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Chapter 7
Generalized (topological) metric space. From nowhere density to infinite games

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7.1 Introduction and basic notations and denotations

In many considerations connected with pure mathematics as well as with its applications, topological structures play an important role. Without them, applying of many mathematical tools would be impossible. For that reason, having defined some set (sometimes in very practical situations connected for example with information flow theory, graph theory etc.) we tend to equip this set with some topological structure (e.g. topology, metric, pseudometric, uniformity etc.). It is also a natural action to enrich possibility of creating such structures.

At the end of XX century Á. Császár introduced new structures, *generalized topologies* ([9, 10]). In the paper [21] the possibility of applying these structures in research connected with information flow has been noticed for the first time.

From the point of view of pure mathematics, considering generalized topological spaces gives new research possibilities, which often do not have its analogues in the case of classical topological spaces.

The authors of this chapter have focused their examinations on the notion of nowhere density of sets. The reason for that is the fact that adopting the two definitions which give equivalent notions in classical topological spaces, leads to nonequivalent notions in the case of generalized topological spaces. In consequence, examining of different notions connected with analogues of meager sets and further with Baire spaces, is desirable. Analysis of properties of these sets or spaces gives completely new possibilities and allows to obtain new theorems which are unknown in classical case. The authors of this chapter have devoted to this issue, among others, the papers: [24, 22].

Let us return for a moment to classical topological spaces. The notion of a Baire space is closely related to the Baire Theorem for complete spaces. From this fact, the new challenge arises: building theory of generalized metric spaces and considering in it the issue of nowhere density of sets and, in consequence, complete spaces, Baire spaces and analogues of known theorems for complete metric spaces, especially connected with the Cantor Theorem and infinite games. Initial results within the scope of this issue were published in [19]. Significant development of this theory is submitted in, unpublished yet, paper [25].

The above introduction justifies the fact that we will start our considerations here with presenting basic facts connected with nowhere density in generalized topological space and Baire generalized topological space. It is not our aim to extend excessively all the research directions connected with this issue but only to signalize basic definitions and theorems. By contrast, we will discuss more precisely the issue related to generalized metric spaces. Particular attention will be paid to infinite games. We will start with the generalization of known Banach-Mazur Game and then we will show a new original game which has been presented for the first time in [19].

In order to avoid excessive lengthening of this section we will not present the proofs of theorems as well as examples. All the facts presented here one can find in [19, 24, 22].

Throughout the paper $\mathbb{N}$ denotes the set of positive integers. The symbol $\mathbb{N}_0$ stands for the set $\mathbb{N} \cup \{0\}$. We will write ρ_E for the Euclidean metric for real line. The power set of a nonempty set X will be denoted by $\mathscr{P}(X)$. Moreover, we will denote by $\Gamma(f)$ the graph of a function $f\colon X \to X$. The symbol $\Theta_f(x_0)$ stands for the orbit of f at x_0 i. e. $\Theta_f(x_0) = \{x_0, f(x_0), f^2(x_0), \dots\}$.

Let $\{F_n\}_{n\in\mathbb{N}} \subset \mathscr{P}(X)$. If $\bigcup_{n=1}^{\infty}\bigcap_{k=1}^{\infty} F_{n+k} = \bigcap_{n=1}^{\infty}\bigcup_{k=1}^{\infty} F_{n+k}$, then we will say that $\bigcap_{n=1}^{\infty}\bigcup_{k=1}^{\infty} F_{n+k}$ is a *limit of the sequence* $\{F_n\}_{n\in\mathbb{N}}$ (denoted by $\operatorname{Lim}_{n\to\infty} F_n$).

Let (X,ρ) be a metric space. We will use the symbols $\operatorname{diam}_\rho(A)$, $\operatorname{int}_\rho(A)$ and $\operatorname{cl}_\rho(A)$ to denote the diameter, the interior and the closure of the set $A \subset X$, respectively. Moreover, we will write $\rho - \lim_{n\to\infty} x_n = x$ if the sequence $\{x_n\}_{n\in\mathbb{N}} \subset X$ converges to $x \in X$ with respect to the metric ρ.

In our consideration, set valued functions (known also under the name multifunctions) will play an important role. From now on, we will consider only set valued functions $\mathfrak{F}\colon X \multimap X$ such that $\mathfrak{F}(x) \neq \emptyset$ for each $x \in X$.

If $A \subset X$ and $\mathfrak{F}\colon X \multimap X$, we set $\mathfrak{F}(A) = \bigcup_{a\in A} \mathfrak{F}(a)$. Moreover, we put $\mathfrak{F}^0(x) = \{x\}$ and $\mathfrak{F}^i(x) = \mathfrak{F}(\mathfrak{F}^{i-1}(x))$ for $i \in \mathbb{N}$.

The notation $\mathfrak{F} \sqsubset \mathfrak{F}_1$ (where $\mathfrak{F},\mathfrak{F}_1\colon X \multimap X$) means that $\mathfrak{F}(x) \subset \mathfrak{F}_1(x)$ for any $x \in X$. We will say that a sequence of set valued functions $\{\mathfrak{F}_n\}_{n\in\mathbb{N}}$ is *decreasing* if $\mathfrak{F}_{n+1} \sqsubset \mathfrak{F}_n$ for $n \in \mathbb{N}$.

