

Chapter 8

Path continuity connected with density and porosity

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8.1 Preliminary

The notion of path continuity appeared in the theory of real functions at the beginning of XXth century. Path continuity means that at every $x \in \mathbb{R}$ a family \mathcal{B}_x of subsets of \mathbb{R} , „big” in some sense near x and containing x , is defined and $f: \mathbb{R} \rightarrow \mathbb{R}$ is continuous at x_0 relative to $\{\mathcal{B}_x: x \in \mathbb{R}\}$ if there exists $E \in \mathcal{B}_{x_0}$ such that $f|_E$ is continuous at x_0 . Taking different families of sets we get different kinds of path continuity.

The first and most important, without a doubt, is an approximate continuity, where as \mathcal{B}_x we take a family of measurable sets for which x is a point of density. In [6] a lot types of path continuity is described, including preponderant continuity, qualitative continuity, Császár’s continuity and congruent continuity. Applying system of paths other properties of functions can be generalized, including derivative. Deep studying of generalized path derivatives can be found in [7]. In [27] majority of notions of real analysis is defined based on system of paths. In [1, 2, 3] basic properties of path continuity

were studied. Other problems connected with path continuity were studied in [4, 11, 15, 20, 21, 22].

In our paper we concentrate on path continuity related to the notions of porosity and lower and upper density.

We will use standard notations. \mathbb{R} denotes the set of reals and \mathbb{N} denotes the set of positive integers. For $f: \mathbb{R} \rightarrow \mathbb{R}$, let $\mathcal{C}(f)$, $D(f)$, $\mathcal{Q}(f)$ and $D_{ap}(f)$ denote the sets of points at which f is continuous, discontinuous, quasicontinuous and approximately discontinuous, respectively. The symbols \mathcal{C} , \mathcal{Q} and \mathcal{A} stand for the sets of continuous functions, quasicontinuous functions and approximately continuous functions, respectively. Moreover, \mathcal{C}^\pm is the set of all functions $f: \mathbb{R} \rightarrow \mathbb{R}$ such that at every $x \in \mathbb{R}$, f is continuous from the left or from the right (obviously $\mathcal{C} \subsetneq \mathcal{C}^\pm$). Finally, μ denotes the Lebesgue measure on \mathbb{R} and $N_f = \{x: f(x) = 0\}$.

8.2 Path continuity connected with density

Path continuities defined via lower and upper density were introduced in [7, 13, 23]. They are generalizations of preponderant density which was defined by Denjoy in [8].

Let E be a measurable subset of \mathbb{R} and let $x \in \mathbb{R}$. The numbers

$$\underline{d}^+(E, x) = \liminf_{t \rightarrow 0^+} \frac{\mu(E \cap [x, x+t])}{t} \quad \text{and} \quad \bar{d}^+(E, x) = \limsup_{t \rightarrow 0^+} \frac{\mu(E \cap [x, x+t])}{t}$$

are called the right lower density of E at x and right upper density of E at x , respectively. The left lower and upper densities of E at x are defined analogously. If

$$\underline{d}^+(E, x) = \bar{d}^+(E, x) \quad (\underline{d}^-(E, x) = \bar{d}^-(E, x))$$

then we call this number the right density (left density) of E at x and denote it by $d^+(E, x)$ ($d^-(E, x)$). The numbers

$$\bar{d}(E, x) = \limsup_{\substack{t \rightarrow 0^+ \\ k \rightarrow 0^+}} \frac{\mu(E \cap [x-t, x+k])}{k+t}, \quad \underline{d}(E, x) = \liminf_{\substack{t \rightarrow 0^+ \\ k \rightarrow 0^+}} \frac{\mu(E \cap [x-t, x+k])}{k+t}$$

are called the upper and lower density of E at x , respectively. If $\bar{d}(E, x) = \underline{d}(E, x)$, we call this number the density of E at x and denote it by $d(E, x)$.

Let us observe that

$$\bar{d}(E, x) = \max \{ \bar{d}^+(E, x), \bar{d}^-(E, x) \} \text{ and } \underline{d}(E, x) = \min \{ \underline{d}^+(E, x), \underline{d}^-(E, x) \}.$$

Moreover, it is clear that

$$\bar{d}^+(E, x) + \underline{d}^+(\mathbb{R} \setminus E, x) = 1 \quad \text{and} \quad \bar{d}^-(E, x) + \underline{d}^-(\mathbb{R} \setminus E, x) = 1.$$

Definition 8.1. [13, 14] Let E be a measurable subset of \mathbb{R} . Let $x \in \mathbb{R}$ and $0 < \rho \leq 1$. We say that the point x is a point of ρ -type upper density of E if $\bar{d}(E, x) > \rho$ when $\rho < 1$ or if $\bar{d}(E, x) = 1$ when $\rho = 1$.

Definition 8.2. [13, 14] Let $\rho \in (0, 1]$. A real-valued function f defined on \mathbb{R} is called ρ -upper continuous at x provided that there is a measurable set $E \subset \mathbb{R}$ such that the point x is a point of ρ -type upper density of E , $x \in E$ and $f|_E$ is continuous at x . If f is ρ -upper continuous at every point of \mathbb{R} we say that f is ρ -upper continuous.

We will denote the class of all ρ -upper continuous functions by \mathcal{UC}_ρ .

Corollary 8.1. Let $0 < \rho < \rho_1 \leq 1$, $f: \mathbb{R} \rightarrow \mathbb{R}$ and $x \in \mathbb{R}$. If f is ρ_1 -upper continuous at x , then f is ρ -upper continuous at x .

Theorem 8.1. [13] Let $\rho, \rho_1 \in (0, 1]$. Then $\mathcal{UC}_\rho \subset \mathcal{UC}_{\rho_1}$ if and only if $\rho_1 \leq \rho$. Moreover, if $\rho_1 < \rho$ then $\mathcal{UC}_\rho \subsetneq \mathcal{UC}_{\rho_1}$.

In an obvious way we define one-sided ρ -upper continuity, and f is ρ -upper continuous at x if and only if it is ρ -upper continuous at x from the right or from the left. Moreover, if a function is continuous from the right or from the left at some point then it is ρ -upper continuous at this point.

Next definitions are based on the notion of preponderant continuity in O'Malley sense [25, 12].

Definition 8.3. [17] Let $\rho \in (0, 1)$. A point x is said to be a point of ρ -type upper density in O'Malley sense of a measurable set E if for each $\varepsilon > 0$ there exists $y \in (x - \varepsilon, x + \varepsilon)$ such that for the closed interval J with endpoints x and y the inequality $\frac{\mu(E \cap J)}{|x - y|} > \rho$ holds.

Definition 8.4. [17] Let $\rho \in (0, 1)$. A function $f: \mathbb{R} \rightarrow \mathbb{R}$ is said to be ρ -upper continuous in O'Malley sense at $x \in \mathbb{R}$ (f is O'Malley ρ -upper continuous at x , in abbreviation) if there exists a measurable set $E \subset \mathbb{R}$ containing x such that x is a point of ρ -type upper density in O'Malley sense of E and $f|_E$ is continuous at x . A function f is said to be O'Malley ρ -upper continuous if it is O'Malley ρ -upper continuous at each point of \mathbb{R} .

We denote the class of O'Malley ρ -upper continuous functions by $\mathcal{O}\mathcal{U}\mathcal{C}_\rho$.

Corollary 8.2. *Let $0 < \rho < \rho_1 < 1$, $f: \mathbb{R} \rightarrow \mathbb{R}$ and $x \in \mathbb{R}$. If f is O'Malley ρ_1 -upper continuous at x , then f is O'Malley ρ -upper continuous at x .*

Theorem 8.2. [17] *Let $\rho, \rho_1 \in (0, 1)$. Then $\mathcal{O}\mathcal{U}\mathcal{C}_\rho \subset \mathcal{O}\mathcal{U}\mathcal{C}_{\rho_1}$ if and only if $\rho_1 \leq \rho$. Moreover, if $\rho_1 < \rho$ then $\mathcal{O}\mathcal{U}\mathcal{C}_\rho \subsetneq \mathcal{O}\mathcal{U}\mathcal{C}_{\rho_1}$.*

In an obvious way we define one-sided O'Malley ρ -upper continuity and f is O'Malley ρ -upper continuous at x if and only if it is O'Malley ρ -upper continuous at x from the right or from the left. Moreover, if a function is continuous from the right or from the left at some point then it is O'Malley ρ -upper continuous at this point.

Now, we give next definition of path continuity connected with lower and upper density simultaneously.

