13.5) and (13.6) for $n = 2, 3, \ldots$ sharp?

(The Authors suppose that they are. Moreover, the Authors think that $\mathfrak{C}_n \neq \mathfrak{D}_n$, $n \in \mathbb{N}$.)

Remark 13.1. The following result refers also to inclusions (13.4), (13.5) and (13.6) (we treat this result rather as a loyal supplement for our discussion – that is the answer to a problem: what if, in spite of everything, our supposition concerning the inclusions (13.5) and (13.6) and relation $\mathfrak{C}_n \neq \mathfrak{D}_n$, $n \in \mathbb{N}$, is wrong.)

Theorem 13.4. *(see [19])*

a) If $\mathfrak{A}_n = \mathfrak{A}_{n+1}$ for some $\mathfrak{A} \in \{\mathfrak{C}, \mathfrak{D}\}$, $n \in \mathbb{N}$, then $\mathfrak{G} = \mathfrak{A}_n$.
b) If for some $n \in \mathbb{N}$ either $\mathfrak{C}_n \cup \mathfrak{D}_n = \mathfrak{C}_{n+1}$ or $\mathfrak{C}_n \cup \mathfrak{D}_n = \mathfrak{D}_{n+1}$, then we have

$$\mathfrak{G} = \mathfrak{D}_{n+1} = \mathfrak{C}_{n+2} \quad \text{or} \quad \mathfrak{G} = \mathfrak{C}_{n+1} = \mathfrak{D}_{n+2},$$

respectively.
c) If for some $n \in \mathbb{N}$ either $\mathfrak{C}_n \subseteq \mathfrak{D}_n$ or $\mathfrak{D}_n \subseteq \mathfrak{C}_n$, then we have

$$\mathfrak{G} = \mathfrak{D}_{n+1} = \mathfrak{C}_{n+2} \quad \text{or} \quad \mathfrak{G} = \mathfrak{C}_{n+1} = \mathfrak{D}_{n+2},$$

respectively.

Sketch of the proof.
b) Let $n \in \mathbb{N}$ be such that $\mathfrak{C}_n \cup \mathfrak{D}_n = \mathfrak{C}_{n+1}$. Then, with respect to (13.3), we obtain

$$\mathfrak{D}_{n+2} = \mathfrak{C}_{n+1} \circ \mathfrak{D}\mathfrak{C} = (\mathfrak{C}_n \circ \mathfrak{D}\mathfrak{C}) \cup (\mathfrak{D}_n \circ \mathfrak{D}\mathfrak{C}) = \mathfrak{D}_{n+1} \cup \mathfrak{D}_n = \mathfrak{D}_{n+1},$$

from which, in view of *a*), we get $\mathfrak{G} = \mathfrak{D}_{n+1}$.

Moreover, we have

$$\mathfrak{C}_{n+2} = \mathfrak{D}\mathfrak{C} \circ \mathfrak{C}_{n+1} = (\mathfrak{D}\mathfrak{C} \circ \mathfrak{C}_n) \cup (\mathfrak{D}\mathfrak{C} \circ \mathfrak{D}_n) = \mathfrak{C}_n \cup \mathfrak{D}_{n+1} = \mathfrak{D}_{n+1},$$

if only $n \in 2\mathbb{N}-1$, and

$$\mathfrak{C}_{n+2} = \mathfrak{C}\mathfrak{D} \circ \mathfrak{C}_{n+1} = (\mathfrak{C}\mathfrak{D} \circ \mathfrak{C}_n) \cup (\mathfrak{C}\mathfrak{D} \circ \mathfrak{D}_n) = \mathfrak{C}_n \cup \mathfrak{D}_{n+1} = \mathfrak{D}_{n+1},$$

if only $n \in 2\mathbb{N}$. Thus, we have also $\mathfrak{C}_{n+2} = \mathfrak{G}$.

