Mock theta functions and asymptotics for partition-theoretic functions

I n a u g u r a l - D i s s e r t a t i o n

zur
Erlangung des Doktorgrades
der Mathematisch-Naturwissenschaftlichen Fakultät
der Universität zu Köln

vorgelegt von
Min-Joo Jang
aus
Jeonju

Köln, 2018

Berichterstatter: Prof. Dr. Kathrin Bringmann
Prof. Dr. Sander Zwegers

Tag der mündlichen Prüfung: 8. März 2018

Kurzzusammenfassung

Diese Dissertation umfasst Arbeiten zu verschiedenen Themen aus der Theorie der Modulformen und ganzzahligen Partitionen. Als erstes greifen wir Ramanujans ursprüngliche Definition einer Mock-Thetafunktion auf. Wir lösen das allgemeine Problem, Ramanujans Definition explizit zu verstehen, für die universelle Mock-Thetafunktion g_3 und beantworten damit eine Frage von Rhoades. Danach analysieren wir eine neue spt-Funktion und ihre Crankfunktion. Wir untersuchen asymptotische Aspekte dieser Crankfunktion und beweisen eine Vermutung über die Positivität des Cranks. Weiter untersuchen wir ein Muster für das Vorzeichen des Cranks und erhalten lineare Kongruenzen der spt-Funktion durch ihre Mockmodularität. Schließlich leiten wir eine asymptotische Formel für sogenannte odd-even Partitionen her, deren Erzeugendenfunktion in Ramanujans Identitäten vorkommt. Wir untersuchen auch ihre entsprechenden Überpartitionen, die odd-even Überpartitionen.

Abstract

This thesis contains research articles on various topics in the theory of modular forms and integer partitions. First we revisit Ramanujan's original definition of a mock theta function. We solve the general problem of understanding Ramanujan's definition explicitly for the universal mock theta function g_3, answering a question of Rhoades. After that we study a new spt function and its crank function. We investigate asymptotic aspects of this crank function and confirm a positivity conjecture of the crank. We further analyze a sign pattern of the crank and obtain linear congruences of the spt function via its mock modularity. Finally we provide the asymptotic formula for so-called odd-even partitions whose generating function appears in Ramanujan's identities. We also study their overpartition analogue odd-even overpartitions.

Acknowledgements

This thesis would not have been possible without the support and guidance of my advisor, Kathrin Bringmann. I appreciate all she has done for me and my career. It was an honor for me to be her student and to start my mathematical career with her.

I would like to thank Sander Zwegers not only for being the second advisor of my thesis but also for always being willing to help me.

For their support, kindness and inspiration, I would like to express my sincere gratitude to George Andrews, Amanda Folsom, and Jeremy Lovejoy. Also, I would like to thank my Master advisor Jaebum Sohn and Byungchan Kim. They have always stood behind me, and provided me with permanent assistance. I owe my deepest gratitude to them.

I am grateful to my former and current colleagues for the numerous mathematical conversations and comments as well as friendship. Thanks to them, I could enjoy the time in the office and laugh.

Finally, I would like to thank my friends, my family, and Fabian: thank you very much for your endless support, encouragement, and literally everything. I love you all.

I am deeply grateful to everyone and everything for making me happy and realize that I am really lucky person.

Table of Contents

I

Introduction

Modular forms play a significant role in modern number theory as well as a number of other areas, such as arithmetic geometry, representation theory, mathematical physics, and combinatorics. In particular, modular forms are closely related to the theory of integer partitions. In this chapter, we aim to introduce preliminary definitions and give an overview of this thesis.

I.1 Harmonic Maass forms and Mock Theta functions

I.1.1 Harmonic Maass forms

Harmonic Maass forms, introduced by Brunier and Funke [18], are a natural extension of modular forms. Before discussing harmonic Maass forms, we first set some notation. Throughout, we let $\tau = x + iy \in \mathbb{H}$, where $x, y \in \mathbb{R}$ and $\mathbb{H} := \{\tau = x + iy \in \mathbb{C} \mid y > 0\}$ the complex upper half-plane, and $k \in \frac{1}{2}\mathbb{Z}$. Let for $N \in \mathbb{N}$

$$\Gamma_0(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \,\middle|\, c \equiv 0 \pmod{N} \right\}$$

be the *Hecke congruence subgroup of lever* N. We further define *the weight* k *hyperbolic Laplacian operator* Δ_k by

$$\Delta_k := -y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) + iky \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right),$$

and for odd integer d set

$$\varepsilon_d := \begin{cases} 1 & \text{if } d \equiv 1 \pmod{4}, \\ i & \text{if } d \equiv 3 \pmod{4}. \end{cases}$$

Following [17, Definition 4.2], we may now define harmonic Maass forms of weights k.