Let $\mathfrak{F}\colon X \multimap X$ and $\mathfrak{F}_n\colon X \multimap X (n \in \mathbb{N})$ be set valued functions. A sequence $\{\mathfrak{F}_n\}_{n\in\mathbb{N}}$ is said to be *s-convergent* to a set valued function $\mathfrak{F}$ (denoted $\mathfrak{F} = \operatorname{LIM}_{n\to\infty} \mathfrak{F}_n$), if $\mathfrak{F}(x) = \operatorname{Lim}_{n\to\infty} \mathfrak{F}_n(x)$ for any $x \in X$.

7.2 GTS and GMS

As it has been already mentioned, the notion of a generalized topological space was introduced by A. Császár in [9]. Generalized metric spaces were first considered in [19]. In this section we will recall basic definitions and facts connected with these notions.

7.2.1 Generalized topological space

Let X be a nonempty set. We shall say that a family $\mathscr{G} \subset \mathscr{P}(X)$ is a *generalized topology* in X iff $\emptyset \in \mathscr{G}$ and $\bigcup_{t\in T} G_t \in \mathscr{G}$ whenever $\{G_t : t \in T\} \subset \mathscr{G}$. In further considerations we will assume that $\mathscr{G}$ contains at least one nonempty set. The pair $(X,\mathscr{G})$ will be called a *generalized topological space* (briefly GTS). Moreover, if $X \in \mathscr{G}$ we shall say that $(X,\mathscr{G})$ is a *strong generalized topological space* (sGTS for short)) and $\mathscr{G}$ is a *strong generalized topology*.

Let us say that $\mathscr{B} \subset \mathscr{G}$ is a base for $\mathscr{G}$ if every $A \in \mathscr{G}$ is a union of elements of $\mathscr{B}$ ([13]). If $(X, \mathscr{G}_X)$ and $(Y, \mathscr{G}_Y)$ are GTS, then a *product generalized topology* $\mathscr{G}_{X\times Y}$ in $X \times Y$ is a collection of all sets being a union of sets of the form $M_1 \times M_2$ where $M_1 \in \mathscr{G}_X$ and $M_2 \in \mathscr{G}_Y$ ([15]).

From now on, if we will consider a generalized topological space $(X, \mathscr{G})$, then we will use the symbol $\widetilde{\mathscr{G}}$ to denote the family $\mathscr{G} \setminus \{\emptyset\}$.

Generalized topological spaces were studied by many mathematicians (e.g. [10] - [15], [5, 19, 21, 27]). These studies are associated with pure mathematics, as well as with the applications, e.g. in the theory of information flow.

It seems interesting to note that every generalized topology in X can be associated with a monotonic map $\Psi\colon \mathscr{P}(X) \to \mathscr{P}(X)$ (i.e a map such that $\Psi(A) \subset \Psi(B)$ if $A \subset B \subset X$). More precisely, in [10] one can find that every generalized topology $\mathscr{G}$ in X can be generated by some monotonic map $\Psi\colon \mathscr{P}(X) \to \mathscr{P}(X)$ in the following way $\mathscr{G} = \{A \subset X \colon A \subset \Psi(A)\}$. On the other hand, if $\Psi\colon \mathscr{P}(X) \to \mathscr{P}(X)$ is a monotonic map then $\mathscr{G}_\Psi = \{A \subset X \colon A \subset \Psi(A)\}$ is a generalized topology ([9]).

In the theory of a generalized topological space almost all notions are defined similarly as for a standard topological space. We recall some of them since they will be useful in the next part of this note. We shall follow the terminology of [9, 10, 22].

Let $(X, \mathscr{G})$ be a generalized topological space. The $\mathscr{G}$-closure ($\mathscr{G}$-interior) of $A \subset X$ will be denoted by $\mathrm{cl}(A)$ ($\mathrm{int}(A)$). A set $A \subset X$ is called *dense* if $\mathrm{cl}(A) = X$. It is easily seen that A is a dense set iff for any $U \in \widetilde{\mathscr{G}}$ we have that $A \cap U \neq \emptyset$.

The space $(X, \mathscr{G})$ is said to be *thick* if for any $U \in \widetilde{\mathscr{G}}$ and any finite set $A \subset U$ there exists $V \in \widetilde{\mathscr{G}}$ such that $V \subset U \setminus A$.

However, despite identical definitions, the properties of some mathematical objects in the case of usual topological space may be quite different from the properties of respective objects in generalized topology. The examples of such situation are the notions of nowhere dense sets.

Let $(X, \mathscr{G})$ be GTS. If $\mathrm{int}(\mathrm{cl}(A)) = \emptyset$ then we shall say that A is a *nowhere dense set*. In the case of topological space the above definition is equivalent to the fact that every nonempty open set U contains nonempty open subset V such that $V \cap A \neq \emptyset$. A simple example (see [19]) leads us to the conclusion that in the case of GTS this equivalence is false. Consequently, we have a second notion connected with nowhere density. We shall say that $A \subset X$ is a *strongly nowhere dense set* if for any $U \in \widetilde{\mathscr{G}}$ there exists $V \in \widetilde{\mathscr{G}}$ such that $V \subset U$ and $V \cap A = \emptyset$.

Of course, if A is a strongly nowhere dense set then A is a nowhere dense set. Let us note the basic difference between these concepts.