$c)$ Let $n \in \mathbb{N}$ be such that $\mathfrak{C}_n \subseteq \mathfrak{D}_n$. Then we have

$$\mathfrak{C}_{n+1} = \mathfrak{C}\mathfrak{D} \circ \mathfrak{C}_n \subseteq \mathfrak{C}\mathfrak{D} \circ \mathfrak{D}_n = \mathfrak{D}_n,$$

if $n \in 2\mathbb{N}$, otherwise we have

$$\mathfrak{C}_{n+1} = \mathfrak{D}\mathfrak{C} \circ \mathfrak{C}_n \subseteq \mathfrak{D}\mathfrak{C} \circ \mathfrak{D}_n = \mathfrak{D}_n.$$

So, we get $\mathfrak{C}_{n+1} = \mathfrak{D}_n$ and $\mathfrak{C}_n \cup \mathfrak{D}_n = \mathfrak{C}_{n+1}$ which, by $b)$, implies $\mathfrak{G} = \mathfrak{C}_{n+2} = \mathfrak{D}_{n+1}$. □

Corollary 13.1. *If $\mathfrak{G} \neq \mathfrak{C}_n$ for every $n \in \mathbb{N}$ or equivalently $\mathfrak{G} \neq \mathfrak{D}_n$ for every $n \in \mathbb{N}$, then the following relations hold*

$$\mathfrak{C}_n \setminus \mathfrak{D}_n \neq \emptyset, \quad \mathfrak{D}_n \setminus \mathfrak{C}_n \neq \emptyset,$$

$$\mathfrak{C}_{n+1} \setminus (\mathfrak{C}_n \cup \mathfrak{D}_n) \neq \emptyset \quad \text{and} \quad \mathfrak{D}_{n+1} \setminus (\mathfrak{C}_n \cup \mathfrak{D}_n) \neq \emptyset,$$

for every $n \in \mathbb{N}$.

Problem 13.2. Is it true that if the equality holds

$$\mathfrak{C}_n \cup \mathfrak{D}_n = \mathfrak{C}_{n+1} \cap \mathfrak{D}_{n+1},$$

for some $n = n_0 \in \mathbb{N}$, then it means that

$$\mathfrak{C}_n = \mathfrak{C}_{n+1} \quad \text{and} \quad \mathfrak{D}_n = \mathfrak{D}_{n+1},$$

for every $n \geq n_0$?

Remark 13.2. Let $\mathfrak{S}$ denote the family of all permutations p on $\mathbb{N}$ preserving the sum of series (i.e. satisfying the following condition: for every convergent series $\sum a_n$ of real terms, if the series $\sum a_{p(n)}$ is also convergent then $\sum a_{p(n)} = \sum a_n$). The elements of family

$$\mathfrak{I} := \mathfrak{P} \setminus \mathfrak{S}$$

are called, after A. S. Kronrod [5], the substantially singular permutations p of $\mathbb{N}$ (singularity of this permutation consists in the existence of a convergent series $\sum a_n$ of real terms such that the series $\sum a_{p(n)}$ is also convergent but the sums $\sum a_n$ and $\sum a_{p(n)}$ are different). It is known that (see [23]):

$$\mathfrak{C}_2 \cap \mathfrak{I} \neq \emptyset^2, \quad \mathfrak{D}_2 \subset \mathfrak{S} \quad \text{and} \quad \mathfrak{D}_3 \cap \mathfrak{I} \neq \emptyset.$$

Sketch of the proof. The inclusion $\mathfrak{D}_2 \subset \mathfrak{S}$ follows from the definition of $\mathfrak{D}_2$ and the following characterization of sum preserving permutations (see Witula's works [22, 21]):

The permutation $p \in \mathfrak{P}$ belongs to $\mathfrak{S}$ iff there exists a natural number $k = k(p)$ such that for each $n \in \mathbb{N}$ the nonempty finite sets $A_n, B_n \subset \mathbb{N}$ exist and satisfy the conditions:

1) $p(A_n) = B_n$,
2) $[1,n] \subset A_n$,
3) each of sets A_n and B_n is a union of k **MSI**.

The relation $\mathfrak{C}_2 \cap \mathfrak{I} \neq \emptyset$ results from the following relations (see [23]):

$$\mathfrak{F} \subseteq \mathfrak{C}_2 \quad \text{and} \quad \mathfrak{F} \cap \mathfrak{I} \neq \emptyset,$$

where $\mathfrak{F}$ denotes the family of all permutations $p \in \mathfrak{P}$, for which there exists the finite partition $N_1, N_2, \ldots, N_k$ of the set of natural numbers such that the restriction $p|_{N_i}$ is an increasing map for every $i = 1, 2, \ldots, k$. The respective example of permutation $p \in \mathfrak{C}_2 \cap \mathfrak{I}$ is given also in paper [23]. Then the relation $\mathfrak{D}_3 \cap \mathfrak{I} \neq \emptyset$ results from the inclusion (13.6) for $n = 2$. □