Definition 8.5. [23] Let E be a measurable subset of \mathbb{R} and let $0 < \lambda \leq \rho \leq 1$. We say that a point $x \in \mathbb{R}$ is a point of $[\lambda, \rho]$ -density of E if $\underline{d}(E, x) > \lambda$ and $\bar{d}(E, x) > \rho$ for $\rho < 1$ or if $\underline{d}(E, x) > \lambda$ and $\bar{d}(E, x) = 1$ for $\rho = 1$ and $\lambda < 1$ or if $\underline{d}(E, x) = 1$ for $\lambda = 1$.

Definition 8.6. [23] Let $0 < \lambda \leq \rho \leq 1$. A function $f: \mathbb{R} \rightarrow \mathbb{R}$ is called $[\lambda, \rho]$ -continuous at $x \in \mathbb{R}$, provided that there is a measurable set $E \subset \mathbb{R}$ such that x is a point of $[\lambda, \rho]$ -density of E , $x \in E$ and $f|_E$ is continuous at x . If f is $[\lambda, \rho]$ -continuous at each point of \mathbb{R} , we say that f is $[\lambda, \rho]$ -continuous.

We will denote the class of all $[\lambda, \rho]$ -continuous functions by $\mathcal{C}_{[\lambda, \rho]}$.

Remark 8.1. $\mathcal{C}_{[1, 1]}$ is the class of approximately continuous functions and $\mathcal{C}_{[\frac{1}{2}, \frac{1}{2}]}$ is the class of preponderantly continuous functions. Approximately continuous functions and preponderantly continuous functions were studied much earlier, see [8, 28]. Since approximately continuous functions were very widely studied, we will consider only classes $\mathcal{C}_{[\lambda, \rho]}$, where $\lambda < 1$.

Corollary 8.3. [23] *Let $0 < \lambda \leq \rho \leq 1$, $0 < \lambda_1 \leq \rho_1 \leq 1$, $\rho \leq \rho_1$, $\lambda \leq \lambda_1$, $f: \mathbb{R} \rightarrow \mathbb{R}$ and $x \in \mathbb{R}$. If f is $[\lambda_1, \rho_1]$ -continuous at x , then f is $[\lambda, \rho]$ -continuous at x .*

Theorem 8.3. [23] *Let $0 < \lambda \leq \rho \leq 1$, $0 < \lambda_1 \leq \rho_1 \leq 1$. Then $\mathcal{C}_{[\lambda_1, \rho_1]} \subset \mathcal{C}_{[\lambda, \rho]}$ if and only if $\rho \leq \rho_1$ and $\lambda \leq \lambda_1$. If, moreover, if $\rho < \rho_1$ or $\lambda < \lambda_1$ then $\mathcal{C}_{[\lambda_1, \rho_1]} \subsetneq \mathcal{C}_{[\lambda, \rho]}$.*

Clearly, if a function is continuous at some point then it is $[\lambda, \rho]$ -continuous at this point. It is worth to mention that functions which are continuous from the left or from the right at some point need not be $[\lambda, \rho]$ -continuous at this point. In an obvious way we define one-sided $[\lambda, \rho]$ -continuity. Then, if f is $[\lambda, \rho]$ -continuous at x then f is $[\lambda, \rho]$ -continuous at x from the right **or** from the left. On the other hand, if f is $[\lambda, \rho]$ -continuous at x from the right **and** from the left then f is $[\lambda, \rho]$ -continuous at x .

Example 8.1. Take any $0 < \lambda < \rho \leq 1$. Let $([a_n, b_n])_{n \in \mathbb{N}}$ be a sequence of closed intervals such that $0 < \dots b_{n+1} < a_n < b_n \dots$ and $\bar{d}^+ \left(\bigcup_{n=1}^{\infty} [a_n, b_n], 0 \right) = \underline{d}^+ \left(\bigcup_{n=1}^{\infty} [a_n, b_n], 0 \right) = \frac{\rho + \lambda}{2}$. Let $I = \bigcup_{n=1}^{\infty} [a_n, b_n]$. Let $([c_n, d_n])_{n \in \mathbb{N}}$ be a sequence of closed pairwise disjoint intervals such that $[a_n, b_n] \subset (c_n, d_n)$ for $n \geq 1$ and $\bar{d}^+ \left(\bigcup_{n=1}^{\infty} ([c_n, d_n] \setminus [a_n, b_n]), 0 \right) = 0$. Define $f: \mathbb{R} \rightarrow \mathbb{R}$ by

$$f(x) = \begin{cases} 0, & x \in (-\infty, 0] \cup I, \\ 1, & x \in [d_1, \infty) \cup \bigcup_{n=1}^{\infty} [d_{n+1}, c_n], \\ \text{linear} & \text{on each interval } [c_n, a_n] \text{ and } [b_n, d_n], \quad n \geq 1. \end{cases}$$

Obviously, f is continuous at each point except 0. Let $E = (-\infty, 0] \cup I$. Then $f|_E$ is constant, $0 \in E$, $\bar{d}(E, 0) = \bar{d}^-(E, 0) = 1$ and $\underline{d}(E, 0) = \underline{d}^+(E, 0) = \frac{\rho + \lambda}{2} > \lambda$. Thus f is $[\lambda, \rho]$ -continuous at 0 and $f \in \mathcal{C}_{[\lambda, \rho]}$. On the other hand,

$$\bar{d}^+(\{x: |f(x) - f(0)| < \frac{1}{2}\}, 0) \leq \bar{d}^+ \left(\bigcup_{n=1}^{\infty} [c_n, d_n], 0 \right) = \bar{d}^+(I, 0) = \frac{\rho + \lambda}{2} < \rho.$$

Thus f is not $[\lambda, \rho]$ -continuous at 0 from the right.

Example 8.2. Take any $0 < \lambda < \rho \leq 1$. Define $f: \mathbb{R} \rightarrow \mathbb{R}$ by

$$f(x) = \begin{cases} 0, & x \in [0, \infty), \\ 1, & x \in (-\infty, 0). \end{cases}$$

Clearly, f is continuous at each point except 0 and is continuous from the right at 0. Let $E = [0, \infty)$. Then $f|_E$ is constant, $0 \in E$ and $\bar{d}^+(E, 0) = \underline{d}^+(E, 0) = 1$. Thus f is $[\lambda, \rho]$ -continuous at 0 from the right. On the other hand

$$\underline{d}(\{x: |f(x) - f(0)| < \frac{1}{2}\}, 0) = \underline{d}^-(\{x: |f(x) - f(0)| < \frac{1}{2}\}, 0) = 0.$$

Thus f is not $[\lambda, \rho]$ -continuous at 0.

8.3 Porouscontinuity and v -porouscontinuity

In [5] J. Borsík and J. Holos introduced path continuity connected with the notion of porosity. These notions were generalized in [19].

For a set $A \subset \mathbb{R}$ and an interval $I \subset \mathbb{R}$ let $\Lambda(A, I)$ denote the length of the largest open subinterval of I having an empty intersection with A . Let $x \in \mathbb{R}$. Then, according to [5, 29, 9], the right-porosity of the set A at x is defined as

$$p^+(A, x) = \limsup_{h \rightarrow 0^+} \frac{\Lambda(A, (x, x+h))}{h},$$

the left-porosity of the set A at x is defined as

$$p^-(A, x) = \limsup_{h \rightarrow 0^+} \frac{\Lambda(A, (x-h, x))}{h},$$

and the porosity of A at x is defined as

$$p(A, x) = \max \{p^-(A, x), p^+(A, x)\}.$$

Definition 8.7. [5] A point $x \in \mathbb{R}$ will be called a point of π_r -density of the set $A \subset \mathbb{R}$ for $r \in [0, 1]$ (μ_r -density of a set A for $r \in (0, 1]$) if $p(\mathbb{R} \setminus A, x) > r$ ($p(\mathbb{R} \setminus A, x) \geq r$).

Definition 8.8. [5] Let $r \in [0, 1]$. The function $f: \mathbb{R} \rightarrow \mathbb{R}$ will be called

- \mathcal{P}_r -continuous at x if there exists a set $A \subset \mathbb{R}$ such that $x \in A$, x is a point of π_r -density of A and $f|_A$ is continuous at x .
- \mathcal{S}_r -continuous at x if for each $\varepsilon > 0$ there exists a set $A \subset \mathbb{R}$ such that $x \in A$, x is a point of π_r -density of A and $f(A) \subset (f(x) - \varepsilon, f(x) + \varepsilon)$.

Let $r \in (0, 1]$. The function $f: \mathbb{R} \rightarrow \mathbb{R}$ will be called

- \mathcal{M}_r -continuous x if there exists a set $A \subset \mathbb{R}$ such that $x \in A$, x is a point of μ_r -density of A and $f|_A$ is continuous at x ;
- \mathcal{N}_r -continuous x if for each $\varepsilon > 0$ there exists a set $A \subset \mathbb{R}$ such that $x \in A$, x is a point of μ_r -density of A and $f(A) \subset (f(x) - \varepsilon, f(x) + \varepsilon)$.

