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Chapter 3

Convolution operators on some spaces of functions and distributions in the theory of circuits

ANDRZEJ BORYS, ANDRZEJ KAMIŃSKI AND SŁAWOMIR SOREK

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3.1 Introduction

For the theory of linear systems and circuits, investigated in telecommunications, electronics and signal processing [10, 12], the Dirac delta impulse δ is a natural and useful object but no function in the classical sense corresponds to it. The notion can be mathematically justified on the base of the theory of distributions created by L. Schwartz [21] and appears to be very fruitful in various fields of applications and in mathematics itself. In particular, the signal $\delta = \delta(t)$, meant as a distribution (generalized function) of time t on the real line $\mathbb{R}$, allows one to determine in some cases the input-output characteristics of a non-autonomous linear system as well as its impulse response in the theory of systems and circuits.

We present here our attempt to deliver a strict mathematical basis for some aspects of the theory of linear and nonlinear systems and circuits extending the domain of objects in use from functions to distributions to embrace δ, in particular. The presented work was inspired by the talk [5] delivered by the first author during the conference on generalized functions in Będlewo in 2007. We recall the results of the present authors given in [4] and in [16], extending them in section 7 by one of the results of the third author which are going to be published separately (see [4]).

The basic notation and the definition of the convolution of kth order, the notion crucial for our considerations, are given in section 2.

In section 3, the two types of linear circuits are discussed: circuits with decaying memory and memoryless ones. We describe them in terms of linear operators defined for functions and extended suitably for distributions. This extension allows one to represent both types of circuits as convolution operators determined by the impulse response distributions.

In section 4, we extend the theory to the nonlinear Volterra systems described by Volterra and Taylor series and discuss conditions under which the corresponding nonlinear operators are well defined on certain spaces of functions. A possibility of defining these nonlinear operators for the Dirac delta impulse, desirable for applications but not attainable in the standard sense of operations on distributions, was posed as a problem in [5].

Two aspects of the problem, concerning the product of k distributions (in particular, the kth power of the Dirac delta) and the convolution of kth order of distributions (in particular, the kth convolution of the Dirac delta) are discussed and solved in sections 6 and 7 by means of the notion of neutrix. A general concept of neitrix was introduced by J. G. van der Corput in [7] and then it was adapted in a particular form to the product and the convolution of distributions by B. Fisher and his co-authors in numerous papers, but their approach contains certain mathematical incoherences (see Remarks 3.4 and 3.5).

Therefore we discuss in section 5 some aspects of the theory more carefully and remove its drawbacks due to certain essential modifications and generalizations. In particular, we replace Fisher's neutrix of sequences by the corresponding neutrix of nets. In our opinion, this is a good example of the situation where nets appear to be a more adequate tool in the theory of the product of generalized functions (see also [6, 20]).

In section 6, we present a solution to the first part of the problem, concerning the product of distributions. Following the ideas of E. L. Koh and C. K. Li in [17], we show how to define the kth power of the Dirac delta distribution and, more generally, the product of k distributions, in the sense of the notion of neutrix suitably modified in section 5. We prove the result of Koh and Li for a certain net neutrix.

Theorem 3.2 given in section 7 is an answer to the second part of the problem, concerning the kth convolution of distributions, also in the sense of net neutrix discussed in section 5. A complete proof of Theorem 3.2 and other aspects of the theory are discussed in [22].

3.2 Basic definitions and notation

The symbols $\mathbb{N}$, $\mathbb{N}_0$ and $\mathbb{R}$ denote the sets of all positive integers, all non-negative integers and all real numbers, respectively. For given $j \in \mathbb{N}$ the symbols $\mathbb{N}_0^j$ and $\mathbb{R}^j$ denote the Cartesian products of j copies of the sets $\mathbb{N}_0$ and $\mathbb{R}$, respectively; in particular, the symbol $\mathbb{R}^{jk}$ for $j,k \in \mathbb{N}$ means the Cartesian product of k copies of $\mathbb{R}^j$. The expressions: *measurable functions*, *almost everywhere*, *almost all* are meant in the sense of Lebesgue.

We will start with considering certain convolution operators on the spaces $L^1(\mathbb{R}^j)$ and $L^1(\mathbb{R}^{jk})$ of integrable functions on $\mathbb{R}^j$ and $\mathbb{R}^{jk}$ as well as on the spaces $L^\infty(\mathbb{R}^j)$ and $L^\infty(\mathbb{R}^{jk})$ of essentially bounded functions on $\mathbb{R}^j$ and $\mathbb{R}^{jk}$, respectively, but later we will extend our considerations for spaces $\mathscr{D}'$ of distributions and $\mathscr{S}'$ of tempered distributions defined on the Euclidean space of a suitable dimension.

Let us recall that the space $\mathscr{D}' = \mathscr{D}'(\mathbb{R}^j)$ of distributions on $\mathbb{R}^j$ is the strong dual of the space $\mathscr{D} = \mathscr{D}(\mathbb{R}^j)$ of test functions on $\mathbb{R}^j$, i.e. smooth (infinitely differentiable) functions of compact support, endowed with the respective inductive limit topology, while the space $\mathscr{S}' = \mathscr{S}'(\mathbb{R}^j)$ of tempered distributions, a subspace of $\mathscr{D}' = \mathscr{D}'(\mathbb{R}^j)$, is the dual of the space $\mathscr{S} = \mathscr{S}(\mathbb{R}^j)$ of smooth functions rapidly decreasing together with all derivatives at infinity, endowed with the respective metric topology (see [21]; see also [1] and [13]).