Since $\mathfrak{I} \subset \mathfrak{D}\mathfrak{D}$ and $\mathfrak{D}_2 \subset \mathfrak{S}$, therefore by (13.1) and (13.2) we get

$$(\mathfrak{C}\mathfrak{D} \cup \mathfrak{D}\mathfrak{C}) \circ \mathfrak{I} \subset \mathfrak{D}\mathfrak{D}. \tag{13.7}$$

For the contrast let us notice that $(\mathfrak{I} \circ \mathfrak{D}\mathfrak{C}) \cap \mathfrak{D}\mathfrak{C} \neq \emptyset$ since $\mathfrak{C}_2 \cap \mathfrak{I} \neq \emptyset$. From the combinatoric characterization of the permutations preserving the sum the following equality results

$$\mathfrak{C}\mathfrak{C} \circ \mathfrak{S} = \mathfrak{S} \circ \mathfrak{C}\mathfrak{C} = \mathfrak{S}.$$

In consequence, with respect to definition $\mathfrak{I} = \mathfrak{P} \setminus \mathfrak{S}$ we also have

$$\mathfrak{C}\mathfrak{C} \circ \mathfrak{I} = \mathfrak{I} \circ \mathfrak{C}\mathfrak{C} = \mathfrak{I}.$$

Thus, the relation (13.7) takes the more general form

$$(\mathfrak{C} \cup \mathfrak{C}^{-1}) \circ \mathfrak{I} \subset \mathfrak{D}\mathfrak{D}. \tag{13.8}$$

[2] We are troubled by the following problem: is the family $\mathfrak{C}_2 \cap \mathfrak{I} = \mathfrak{C}_2 \setminus \mathfrak{S}$ algebraically big (definition of the algebraically big subset of $\mathfrak{P}$ is given in Section 4 of this paper)?

Historical remark

Families $\mathfrak{C}$, $\mathfrak{D}$, $\mathfrak{CC}$, $\mathfrak{CD}$, $\mathfrak{DC}$, $\mathfrak{DD}$, $\mathfrak{C}_2$ and $\mathfrak{D}_2$ have been introduced for the first time by A. S. Kronrod [5] in 1946. He noticed that

$$\mathfrak{CC} \circ \mathfrak{A} = \mathfrak{A} \circ \mathfrak{CC} = \mathfrak{A} \tag{13.9}$$

for each listed above subfamily $\mathfrak{A}$ of family $\mathfrak{P}$. However, Kronrod did not discover any essential algebraic connections between these families of permutations.

Additionally, let us notice that equality (13.9) holds also for $\mathfrak{A} = \mathfrak{C}_n$, $\mathfrak{D}_n$, $n \in \mathbb{N}$, and for $\mathfrak{A} = \mathfrak{G}$.

13.3 Families $\mathfrak{D}(k)$

We denote by $\mathfrak{D}(k)$, for every $k \in \mathbb{N}$, the set of all divergent permutations p for which there exists an increasing sequence $\{r_n(p)\}_{n=1}^{\infty}$ of positive integers such that the set $p^{-1}([1, r_n(p)])$ is a union of at most k **MSI** for every $n \in \mathbb{N}$. Certainly $\bigcup\limits_{k\in\mathbb{N}} \mathfrak{D}(k) \subset \mathfrak{G}$ (see Remark 13.2).

Remark 13.3. To prove that the permutation p does not belong to $\mathfrak{D}(k)$, for given $k \in \mathbb{N}$, we need to show that there exists $N \in \mathbb{N}$ such that the set $p^{-1}([1,n])$ is a union of at least $k+1$ **MSI** for every $n \in \mathbb{N}$, $n \geq N$. This fact will be used here in the proof of Theorem 13.6 and in Examples 13.2 and 13.3.

Families $\mathfrak{D}(k)$ are related, in a very interesting way, to the Riemann Derangement Theorem. For the fact, that for the given conditionally convergent series $\sum a_n$ and an interval $I = [\beta, \gamma] \subset [-\infty, \infty]$ there exists $p \in \mathfrak{D}(k)$ such that the set of limit points of the $p-$rearranged series $\sum a_{p(n)}$ is equal to I, it is necessary that

$$\begin{aligned}\sum a_n \in &\left[-(2k-1)\tfrac{\gamma-\beta}{2} + \tfrac{\gamma+\beta}{2}, (2k-1)\tfrac{\gamma-\beta}{2} + \tfrac{\gamma+\beta}{2}\right] \\ &= [-(k-1)\gamma + k\beta, k\gamma - (k-1)\beta]\end{aligned}$$

(see [25, 24]).