All of these functions are called porouscontinuous functions.

Symbols $\mathcal{P}_r(f)$, $\mathcal{S}_r(f)$, $\mathcal{M}_r(f)$ and $\mathcal{N}_r(f)$ denote the set of all points at which f is \mathcal{P}_r -continuous, \mathcal{S}_r -continuous, \mathcal{M}_r -continuous and \mathcal{N}_r -continuous, respectively. In [5] the equality $\mathcal{M}_r(f) = \mathcal{N}_r(f)$ for every f and every $r \in (0, 1]$ was proved.

Theorem 8.4. [18] Let $r \in [0, 1)$, $x \in \mathbb{R}$ and $f: \mathbb{R} \rightarrow \mathbb{R}$. Then $x \in P_r(f)$ if and only if $\lim_{\varepsilon \rightarrow 0^+} p(\mathbb{R} \setminus \{t: |f(x) - f(t)| < \varepsilon\}, x) > r$.

In an obvious way we define one-sided porouscontinuity, and f is porouscontinuous at x if and only if it is porouscontinuous at x from the right or from the left. Moreover, if f is continuous from the right or from the left at some x then f is porouscontinuous at this point.

Theorem 8.5. [5] Let $0 < r < s < 1$ and $f: \mathbb{R} \rightarrow \mathbb{R}$. Then

$$\mathcal{C}(f) \subset \mathcal{M}_1(f) \subset \mathcal{P}_s(f) \subset \mathcal{S}_s(f) \subset \mathcal{M}_s(f) \subset \mathcal{P}_r(f) \subset \mathcal{P}_0(f) \subset \mathcal{S}_0(f) \subset \mathcal{Q}(f).$$

Following [5], we introduce the following denotations:

- for $r \in (0, 1]$ let $\mathcal{M}_r = \{f: \mathcal{M}_r(f) = \mathbb{R}\}$,
- for $r \in [0, 1)$ let $\mathcal{P}_r = \{f: \mathcal{P}_r(f) = \mathbb{R}\}$ and $\mathcal{S}_r = \{f: \mathcal{S}_r(f) = \mathbb{R}\}$.

Theorem 8.6. [5] Let $0 < r < s < 1$. Then

$$\mathcal{C} \subset \mathcal{M}_1 \subset \mathcal{P}_s \subset \mathcal{S}_s \subset \mathcal{M}_s \subset \mathcal{P}_r \subset \mathcal{M}_r \subset \mathcal{P}_0 \subset \mathcal{S}_0 \subset \mathcal{Q}.$$

All inclusions are proper.

In [18] these results are improved. Let us consider the topology of uniform convergence, which is generated by the metric

$$\theta(f, g) = \min \{1, \sup \{|f(x) - g(x)|: x \in \mathbb{R}\}\}$$

in the space of all functions from \mathbb{R} to \mathbb{R} . Then we can studied topological structure of presented inclusions.

Theorem 8.7. [18]

- (1) \mathcal{C}^\pm is nowhere dense and closed in \mathcal{M}_1 .
- (2) For $s \in [0, 1)$, \mathcal{M}_1 is nowhere dense and closed in \mathcal{P}_s .
- (3) For $s \in (0, 1)$, \mathcal{S}_s is nowhere dense and closed in \mathcal{M}_s .
- (4) For $0 \leq r < s \leq 1$, \mathcal{M}_s is nowhere dense and closed in \mathcal{P}_r .
- (5) For $r \in (0, 1]$, \mathcal{M}_r is nowhere dense and closed in \mathcal{P}_0 .
- (6) \mathcal{S}_0 is nowhere dense and closed in \mathcal{Q} .

Example 8.3. [18] Let \mathcal{F} be any subfamily of \mathcal{Q} . Take an $h \in \mathcal{F}$ and $q \in [0, 1)$. Since $h \in \mathcal{Q}$, we have $\mathcal{C}(h) \neq \emptyset$. Let $x_0 \in \mathcal{C}(h)$. There exist $\varepsilon > 0$ and $\delta > 0$ such that $|h(x) - h(x_0)| < \frac{\varepsilon}{8}$ for $x \in (x_0 - \delta, x_0 + \delta)$. Let $\bigcup_{n=1}^{\infty} [a_n, b_n]$ be “an interval set” with properties $x_0 < \dots < b_{n+1} < a_n < \dots < b_1 < x_0 + \delta$ for

each $n \in \mathbb{N}$, $\lim_{n \rightarrow \infty} a_n = x_0$ and $p^+ \left(\mathbb{R} \setminus \bigcup_{n=1}^{\infty} [a_n, b_n], x_0 \right) = q$. Define a function $h_{\varepsilon, \delta, q, x_0}: \mathbb{R} \rightarrow \mathbb{R}$ by

$$h_{\varepsilon, \delta, q, x_0}(x) = \begin{cases} h(x) - \varepsilon, & x \in (-\infty, x_0), \\ h(x_0), & x \in \{x_0\} \cup \bigcup_{n=1}^{\infty} [a_n, b_n], \\ h(x) + \varepsilon, & x \in \bigcup_{n=1}^{\infty} (b_{n+1}, a_n) \cup (b_1, \infty). \end{cases}$$

Observe that $\theta(h_{\varepsilon, \delta, q, x_0}, h) = \varepsilon$. Now, let us take any function $f: \mathbb{R} \rightarrow \mathbb{R}$ such that $\theta(f, h_{\varepsilon, \delta, q, x_0}) < \frac{\varepsilon}{8}$. Then

$$\begin{aligned} f(x) - h(x) &< \frac{-7\varepsilon}{8} & \text{if } x \in (-\infty, x_0), \\ f(x) - h(x) &\in \left(\frac{-\varepsilon}{8}, \frac{\varepsilon}{8} \right) & \text{if } x \in \{x_0\} \cup \bigcup_{n=1}^{\infty} [a_n, b_n], \\ f(x) - h(x) &> \frac{7\varepsilon}{8} & \text{if } x \in \bigcup_{n=1}^{\infty} (b_{n+1}, a_n) \cup (b_1, \infty). \end{aligned}$$

Therefore

$$\begin{aligned} f(x) &\in (-\infty, h(x_0) - \frac{3\varepsilon}{4}) & \text{if } x \in (x_0 - \delta, x_0), \\ f(x) &\in (h(x_0) - \frac{\varepsilon}{4}, h(x_0) + \frac{\varepsilon}{4}) & \text{if } x \in \{x_0\} \cup \bigcup_{n=1}^{\infty} [a_n, b_n], \\ f(x) &\in (h(x_0) + \frac{3\varepsilon}{4}, \infty) & \text{if } x \in \bigcup_{n=1}^{\infty} (b_{n+1}, a_n) \cup (b_1, x_0 + \delta). \end{aligned}$$

Thus for every $\eta \in (0, \frac{\varepsilon}{2})$, $\{x: |f(x) - f(x_0)| < \eta\} \subset \{x_0\} \cup \bigcup_{n=1}^{\infty} [a_n, b_n]$. It follows that, $f \notin \mathcal{S}_s$ for every $s \geq q$.

Taking suitable values of q we can find required functions, witnessing statements (1) – (6).

Theorem 8.8. [18] \mathcal{P}_s is first category subset of \mathcal{S}_s for every $s \in [0, 1)$.

There are different nonequivalent definitions of porosity. Many of them are described in [29]. Considering different notions of porosity, we obtain different notions of porouscontinuity. In [19], so called, v -porosity and v -porouscontinuity is studied.

The right (left) v-porosity of the set A at x is defined as

$$vp^+(A, x) = \liminf_{h \rightarrow 0^+} \frac{\Lambda(A, (x, x+h))}{h} \quad \left(vp^-(A, x) = \liminf_{h \rightarrow 0^+} \frac{\Lambda(A, (x-h, x))}{h} \right)$$

and the v-porosity of A at x is defined as

$$vp(A, x) = \max \{ vp^-(A, x), vp^+(A, x) \}.$$

The set $A \subset \mathbb{R}$ is called right v-porous at $x \in \mathbb{R}$ if $vp^+(A, x) > 0$, left v-porous at x if $vp^-(A, x) > 0$ and v-porous at x if $vp(A, x) > 0$. It is clear, that A is v-porous at x if and only if it is v-porous from the right **or** from the left at x .

The set A is called v-porous if A is v-porous at each point $x \in A$.