The space $\mathscr{D}'$ of distributions contains *regular distributions* corresponding to locally integrable functions and, in particular, to members of the spaces L^1 and L^∞. Important examples of (tempered) distributions on $\mathbb{R}^j$ (which are not represented by usual functions) are the Dirac delta, that we denote by δ or by $\delta_{(j)}$ to mark the dimension of $\mathbb{R}^j$, defined as follows:

$$< \delta, \varphi > = < \delta_{(j)}, \varphi > := \varphi(0), \qquad \varphi \in \mathscr{D}(\mathbb{R}^j) \ \ (\varphi \in \mathscr{S}(\mathbb{R}^j)) \tag{3.1}$$

as well as its distributional derivatives $\delta^{(l)} = \delta^{(l)}_{(j)}$ defined by

$$< \delta^{(l)}, \varphi > := (-1)^l \varphi^{(l)}(0), \qquad \varphi \in \mathscr{D}(\mathbb{R}^j) \ \ (\varphi \in \mathscr{S}(\mathbb{R}^j))$$

for arbitrary $l \in \mathbb{N}_0^j$, according to the standard multidimensional notation.

The following modification of the convolution of functions plays an important role in further considerations concerning nonlinear circuits.

Definition 3.1. Let $j,k \in \mathbb{N}$. Assume that $f\colon \mathbb{R}^j \to \mathbb{R}$ and $h\colon \mathbb{R}^{jk} \to \mathbb{R}$ are measurable functions. By the *convolution of kth order* or shortly the *kth convolution* of the functions h and f we mean the measurable function $h \overset{k}{*} f\colon \mathbb{R}^j \to \mathbb{R}$ defined almost everywhere on $\mathbb{R}^j$ by the following k-multiple integral:

$$(h \overset{k}{*} f)(x) := \int_{\mathbb{R}^{jk}} h(\eta_1,\ldots,\eta_k)\, f(x-\eta_1)\cdot\ldots\cdot f(x-\eta_k)\; d\eta_1\ldots d\eta_k, \quad (3.2)$$

where $x,\eta_1,\ldots,\eta_k \in \mathbb{R}^j$, under the condition that

$$\int_{\mathbb{R}^{jk}} |h(\eta_1,\ldots,\eta_k)\, f(x-\eta_1)\cdot\ldots\cdot f(x-\eta_k)|\; d\eta_1\ldots d\eta_k < \infty$$

for almost all $x \in \mathbb{R}^j$.

Remark 3.1. Assume that $h \in L^1(\mathbb{R}^{jk})$. If $f \in L^1(\mathbb{R}^j)$, then $h \overset{k}{*} f \in L^1(\mathbb{R}^j)$, due to the Fubini theorem. On the other hand, if $f \in L^\infty(\mathbb{R}^j)$, then $h \overset{k}{*} f \in L^\infty(\mathbb{R}^j)$.

Assume now that $h \in L^\infty(\mathbb{R}^{jk})$. If $f \in L^1(\mathbb{R}^j)$, then $h \overset{k}{*} f \in L^\infty(\mathbb{R}^j)$. But if $f \in L^\infty(\mathbb{R}^j)$, then the convolution of kth order $h \overset{k}{*} f$ need not exist, e.g. in case f and h are constantly equal to 1 on $\mathbb{R}^j$ and $\mathbb{R}^{jk}$, respectively.

Clearly, $h \overset{1}{*} f = h * f$, where $h * f$ means the classical convolution of the functions h and f.

3.3 Linear circuits

For simplicity we will assume further on that $j = 1$ and $k \in \mathbb{N}$.

To describe a linear circuit one usually assumes that an input signal $x = x(t)$ and an output signal $y = y(t)$, functions of time $t \in \mathbb{R}$, are related to each other by a black box linear operator L, i.e. a convolution operator of the form: $y = Lx = h * x$ for a certain function $h = h(t)$, interpreted as a circuit impulse response. This is schematically shown on Fig. 1.

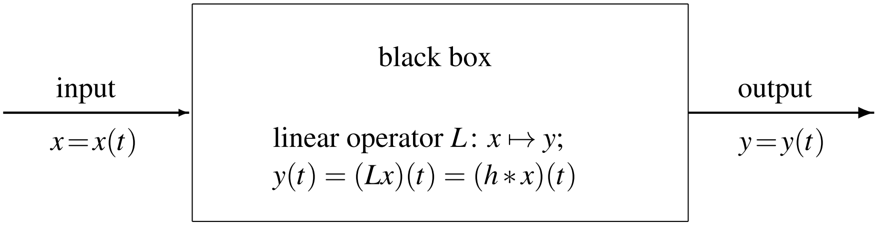

Fig. 1. Scheme of linear circuit

In linear circuits with decaying memory, one usually assumes that h is an integrable function on $\mathbb{R}$ and $L = L_m$ is the convolution operator given in the two cases: (a) for $x \in L^1(\mathbb{R})$, (b) for $x \in L^\infty(\mathbb{R})$, by the same formula:

$$y(t) = (L_m x)(t) = (h * x)(t) = \int_{-\infty}^{+\infty} h(\tau)x(t-\tau)\,d\tau, \qquad t \in \mathbb{R}, \tag{3.3}$$

i.e. L_m maps the input signals: (a) $x \in L^1(\mathbb{R})$, (b) $x \in L^\infty(\mathbb{R})$ to the output signals: (a) $y \in L^1(\mathbb{R})$, (b) $y \in L^\infty(\mathbb{R})$, respectively.

In other words, L_m is the convolution operator acting in the two considered cases as follows:

$$(a) \quad L_m\colon L^1(\mathbb{R}) \to L^1(\mathbb{R}), \qquad (b) \quad L_m\colon L^\infty(\mathbb{R}) \to L^\infty(\mathbb{R}).$$

It is known that $L^1(\mathbb{R})$ is a convolution algebra without unit, but the Dirac delta plays the role of the convolution unit in the wider space $\mathscr{D}'$ of distributions with the convolution meant in the more general distributional sense (see [1, 21]):

$$f * \delta = \delta * f = f, \qquad f \in \mathscr{D}'. \tag{3.4}$$

The operator L_m, defined in (3.3) in case the input signal x and the impulse response h are functions, can be extended to include both $x = \delta$ and $h = \delta$. If $h = \delta$, due to (3.4), one extends L_m to the linear operator $L_m\colon \mathscr{D}' \to \mathscr{D}'$ of the form:

$$y = L_m x = \delta * x = x, \qquad x \in \mathscr{D}'. \tag{3.5}$$

If $x = \delta$, the extension of L_m makes sense for every $h \in \mathscr{D}'$ and has the form:

$$y = L_m \delta = h * \delta = h, \qquad h \in \mathscr{D}', \tag{3.6}$$

which is particularly useful, because the output signal and the impulse response are then equal, i.e. a system is fully described by its impulse response. In particular, if $x = \delta$ and $h = \delta$, then both (3.5) and (3.6) yield $L_m\delta = \delta * \delta = \delta$. We use the same symbol L_m for the extended linear operator given by (3.5) and (3.6) and for that originally defined in (3.3), because the extension is consistent.