The basic algebraic properties of families $\mathfrak{D}(k)$, $k \in \mathbb{N}$, are given in the theorem presented below (the proof is omitted here).

Theorem 13.5. *The following relations hold*

$$\mathfrak{DC} \cap \bigcup_{k\in\mathbb{N}} \mathfrak{D}(k) = \mathfrak{DC},$$

$$\mathfrak{DD} \setminus \bigcup_{k\in\mathbb{N}} \mathfrak{D}(k) \neq \emptyset,$$

$$\mathfrak{DC} \cap \left(\mathfrak{D}(k) \setminus \bigcup_{l<k} \mathfrak{D}(l)\right) \neq \emptyset, \tag{13.10}$$

$$\mathfrak{DD} \cap \left(\mathfrak{D}(k) \setminus \bigcup_{l<k} \mathfrak{D}(l)\right) \neq \emptyset. \tag{13.11}$$

Remark 13.4. The respective examples illustrating the relations (13.10) and (13.11) are given at the end of this paper, in section containing the examples.

Remark 13.5. For every $k \in \mathbb{N}$, $k \geq 2$, there exists a permutation $p \in (\mathfrak{D}(k) \setminus \bigcup_{l<k} \mathfrak{D}(l))$ and such that $p^{-1} \notin \bigcup_{l\in\mathbb{N}} \mathfrak{D}(l)$, equivalently meaning that

$$\lim_{n\to\infty} c(p,n) = \infty, \tag{13.12}$$

where $c(p,n)$ denotes the number of **MSI** forming the partition of set $p([1,n])$ for every $n \in \mathbb{N}$ (the respective example is given in [19]).

It is worth to mention that the condition (13.12) is necessary and sufficient for the existence of a conditionally convergent series $\sum a_n$ such that the series $\sum a_{p(n)}$ is divergent to ∞ (see [4, 20]).

In paper [4] the example of permutation $p \in \mathfrak{S}$ is given such that

$$\lim_{n\to\infty} c(p,n) = \lim_{n\to\infty} c(p^{-1},n) = \infty.$$

At the end of this section we present one more theorem (our new result) connected with the decomposition of permutation p on $\mathbb{N}$ with respect to families $\mathfrak{D}(k)$.

Theorem 13.6. *Let $p \in \mathfrak{P}$ and let k be a limit point of the sequence*

$$\{c(p^{-1},n)\}_{n=1}^{\infty}.$$

Then there exists the permutations $\varphi \in \mathfrak{D}(k) \setminus \bigcup_{l<k} \mathfrak{D}(l)$ and $\psi \in \mathfrak{D}(1)$ such that $p = \psi\varphi$. In other words, we have then $p \in \mathfrak{D}(1) \circ (\mathfrak{D}(k) \setminus \bigcup_{l<k} \mathfrak{D}(l))$.

Sketch of the proof. Let us fix the increasing sequence $\{r_j\}_{j=1}^{\infty}$ of natural numbers such that

(1) each of the sets $p^{-1}([1,r_j])$, $j \in \mathbb{N}$ is a union of k **MSI**;
(2) $[1, 1+m_j] \subset p^{-1}([1,r_{j+1}])$ for every $j \in \mathbb{N}$, where $m_j := \max p^{-1}([1,r_j])$.

We define permutation φ in the following way.

At first we fix, for every $j \geq 2$, the increasing sequences $\{s_i^{(j)} : i = 1,2,\dots,k\}$ of natural numbers from the set $p^{-1}([1,r_j])$ such that

$$s_1^{(j)} > m_{j-1},$$

$$(s_i^{(j)}, s_{i+1}^{(j)}) \setminus p^{-1}([1,r_j]) \neq \emptyset,$$

for each index $i = 1,2,\dots,k-1$.

We take that the restriction of φ to the set $p^{-1}([1,r_1])$ is an increasing map on interval $[1,r_1]$. Next, we define φ as the increasing map:

- of the set $\{s_i^{(2)} : i = 1,2,\dots,k\}$ on interval $(r_1, r_1 + k]$;
- of the set $p^{-1}([1,r_2]) \setminus \left(p^{-1}([1,r_1]) \cup \{s_i^{(2)} : i = 1,2,\dots,k\}\right)$ on interval $(r_1 + k, r_2]$;
- of the set $\{s_i^{(3)} : i = 1,2,\dots,k\}$ on interval $(r_2, r_2 + k]$;
- of the set $p^{-1}([1,r_3]) \setminus \left(p^{-1}([1,r_2]) \cup \{s_i^{(3)} : i = 1,2,\dots,k\}\right)$ on interval $(r_2 + k, r_3]$;

and so on.