Proposition 7.1 (Property 2.4 [19]). *There exists GTS $(X,\mathcal{G})$ and nowhere dense sets $A,B \subset X$ such that $A \cup B$ is not a nowhere dense set but for every two strongly nowhere dense sets A and B in an arbitrary GTS $(X,\mathcal{G})$ the union $A \cup B$ is a strongly nowhere dense set.*

An obvious consequence of the foregoing is the fact that there are two types of definitions corresponding to a meager set in usual topology. We shall say that $A \subset X$ is a *meager* (*s-meager*) set if there exists a sequence $\{A_n\}_{n\in\mathbb{N}}$ of nowhere dense (strongly nowhere dense) sets such that $A = \bigcup_{n\in\mathbb{N}} A_n$. A set A is called a *second category* (*s-second category*) set if it is not a meager (s-meager) set. A set A is said to be *residual* (*s-residual*) if $X \setminus A$ is meager (s-meager).

A further consequence is a distinction of three types of notions corresponding to a Baire space in the case of topological spaces. We will say that GTS $(X,\mathcal{G})$ is

- a *weak Baire space* if each set $U \in \widetilde{\mathcal{G}}$ is an s-second category set;
- a *Baire space* if each $U \in \widetilde{\mathcal{G}}$ is a second category set;
- a *strong Baire space* if $V_1 \cap \cdots \cap V_n$ is a second category set for any $V_1, V_2, \ldots, V_n \in \mathcal{G}$ such that $V_1 \cap \cdots \cap V_n \neq \emptyset$.

We have

Theorem 7.1 (Property 2.7 [19]). *If GTS $(X,\mathcal{G})$ is a strong Baire space, then it is a Baire space. If GTS $(X,\mathcal{G})$ is a Baire space, then it is a weak Baire space.* The converse implications do not hold.

The definition of a strong Baire space inspires us to consider a new property of GTS $(X,\mathcal{G})$:

(INT-GTS) $\operatorname{int}(V_1 \cap V_2 \cap \cdots \cap V_m) \neq \emptyset$ for any $m \in \mathbb{N}$ and $V_1, V_2, \ldots, V_m \in \mathcal{G}$ such that $V_1 \cap V_2 \cap \cdots \cap V_m \neq \emptyset$.

Taking into account the above condition we have two dual theorems.

Theorem 7.2 (Property 2.2 [19]). *If GTS $(X,\mathcal{G})$ satisfies the condition* (INT-GTS) *then a set A is nowhere dense if and only if it is strongly nowhere dense.*

Theorem 7.3 (Property 2.8 [19]). *If GTS $(X,\mathcal{G})$ satisfies the condition* (INT-GTS) *then three notions: a strong Baire space, a Baire space and a weak Baire space are equivalent.*

Certainly, since each topological space satisfies the condition (INT-GTS), then it follows that in the case of topological space the above three notions are equivalent.

In this paper we will concentrate mostly on generalized metric spaces and their properties. Therefore we will only signal some facts connected with Baire generalized topological spaces. Let us start with recalling some notions.

Let $(X,\mathscr{G})$ be GTS. If $f\colon X\to X$, then the set of all its continuity points will be denoted by $\mathbb{C}(f)$. We shall say that f is a *cm-function* if the set $X\setminus\mathbb{C}(f)$ is a countable set and $f^{-m}(x)=\{z\in X\colon f^m(z)=x\}$ is a meager set for any $x\notin\mathbb{C}(f)$ and $m\in\mathbb{N}_0$.

Let $\mathfrak{F}\colon X\multimap X$ be a set valued function. We will say that $\mathfrak{F}$ is *lower semi-continuous at a point* $x\in X$ if for every set $U\in\mathscr{G}$ such that $\mathfrak{F}(x)\cap U\neq\emptyset$ there exists $V\in\mathscr{G}$ such that $x\in V$ and $\mathfrak{F}(t)\cap U\neq\emptyset$ for any $t\in V$. A set valued function $\mathfrak{F}$ is lower semicontinuous if it is lower semicontinuous at each point $x\in X$.

An *orbit of* x_0 *under* $\mathfrak{F}$ is a set (sequence[1]) $\Theta_{\mathfrak{F}}(x_0)=\{x_0,x_1,x_2,\dots\}$ such that $x_i\in\mathfrak{F}(x_{i-1})$ for any $i=1,2,3,\dots$. Clearly, there may exist a lot of different orbits of x_0 under $\mathfrak{F}$. Let us denote by $\Theta^a_{\mathfrak{F}}(x_0)$ the family of all orbits $\Theta_{\mathfrak{F}}(x_0)$ of x_0 under $\mathfrak{F}$.

A set valued function $\mathfrak{F}$ is *transitive* if, for any pair $U,V\in\widetilde{\mathscr{G}}$, there exists a positive integer n such that $V\cap\mathfrak{F}^n(U)\neq\emptyset$.

As it happens often in the case of GTS one can consider also a dual notion.

A set valued function $\mathfrak{F}$ is *strongly transitive* if, for any pair $U,V\in\widetilde{\mathscr{G}}$, the set $\{x\in U\colon \underset{\Theta_{\mathfrak{F}}(x)}{\exists}\Theta_{\Phi}(x)\cap V\neq\emptyset\}$ is of the second category.

Then we have two interesting theorems.