Another type of linear systems, without memory, considered in electrical engineering and telecommunications as linear memoryless systems (circuits), may be described by a linear operator $L = L_{nm}$ of the form:

$$y = L_{nm}x = \alpha x, \qquad \alpha \in \mathbb{R}, \tag{3.7}$$

where the input signals x are, as in (3.3), functions from a given space or, as in (3.5), distributions. The latter is more general and $L_{nm} : \mathscr{D}' \to \mathscr{D}'$ given by (3.7) can be expressed in the form:

$$y = L_{nm}x = \alpha x = h^0 * x, \qquad \alpha \in \mathbb{R},\ x \in \mathscr{D}', \tag{3.8}$$

where $h^0 := \alpha\delta$, because

$$\alpha x = \alpha(\delta * x) = (\alpha\delta) * x, \qquad x \in \mathscr{D}',$$

in view of (3.4). Hence L_{nm} in (3.8) can be treated as an input-output description of a memoryless circuit in the form of the extended convolution operator with the impulse response $h^0 = \alpha\delta \in \mathscr{D}'$.

An example of a memoryless circuit is a simple resistive voltage divider, consisting of two resistors R_1 and R_2, presented on Fig. 2.

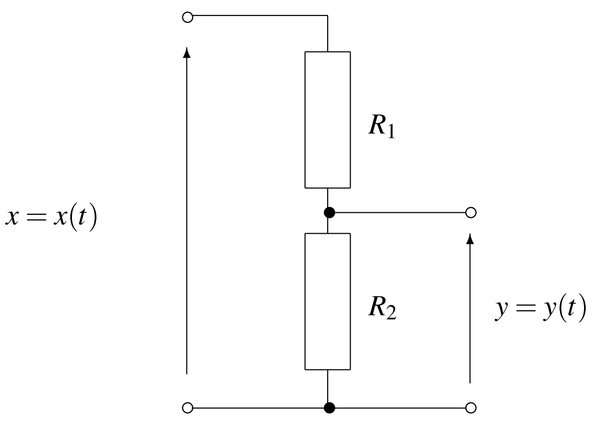

Fig. 2. Example of resistive voltage divider

The linear memoryless operator L_{nm} has the form:

$$y = L_{nm}x = \frac{R_2}{R_1 + R_2}x = h^0 * x,$$

where

$$h^0 := \frac{R_2}{R_1 + R_2}\,\delta.$$

Combining a memoryless circuit and a circuit with (decaying) memory, described by L_{nm} and L_m, we see that the linear operator $L_{ov} := L_{nm} + L_m$ describing the overall circuit is, by (3.3), (3.5) and (3.8), of the following form:

$$y = L_{ov}x = L_{nm}x + L_m x = \alpha x + h * x = h_{ov} * x, \qquad x \in \mathscr{D}', \tag{3.9}$$

where

$$h_{ov} := h^0 + h = \alpha\delta + h \in \mathscr{D}'$$

is the impulse response of the overall circuit.

3.4 Nonlinear circuits

An important class of nonlinear circuits, studied e.g. in [2, 3, 11], is described by the *Taylor power series* and *Volterra series*, i.e. the formal series of the form:

$$y = T_{nm}x = \sum_{k=1}^{\infty} \alpha_k x^k, \qquad y = T_m x = \sum_{k=1}^{\infty} h_k \overset{k}{*} x, \tag{3.10}$$

where $\alpha_k \in \mathbb{R}$, $x = x(t)$ is a function on $\mathbb{R}$, $h_k = h_k(t_1, \ldots, t_k)$ are functions on $\mathbb{R}^k$ and $h_k \overset{k}{*} x$ are functions on $\mathbb{R}$ described in Definition 3.1 such that all expressions in (3.10) and (3.2) are well defined. The nonlinear mappings T_{nm} and T_m, defined by (3.10), as well as T_{ov} of the form:

$$T_{ov}x = (T_{nm} + T_m)x = \sum_{k=1}^{\infty} \alpha_k x^k + \sum_{k=1}^{\infty} h_k \overset{k}{*} x, \tag{3.11}$$

are extensions of the linear operators L_{nm}, L_m and L_{ov}, defined for suitable functions x in (3.3), (3.7) and (3.9), respectively (and coincide with them, respectively, if $\alpha_k = 0$ and $h_k = 0$ for $k \geq 2$). The functions h_k on $\mathbb{R}^k$ are called the *linear* (for $k = 1$) and *nonlinear* (for $k > 1$) *impulse responses* of the kth order.

Obviously, to have the mappings T_{nm}, T_m and T_{ov} well defined, one has to impose some assumptions on the right-hand sides of (3.10) and (3.11). Assume that $h_k \in L^1(\mathbb{R}^k)$ for $k \in \mathbb{N}$ and consider the two cases: (a) $x \in L^1(\mathbb{R})$, (b) $x \in L^\infty(\mathbb{R})$. In both cases, each member of the second series in (3.10) is well defined and, by the Fubini theorem, $h_k \overset{k}{*} x \in L^1(\mathbb{R})$ in case (a), and $h_k \overset{k}{*} x \in L^\infty(\mathbb{R})$ in case (b). Additional assumptions concerning convergence of the two series in (3.10) are necessary. In case (b), the mappings T_{nm}, T_m and T_{ov} are well defined if both series are convergent uniformly, i.e. in L^∞. In case (a), T_m is well defined if the second series in (3.10) is convergent in $L^1(\mathbb{R})$, but to have the mapping T_{nm} well defined assume, in addition, that $x \in L^k(\mathbb{R})$ for all $k \in \mathbb{N}$ and the first series in (3.10) is convergent in $L^1(\mathbb{R})$.