Certainly, permutation ψ is defined by the relation $p = \psi\varphi$. Let us only notice that $\psi([1,r_j]) = [1,r_j]$, $j \in \mathbb{N}$, so $\psi \in \mathfrak{D}(1)$. Next, $\varphi^{-1}([1,n])$ is a union of at least k **MSI** for every $n \geq r_1 + k$. Moreover, $\varphi^{-1}([1,r_n])$ is a union of k **MSI**, for every $n = 1,2,\dots$ $\square$

Remark 13.6. We think (?) that the following relation holds true as well

$$\mathfrak{I} \circ \left(\mathfrak{D}(k) \setminus \bigcup_{l<k} \mathfrak{D}(l)\right) = \left(\mathfrak{D}(k) \setminus \bigcup_{l<k} \mathfrak{D}(l)\right) \circ \mathfrak{I} = \mathfrak{P},$$

for every $k \in \mathbb{N}$.

13.4 Algebraically big subsets of $\mathfrak{P}$

We say that the nonempty family $\mathfrak{A} \subset \mathfrak{P}$ is algebraically big if $\mathfrak{A} \circ \mathfrak{A} = \mathfrak{P}$. Many of the subsets of $\mathfrak{P}$, discussed in this paper, are algebraically big or algebraically small (it means that the group generated by a given set of $\mathfrak{P}$ is different than $\mathfrak{P}$). Simultaneously, let us notice that there exists the subsets of set $\mathfrak{P}$ which are neither algebraically big nor algebraically small (see [18]).

It appears that $\mathfrak{D}(1)$ is algebraically big (see G. Stoller [15])[3] as well as the sets $\mathfrak{D}\mathfrak{D}$, $\mathfrak{S}$ (see P.A.B. Pleasants [10] and independently F.W. Levi [6]) and $\mathfrak{I} = \mathfrak{D} \setminus \mathfrak{S}$. Many other algebraically big subsets of $\mathfrak{P}$ are also distinguished in paper [18].

In paper [17] we have noticed that if G is a proper subgroup of the group $\mathfrak{P}$ and there exists an element $g \in \mathfrak{P} \setminus G$ such that also $g^2 \in \mathfrak{P} \setminus G$, then the set $\mathfrak{P} \setminus G$ is algebraically big. In particular, the family $\mathfrak{P} \setminus \mathfrak{G}$ is algebraically big since, as it is proven in paper [27], the following fact holds true.

Theorem 13.7. *For every $k \in \mathbb{N}$, $k > 1$, the equation $p^k = \mathrm{id}_{\mathbb{N}}$ possesses a solution $p \in \mathfrak{P} \setminus \mathfrak{G}$ such that $p^i \in \mathfrak{P} \setminus \mathfrak{G}$ for every $i = 1, 2, \ldots, k-1$.*

Moreover, in [27] we prove the following surprising result.

Theorem 13.8. *If $p \in \mathfrak{P} \setminus \mathfrak{G}$ and $p^2 \in \mathfrak{P} \setminus \mathfrak{G}$, then there exists a family $\mathfrak{B} = \mathfrak{B}(p) \subset \mathfrak{P} \setminus \mathfrak{G}$ such that* $\operatorname{card}\mathfrak{B} = \mathfrak{c}$ *and for every permutation $q \in \mathfrak{B}$ we have*

$$pq \in \mathfrak{C} \quad \text{and} \quad qp \in \mathfrak{C}.$$

Remark 13.7. Theorems 13.7 and 13.8 are more general in comparison with the cited before Pleasants' result that $\mathfrak{G} \neq \mathfrak{P}$ (see [10]).

One more important result, proven in paper [27], refers to the above relation.

Theorem 13.9. *If we fix an increasing sequence $\{n_k\}_{k=0}^{\infty}$ of natural numbers such that $n_0 = 1$ and*

$$\limsup(n_{k+1} - n_k) = \infty, \tag{13.13}$$

then the family $G = G\left(\{n_k\}_{k=0}^{\infty}\right)$ of permutations $p \in \mathfrak{P}$, such that the permutation p maps every interval $[n_{k-1}, n_k)$, $k = 1, 2, \ldots$, onto itself, satisfies the following conditions:

- *it is a group of permutations on $\mathbb{N}$,*
- *$G \cap \mathfrak{D} = G \cap \mathfrak{D}(1)$ (certainly the remaining elements from G belong to $\mathfrak{C}$),*