Theorem 7.4 (Theorem 5 [22]). *Let $(X,\mathscr{G})$ be a Baire generalized topological space with a countable base. Let $\mathfrak{F}\colon X\multimap X$ be a lower semicontinuous set valued function. The following conditions are equivalent*

(i) $\mathfrak{F}$ *is transitive,*
(ii) *the set* $\{x\in X\colon \mathrm{cl}(\bigcup\Theta^a_{\mathfrak{F}}(x))=X\}$ *is residual.*

Theorem 7.5 (Theorem 10 [22]). *Let $(X,\mathscr{G})$ be a thick, strong Baire generalized topological space with countable base. Let $f\colon X\to X$ be a cm-function and $\bar f\colon X\multimap X$ be a set valued function such that $\bar f(x)=\{\alpha\in X\colon (x,\alpha)\in cl(\Gamma(f))\}$ for any $x\in X$. The following conditions are equivalent:*

(a) f *is strongly transitive,*

[1] In the literature the notion of orbit is used interchangeably in both senses: as a set and as a sequence.

(b) *there exists* $x_0 \in X$ *such that* $\Theta_f(x_0)$ *is a dense set and* $\Theta_f(x_0) \subset \mathbb{C}(f)$*,*
(c) $\bar{f}$ *is strongly transitive,*
(d) *there exists* $x_0 \in X$ *such that there exists an orbit* $\Theta_{\bar{f}}(x_0)$ *which is a dense set and* $\Theta_{\bar{f}}(x_0) \subset \mathbb{C}(f)$*.*

More information about Baire GTS one can find in [22, 19] and [25].

7.2.2 Generalized metric space

In the case of topological spaces, a special role is played by metrizable spaces and, therefore, by metric spaces. A possibility of considering abstract distances between elements of a space, allows to make a detailed analysis of many important problems in theoretical context as well as in practical issues. On the other hand, use of techniques which are specific to metric spaces, makes many considerations more simple (a proper example here is completeness of a space) or easier in description.

In 2013, there was published the paper [19] in which the notion of a generalized metric space being an analogue to metric spaces for GTS was introduced. Expanding of these issues can be found in [25].

Now, we will present briefly some facts connected with this theory.

Let $X \neq \emptyset$. The symbol π stands for the family of metrics defined on subsets of X, i.e. if $\rho \in \pi$ it means that one can find a nonempty set $A_\rho \subset X$ such that ρ is a metric on A_ρ. The set A_ρ is named a domain of ρ. We will use the symbol $\mathrm{dom}(\rho)$) to denote the domain of a metric ρ. The space (X,π) is called a *generalized metric space* (GMS for short). If we will write π_X it means that for each metric $\rho \in \pi_X$ we have that $\mathrm{dom}(\rho) = X$.

We will say that a set $A \subset X$ is π-open if for each $x \in A$ there exist $\rho \in \pi$ and $\varepsilon > 0$ such that $x \in \mathrm{dom}(\rho)$ and the set $B_\rho(x,\varepsilon) = \{y \in \mathrm{dom}(\rho) \colon \rho(x,y) < \varepsilon\}$ is contained in A. We will denote by $\mathscr{G}_\pi$ the family of all π-open sets in (X,π). It is easy to check that if (X,π) is GMS then $(X,\mathscr{G}_\pi)$ is GTS.

For our further considerations, the notion of *kernel* of GMS will be particularly important.

Let (X,π) be GMS. A *kernel* of the space (X,π) is a finite family $\pi_0 \subset \pi$ such that for any set $V \in \widetilde{\mathscr{G}_\pi}$ there exist $\rho \in \pi_0$ with property $\mathrm{int}_\rho(V) \neq \emptyset$. If a finite family $\pi_0 \subset \pi$ have the property: for any $V_1,\dots,V_m \in \mathscr{G}_\pi$ such that $V_1 \cap \dots \cap V_m \neq \emptyset$ there exists $\rho \in \pi_0$ such that $\mathrm{int}_\rho(V_1 \cap \dots \cap V_m) \neq \emptyset$, then we call it a *perfect kernel* of the space (X,π). The set of all kernels (perfect kernels) of the space (X,π) will be denoted by $\mathrm{Ker}(X,\pi)$ ($\mathrm{Ker}_p(X,\pi)$).

Obviously, each perfect kernel of the space (X,π) is a kernel of this space. Moreover, if π_0 is a perfect kernel of the space (X,π) and π_1 is a finite family such that $\pi_0 \subset \pi_1 \subset \pi$ then π_1 is a perfect kernel of the space (X,π).

The examples of kernels of generalized metric spaces one can find in [19] and [25].

The definitions introduced for GTS may be adopted for GMS. So we have:

Theorem 7.6 (Lemma 4.4 [19]). *If GMS (X,π) has a perfect kernel then it fulfills the condition* (INT-GTS).

We have also

Theorem 7.7 (Proposition 4.5 [19]). *Let (X,π) be GMS with a perfect kernel. The following conditions are equivalent:*

(i) (X,π) *is a strong Baire space.*
(ii) (X,π) *is a Baire space.*
(iii) (X,π) *is a weak Baire space.*

7.2.3 Complete spaces

In the theory of metric spaces, a particular role is played by *complete spaces*. The Baire Theorem (vide definition of a Baire space), the Cantor Theorem and the Banach Fixed-Point Theorems and Banach spaces are classical examples of a wide usage of these spaces in considerations of various problems. Within the range of GTS theory one can also consider complete spaces, wherein in this case we need to consider several types of these spaces (similarly to Baire spaces). At the current stage of research, many questions related to these spaces have not been answered yet. Therefore, we will only signal the existing results concerning these issues.