In [29] Zajíček investigate similar notion, which we remind. Let X be a metric space. The open ball with the center $x \in X$ and with the radius R will be denoted by $B(x, R)$. Let $M \subset X$, $x \in X$ and $R > 0$. Then, according to [29], by $\gamma(x, R, M)$, we denote the supremum of the set of all $r > 0$ for which there exists $z \in X$ such that $B(z, r) \subset B(x, R) \setminus M$. The number $2 \limsup_{R \rightarrow 0^+} \frac{\gamma(x, R, M)}{R}$ is called the porosity of M at x . We say that the set M is very porous at x if $\liminf_{R \rightarrow 0^+} \frac{2\gamma(x, R, M)}{R} > 0$.

The definition of v-porosity of the set is very similar to the definition of very porosity in the space \mathbb{R} with the euclidian metric, but they are not equivalent.

Corollary 8.4. *Let $A \subset \mathbb{R}$, $x \in \mathbb{R}$. If the set A is v-porous at x then A is very porous at x .*

The next example shows that the converse implication is not true.

Example 8.4. [19] Let $A = \bigcup_{n=1}^{\infty} \left[\frac{1}{(2n)!}, \frac{1}{(2n-1)!} \right] \cup \left[-\frac{1}{(2n)!}, -\frac{1}{(2n+1)!} \right]$. Then

$$vp^+(A, 0) = \liminf_{h \rightarrow 0^+} \frac{\Lambda(A, (0, h))}{h} \leq \lim_{n \rightarrow \infty} \frac{\frac{1}{(2n)!} - \frac{1}{(2n+1)!}}{\frac{1}{(2n-1)!}} = \lim_{n \rightarrow \infty} \frac{1}{2n+1} = 0$$

and

$$vp^-(A, 0) = \liminf_{h \rightarrow 0^+} \frac{\Lambda(A, (-h, 0))}{h} \leq \lim_{n \rightarrow \infty} \frac{-\frac{1}{(2n+2)!} + \frac{1}{(2n+1)!}}{\frac{1}{(2n)!}} = \lim_{n \rightarrow \infty} \frac{1}{2n+2} = 0.$$

Hence $vp(A, 0) = 0$.

Denote $t_n = \frac{1}{n!} + \left(\frac{1}{n!} - \frac{1}{(n+1)!} \right) = \frac{2n+1}{(n+1)!}$ for each $n \in \mathbb{N}$. Let $h \in (0, 1)$. There exists $n \in \mathbb{N}$ such that $h \in \left[\frac{1}{(2n+1)!}, \frac{1}{(2n-1)!} \right)$.

If $h \in \left[\frac{1}{(2n+1)!}, t_{2n+1} \right)$ then $\left(-\frac{1}{(2n+1)!}, -\frac{1}{(2n+2)!} \right) \cap A = \emptyset$ and

$$\frac{2\gamma(0, h, A)}{h} \geq \frac{\frac{1}{(2n+1)!} - \frac{1}{(2n+2)!}}{t_{2n+1}} = \frac{\frac{2n+1}{(2n+2)!}}{\frac{4n+3}{(2n+2)!}} = \frac{2n+1}{4n+3}.$$

If $h \in \left[t_{2n+1}, \frac{1}{(2n)!} \right)$ then $\left(\frac{1}{(2n+1)!}, h \right) \cap A = \emptyset$ and

$$\frac{2\gamma(0, h, A)}{h} \geq \frac{h - \frac{1}{(2n+1)!}}{h} \geq \frac{t_{2n+1} - \frac{1}{(2n+1)!}}{t_{2n+1}} = \frac{\frac{2n+1}{(2n+1)!}}{\frac{4n+3}{(2n+2)!}} = \frac{2n+1}{4n+3}.$$

If $h \in \left[\frac{1}{(2n)!}, t_{2n} \right)$ then $\left(\frac{1}{(2n+1)!}, \frac{1}{(2n)!} \right) \cap A = \emptyset$ and

$$\frac{2\gamma(0, h, A)}{h} \geq \frac{\frac{1}{(2n)!} - \frac{1}{(2n+1)!}}{h} \geq \frac{\frac{2n}{(2n+1)!}}{t_{2n}} = \frac{\frac{2n}{(2n+1)!}}{\frac{4n+1}{(2n+1)!}} = \frac{2n}{4n+1}.$$

Finally, if $h \in \left[t_{2n}, \frac{1}{(2n-1)!} \right)$ then $\left(-h, -\frac{1}{(2n)!} \right) \cap A = \emptyset$ and

$$\frac{2\gamma(0, h, A)}{h} \geq \frac{h - \frac{1}{(2n)!}}{h} \geq \frac{t_{2n} - \frac{1}{(2n)!}}{t_{2n}} = \frac{\frac{2n}{(2n+1)!}}{\frac{4n+1}{(2n+1)!}} = \frac{2n}{4n+1}.$$

Therefore $\liminf_{h \rightarrow 0^+} \frac{2\gamma(0, h, A)}{h} = \frac{1}{2}$ and A is very porous at 0.

Moreover,

$$p(A, 0) \geq p^+(A, 0) \geq \lim_{n \rightarrow \infty} \frac{\Lambda \left(A, \left(0, \frac{1}{(2n)!} \right) \right)}{\frac{1}{(2n)!}} = \lim_{n \rightarrow \infty} \frac{\frac{1}{(2n)!} - \frac{1}{(2n+1)!}}{\frac{1}{(2n)!}} = 1.$$

Definition 8.9. [19] A point $x \in \mathbb{R}$ will be called a point of $v\pi_r$ -density of a set $A \subset \mathbb{R}$ for $r \in [0, 1)$ ($v\mu_r$ -density of a set A for $r \in (0, 1]$) if $v\rho(\mathbb{R} \setminus A, x) > r$ ($v\rho(\mathbb{R} \setminus A, x) \geq r$).

In an obvious way, we may define one-sided $v\pi_r$ -densities and $v\mu_r$ -densities.

Definition 8.10. [19] Let $r \in [0, 1)$. The function $f: \mathbb{R} \rightarrow \mathbb{R}$ will be called

- $v\mathcal{P}_r$ -continuous at x if there exists a set $A \subset \mathbb{R}$ such that $x \in A$, x is a point of $v\pi_r$ -density of A and $f|_A$ is continuous at x ;
- $v\mathcal{S}_r$ -continuous at x if for each $\varepsilon > 0$ there exists a set $A \subset \mathbb{R}$ such that $x \in A$, x is a point of $v\pi_r$ -density of A and $f(A) \subset (f(x) - \varepsilon, f(x) + \varepsilon)$.

Let $r \in (0, 1]$. The function $f: \mathbb{R} \rightarrow \mathbb{R}$ will be called

- $v\mathcal{M}_r$ -continuous at x if there exists a set $A \subset \mathbb{R}$ such that $x \in A$, x is a point of $v\mu_r$ -density of A and $f|_A$ is continuous at x ;
- $v\mathcal{N}_r$ -continuous at x if for each $\varepsilon > 0$ there exists a set $A \subset \mathbb{R}$ such that $x \in A$, x is a point of $v\mu_r$ -density of A and $f(A) \subset (f(x) - \varepsilon, f(x) + \varepsilon)$.

The symbols $v\mathcal{P}_r(f)$, $v\mathcal{S}_r(f)$, $v\mathcal{M}_r(f)$ and $v\mathcal{N}_r(f)$ denote the sets of all points at which the function f is $v\mathcal{P}_r$ -continuous, $v\mathcal{S}_r$ -continuous, $v\mathcal{M}_r$ -continuous and $v\mathcal{N}_r$ -continuous, respectively. These sets will be called the sets of v -porouscontinuity points of the function f .

In an obvious way, we may define one sided v -porouscontinuity of the function $f: \mathbb{R} \rightarrow \mathbb{R}$ at a point. Symbols $v\mathcal{P}_r^+(f)$, $(v\mathcal{P}_r^-(f))$, $v\mathcal{S}_r^+(f)$, $(v\mathcal{S}_r^-(f))$, $v\mathcal{M}_r^+(f)$, $(v\mathcal{M}_r^-(f))$ and $v\mathcal{N}_r^+(f)$, $(v\mathcal{N}_r^-(f))$ will denote the sets of all points at which the function f is $v\mathcal{P}_r$ -continuous from the right (from the left), $v\mathcal{S}_r$ -continuous from the right (from the left), $v\mathcal{M}_r$ -continuous from the right (from the left) and $v\mathcal{N}_r$ -continuous from the right (from the left), respectively. Obviously, $v\mathcal{P}_r(f) = v\mathcal{P}_r^+(f) \cup v\mathcal{P}_r^-(f)$, $v\mathcal{S}_r(f) = v\mathcal{S}_r^+(f) \cup v\mathcal{S}_r^-(f)$ and $v\mathcal{M}_r(f) = v\mathcal{M}_r^+(f) \cup v\mathcal{M}_r^-(f)$. If f is continuous from the right or from the left at x then $x \in v\mathcal{P}_r(f) \cap v\mathcal{S}_r(f) \cap v\mathcal{M}_s(f)$ for every $r \in [0, 1)$ and every $s \in (0, 1]$.