[3] We also expect the following relation

$$\mathfrak{D}(1) \circ (\mathfrak{D}(k) \setminus \bigcup_{i<k} \mathfrak{D}(i)) = (\mathfrak{D}(k) \setminus \bigcup_{i<k} \mathfrak{D}(i)) \circ \mathfrak{D}(1) = \mathfrak{P},$$

for every $k = 2, 3, \ldots$ Slightly more sophisticated seems to be a question whether also

$$(\mathfrak{D}(k) \setminus \bigcup_{i<k} \mathfrak{D}(i)) \circ (\mathfrak{D}(l) \setminus \bigcup_{i<l} \mathfrak{D}(i)) = \mathfrak{P}$$

for any $k, l \in \{2, 3, \ldots\}$?

- *$G \subset \mathfrak{S}$,*
- *$G \cap \mathbf{X} \neq \emptyset$ for every $\mathbf{X} \in \{\mathfrak{CC}, \mathfrak{CD}, \mathfrak{DC}, \mathfrak{DD}\}$,*
- *$G \setminus \mathfrak{G} \neq \emptyset$ (which gives an essential strengthening of Pleasants' result that $\mathfrak{G} \neq \mathfrak{P}$),*
- *if $\lim\limits_{k\to\infty} \frac{n_{k+1}}{n_k} = 1$ then G is a subgroup of the, so called, Levy group (see [9]).*

Corollary 13.2. *We have*
$$\mathfrak{D}(1) \setminus \mathfrak{G} \neq \emptyset,$$
that is we also have
$$\mathfrak{D}(1) \setminus (\mathfrak{D}_n \cup \mathfrak{C}_n) \neq \emptyset$$
for every $n \in \mathbb{N}$ and
$$\mathfrak{D}(1) \subset \mathfrak{S}.$$

Moreover, let us notice that the last inclusion results also from the combinatoric characterization of permutation preserving the sum of rearranged series (see [21], [8] and [22]).

Remark 13.8. As it follows from the original Sierpiński's construction of the, so called, Sierpiński family S of the increasing sequences of natural numbers, pairwise almost disjoint, if $\{n_k\}_{k=1}^{\infty} \in S$ then
$$\lim_{k\to\infty} (n_{k+1} - n_k) = \infty. \tag{13.14}$$

Furthermore, card$S = \mathfrak{c}$. Thus, the family of increasing sequences of natural numbers $\{n_k\}_{k=1}^{\infty}$, satisfying condition (13.14), possesses the cardinality of the continuum. In result we obtain the existence of continuum many subgroups G of group $\mathfrak{P}$, such as the ones described above, for which, among others, the condition $G \setminus \mathfrak{G} \neq \emptyset$ holds. We manage to make this selection of subgroups of group $\mathfrak{P}$ more subtle and to connect it with the concept of incomparability of the convergence classes of permutations (it is important because we do not want to get too far away with the current considerations from the roots of this research, that is from the discussion devoted to the rearranged scalar series). We received the following result (see [28]).

Theorem 13.10. *There exists a family $\{G_x \colon x \in \mathbb{R}\}$ of subgroups of the group $\mathfrak{P}$ satisfying the conditions from Theorem 13.9 and, what is more, such that for every pair $x, y \in \mathbb{R}$, $x \neq y$, there exist the subsets $G'_x \subset G_x$ and $G'_y \subset G_y$, both of the power of the continuum and such that if $p \in G'_x$, $q \in G'_y$ then p and q are incomparable, that is*
$$\sum(p) \setminus \sum(q) \neq \emptyset \quad \text{and} \quad \sum(q) \setminus \sum(p) \neq \emptyset,$$

where $\sum(\lambda)$ denotes the convergence class of permutation $\lambda \in \mathfrak{P}$, i.e. the family of all convergent real series $\sum a_n$ for which the $\lambda-$rearranged series $\sum a_{\lambda(n)}$ is also convergent.

13.5 Examples

Example 13.2. Let $k \in \mathbb{N}$. We give an example of permutation

$$p \in \mathfrak{D}\mathfrak{C} \cap \left(\mathfrak{D}(k) \setminus \bigcup_{l<k} \mathfrak{D}(l) \right),$$

more precisely, of the permutation p on $\mathbb{N}$ such that $\sigma(p^{-1}) = \{k, k+1\}$ and $\sigma(p) = [k, +\infty]$, where $\sigma(p)$ denotes the set of limit points of the sequence $\{c(p,n)\}_{n=1}^{\infty}$.