We shall say that GMS (X,π) is *weakly complete* (*complete*) if there exists $\pi_0 \in \mathrm{Ker}(X,\pi)$ ($\pi_0 \in \mathrm{Ker}_p(X,\pi)$) consisting of complete metrics. Moreover, if (X,π) is a complete space and π is a finite family consisting of complete metrics then we shall say that (X,π) is *strongly complete*.

Obviously if (X,π) is a strongly complete space then it is a complete space, and if it is a complete space then it is a weakly complete space. Moreover, these implications can not be inverted. One can find relevant examples in [19].

The basic question one can ask, concerns the possibility of transfering the Baire Theorem for the case of GMS. In our situation, it refers to establishing the relation between complete spaces and Baire spaces in GMS. This relation is established by the following theorem.

Theorem 7.8 (Theorem 4.11 and Corollary 4.12 [19]).

(i) *If GMS* (X,π_X) *is weakly complete then* $(X,\mathscr{G}_{\pi_X})$ *is a weak Baire GTS.*
(ii) *If GMS* (X,π_X) *is complete then* $(X,\mathscr{G}_{\pi_X})$ *is a strong Baire GTS.*

Another interesing question concerns analogue of the Cantor Theorem. In this case we have:

Theorem 7.9 (Theorem 4.9 [19]). *Let* (X,π) *be GMS. The space* (X,π) *is weakly complete if and only if there exists* $\pi_0 \in \mathrm{Ker}(X,\pi)$ *such that for any sequence of metrics* $\{\rho_n\}_{n\in\mathbb{N}} \subset \pi_0$ *and for any decreasing sequence of sets* $\{F_n\}_{n\in\mathbb{N}}$ *such that* $F_n = \mathrm{cl}_{\rho_n}(F_n)$ *for* $n \in \mathbb{N}$ *and* $\rho_E - \lim_{n\to\infty} \mathrm{diam}_{\rho_n}(F_n) = 0$ *we have that* $\bigcap_{n=1}^{\infty} F_n$ *is a singleton.*

7.3 Infinite games

The history of infinite games is quite rich. Undoubtedly, its background are finite games being considered in XVII century. The basis of strategic and positional games have been developed by Borel [2, 3, 4], von Neumann [23] and Steinhaus [28]. We do not tend to present all aspects of this issue (it would be impossible in view of a very rich literature), however it should be emphasized that it is still examined by many scientists. As examples we can mention here the following papers: [6, 7, 8, 16, 17, 18, 20, 31, 32], however mentioning these few items definitely does not exhaust the subject.

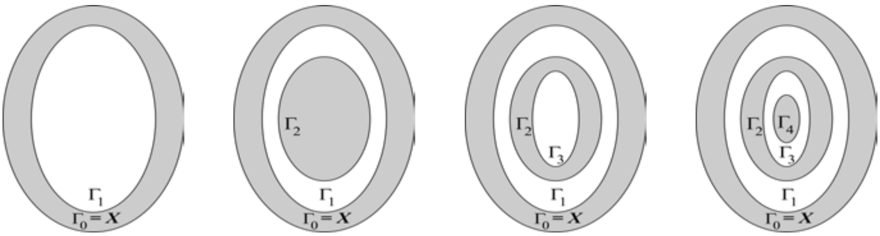
In our case, we will refer the considerations connected with infinite games exclusively to GMS (X,π_X) having perfect kernel π_0.

7.3.1 A B-M game

In the period 1935-1941 in the town Lwów (which at that time was in Poland), so-called Scottish Book ([30]) was created. A group of mathematicians (among others St. Banach, H. Steinhaus, S. Mazur, S. Ulam) used to meet and discuss on mathematics in the Scotish Caffé. They had written down mathematical problems in a thick notebook which was a gift from the wife of Stefan Banach.

After some time, the notebook was named a Scotish Book (after the name of cafe). Problem 43 in that book (formulated by S. Mazur) was connected with (using modern terminology) Banach-Mazur game (we will denote it briefly B-M game). An interesting history of research dealing with Banach-Mazur game can be found in [26] and [29].

Now, we will present B-M game for GMS. Let us fix GMS (X, π_X) which has a perfect kernel. Put $\Gamma_0 = X$. Two players take part in the game. Let us denote them by $\mathscr{A}$ and $\mathscr{B}$ (similarly as S. Mazur had done it). The players choose sets succesively according to the following rules:



The first player chooses a set $\Gamma_1 \in \widetilde{\mathscr{G}_\pi}$ such that $\Gamma_1 \subset \Gamma_0$.

The second player chooses a set $\Gamma_2 \in \widetilde{\mathscr{G}_\pi}$ such that $\Gamma_2 \subset \Gamma_1$.

The first player chooses a set $\Gamma_3 \in \widetilde{\mathscr{G}_\pi}$ such that $\Gamma_3 \subset \Gamma_2$.

The second player chooses a set $\Gamma_4 \in \widetilde{\mathscr{G}_\pi}$ such that $\Gamma_4 \subset \Gamma_3$, etc.