Theorem 8.9. [19] *Let $A \subset \mathbb{R}$, $x \in A$. If there exists a decreasing sequence $(x_n)_{n \in \mathbb{N}} \subset A$ converging to x , then $vp^+(A, x) \leq \frac{1}{2}$.*

Corollary 8.5. $v\mathcal{P}_r(f) = v\mathcal{S}_r(f) = \mathcal{C}(f)$ for each $r \in [\frac{1}{2}, 1)$. Similarly, $v\mathcal{M}_r(f) = v\mathcal{N}_r(f) = \mathcal{C}(f)$ for each $r \in (\frac{1}{2}, 1]$.

Theorem 8.10. [19] *Let $r \in (0, \frac{1}{2}]$ and $f: \mathbb{R} \rightarrow \mathbb{R}$. Then $v\mathcal{M}_r(f) = v\mathcal{N}_r(f)$.*

We introduce the following denotations:

- for $r \in (0, 1]$ let $v\mathcal{M}_r = \{f: v\mathcal{M}_r(f) = \mathbb{R}\}$,
- for $r \in [0, 1)$ let $v\mathcal{P}_r = \{f: v\mathcal{P}_r(f) = \mathbb{R}\}$ and $v\mathcal{S}_r = \{f: v\mathcal{S}_r(f) = \mathbb{R}\}$.

Corollary 8.6. $v\mathcal{M}_r = v\mathcal{N}_r$ for every $r \in (0, \frac{1}{2}]$.

Theorem 8.11. [19] *Let $r \in [0, \frac{1}{2})$, $x \in \mathbb{R}$ and $f: \mathbb{R} \rightarrow \mathbb{R}$. Then $x \in v\mathcal{P}_r(f)$ if and only if $\lim_{\varepsilon \rightarrow 0^+} vp(\mathbb{R} \setminus \{t: |f(x) - f(t)| < \varepsilon\}, x) > r$.*

Theorem 8.12. [19] *Let $0 < r < s < \frac{1}{2}$ and $f: \mathbb{R} \rightarrow \mathbb{R}$. Then*

$$\mathcal{C}^\pm(f) \subset v\mathcal{M}_{\frac{1}{2}}(f) \subset v\mathcal{P}_s(f) \subset v\mathcal{S}_s(f) \subset v\mathcal{M}_s(f) \subset v\mathcal{P}_r(f) \subset v\mathcal{P}_0(f) \subset v\mathcal{S}_0(f).$$

Theorem 8.13. [19] Let $0 < r < s < \frac{1}{2}$. Then

$$\mathcal{C}^\pm \subset v.\mathcal{M}_{\frac{1}{2}} \subset v.\mathcal{P}_s \subset v.\mathcal{S}_s \subset v.\mathcal{M}_s \subset v.\mathcal{P}_r \subset v.\mathcal{P}_0 \subset v.\mathcal{S}_0 \subset \mathcal{S}_0.$$

Moreover, all inclusions are proper.

8.4 Relationship

In this section we describe primary relationships between discussed kinds of path continuity.

Theorem 8.14. [18] $\mathcal{P}_r \subsetneq \mathcal{UC}_r$ for every $r \in (0, 1)$.

Theorem 8.15. [17] $\mathcal{UC}_r \subsetneq \mathcal{OUC}_r$ for every $r \in (0, 1)$.

Theorem 8.16. [17] $\mathcal{OUC}_r \subsetneq \mathcal{UC}_s$ for every $0 < s < r < 1$.

Theorem 8.17. $\mathcal{C}_{[\rho, \rho]} \subsetneq \mathcal{UC}_\rho$ for every $\rho \in (0, 1]$.

Theorem 8.18. [19] $v.\mathcal{P}_r \subsetneq \mathcal{P}_{\frac{r}{1-r}}$ and $v.\mathcal{S}_r \subsetneq \mathcal{S}_{\frac{r}{1-r}}$ for $r \in [0, \frac{1}{2})$.

Theorem 8.19. [19] $v.\mathcal{M}_r \subsetneq \mathcal{M}_{\frac{r}{1-r}}$ for $r \in (0, \frac{1}{2})$.

Inclusions in Theorem 8.14 – 8.19 are easy to prove. We will show that these inclusions are proper.

Example 8.5. Let $x_n = \frac{1}{2^n}$ for $n \geq 1$. For each $n \geq 1$ there are pairwise disjoint closed intervals $[a_1^n, b_1^n], \dots, [a_n^n, b_n^n] \subset (x_{n+1}, x_n)$ such that

$$\mu\left([x_{n+1}, x_n] \setminus \bigcup_{i=1}^n [a_i^n, b_i^n]\right) < \frac{x_{n+1}}{n} \quad \text{and} \quad b_i^n - a_i^n = \frac{x_n - x_{n+1}}{n+1} \quad \text{for } i = 1, \dots, n.$$

Let $E = \{0\} \cup \bigcup_{n=1}^{\infty} \bigcup_{i=1}^{k_n} [a_i^n, b_i^n]$ and let χ_E be the characteristic function of the set E . Clearly, χ_E is continuous from the right or from the left at every point except 0. Since $\bar{d}^+(E, 0) = 1$, we conclude that $\chi_E \in \mathcal{UC}_r$ for every $r \in (0, 1]$ and since $p^+(\mathbb{R} \setminus E, 0) = 0$, we conclude $\chi_E \notin \mathcal{P}_r$ for any $r \in (0, 1]$.

Example 8.6. Fix $0 < r_0 < r < r_1 < 1$. Let $x_n = \frac{1}{2^n}$ and $y_n \in (x_{n+1}, x_n)$ be such that $x_n - y_n = r(x_n - x_{n+1})$ for every $n \geq 1$. Let $I_n = [y_n, x_n]$. Then $\frac{\mu\left(\bigcup_{n=1}^{\infty} I_n \cap [0, x]\right)}{x} \leq r$ for each $x > 0$ and $\frac{\mu\left(\bigcup_{n=1}^{\infty} I_n \cap [0, x_k]\right)}{x_k} = r$ for every $k \in \mathbb{N}$.

Hence $\bar{d}^+\left(\bigcup_{n=1}^{\infty} I_n, 0\right) = r$. Let $\{[c_n, d_n]\}_{n \geq 1}$ be a sequence of pairwise disjoint

closed intervals such that $[y_n, x_n] \subset (c_n, d_n)$ for every $n \geq 1$ and, moreover, $\bar{d}^+ \left(\bigcup_{n=1}^{\infty} ([c_n, d_n] \setminus [y_n, x_n]), 0 \right) = 0$. Finally, let $J_n = [c_n, d_n]$ for every $n \geq 1$. Define $f: \mathbb{R} \rightarrow \mathbb{R}$ by

$$f(x) = \begin{cases} 0, & x \in \{0\} \cup \bigcup_{n=1}^{\infty} I_n, \\ 1, & x \in (-\infty, 0) \cup \bigcup_{n=1}^{\infty} [d_{n+1}, c_n] \cup [d_1, \infty), \\ \text{linear on each interval } [c_n, y_n], [x_n, d_n], & n = 1, 2, \dots \end{cases}$$

It is easily seen that f is continuous at every point except 0. Since

$$\begin{aligned} \bar{d}^+ \left(\{x: |f(x)| < 1\}, 0 \right) &\leq \bar{d}^+ \left(\bigcup_{n=1}^{\infty} J_n, 0 \right) \leq \\ &\leq \bar{d}^+ \left(\bigcup_{n=1}^{\infty} I_n, 0 \right) + \bar{d}^+ \left(\bigcup_{n=1}^{\infty} (J_n \setminus I_n), 0 \right) = r, \end{aligned}$$

we conclude that f is not r -upper continuous at 0. Hence $f \notin \mathcal{UC}_r$.