For this aim let us set the increasing sequence $\{r_n\}$ of natural numbers, where $r_1 = 1$, satisfying the conditions:

1^0 $\quad r_{n+2} - r_{n+1} > r_{n+1} - r_n$
and
2^0 $\quad (r_{n+1} - r_n - 1) \in 2\mathbb{N}$,

for every $n \in \mathbb{N}$. Let us take

$$s_n := \frac{1}{2}(r_{n+1} - r_n - 1),$$

for $n = 1, 2, \ldots$

For clarity of this construction, instead of permutation p we define below the permutation p^{-1}, that is the inverse of permutation p.

Construction of permutation p^{-1} will be carried out in the successive, countably many steps, characterized by the same scenario. With reference to the last statement we describe only three initial steps of constructing permutation p^{-1}.

We take that the permutation p^{-1} is

1) the increasing map of interval $[1, k+1]$ onto the set $\{r_i \colon i \in [1, k+1]\}$;
2*a*) the increasing map of set $\{k+2, k+4, \ldots, k+2s_1\}$ onto the interval $(r_1, r_1+s_1]$;
2*b*) the decreasing map of set $\{k+3, k+5, \ldots, k+2s_1+1\}$ onto the interval (r_1+s_1, r_2);
2*c*) $p^{-1}(t_1) = r_{k+2}$, where $t_1 := k + 2s_1 + 2$;

$3a)$ the increasing map of set $\{t_1+1, t_1+3, \ldots, t_1+2s_2-1\}$ onto the interval $(r_2, r_2+s_2]$;
$3b)$ the decreasing map of set $\{t_1+2, t_1+4, \ldots, t_1+2s_2\}$ onto the interval (r_2+s_2, r_3);
$3c)$ $p^{-1}(t_2) = r_{k+3}$, where $t_2 := t_1+2s_2+1$;
$4a)$ the increasing map of set $\{t_2+1, t_2+3, \ldots, t_2+2s_3-1\}$ onto the interval $(r_3, r_3+s_3]$;
$4b)$ the decreasing map of set $\{t_2+2, t_2+4, \ldots, t_2+2s_3\}$ onto the interval (r_3+s_3, r_4);
$4c)$ $p^{-1}(t_3) = r_{k+4}$, where $t_3 := t_2+2s_3+1$;

and so on.

Let us notice that

(i) with respect to conditions $2a)$, $3a)$, $\ldots$, as well as $2b)$, $3b)$, $\ldots$, p is the divergent permutation;
(ii) conditions $1), 2a)-c), 3a)-c), \ldots$ imply the relation

$$\sigma(p^{-1}) = \{k, k+1\};$$

(iii) conditions 1^0 and $2a), 3a), \ldots$ imply the relation

$$\limsup_{n\to\infty} c(p,n) \geq \limsup_{n\to\infty} s_n = +\infty,$$

that is

$$\limsup_{n\to\infty} c(p,n) = +\infty;$$

(iv) each of the sets $p([1, r_n])$, $n \in \mathbb{N}$, $n \geq k+1$ is a union of k **MSI** and, what is more,

$$\liminf_{n\to\infty} c(p,n) = k.$$

Example 13.3. We give now an example of permutation

$$p \in \mathfrak{DD} \cap \left(\mathfrak{D}(k) \setminus \bigcup_{l<k} \mathfrak{D}(l) \right).$$

Let $\{r_n\}$ be any increasing sequence of natural numbers such that

$$r_{n+1} - r_n > r_n - r_{n-1} > 2, \tag{13.15}$$

for every $n \in \mathbb{N}$, $n > 1$. Similar as in Example 13.2, for clarity of the discussion, we define p^{-1} instead of p. Let us describe only few initial steps of the inductive construction of p^{-1}.

We take that p^{-1} is the increasing map

$1a)$ of interval $[1,k]$ onto the set $\{r_i\colon i\in[1,k]\}$ (spreading out);
$1b)$ of set $\{k+1,k+3,\dots,k+2(r_2-r_1-1)-1\}$ onto the interval (r_1,r_2) (filling in the first free gap);
$1c)$ of set $\{k+2,k+4,\dots,k+2(r_2-r_1-1)\}$ onto the set $\{r_i\colon i\in[k+1,k+r_2-r_1)\}$ (spreading out);
$1d)$ of interval $(k+2(r_2-r_1-1),t_1]$ onto the set $\bigcup_{i=2}^{r_2-r_1-1}(r_i,r_{i+1})$, where