Obviously $\{\Gamma_n\}_{n\in\mathbb{N}_0}$ is a decreasing sequence of nonempty $\mathscr{G}_\pi$-open sets. It is easy to see that

$$\bigcap_{n=0}^{\infty} \Gamma_n = \bigcap_{n=0}^{\infty} \Gamma_{2n} = \bigcap_{n=1}^{\infty} \Gamma_{2n-1}.$$

There are two possibilities either $\bigcap_{n=0}^{\infty} \Gamma_n \neq \emptyset$ and then the player $\mathscr{A}$ wins or $\bigcap_{n=0}^{\infty} \Gamma_n = \emptyset$ and in this case, the player $\mathscr{B}$ is a winner.

In order to establish a definition of a strategy and a winning strategy, we will formulate first, as in the literature, a definition of a partial play.

The partial play in B-M game for the player $\mathscr{A}$ ($\mathscr{B}$) is a finite sequence of sets $\{\Gamma_0, \Gamma_1, ..., \Gamma_{n-1}, \Gamma_n\} \subset \widetilde{\mathscr{G}_\pi}$ such that

$$\Gamma_0 \supset \Gamma_1 \supset \Gamma_2 \supset \Gamma_3 \supset ... \supset \Gamma_{n-1} \supset \Gamma_n$$

and Γ_n was chosen by the player $\mathscr{B}$ ($\mathscr{A}$). To facilitate of definitions and formulas let us assume that, if the player $\mathscr{A}$ ($\mathscr{B}$) chooses first then the sequence $\{\Gamma_0\}$ is the partial play in B-M for the player $\mathscr{A}$ ($\mathscr{B}$).

The set of all partial plays in B-M game for the player $\mathscr{A}$ ($\mathscr{B}$) will be denoted by $P(\mathscr{A})$ ($P(\mathscr{B})$).

The strategy in B-M game for the player $\mathscr{A}$ ($\mathscr{B}$) is a function $\eta\colon P(\mathscr{A}) \to \widetilde{\mathscr{G}_\pi}$ ($\eta\colon P(\mathscr{B}) \to \widetilde{\mathscr{G}_\pi}$), such that $\eta(\{\Gamma_0, \Gamma_1, ..., \Gamma_{n-1}, \Gamma_n\}) \subset \Gamma_n$.

In the theory of infinite games, the existence of winning strategy for a given player is one of the most essential questions. It may be dependent (except of the case of determined games) on properties of some mathematical objects. Roughly speaking a winning strategy is a possibility of such activity of the player that it determines its victory independently of the reaction of the other player. So, let us start with the definitions.

We shall say that a strategy $\eta\colon P(\mathscr{A}) \to \widetilde{\mathscr{G}_\pi}$ ($\eta\colon P(\mathscr{B}) \to \widetilde{\mathscr{G}_\pi}$) is *winning in B-M game for the player* $\mathscr{A}$ ($\mathscr{B}$) if for any decreasing sequence of sets $\{\Gamma_n\}_{n\in\mathbb{N}_0} \subset \widetilde{\mathscr{G}_\pi}$ with the property:

for any $i \in \mathbb{N}$, if $\{\Gamma_0, \Gamma_1, ..., \Gamma_{i-1}\} \in P(\mathscr{A})$ ($\{\Gamma_0, \Gamma_1, ..., \Gamma_{i-1}\} \in P(\mathscr{B})$)
then $\Gamma_i = \eta(\{\Gamma_0, \Gamma_1, ..., \Gamma_{i-1}\})$

we have that $\bigcap_{n=0}^{\infty} \Gamma_n \neq \emptyset$ ($\bigcap_{n=0}^{\infty} \Gamma_n = \emptyset$).

Now, we will present two theorems connected with B-M game in GMS (X, π_X) (let us recall that we consider here exclusively GMS (X, π_X) having a perfect kernel π_0). Although the theorems have their analogous in the case of classical spaces, the proofs differ significantly from the earlier results.

Theorem 7.10 (Theorem 5.1 [19]). *A space (X, π_X) is a Baire space if and only if there is no winning strategy in B-M game for the player $\mathscr{A}$ whenever $\mathscr{A}$ chooses first.*

Theorem 7.11 (Proposition 5.2 [19]). *There is no winning strategy in B-M game for the player $\mathscr{B}$.*

7.3.2 An S-F game

This part contains considerations connected with original game described in [19]. This game does not have its equivalent in earlier research.

Obviously, in the literature one can find infinite games connected with set valued function, e.g. [1] (the notion *topological game* has been introduced there) or [26]. However, they are of different character then the game presented below. Additional thing that distinguishes this game is the fact that there are three players taking part in it, however it is not connected with the number of its authors. Nevertheless, in the below notations the authors' names will be reflected.

So, let us fix GMS (X,π_X) which has a perfect kernel and assume that three players take part in our game: $\mathscr{K}$, $\mathscr{L}$ and $\mathscr{P}$ (the first letters of the names of the authors of this chapter), who choose alternately set valued functions according to fixed rules.

Player $\mathscr{K}$ either choose first or after the player $\mathscr{P}$.
Player $\mathscr{P}$ either choose first or after the player $\mathscr{L}$.
Player $\mathscr{L}$ either choose first or after the player $\mathscr{K}$.