On the other hand, for every $\varepsilon > 0$ and for every $y > 0$ we have

$$\mu \left(\left\{ x \in [0, y] \setminus \bigcup_{n=1}^{\infty} I_n : |f(x)| < \varepsilon \right\} \right) > 0.$$

Hence

$$\begin{aligned} \frac{\mu \left(\{x: |f(x)| < \varepsilon\} \cap [0, x_k] \right)}{x_k} &= \frac{\mu \left(\left\{ x \in \bigcup_{n=1}^{\infty} I_n : |f(x)| < \varepsilon \right\} \cap [0, x_k] \right)}{x_k} + \\ &+ \frac{\mu \left(\left\{ x \notin \bigcup_{n=1}^{\infty} I_n : |f(x)| < \varepsilon \right\} \cap [0, x_k] \right)}{x_k} = \\ &= r + \frac{\mu \left(\left\{ x \notin \bigcup_{n=1}^{\infty} I_n : |f(x)| < \varepsilon \right\} \cap [0, x_k] \right)}{x_k} > r. \end{aligned}$$

Since $\lim_{k \rightarrow \infty} x_k = 0$, we conclude that f is r -upper continuous in O'Malley sense at 0 and $f \in \mathcal{OU}\mathcal{C}_r$.

Moreover, it is obvious that $f \in \mathcal{UC}_{r_0}$ and $f \notin \mathcal{OU}\mathcal{C}_{r_1}$.

Since $\underline{d}(\{x: |f(x) - f(0)| < \varepsilon\}, 0) = \underline{d}^-(\{x: |f(x) - f(0)| < \varepsilon\}, 0) = 0$ for $\varepsilon \in (0, 1)$, we conclude that $f \notin \mathcal{C}_{[r_0, r_0]}$.

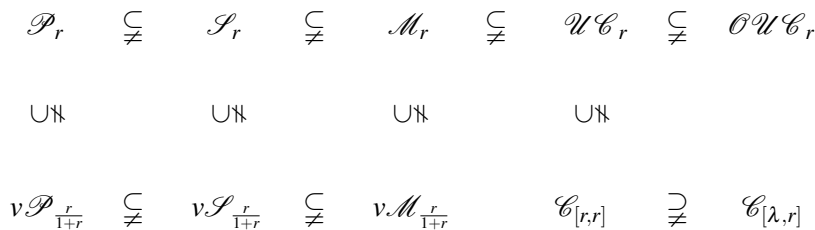
This example shows that inclusions in Theorem 8.15, Theorem 8.16 and Theorem 8.17 are proper.

Example 8.7. Let A be the set from Example 8.4. Then the characteristic function of the set $\mathbb{R} \setminus A$ belongs to \mathcal{M}_1 and does not belong to $v\mathcal{S}_0$.

Theorem 8.20. $\mathcal{UC}_{\rho_1} \not\subset \mathcal{C}_{[\lambda, \rho]}$ for any $\rho, \rho_1 \in (0, 1], \lambda \in (0, \rho]$.

Proof. Let $\chi_{(-\infty, 0]}$ be the characteristic function of $(-\infty, 0]$. Then $f \in \mathcal{UC}_{\rho_1}$ but $f \notin \mathcal{C}_{[\lambda, \rho]}$ for any $\rho, \rho_1 \in (0, 1], \lambda \in (0, \rho]$. \square

Relationship between discussed classes of function for $r \in (0, 1)$ and $\lambda \in (0, r)$ can be represented in the following diagram.



8.5 Properties of path continuous functions

In this section we will consider measurability, connections with Baire one functions, uniform convergence, adding and multiplying of functions from defined classes.

Theorem 8.21. [13] Every function from $\bigcup_{r \in (0, 1)} \mathcal{OUC}_r$ is measurable.

Applying Theorem 8.21 and inclusions showed in section 8.4, we obtain

Theorem 8.22. Every function from

$$\begin{aligned}
 & \bigcup_{r \in (0, 1]} (\mathcal{UC}_r \cup \mathcal{M}_r) \cup \bigcup_{r \in [0, 1)} (\mathcal{P}_r \cup \mathcal{S}_r) \cup \bigcup_{r \in [0, \frac{1}{2})} (v\mathcal{P}_r \cup v\mathcal{S}_r) \cup \\
 & \cup \bigcup_{r \in (0, \frac{1}{2}]} v\mathcal{M}_r \cup \bigcup_{0 < \lambda \leq \rho \leq 1} \mathcal{C}_{[\lambda, \rho]}
 \end{aligned}$$

is measurable.

Theorem 8.23. [19] For every $r \in (0, \frac{1}{2}]$ there exists a function from $v\mathcal{P}_r$ which does not belong to Baire class one.

Let

$$\mathfrak{Y} = \{\mathcal{UC}_r : r \in (0, 1]\} \cup \{\mathcal{OUC}_r : r \in (0, 1)\} \cup \{\mathcal{M}_r : r \in (0, 1]\} \cup \{\mathcal{P}_r : r \in [0, 1)\} \cup \{\mathcal{S}_r : r \in [0, 1)\}.$$

In section 8.4 we showed that for every $\mathcal{Y} \in \mathfrak{Y}$ there exists $r \in (0, \frac{1}{2}]$ such that $v\mathcal{P}_r \subset \mathcal{Y}$. Thus we have

Theorem 8.24. [19] For every $\mathcal{Y} \in \mathfrak{Y}$ there exists $f \in \mathcal{Y}$ which does not belong to Baire class one.

Theorem 8.25. [6] Every preponderantly continuous function belongs to Baire class one.

Corollary 8.7. For $\frac{1}{2} \leq \lambda \leq \rho \leq 1$ the class $\mathcal{C}_{[\lambda, \rho]}$ consists of functions belonging to Baire class one.

Question 8.1. Let $0 < \lambda \leq \rho < 1$, $\lambda < \frac{1}{2}$. Does the class $\mathcal{C}_{[\lambda, \rho]}$ consist of functions belonging to Baire class one?

Theorem 8.26.

1. For every $\rho \in (0, 1)$, \mathcal{OUC}_ρ is closed under uniform convergence.
2. \mathcal{UC}_1 is closed under uniform convergence.
3. For every $r \in (0, 1]$, \mathcal{M}_r is closed under uniform convergence, [18].
4. For every $r \in [0, 1)$, \mathcal{S}_r is closed under uniform convergence, [18].
5. For every $r \in (0, \frac{1}{2}]$, $v\mathcal{M}_r$ is closed under uniform convergence, [19].
6. For every $r \in [0, \frac{1}{2})$, $v\mathcal{S}_r$ is closed under uniform convergence, [19].

Proof. All assertions follows directly from the fact that if $\rho(f, f_n) < \frac{\varepsilon}{3}$ then $\{x: |f(x) - f(x_0)| < \frac{\varepsilon}{3}\} \subset \{x: |f_n(x) - f(x_0)| < \varepsilon\}$ for every x_0 . \square

Theorem 8.27.

1. For every $\rho \in (0, 1)$, \mathcal{UC}_ρ is not closed under uniform convergence, [24].
2. For every $0 < \lambda \leq \rho \leq 1$, $\lambda < 1$, $\mathcal{C}_{[\lambda, \rho]}$ is not closed under uniform convergence.
3. For every $r \in [0, 1)$, \mathcal{P}_r is not closed under uniform convergence, [18].
4. For every $r \in [0, \frac{1}{2})$, $v\mathcal{P}_r$ is not closed under uniform convergence, [19].

Example 8.8. Fix $0 < \lambda \leq \rho \leq 1$, $\lambda < 1$. We shall construct a sequence of functions from $\mathcal{C}_{[\lambda, \rho]}$ uniformly convergent to a function which does not belong to $\mathcal{C}_{[\lambda, \rho]}$. Let $x_n = \frac{1}{2^n}$, $y_n \in (x_{n+1}, x_n)$ be such that $y_n - x_{n+1} = \lambda(x_n - x_{n+1})$ and $z_n = \frac{y_n + x_n}{2}$ for $n \geq 1$. Then $\mu\left(\bigcup_{k=n}^{\infty} [x_{k+1}, y_k]\right) = \lambda \cdot x_n$ for $n \geq 1$. Define $f: \mathbb{R} \rightarrow \mathbb{R}$ by

$$f(x) = \begin{cases} 0, & x \in (-\infty, 0] \cup \bigcup_{n=1}^{\infty} [x_{n+1}, y_n] \cup (x_1, \infty), \\ 1, & x = z_n, \quad n = 1, 2, \dots \\ \text{linear} & \text{on intervals } [y_n, z_n] \text{ and } [z_n, x_n], \quad n = 1, 2, \dots \end{cases}$$