We may try to extend the nonlinear mappings T_{nm}, T_m and T_{ov}, as it was done for their linear counterparts in the preceding section, from the above particular spaces of functions to distributions, at least in the special case of the input signal x and the impulse response h_k are the Dirac delta distributions: $x = \delta = \delta_{(1)}$ on $\mathbb{R}$ and $h_k = \delta_{(k)}$ on $\mathbb{R}^k$, respectively (according to the notation introduced in (3.1)).

However, we then encounter mathematical difficulties: putting $x = \delta$ in the first series in (3.10) for $k > 1$ is not allowed, because the power δ^k of δ for $k > 1$ does not exist in the standard sense of the theory of distributions (for $k = 2$ see e.g. [1], pp. 243-244); a similar difficulty concerns $\delta_{(k)} \overset{k}{*} \delta$ in the second series, where $\delta_{(k)}$ is the Dirac delta on $\mathbb{R}^k$.

How to overcome these two difficulties was asked in [5]. In section 3.6, we present a solution to the first part of the problem and in section 3.7 to the second one: both the power δ^k of δ and the convolution of kth order $\delta_{(k)} \overset{k}{*} \delta$ exist in the sense of net neutrix described in the next section.

3.5 Neutrices

We start from recalling van der Corput's general definition of neutrix given in [7]. Then we impose Assumptions 3.1 and 3.2 used in the sequel and specify the form of neutrices used in the theory of the product of distributions.

Definition 3.2. Let N' be an arbitrary nonempty set and N'' be a commutative additive group. By a *neutrix* (of type (N', N'')) one means a commutative additive group N of functions ν: $N' \to N''$ (called *negligible functions*) such that

$$(*) \quad \textit{the only constant function } \nu \textit{ in } N \textit{ is } \nu \equiv 0.$$

Condition $(*)$ guarantees the uniqueness of *N–limits* in the sense of the *N–convergence*, defined by means of the neutrix N in the following way:

Definition 3.3. Let N' be a nonempty subset of a certain set $\overline{N'}$, let a be a fixed element of $\overline{N'}$ and assume that $\xi \to a$ is well defined for $\xi \in N'$, e.g. N' is a subset of a topological space $\overline{N'}$ and $a \in \overline{N'}$ is a limit point of N'. Moreover, assume that N'' is a commutative additive group and N is a neutrix in the sense of Definition 3.2. For $\nu\colon N' \to N''$ and $l \in N''$, we define

$$N\text{–}\lim_{\xi\to a} \nu(\xi) = l, \quad \text{if } \nu_0 \in N, \tag{3.12}$$

where

$$\nu_0(\xi) := \nu(\xi) - l, \qquad \xi \in N'.$$

Clearly, if N_1 and N_2 are neutrices as in Definition 3.3 such that $N_1 \subseteq N_2$, then N_1*–convergence* implies N_2*–convergence*.

Proposition 3.1. *Let $N' := (0,1)$, $\overline{N'} := [0,1]$ and $a := 0 \in \overline{N'}$ be a limit point of $(0,1)$ in the standard topology of $[0,1]$. Assume that $N'' := X$ is a topological vector space (over $\mathbb{R}$) and fix a neutrix $N := N^X$ of type $((0,1),X)$, i.e. a commutative additive group of $\gamma \in X^{(0,1)}$ satisfying $(*)$. We call $\gamma = (\gamma^\tau) = (\gamma^\tau)_{\tau\in(0,1)}$ nets in X. If N' and $\overline{N'}$ above are replaced by $N' := \mathbb{N}$, $\overline{N'} := \mathbb{N}\cup\{\infty\}$, a fixed neutrix N of type $(\mathbb{N},X)$ of sequences $(\gamma_n) \in X^{\mathbb{N}}$ satisfying $(*)$ will be denoted by N_X. Formula (3.12) defines the neutrix limits $N^X\text{–}\lim_{\tau\to 0} \gamma^\tau$ and $N_X\text{–}\lim_{n\to\infty} \gamma_n$ in X for all nets $\gamma = (\gamma^\tau) \in X^{(0,1)}$ and all sequences $(\gamma_n) \in X^{\mathbb{N}}$, respectively.*

Denote by $c^0(X)$ the set of all nets $\alpha = (\alpha^\tau) \in X^{(0,1)}$ convergent to 0 as $\tau \to 0$ and by $c_0(X)$ the set of all sequences $(\alpha_n) \in X^{\mathbb{N}}$ convergent to 0 as $n \to \infty$ in the topology of X; if $X = \mathbb{R}$ we write $c^0 := c^0(\mathbb{R})$ and $c_0 := c_0(\mathbb{R})$. By d^∞ denote the set of all nets $\beta = (\beta^\tau) \in \mathbb{R}^{(0,1)}$ divergent to ∞ as $\tau \to 0$ and by d_∞ the set of all sequences $(\beta_n) \in \mathbb{R}^{\mathbb{N}}$ divergent to ∞ as $n \to \infty$. Clearly, $\alpha = (\alpha^\tau) \in c^0(X)$ iff $(\alpha_n) \in c_0(X)$ for all (α_n) of the form $\alpha_n := \alpha^{\tau_n}$, $\tau_n \in (0,1)$, $\tau_n \to 0$ and $\beta = (\beta^\tau) \in d^\infty$ iff $(\beta_n) \in d_\infty$ for all (β_n) of the form $\beta_n := \beta^{\tau_n}$, $\tau_n \in (0,1)$, $\tau_n \to 0$.