In this game, players use a special kind of set valued functions i. e. set valued functions having a fixed set. We call $U \in \widetilde{\mathscr{G}_{\pi_X}}$ a *fixed set* for a set valued function $\mathfrak{F}\colon X \multimap X$ if $U \subset \mathfrak{F}(x)$ for each $x \in U$. The family of all fixed sets for a set valued function $\mathfrak{F}$ will be denoted by $\mathbb{F}(\mathfrak{F})$. The symbol $\mathbf{FIX}_{\mathbb{F}}(X)$ stands for the family of all set valued functions $\mathfrak{F}\colon X \multimap X$ such that $\mathbb{F}(\mathfrak{F}) \neq \emptyset$.

We start by putting $\mathfrak{F}_0(x) = X$ for $x \in X$ and fixing the first player. The game follows by the rules

The first player chooses $\mathfrak{F}_1 \in \mathbf{FIX}_{\mathbb{F}}(X)$, such that $\mathfrak{F}_1 \sqsubset \mathfrak{F}_0$.
The second player chooses $\mathfrak{F}_2 \in \mathbf{FIX}_{\mathbb{F}}(X)$, such that $\mathfrak{F}_2 \sqsubset \mathfrak{F}_1$.
The third player chooses $\mathfrak{F}_3 \in \mathbf{FIX}_{\mathbb{F}}(X)$, such that $\mathfrak{F}_3 \sqsubset \mathfrak{F}_2$.
The first player chooses $\mathfrak{F}_4 \in \mathbf{FIX}_{\mathbb{F}}(X)$, such that $\mathfrak{F}_4 \sqsubset \mathfrak{F}_3$.
The second player chooses $\mathfrak{F}_5 \in \mathbf{FIX}_{\mathbb{F}}(X)$, such that $\mathfrak{F}_5 \sqsubset \mathfrak{F}_4$, etc.

In view of the fact that players in this game choose set valued functions, we will call it a *set valued function game* (S-F game for short). In order to make further notation clear, we will sometimes use upper index $\mathscr{K}$ ($\mathscr{P}$ or $\mathscr{L}$) to denote a set valued function chosen by $\mathscr{K}$ ($\mathscr{P}$ or $\mathscr{L}$), i.e. the notation $\mathfrak{F}_n = \mathfrak{F}_n^{\mathscr{K}}$ means that the set valued function $\mathfrak{F}_n$ was chosen by the player $\mathscr{K}$.

Now, we need to define the rules of wins for individual players.

The player $\mathscr{K}$ wins in S-F game if the sequence $\{\mathfrak{F}_n\}_{n\in\mathbb{N}_0}$ is s-convergent to a set valued function $\mathfrak{F} \in \mathbf{FIX}_{\mathbb{F}}(X)$.

The player $\mathscr{L}$ wins in S-F game if the sequence $\{\mathfrak{F}_n\}_{n\in\mathbb{N}_0}$ is s-convergent to a set valued function $\mathfrak{F} \notin \mathbf{FIX}_{\mathbb{F}}(X)$.

The player $\mathscr{P}$ wins in S-F game if the sequence $\{\mathfrak{F}_n\}_{n\in\mathbb{N}_0}$ is not s-convergent.

Similarly to the case of B-M game, we will define now the notions of a partial play and strategy in S-F game.

The partial play in S-F game for the player $\mathscr{K}$ ($\mathscr{L}$ or $\mathscr{P}$) is a finite sequence of set valued functions $\{\mathfrak{F}_0, \mathfrak{F}_1, ..., \mathfrak{F}_n\} \subset \mathbf{FIX}_{\mathbb{F}}(X)$ such that

$$\mathfrak{F}_n \sqsubset \mathfrak{F}_{n-1} \sqsubset ... \sqsubset \mathfrak{F}_1 \sqsubset \mathfrak{F}_0$$

and $\mathfrak{F}_n = \mathfrak{F}_n^{\mathscr{P}}$ ($\mathfrak{F}_n = \mathfrak{F}_n^{\mathscr{K}}$ or $\mathfrak{F}_n = \mathfrak{F}_n^{\mathscr{L}}$). Moreover, if the player $\mathscr{K}$ ($\mathscr{L}$ or $\mathscr{P}$) chooses first then the sequence $\{\mathfrak{F}_0\}$ is the partial play in S-F game for the player $\mathscr{K}$ ($\mathscr{L}$ or $\mathscr{P}$). The set of all partial plays in S-F game for the player $\mathscr{K}$ ($\mathscr{L}$ or $\mathscr{P}$) will be denoted by $S(\mathscr{K})$ ($S(\mathscr{L})$ or $S(\mathscr{P})$).

The strategy in S-F game for the player $\mathscr{K}$ ($\mathscr{L}$ or $\mathscr{P}$) is a function $\xi : S(\mathscr{K}) \to \mathbf{FIX}_{\mathbb{F}}(X)$ ($\xi : S(\mathscr{L}) \to \mathbf{FIX}_{\mathbb{F}}(X)$ or $\xi : S(\mathscr{P}) \to \mathbf{FIX}_{\mathbb{F}}(X)$), such that
$\xi(\{\mathfrak{F}_0, \mathfrak{F}_1, ..., \mathfrak{F}_n\}) \sqsubset \mathfrak{F}_n$.