$$t_1 := k+2(r_2-r_1-1)+\sum_{i=2}^{r_2-r_1-1}(r_{i+1}-r_i-1)$$

(we fill in, successively, all the remaining gaps, except the $(k-1)$ last of them);
$2a)$ of interval $(t_1,t_1+k+1]$ onto the set $\{r_i\colon i\in[j_1+k,j_1+2k]\}$, where $j_1:=r_2-r_1$ (spreading out);
$2b)$ of set $\{t_1+k+2,t_1+k+4,\dots,t_1+k+2(r_{j_1+1}-r_{j_1}-1)\}$ onto the interval (r_{j_1},r_{j_1+1}) (filling in the first free gap – see item $1d)$);
$2c)$ of set $\{t_1+k+3,t_1+k+5,\dots,t_1+k+2(r_{j_1+1}-r_{j_1}-1)+1\}$ onto the set $\{r_i\colon i\in(j_1+2k,l_1)\}$, where $l_1:=j_1+2k+r_{j_1+1}-r_{j_1}$ (spreading out);
$2d)$ of interval $[t_1+k+2(r_{j_1+1}-r_{j_1}-1)+2,t_2]$ onto the set $\bigcup_{i=j_1+1}^{l_1-k-1}(r_i,r_{i+1})$, where

$$t_2 := t_1+k+2(r_{j_1+1}-r_{j_1}-1)+1+\sum_{i=j_1+1}^{l_1-k-1}(r_{i+1}-r_i-1)$$

(we fill in the successive gaps, except the $(k-1)$ last of the gaps which we leave unchanged);
$3a)$ of interval $(t_2,t_2+k+2]$ onto the set $\{r_i\colon i\in[l_1,l_1+k+1]\}$ (spreading out);
$3b)$ of set $\{t_2+k+3,t_2+k+5,\dots,t_2+k+2(r_{j_2+1}-r_{j_2}-1)+1\}$ onto the interval (r_{j_2},r_{j_2+1}), where $j_2:=l_1-k$ (shortly speaking, we fill in the first free gap);
$3c)$ of set $\{t_2+k+4,t_2+k+6,\dots,t_2+k+2(r_{j_2+1}-r_{j_2}-1)+2\}$ onto the set $\{r_i\colon i\in[l_1+k+2,l_2]\}$, where $l_2:=l_1+k+r_{j_2+1}-r_{j_2}$ (spreading out);
$3d)$ of interval $[t_2+k+2(r_{j_2+1}-r_{j_2}-1)+3,t_3]$ onto the set $\bigcup_{i=j_2+1}^{l_2-k}(r_i,r_{i+1})$,

where

$$t_3 := t_2 + k + 2(r_{j_2+1} - r_{j_2} - 1) + 2 + \sum_{i=j_2+1}^{l_2-k} (r_{i+1} - r_i - 1)$$

(filling in all the gaps, except the $(k-1)$ last of the gaps which we leave untouched);

and so on.

Let us notice that

(i) with respect to assumption (13.15) the conditions $1b)$, $2b)$, $3b)$, ..., guarantee the divergence of permutation p;

(ii) conditions $2a)$, $3a)$, $4a)$, ..., guarantee the divergence of the inverse of permutation p;

(iii) conditions $1d)$, $2d)$, $3d)$, ..., imply the existence of increasing sequence $\{t_n\}_{n=1}^{\infty}$ of natural numbers such that each set $p^{-1}([1,t_n])$ is a union of k **MSI**;

(iv) pairs of conditions $1b)$ and $1c)$, $2b)$ and $2c)$, $3b)$ and $3c)$, ..., as well as $1d)$ and $2a)$, $2d)$ and $3a)$, ..., guarantee that

$$\liminf_{n\to\infty} c(p^{-1}, n) \geq k.$$

Final comments

We recommend to the Readers, interested in extending the spectrum of their knowledge concerning the issues presented in this paper, also the paper [21], which is especially valuable considering the combinatoric-analytical and typical set-theoretic circle of interests. We want to mention that, in spite of everything, we focused in the current paper on several selected aspects of algebraic problems. We omitted, among others, the problem of commutation (see [26]), the problem of decomposing the permutation into cycles (see [18]), the subject-matter concerning the semigroups of $\mathfrak{C}$ (subgroups of $\mathfrak{CC}$, respectively) and so on. We did not sacrifice either any space for the topic of connections with the respective topologies. However, we believe that the issues undertaken in this paper may be still intriguing and we also count on the possibility of adapting them in some wider context.

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