Remark 3.2. The convergence of nets (sequences) in the topology of X implies the N^X–convergence (resp. N_X–convergence) to the same limit iff $N^X \supseteq c^0(X)$ (resp. $N_X \supseteq c_0(X)$); they coincide if the equality holds in the inclusion, so to extend essentially the respective neutrix convergence one has to add nets (sequences) not convergent to 0 in X to the neutrix. For example, if $X = \mathbb{R}$, it is standard to assume [7, 9] that a given neutrix $N^{\mathbb{R}}$ (resp. $N_{\mathbb{R}}$) contains c^0 (resp. c_0) and a certain subclass d^* of d^∞ (resp. d_* of d_∞) which determines the neutrix $N^{\mathbb{R}} = N^{d^*}$ (resp. $N_{\mathbb{R}} = N_{d_*}$) in the following way: all negligible functions in N^{d^*} (resp. N_{d_*}) are finite linear sums of elements of c^0 and d^* (resp. c_0 and d_*). That means,

$$N^{d^*} := \mathrm{span}(c^0 \cup d^*); \qquad N_{d_*} := \mathrm{span}(c_0 \cup d_*). \tag{3.13}$$

The range of the extensions of the N^{d^*}–convergence and N_{d_*}–convergence depends essentially on the selection of the subclasses d^* and d_* of d^∞ and d_∞, respectively.

If $X = E'$ is the dual of a topological vector space E (over $\mathbb{R}$) endowed with the weak topology, then it is natural to define the corresponding neutrices N^X and N_X via given neutrices $N^{\mathbb{R}}$ and $N_{\mathbb{R}}$, by means of values of $x' \in E'$ on $x \in E$.

Proposition 3.2. *Assume that E is a topological vector space (over $\mathbb{R}$) and $X := E'$ is its dual endowed with the weak topology. Under Assumption 3.1, define the neutrix N^X generated by $N^{\mathbb{R}}$ as follows: $(\gamma^\tau) \in N^X$ if $(\langle \gamma^\tau, x\rangle) \in N^{\mathbb{R}}$ for $x \in E$. Obviously, for $\gamma = (\gamma^\tau) \in X^{(0,1)}$ and $\gamma^* \in X$, we have $N^X\!-\lim_{\tau\to 0} \gamma^\tau = \gamma^*$ iff $N^{\mathbb{R}}\!-\lim_{\tau\to 0}\langle \gamma^\tau, x\rangle = \langle \gamma^*, x\rangle$ for $x \in E$. Denote, in particular, by N^{X,d^*} the neutrix generated by N^{d^*} of the form (3.13). Similarly, we define the neutrices N_X, N_{X,d_*}, generated by given neutrices $N_{\mathbb{R}}$, $N_{\mathbb{R},d_*}$, and the respective neutrix convergences.*

Remark 3.3. In particular, if $X = \mathscr{D}'$ ($X = \mathscr{S}'$) in Assumption 3.2, we have $(f^\tau) \in N^{X,d^*}$ iff $(\langle f^\tau, \varphi\rangle) \in N^{d^*}$ for $\varphi \in E$ and, consequently,

$$N^{X,d^*}\!-\lim_{\tau\to 0} f^\tau = f \quad \text{iff} \quad N^{d^*}\!-\lim_{\tau\to 0}\langle f^\tau, \varphi\rangle = \langle f, \varphi\rangle \quad \text{for} \quad \varphi \in E,$$

where $E = \mathscr{D}$ ($E = \mathscr{S}$), respectively. Analogously, we define the neutrix convergence $N_{d_*}\!-\lim_{n\to\infty} f_n$ in $\mathscr{D}'$ (in $\mathscr{S}'$). Thus the neutrix convergences in $\mathscr{D}'$ and in $\mathscr{S}'$ are determined by a suitable choice of the classes d^* and d_*.

B. Fisher in [9] has chosen and used in his numerous papers on neutrix products and convolutions of distributions the fixed neutrix of sequences defined by the class $d_* := d_F$, where d_F consists of all sequences $(\beta_n)_{n\in\mathbb{N}}$ whose members are of the form:

$$\beta_n := n^\lambda \ln^r n \qquad \text{for } \lambda > 0,\ r \in \mathbb{N}_0 \quad \text{or} \quad \lambda = 0,\ r \in \mathbb{N}. \tag{3.14}$$

Consider the corresponding neutrix of nets defined by $d^* := d^F$, where d^F consists of all nets $(\beta^\tau)_{\tau\in(0,1)}$ whose members are of the form:

$$\beta^\tau := \tau^{-\lambda}(-\ln\tau)^r \quad \text{for } \lambda > 0,\ r\in\mathbb{N}_0 \quad \text{or} \quad \lambda = 0,\ r\in\mathbb{N}. \tag{3.15}$$

To prove Theorem 3.1 and 3.2 in a stronger form, consider also the following narrower class $d^* := d^P$, where d^P consists of all nets $(\beta^\tau)_{\tau\in(0,1)}$ whose members are of the form:

$$\beta^\tau := \tau^{-r} \quad \text{for } r\in\mathbb{N}. \tag{3.16}$$

Definition 3.4. Denote by N_F the sequential neutrix of Fisher defined by the equality on the right hand side of (3.13) with $d_* = d_F$ given by (3.14). On the other hand, denote by N^F and N^P the net neutrices defined by the equality on the left hand side of of (3.13) with $d^* = d^F$ and $d^* = d^P$ given by (3.15) and (3.16), respectively.

Clearly, the neutrix N^P is essentially narrower than the neutrix N^F.

Remark 3.4. The sequential neutrix N_F of Fisher has an essential drawback. Namely, a subsequence of a sequence belonging to N_F does not belong to N_F, in general. Consequently, a subsequence of a sequence N_F– convergent in $\mathscr{D}'$ (in $\mathscr{S}'$) is not N_F–convergent in $\mathscr{D}'$ (in $\mathscr{S}'$). This leads to inconsistency of the definitions of the product and convolution of distributions in the sense of the neutrix N_F in $\mathscr{D}'$ and in $\mathscr{S}'$.