Obviously, f is continuous at every point except 0. Moreover, $\{x: |f(x) - f(0)| < \varepsilon\} = (-\infty, 0] \cup \bigcup_{n=1}^{\infty} (x_{n+1} - \varepsilon(x_{n+1} - z_{n+1}), y_n + \varepsilon(z_n - y_n))$ for $\varepsilon \in (0, 1)$. Therefore

$$\begin{aligned} \underline{d}(\{x: |f(x) - f(0)| < \varepsilon\}, 0) &= \\ &= \lim_{n \rightarrow \infty} \frac{\mu\left(\bigcup_{k=n}^{\infty} (x_{k+1} - \varepsilon(x_{k+1} - z_{k+1}), y_k + \varepsilon(z_k - y_k))\right)}{x_n - \varepsilon(x_n - z_n)} = \\ &= \lim_{n \rightarrow \infty} \frac{\sum_{k=n}^{\infty} (\lambda(x_k - x_{k+1}) + \varepsilon \frac{1-\lambda}{2}(x_{k+1} - x_{k+2}) + \varepsilon \frac{1-\lambda}{2}(x_k - x_{k+1}))}{x_n - \varepsilon \frac{1-\lambda}{2}(x_n - x_{n+1})} = \\ &= \lim_{n \rightarrow \infty} \frac{\sum_{k=n}^{\infty} (x_k - x_{k+1})(\lambda + \varepsilon \frac{1-\lambda}{4} + \varepsilon \frac{1-\lambda}{2})}{x_n - \varepsilon \frac{1-\lambda}{2} \frac{x_n}{2}} = \\ &= \lim_{n \rightarrow \infty} \frac{x_n(\lambda + \frac{3\varepsilon}{4}(1-\lambda))}{x_n(1 - \varepsilon \frac{1-\lambda}{4})} = \frac{\lambda + \frac{3\varepsilon}{4}(1-\lambda)}{1 - \varepsilon \frac{1-\lambda}{4}}. \end{aligned}$$

Since, $\lim_{\varepsilon \rightarrow 0^+} \frac{\lambda + \frac{3\varepsilon}{4}(1-\lambda)}{1 - \varepsilon \frac{1-\lambda}{4}} = \lambda$, f is not $[\lambda, \rho]$ -continuous at 0, by [23, Theorem 2.1].

Let $f_n = \max\{\frac{1}{n}, f\}$. Then $(f_n)_{n \in \mathbb{N}}$ is uniformly convergent to f . Fix $n \in \mathbb{N}$. Clearly, f_n is continuous at every point except 0. Moreover, we obtain $\{x: f_n(x) = f_n(0)\} = \{x: |f(x)| \leq \frac{1}{n}\}$. Therefore $\overline{d}(\{x: f_n(x) = f_n(0)\}, 0) = \overline{d}(\{x: |f(x)| \leq \frac{1}{n}\}, 0) = 1$ and

$$\begin{aligned} \underline{d}(\{x: f_n(x) = f_n(0)\}, 0) &= \underline{d}^+(\{x: f_n(x) = f_n(0)\}, 0) = \\ &= \underline{d}^+(\{x: |f_n(x)| \leq \frac{1}{n}\}, 0) = \frac{\lambda + \frac{3}{4n}(1-\lambda)}{1 - \frac{1-\lambda}{4n}} > \lambda. \end{aligned}$$

Thus $f_n \in \mathcal{C}_{[\lambda,1]} \subset \mathcal{C}_{[\lambda,\rho]}$ for $n \geq 1$.

It is easily seen that result of addition and multiplication of functions from discussed classes of functions, in general, need not belong to these classes. Therefore we studied the following notion.

Definition 8.11. Let \mathcal{F} be a family of real functions defined on \mathbb{R} . A set $\mathfrak{M}_a(\mathcal{F}) = \{g: \mathbb{R} \rightarrow \mathbb{R}: \forall f \in \mathcal{F} f + g \in \mathcal{F}\}$ is called the maximal additive class for \mathcal{F} .

Remark 8.2. Let $f: \mathbb{R} \rightarrow \mathbb{R}$, $f(x) = 0$ for $x \in \mathbb{R}$ be a constant function. Clearly, if $f \in \mathcal{F}$ then $\mathfrak{M}_a(\mathcal{F}) \subset \mathcal{F}$.

Theorem 8.28. [16] $\mathfrak{M}_a(\mathcal{C}_{[\lambda,\rho]}) = \mathcal{A}$ for $0 < \lambda \leq \rho \leq 1$.

In the sequel we will need the notions of sparse sets and T^* -continuity.

Definition 8.12. [26] A measurable set E is called sparse at a point $x \in \mathbb{R}$ from the right if for each $\varepsilon > 0$ there exists $k > 0$ such that any interval $(\alpha, \beta) \subset (x, x+k)$ satisfying condition $\alpha - x < k(\beta - x)$ contains at least one point y such that $\mu(E \cap (x, y)) < \varepsilon(y - x)$.

Analogously, we can define a left-sided sparsity. A family of all measurable sets which are sparse from the right (from the left) at x we denote by $S(x+)$ ($S(x-)$).

Definition 8.13. [26] We say that a measurable set E is sparse at x if $E \in S(x)$, where $S(x) = S(x+) \cap S(x-)$.

Theorem 8.29. [26] Let $x \in \mathbb{R}$ and let E be a measurable subset of \mathbb{R} . The following conditions are equivalent

1. $E \in S(x+)$,
2. if $F \subset \mathbb{R}$ is a measurable set and $\bar{d}^+(F, x) < 1$ then $\bar{d}^+(E \cup F) < 1$,
3. if $F \subset \mathbb{R}$ is a measurable set, $\underline{d}^+(F, x) = 0$ and $\bar{d}^+(F, x) < 1$ then $\underline{d}^+(E \cup F, x) = 0$ and $\bar{d}^+(E \cup F, x) < 1$,
4. if $F \subset \mathbb{R}$ is a measurable set and $\underline{d}^+(F, x) = 0$ then $\underline{d}^+(E \cup F, x) = 0$.

Obviously, if $\bar{d}^+(E, x) = 0$ then E is sparse from the right at x . In [10] it is shown that there exists a measurable set E which is sparse from the right at some point x and $\bar{d}^+(E, x) > 0$. Moreover, if $\underline{d}^+(E, x) > 0$ or $\bar{d}^+(E, x) = 1$ then E is not sparse from the right at x .

According to [10] we have:

Definition 8.14. [10] A real-valued function f defined on \mathbb{R} is called T^* -continuous if for each $a \in \mathbb{R}$ the sets $\{x: f(x) < a\}$ and $\{x: f(x) > a\}$ are complements of sparse sets.

We denote by \mathcal{C}_{T^*} a family of T^* -continuous functions. Clearly, \mathcal{A} is a proper subset of \mathcal{C}_{T^*} .

Since complements of sparse sets form a topology, [10], we can define T^* -continuity locally.

Definition 8.15. [10] We say that a function $f: \mathbb{R} \rightarrow \mathbb{R}$ is T^* -continuous at $x_0 \in \mathbb{R}$ if for each $\varepsilon > 0$ the complement of the set $\{x: |f(x) - f(x_0)| < \varepsilon\}$ is sparse at x_0 .

Corollary 8.8. A function $f: \mathbb{R} \rightarrow \mathbb{R}$ is T^* -continuous if and only if it is T^* -continuous at each point of \mathbb{R} .

Theorem 8.30. [14]

1. $\mathfrak{M}_a(\mathcal{UC}_\rho) = \mathcal{A}$ for $\rho \in (0, 1)$,
2. $\mathfrak{M}_a(\mathcal{UC}_1) = \mathcal{C}_{T^*}$.

Theorem 8.31. [17] $\mathfrak{M}_a(\mathcal{OC}_\rho) = \mathcal{C}$ for $r \in (0, 1)$,

Theorem 8.32. $\mathfrak{M}_a(\mathcal{P}_r) = \mathfrak{M}_a(\mathcal{S}_r) = \mathfrak{M}_a(\mathcal{M}_r) = \mathcal{C}$ for $r \in (0, 1)$.

Proof. Inclusions $\mathcal{C} \subset \mathfrak{M}_a(\mathcal{P}_r)$, $\mathcal{C} \subset \mathfrak{M}_a(\mathcal{S}_r)$, $\mathcal{C} \subset \mathfrak{M}_a(\mathcal{M}_r)$ for $r \in (0, 1)$ are obvious.