We shall say that a strategy $\xi : S(\mathscr{K}) \to \mathbf{FIX}_{\mathbb{F}}(X)$ ($\xi : S(\mathscr{L}) \to \mathbf{FIX}_{\mathbb{F}}(X)$ or $\xi : S(\mathscr{P}) \to \mathbf{FIX}_{\mathbb{F}}(X)$) *is winning in S-F game for the player* $\mathscr{K}$ ($\mathscr{L}$ or $\mathscr{P}$) if for any decreasing sequence of set valued functions $\{\mathfrak{F}_n\}_{n\in\mathbb{N}_0} \subset \mathbf{FIX}_{\mathbb{F}}(X)$ with the property

for any $i \in \mathbb{N}$ if $\{\mathfrak{F}_0, \mathfrak{F}_1, ..., \mathfrak{F}_{i-1}\} \in S(\mathscr{K})$ ($\{\mathfrak{F}_0, \mathfrak{F}_1, ..., \mathfrak{F}_{i-1}\} \in S(\mathscr{L})$ or $\{\mathfrak{F}_0, \mathfrak{F}_1, ..., \mathfrak{F}_{i-1}\} \in S(\mathscr{P})$) then $\mathfrak{F}_i = \xi(\{\mathfrak{F}_0, \mathfrak{F}_1, ..., \mathfrak{F}_{i-1}\})$

we have that the sequence $\{\mathfrak{F}_n\}_{n\in\mathbb{N}_0}$ is s-convergent to $\mathfrak{F} \in \mathbf{FIX}_{\mathbb{F}}(X)$ ($\{\mathfrak{F}_n\}_{n\in\mathbb{N}_0}$ is s-convergent to $\mathfrak{F} \notin \mathbf{FIX}_{\mathbb{F}}(X)$ or $\{\mathfrak{F}_n\}_{n\in\mathbb{N}_0}$ is not s-convergent).

Let us begin with the situation when player $\mathscr{K}$ chooses first in S-F game. Then we have:

Theorem 7.12 (Theorem 5.6 [19]). *If there is an isolated point in the space* (X, π_X) *and the player* $\mathscr{K}$ *chooses first in S-F game then the player* $\mathscr{K}$ *has a winning strategy in S-F game.*

The natural consequence is considering the situation when player $\mathscr{K}$ does not choose first in S-F game. Then:

Theorem 7.13 (Theorem 5.5 [19]). *If the player* $\mathscr{K}$ *does not choose first in S-F game and* $\mathscr{K}$ *has a winning strategy in S-F game then* (X, π_X) *is a strong Baire space.*

In the previous theorem we have assumed that $\mathscr{K}$ does not choose first in S-F game and $\mathscr{K}$ has a winning strategy. In consequence, the next question arises: When does player $\mathscr{K}$ have a winning strategy?

Theorem 7.14 (Theorem 5.4 [19]). *If the player* $\mathscr{K}$ *does not choose first in S-F game then* $\mathscr{K}$ *has a winning strategy in S-F game if and only if the set of all isolated points of* (X, π_X) *is dense.*

Finally, we will refer to considerations connected with players $\mathscr{L}$ and $\mathscr{P}$.

Theorem 7.15 (Theorem 5.3 (ii) [19]). *If the player $\mathscr{L}$ chooses first in S-F game and there is no winning strategy in SFG for the player $\mathscr{L}$ then (X,π_X) is a Baire space.*

Theorem 7.16 (Theorem 5.3 (i) [19]). *If (X,π_X) is a Baire space with no isolated points then there is no winning strategy in S-F game for the player $\mathscr{P}$.*

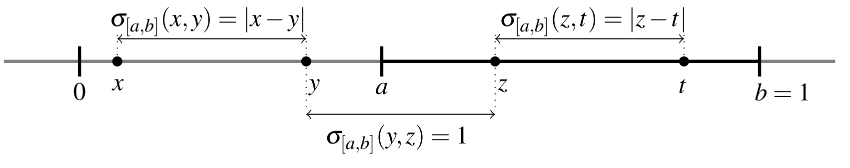
7.4 Recent results

The research concerning GMS are continued by the authors of this chapter in [25]. In view of the fact that this paper has not been published yet, we will only signal a piece of it. A notion of a base consisting of metrics, introduced in this paper, is particularly important. This notion has a close relationship with the notion of kernel presented earlier in this chapter. However, we will concentrate here on the *generalization of the unit interval*.

First, let us define some kind of metric. Let $A \subset [0,1]$. From now on, the symbol σ_A stands for the following metric (called *almost natural metric*):

$$\sigma_A(x,y) = \begin{cases} \rho_E(x,y) & \text{if } x,y \in A \text{ or } x,y \notin A, \\ 1 & \text{otherwise.} \end{cases}$$

Obviously, in the above definition we can consider any nonempty set A. However, if we take into account an interval $[a,b] \subset [0,1]$, we obtain the metric $\sigma_{[a,b]}$ having some special properties (writing $\sigma_{[a,b]}$ we assume that $a < b$). Clearly, in this case we can obtain for example the following situation:



In paper [25], generalized metric spaces connected with metrics of the form $\sigma_{[a,b]}$ were investigated. Such kind of GMS is called π-unit interval. For example we have that GMS $([0,1],\pi)$ is a *π-unit interval* if π consists of a finite number of almost natural metrics of the form $\sigma_{[a,b]}$.

From our point of view, the following theorem is important.

Theorem 7.17 ([25]). *Every π-unit interval is a Baire space.*

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