The net neutrices N^F and N^P are free from such incoherences. For example, if $(\tau_n)_{n\in\mathbb{N}}$ is an arbitrary numerical sequence such that $\tau_n \to 0$, then in the net $(\beta^\tau)_{\tau\in(0,1)}$ of the form (3.15) of the neutrix N^F one can find the corresponding sequence (β_n) of the form $\beta_n := \beta^{\tau_n}$, i.e. of the form (3.14) with n replaced by τ_n^{-1}, while in the neutrix N_F only the sequences (β_n) corresponding to the single sequence (τ_n) of the form $\tau_n = n^{-1}$ are considered.

3.6 Neutrix powers of δ

It will be convenient now to use the following notation for $j\in\mathbb{N}$ and, in particular, for $j = 1$:

$$\mathscr{D}_1(\mathbb{R}^j) := \left\{\varphi\in\mathscr{D}(\mathbb{R}^j) : \int_{\mathbb{R}^j}\varphi(t)\,dt = 1\right\}; \quad \mathscr{D}_1 := \mathscr{D}_1(\mathbb{R}^1). \tag{3.17}$$

We begin with giving the definitions of the product of k distributions in $\mathscr{D}'$ as well as the product and the Gaussian product of k tempered distributions in $\mathscr{S}'$ which are modifications and extensions of the sequential definition of the product of two distributions given in [1] (p. 242), [14] and [18].

Definition 3.5. Fix $k \in \mathbb{N}$. For given $f_1, \ldots, f_k \in \mathscr{D}'$ ($f_1, \ldots, f_k \in \mathscr{S}'$), the *product* $f_1 \cdot \ldots \cdot f_k$ in $\mathscr{D}'$ (in $\mathscr{S}'$) is defined by

$$f_1 \cdot \ldots \cdot f_k := \lim_{\tau \to 0} (f_1 * \delta_\tau) \cdot \ldots \cdot (f_k * \delta_\tau), \tag{3.18}$$

if the above limit exists in $\mathscr{D}'$ (in $\mathscr{S}'$) for all delta-nets (δ_τ) of the form

$$\delta_\tau(x) = \tau^{-1} \sigma(\tau^{-1} x), \qquad \tau \in (0,1),\ \sigma \in \mathscr{D}_1,\ x \in \mathbb{R} \tag{3.19}$$

and does not depend on σ, where $\mathscr{D}_1$ is defined in (3.17).

In particular, if $f \in \mathscr{D}'$ ($f \in \mathscr{S}'$), then formula (3.18) with $f_1 = \ldots = f_k = f$ defines the *kth power* f^k of f in $\mathscr{D}'$ (in $\mathscr{S}'$).

Definition 3.6. Let N be a neutrix in $\mathbb{R}^{(0,1)}$ and $k \in \mathbb{N}$. For given $f_1, \ldots, f_k \in \mathscr{D}'$ ($f_1, \ldots, f_k \in \mathscr{S}'$), the *N–product* $f_1 \cdot \ldots \cdot f_k$ in $\mathscr{D}'$ (in $\mathscr{S}'$) is defined by

$$f_1 \cdot \ldots \cdot f_k := N - \lim_{\tau \to 0} (f_1 * \delta_\tau) \cdot \ldots \cdot (f_k * \delta_\tau), \tag{3.20}$$

if the N–limit on the right hand side exists in $\mathscr{D}'$ (in $\mathscr{S}'$) for all delta-nets (δ_τ) given by (3.19) and does not depend on σ.

In particular, if $f \in \mathscr{D}'$ ($f \in \mathscr{S}'$), then formula (3.20) with $f_1 = \ldots = f_k = f$ defines the *kth N–power* in $\mathscr{D}'$ (in $\mathscr{S}'$).

Definition 3.7. For $f_1, \ldots, f_k \in \mathscr{S}'$, the *Gaussian product* and the *Gaussian N–product* $f_1 \cdot \ldots \cdot f_k$ in $\mathscr{D}'$ (in $\mathscr{S}'$) is defined by (3.18) and by (3.20), respectively, whenever the limits in (3.18) and (3.20) exist in $\mathscr{D}'$ (in $\mathscr{S}'$) for all (δ_τ) of the form (3.19), where σ is replaced by the single $\sigma_0 \in \mathscr{S}$ given by

$$\sigma_0(x) := \pi^{-1/2} e^{-x^2}, \qquad x \in \mathbb{R}. \tag{3.21}$$

In particular, if $f \in \mathscr{S}'$, then formulas (3.18) and (3.20) with $f_1 = \ldots = f_k = f$ (and $\sigma = \sigma_0$ with σ_0 given by (3.21)) define the *kth Gaussian power* and the *kth Gaussian N–power* f^k of f, respectively, in $\mathscr{D}'$ (in $\mathscr{S}'$).

Remark 3.5. Clearly, delta-nets (δ_τ) and the net limits in (3.18) and (3.20) as $\tau \to 0$ can be equivalently replaced by delta-sequences (δ_n) of the form:

$$\delta_n(x) = \tau_n \sigma(\tau_n x), \qquad (\tau_n) \in d_\infty,\ \sigma \in \mathscr{D}_1,\ x \in \mathbb{R}, \tag{3.22}$$

and by the sequential limits as $n \to \infty$, respectively. The class Δ_m of all (δ_n) of the form (3.22) called *model* delta-sequences was introduced in [14]. The product in $\mathscr{D}'$ given by (3.18) for $k = 2$ with (δ_n) instead of (δ_τ) and the respective sequential limit was studied first in [19] and then in [1] for other classes of delta-sequences. If the sequential limit exists in $\mathscr{D}'$ (in $\mathscr{S}'$) for all $(\delta_n) \in \Delta_m$, then it does not depend on $(\tau_n) \in d_\infty$, but it may depend on $\sigma \in \mathscr{D}_1$, as noticed in [15]. The same concerns the neutrix product of distributions in $\mathscr{D}'$ (in $\mathscr{S}'$). The additional assumption in Definitions 3.5 and 3.6 is made just to avoid such a dependence.