Fix $r \in (0, 1)$ and take any $f: \mathbb{R} \rightarrow \mathbb{R}$, $f \in \mathcal{P}_r \setminus \mathcal{C}$ and $c \in (r, \frac{2r}{1+r})$. Then $\frac{c}{2-c} = \frac{1}{\frac{2}{c}-1} < \frac{1}{\frac{2}{2r}-1} = r$. By assumption, f is discontinuous at some x_0 . We may assume that $x_0 = 0$ and f is discontinuous from the right at 0. Let $\varepsilon > 0$ and $(y_k)_{k \in \mathbb{N}}$ be such that $\lim_{k \rightarrow \infty} y_k = 0$, $y_k > 0$ and $|f(y_k) - f(0)| > \varepsilon$ for $k \geq 1$. Let $(x_n)_{n \in \mathbb{N}}$ be a subsequence of $(y_k)_{k \in \mathbb{N}}$ such that $x_{n+1} < x_n(1-c)$ for $n \geq 1$. Let $a_n = \frac{2x_n(1-c)}{2-c}$ and $b_n = \frac{2x_n}{2-c}$ for $n \geq 1$. Then $b_{n+1} < a_n < b_n$, $b_n + a_n = \frac{4x_n - 2cx_n}{2-c} = 2x_n$ and $\frac{b_n - a_n}{b_n} = \frac{2cx_n}{2x_n} = c$ for all $n \geq 1$. Define $g: \mathbb{R} \rightarrow \mathbb{R}$ by

$$g(x) = \begin{cases} 0 & x \in \{0\} \cup \bigcup_{n=1}^{\infty} [a_n, b_n], \\ f(0) - f(x) + \varepsilon, & x \in (-\infty, 0) \cup \bigcup_{n=1}^{\infty} (b_{n+1}, a_n) \cup (b_1, \infty). \end{cases}$$

Obviously, $\mathbb{R} \setminus \{0\} \subset \mathcal{P}_r(g)$. Since $p^+(\mathbb{R} \setminus \bigcup_{n=1}^{\infty} [a_n, b_n], 0) = c > r$, $0 \in \mathcal{P}_r(g)$. Thus $g \in \mathcal{P}_r$. Let $E = \{x: |f(x) + g(x) - f(0) - g(0)| < \varepsilon\}$. Certainly, $f(0) + g(0) = f(0)$ and $|f(x) + g(x) - f(0)| = \varepsilon$ for $x \in (-\infty, 0) \cup \bigcup_{n=1}^{\infty} (b_{n+1}, a_n) \cup (b_1, \infty)$. Therefore

$$\begin{aligned} p(\mathbb{R} \setminus E, 0) &= p^+(\mathbb{R} \setminus E, 0) = \limsup_{n \rightarrow \infty} \frac{x_n - a_n}{x_n} = \limsup_{n \rightarrow \infty} \frac{\frac{1}{2}(b_n - a_n)}{b_n - \frac{1}{2}(b_n - a_n)} = \\ &= \limsup_{n \rightarrow \infty} \frac{1}{\frac{2}{\frac{b_n - a_n}{b_n}} - 1} = \frac{1}{\frac{2}{c} - 1} = \frac{c}{2 - c} < r. \end{aligned}$$

Hence $f + g \notin \mathcal{P}_r$ and $f \notin \mathfrak{M}_a(\mathcal{P}_r)$. It follows that $\mathfrak{M}_a(\mathcal{P}_r) \subset \mathcal{C}$.

Inclusions $\mathfrak{M}_a(\mathcal{M}_r) \subset \mathcal{C}$ and $\mathfrak{M}_a(\mathcal{S}_r) \subset \mathcal{C}$ can be proved similarly. \square

Remark 8.3. Maximal additive classes for \mathcal{S}_0 , \mathcal{P}_0 , \mathcal{M}_1 and for v -porouscontinuous functions are still unknown.

Definition 8.16. Let \mathcal{F} be a family of real functions defined on \mathbb{R} . A set $\mathfrak{M}_m(\mathcal{F}) = \{g: \mathbb{R} \rightarrow \mathbb{R}: \forall f \in \mathcal{F} \ f \cdot g \in \mathcal{F}\}$ is called the maximal multiplicative class for \mathcal{F} .

Remark 8.4. Let $f: \mathbb{R} \rightarrow \mathbb{R}$, $f(x) = 1$ for $x \in \mathbb{R}$ be a constant function. If $f \in \mathcal{F}$ then $\mathfrak{M}_m(\mathcal{F}) \subset \mathcal{F}$,

Definition 8.17. [14] Let $0 < \rho < 1$ and let $\mathbf{Z}(\rho)$ be the family of all measurable functions $f: \mathbb{R} \rightarrow \mathbb{R}$ such that at each $x_0 \in D_{ap}(f)$ the following two conditions hold

(Z1) $f(x_0) = 0$ (in other words $D_{ap}(f) \subset N_f$),

(Z2) for each measurable set E such that $E \supset N_f$ and $\bar{d}(E, x_0) > \rho$ we have

$$\lim_{\varepsilon \rightarrow 0^+} \bar{d}(E \cap \{x: |f(x)| < \varepsilon\}, x_0) > \rho.$$

Corollary 8.9. *The family of approximately continuous functions is a proper subset of $\mathbf{Z}(\rho)$.*

Theorem 8.33. [14] $\mathfrak{M}_m(\mathcal{UC}_\rho) = \mathbf{Z}(\rho)$ for each $0 < \rho < 1$.

Definition 8.18. [14] Let $\mathbf{Z}(1)$ be the family of all measurable functions $f: \mathbb{R} \rightarrow \mathbb{R}$ such that at each x_0 , at which f is not T^* -continuous, the following two conditions hold:

$$(Z3) \quad f(x_0) = 0,$$

(Z4) for each measurable set F such that $\bar{d}(F, x_0) = 1$ and $N_f \subset F$ and for each $\varepsilon > 0$ we have

$$\bar{d}(F \cap \{x: |f(x)| < \varepsilon\}, x_0) = 1.$$

Corollary 8.10. *The class of T^* -continuous functions is a proper subset of $\mathbf{Z}(1)$.*

Theorem 8.34. [14] $\mathfrak{M}_m(\mathcal{UC}_1) = \mathbf{Z}(1)$.

Definition 8.19. [16] Let $0 < \lambda \leq \rho < 1$. Let $\mathbf{P}(\lambda, \rho)$ be a set of all functions $f: \mathbb{R} \rightarrow \mathbb{R}$ satisfying the following conditions

$$(P1) \quad D_{ap}(f) \subset N_f,$$

(P2) for each $x \in D_{ap}(f)$ and for each measurable set E such that $E \supset N_f$ and $\underline{d}(E, x) > \lambda, \bar{d}(E, x) > \rho$ we have

$$\lim_{\varepsilon \rightarrow 0^+} \underline{d}(E \cap \{y: |f(y) - f(x)| < \varepsilon\}, x) > \lambda$$

and

$$\lim_{\varepsilon \rightarrow 0^+} \bar{d}(E \cap \{y: |f(y) - f(x)| < \varepsilon\}, x) > \rho.$$

Corollary 8.11. *Let $0 < \lambda \leq \rho < 1$. Then $\mathcal{A} \subsetneq \mathbf{P}(\lambda, \rho)$.*

Theorem 8.35. [16] $\mathfrak{M}_m(\mathcal{C}_{[\lambda, \rho]}) = \mathbf{P}(\lambda, \rho)$ for each $0 < \lambda \leq \rho < 1$.

Definition 8.20. [17] For each $\rho \in (0, 1)$ let $\mathbf{A}(\rho)$ be the family of all functions from \mathcal{UC}_ρ such that at each $x_0 \in D(f)$ the following two conditions hold

$$(A_\rho 1) \quad f(x_0) = 0,$$

(A $_\rho$ 2) for every $\varepsilon > 0$ there exists $x \in (x_0 - \varepsilon, x_0 + \varepsilon)$ such that $\lambda(N_f \cap J) > \rho \lambda(J)$, where J is a closed interval with endpoints x and x_0 (in other words x_0 is a point of ρ -type upper density in O'Malley sense of N_f).

Definition 8.21. [17] For each $\rho \in (0, 1)$ let $\mathbf{B}(\rho)$ be a family of all functions from \mathcal{UC}_ρ such that at each $x_0 \in D(f)$ the following two conditions hold

$$(B_\rho 1) \quad f(x_0) = 0,$$

(B $_\rho$ 2) for every $\varepsilon > 0$ there exists $x \in (x_0 - \varepsilon, x_0 + \varepsilon)$ such that $\lambda(N_f \cap J) \geq \rho \lambda(J)$, where J is a closed interval with endpoints x and x_0 .

Theorem 8.36. [17]

1. $\mathbf{A}(\rho) \subset \mathfrak{M}_m(\mathcal{O}\mathcal{U}\mathcal{C}_\rho)$ for $\rho \in (0, 1)$.
2. $\mathfrak{M}_m(\mathcal{O}\mathcal{U}\mathcal{C}_\rho) \subset \mathbf{B}(\rho)$ for $\rho \in (0, 1)$.

Problem 8.1. Characterize $\mathcal{M}_m(\mathcal{O}\mathcal{U}\mathcal{C}_\rho)$ for $\rho \in (0, 1)$.

Remark 8.5. Maximal multiplicative classes for porouscontinuous functions and v -porouscontinuous functions are still unknown.

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