Fisher and his followers in their papers on (neutrix) products of distributions [8, 9] use delta-sequences (δ_n) of the form (3.22) with one fixed $\sigma \in \mathscr{D}_1$ (satisfying additional conditions) and one fixed (τ_n), $\tau_n := n$ $(n \in \mathbb{N})$, so their definition of the (neutrix) product of distributions depends, in general, on these particularly fixed σ and (τ_n). Koh and Li [17] use delta-sequences (δ_n) of the form (3.22) with the fixed σ_0 of the form (3.21) instead of $\sigma \in \mathscr{D}_1$ and with the fixed $(\tau_n) \in d_\infty$, this time given by $\tau_n := \sqrt{n}$ $(n \in \mathbb{N})$. The appearance of the two different particular (τ_n) motivates additionally the use of arbitrary $(\tau_n) \in d_\infty$ in equation (3.22) or of delta-nets of the form (3.19).

The product $\delta \cdot \delta$ and, more generally, the kth power δ^k of δ for $k \geq 2$ do not make sense in the standard approach [21] to the theory of distributions and do not exist in the sense of the Mikusiński product of distributions (see [1], pp. 243-244). However Koh and Li proved in [17] that δ^k $(k \geq 2)$ exists in $\mathscr{D}'$ in the sense of Definition 3.7 of the Gaussian N–product (3.20) for $f_1 = \ldots = f_k = \delta$ and $N = N_F$. More exactly, they proved that the Gaussian N_F–power δ^k exists in $\mathscr{D}'$ for (δ_n) of the form (3.22) with $\sigma = \sigma_0$ given by (3.21) and particular $\tau_n := \sqrt{n}$.

We extend below the result of Koh and Li replacing the neutrix $N = N_F$ of sequences by the neutrix $N = N^P$ (in particular $N = N^F$) of nets and the limit in $\mathscr{D}'$ by the stronger limit in $\mathscr{S}'$:

Theorem 3.1. *The kth Gaussian N^P-power (and the more N^F–power) of δ exists in $\mathscr{S}'$ for arbitrary $k \in \mathbb{N}$ and the following formulas hold:*

$$\delta^{2j} = 0 \quad (j \in \mathbb{N}), \qquad \delta^{2j+1} = \frac{1}{(4\pi)^j (2j+1)^{j+1/2} j!}\, \delta^{(2j)} \quad (j \in \mathbb{N}_0). \tag{3.23}$$

Proof. Fix $\psi \in \mathscr{S}$ and a delta-net (δ_τ) of the form (3.19) with $\sigma = \sigma_0$ given by equation (3.21), i.e.

$$\delta_\tau(x) := \tau^{-1}\sigma_0(\tau^{-1}x) = (\tau^2\pi)^{-1/2} e^{-(x/\tau)^2}, \qquad \tau \in (0,1),\ x \in \mathbb{R}.$$

Hence

$$(\delta_\tau^k,\psi)=(\tau^2\pi)^{-k/2}\int_{\mathbb{R}} e^{-k(x/\tau)^2}\psi(x)\,dx,\qquad \psi\in\mathscr{S}. \tag{3.24}$$

By Taylor's formula and equation (3.24), there exists a certain $\xi\in(0,1)$ such that

$$\begin{aligned}(\delta_\tau^k,\psi)&=(\tau^2\pi)^{-k/2}\sum_{i=0}^{k-1}\tfrac{\psi^{(i)}(0)}{i!}\int_{\mathbb{R}} e^{-k(x/\tau)^2}x^i\,dx+\\ &+\ \tfrac{(\tau^2\pi)^{-k/2}}{k!}\int_{\mathbb{R}} e^{-k(x/\tau)^2}\psi^{(k)}(\xi x)x^k\,dx.\end{aligned} \tag{3.25}$$

Putting $t:=(k/\tau^2)^{1/2}x$ we get

$$\int_{\mathbb{R}} e^{-k(x/\tau)^2}x^i\,dx=\left(\tau/\sqrt{k}\right)^{i+1}\int_{\mathbb{R}} e^{-t^2}t^i\,dt\qquad i=0,1,\ldots,k-1$$

and

$$\int_{\mathbb{R}} e^{-k(x/\tau)^2}\psi^{(k)}(\xi x)x^k\,dx=(\tau/\sqrt{k})^{k+1}\int_{\mathbb{R}} e^{-t^2}\psi^{(k)}\left(\frac{\tau\xi t}{\sqrt{k}}\right)t^k\,dt.$$

Since $\psi\in\mathscr{S}(\mathbb{R})$, applying the Lebesgue's dominated convergence theorem we have

$$\lim_{\tau\to0}\int_{\mathbb{R}} e^{-t^2}\psi^{(k)}\left(\frac{\tau\xi t}{\sqrt{k}}\right)t^k\,dt=\psi^{(k)}(0)\int_{\mathbb{R}} e^{-t^2}t^k\,dt,$$

so the second addend in equation (3.25) tends to 0 as $\tau\to0$. Hence, using $(\tau^{i+1-k})_{\tau\in(0,1)}$ as the elements of the neutrix N^P for $i=0,1,\ldots,k-2$, we see from equation (3.25) that

$$N^P-\lim_{\tau\to0}(\delta_\tau^k,\psi)=\frac{\psi^{(k-1)}(0)}{(k\pi)^{k/2}(k-1)!}\int_{\mathbb{R}} e^{-t^2}t^{k-1}\,dt$$

for arbitrarily fixed $\psi\in\mathscr{S}$. Consequently, due to Definition 3.6, δ^k exists in the sense of the kth Gaussian N^P-power in $\mathscr{S}'$ and

$$\delta^k=\frac{(-1)^{k-1}}{(k\pi)^{k/2}(k-1)!}\delta^{(k-1)}\int_{\mathbb{R}} e^{-t^2}t^{k-1}\,dt. \tag{3.26}$$

Clearly

$$\int_{\mathbb{R}} e^{-t^2} t^{2j-1}\, dt = 0 \quad \text{and} \quad \int_{\mathbb{R}} e^{-t^2} t^{2j}\, dt = \frac{(2j-1)!!}{2^j}\sqrt{\pi},$$

where $(2j-1)!! := (2j-1)\cdot(2j-3)\cdot\ldots\cdot 3\cdot 1$ for $j \in \mathbb{N}$. Hence, by equation (3.26), formula (3.23) follows and the proof is completed. □

3.7 Neutrix *k*th convolution of $\delta_{(k)}$ and δ

Definition 3.8. Let $f \in \mathscr{D}'(\mathbb{R})$ ($f \in \mathscr{S}'(\mathbb{R})$) and $h \in \mathscr{D}'(\mathbb{R}^k)$ ($h \in \mathscr{S}'(\mathbb{R}^k)$) for fixed $k \in \mathbb{N}$. Let (δ_τ) and $(\widetilde{\delta}_\tau)$ be delta-nets of the forms

$$\delta_\tau(x) = \tau^{-1}\sigma(\tau^{-1}x), \qquad \tau \in (0,1),\ \ \sigma \in \mathscr{D}_1(\mathbb{R}),\ \ x \in \mathbb{R} \tag{3.27}$$

and

$$\widetilde{\delta}_\tau(y) = \tau^{-1}\widetilde{\sigma}(\tau^{-1}y), \qquad \tau \in (0,1),\ \ \widetilde{\sigma} \in \mathscr{D}_1(\mathbb{R}^k),\ \ y \in \mathbb{R}^k, \tag{3.28}$$

respectively. Denote $f_\tau := f * \delta_\tau$ and $h_\tau := h * \widetilde{\delta}_\tau$ for $\tau \in (0,1)$ and assume that the convolutions $h_\tau \overset{k}{*} f_\tau$ exist in $\mathscr{D}'(\mathbb{R})$ (in $\mathscr{S}'(\mathbb{R})$) for arbitrary delta-nets (δ_τ) and $(\widetilde{\delta}_\tau)$ of the forms (3.27) and (3.28), respectively, and for all $\tau \in (0,1)$.

The *convolution of kth order* $h \overset{k}{*} f$ in $\mathscr{D}'(\mathbb{R})$ (in $\mathscr{S}'(\mathbb{R})$) is defined by

$$h \overset{k}{*} f := \lim_{\tau\to 0} h_\tau \overset{k}{*} f_\tau, \tag{3.29}$$

whenever the limit in (3.29) exists in $\mathscr{D}'(\mathbb{R})$ (in $\mathscr{S}'(\mathbb{R})$) for arbitrary delta-nets (δ_τ) and $(\widetilde{\delta}_\tau)$ of the form (3.27) and (3.28) and does not depend on σ or $\widetilde{\sigma}$.

Definition 3.9. Let N be a neutrix in $\mathbb{R}^{(0,1)}$. Fix $k \in \mathbb{N}$ and let h and f be as in Definition 3.8. The *N–convolution of kth order* $h \overset{k}{*} f$ in $\mathscr{D}'(\mathbb{R})$ (in $\mathscr{S}'(\mathbb{R})$) is defined by

$$h \overset{k}{*} f := N\!-\lim_{\tau\to 0} h_\tau \overset{k}{*} f_\tau, \tag{3.30}$$

whenever the N–limit in (3.30) exists in $\mathscr{D}'(\mathbb{R})$ (in $\mathscr{S}'(\mathbb{R})$) for all delta-nets (δ_τ) and $(\widetilde{\delta}_\tau)$ of the form (3.27) and (3.28), and does not depend on σ or $\widetilde{\sigma}$.

Definition 3.10. For $h \in \mathscr{S}'(\mathbb{R}^k)$ and $f \in \mathscr{S}'(\mathbb{R})$, the *Gaussian convolution of kth order* and the *Gaussian N–convolution of kth order* $h \overset{k}{*} f$ in $\mathscr{D}'(\mathbb{R})$ (in

$\mathscr{S}'(\mathbb{R})$) are defined by (3.29) and by (3.30), respectively, whenever the limits in (3.29) and (3.30) exist in $\mathscr{D}'(\mathbb{R})$ (in $\mathscr{S}'(\mathbb{R})$) for all delta-nets (δ_τ) and $(\widetilde{\delta}_\tau)$ of the form (3.27) and (3.28), where σ and $\widetilde{\sigma}$ are replaced by $\sigma_0 \in \mathscr{S}(\mathbb{R})$ and $\widetilde{\sigma}_0 \in \mathscr{S}(\mathbb{R}^k)$, respectively, given by

$$\sigma_0(x) := \pi^{-1/2} e^{-x^2} \qquad \text{and} \qquad \widetilde{\sigma}_0(y) := \pi^{-k/2} E(y),$$

for $x \in \mathbb{R}$ and $y := (\eta_1, \ldots, \eta_k) \in \mathbb{R}^k$, where $E(y) = e^{-\eta_1^2 - \cdots - \eta_k^2}$.

Theorem 3.2. *For $k \in \mathbb{N}$, the Gaussian N^P−convolution of kth order (and the more Gaussian N^F−convolution of kth order) of the Dirac delta $\delta_{(k)}$ in $\mathbb{R}^k$ and the Dirac delta δ in $\mathbb{R}$ exists in $\mathscr{S}'(\mathbb{R})$ and*

$$\delta_{(k)} \overset{k}{*} \delta = \delta^k \qquad \text{for} \qquad k \in \mathbb{N}, \tag{3.31}$$

where δ^k on the right hand side of (3.31) exists in the sense of the kth Gaussian N^P-power (and the more N^F–power) and is given by (3.23).

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ANDRZEJ BORYS
Faculty of Electrical Engineering, Gdynia Maritime University
Morska 81-87, 81-225 Gdynia, Poland
E-mail: a.borys@we.am.gdynia.pl

ANDRZEJ KAMIŃSKI
Faculty of Mathematics and Natural Sciences, University of Rzeszów
Prof. Pigonia 1, 35-310 Rzeszów, Poland
E-mail: akaminsk@ur.edu.pl

SŁAWOMIR SOREK
Faculty of Mathematics and Natural Sciences, University of Rzeszów
Prof. Pigonia 1, 35-310 Rzeszów, Poland
E-mail: ssorek@ur.edu.pl