Definition I.1. A smooth function $f\colon \mathbb{H} \to \mathbb{C}$ is called a *weight k harmonic Maass form with Nubentypus character χ on a subgroup $\Gamma \subset \Gamma_0(N)$* for some $N \in \mathbb{N}$, where $4 \mid N$ if $k \in \frac{1}{2}\mathbb{Z} \setminus \mathbb{Z}$, if it satisfies the following three conditions:

(1) For all $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$ and all $\tau \in \mathbb{H}$, we have

$$f\left(\frac{a\tau+b}{c\tau+d}\right) = \begin{cases} \chi(d)(c\tau+d)^k f(\tau) & \text{if } k \in \mathbb{Z}, \\ \left(\frac{c}{d}\right)\varepsilon_d^{-2k}\chi(d)(c\tau+d)^k f(\tau) & \text{if } k \in \frac{1}{2}\mathbb{Z} \setminus \mathbb{Z}. \end{cases}$$

Here $\left(\frac{c}{d}\right)$ denotes the extended Legendre symbol, and $\sqrt{\ }$ is the principal branch of the holomorphic square root.

(2) We have that $\Delta_k(f) = 0$.

(3) There is a polynomial $P_f(\tau) \in \mathbb{C}\left[q^{-1}\right]$ such that

$$f(\tau) - P_f(\tau) = O\left(e^{-\varepsilon y}\right)$$

as $y \to \infty$ for some $\varepsilon > 0$. Analogous conditions are required at all cusps.

Remark I.2. Bruinier and Funke [18] also studied vector-valued harmonic Maass forms, which include the scalar-valued forms defined above. In this case, Definition I.1 can be generalized to other finite index subgroups of $\mathrm{SL}_2(\mathbb{Z})$ in the obvious way.

Harmonic Maass forms have Fourier expansions of the form

$$f(\tau) = \sum_{n \gg -\infty} c_f^+(n) q^n + \sum_{n<0} c_f^-(n)\Gamma(k-1, 4\pi|n|y) q^n,$$

where throughout $q := e^{2\pi i\tau}$ and $\Gamma(s,x)$ is the incomplete gamma function defined as

$$\Gamma(s,x) := \int_x^\infty e^{-t} t^{s-1} dt.$$

We refer to

$$f^+(\tau) := \sum_{n \gg -\infty} c_f^+(n) q^n$$

as the *holomorphic part of* $f(\tau)$, and

$$f^-(\tau) := \sum_{n<0} c_f^-(n)\Gamma(k-1, 4\pi|n|y) q^n$$

the *non-holomorphic part of* $f(\tau)$. Note that in particular weakly holomorphic modular forms are harmonic Maass forms with $f^-(\tau) = 0$.

I.1.2 Mock modular forms

Now we turn to mock theta functions, more generally mock modular forms. Mock theta functions were introduced by Ramanujan by means of examples (see Section II.1 below for detail). In the decades after Ramanujan's death, a theoretical framework for them did not arrive until Zwegers discovered the connection between these functions and modular forms in his Ph.D. thesis [47]. While mock theta functions are holomorphic but not modular, they may be "completed" to harmonic Maass forms by adding additional non-holomorphic terms. Before giving a precise definition of mock modular forms, we first define the differential operator for $\kappa \in \frac{1}{2}\mathbb{Z}$

$$\xi_\kappa := 2iy^\kappa \overline{\frac{\partial}{\partial \overline{\tau}}},$$

where $\frac{\partial}{\partial \overline{\tau}} := \frac{1}{2}(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y})$. From a proposition of Bruinier and Funke [18, Proposition 3.2], who originally introduced a notion of harmonic Maass forms, we have a surjective map for $k \in \frac{1}{2}\mathbb{Z} \setminus \{1\}$

$$\xi_{2-k} : H_{2-k}\left(\Gamma_0(N)\right) \to S_k\left(\Gamma_0(N)\right),$$

where $H_k(\Gamma_0(N))$ (resp. $S_k(\Gamma_0(N))$) denotes the space of weight k harmonic Maass forms (resp. cusp forms) on $\Gamma_0(N)$. Moreover, assuming the notation above, we have that for $f(\tau) \in H_{2-k}(\Gamma_0(N))$

$$\xi_{2-k}\left(f(\tau)\right) = \xi_{2-k}\left(f^-(\tau)\right) = -(4\pi)^{k-1} \sum_{n=1}^{\infty} \overline{c_f^-(-n)} n^{k-1} q^n \in S_k\left(\Gamma_0(N)\right). \tag{I.1.1}$$

We now may give the following definition from [17, Definition 5.16], which is due to Zagier.

Definition I.3. Let f be a harmonic Maass form of weight $2-k$.

(1) A *mock modular form of weight* $2-k$ is the holomorphic part f^+ of f for which f^- is nontrivial.

(2) For $f \in H_{2-k}(\Gamma_0(N))$, we refer to the cusp form (I.1.1) as the *shadow* of the mock modular form f^+.

(3) A *mock theta function* is a mock modular form of weight $\frac{1}{2}$ or $\frac{3}{2}$ whose shadow is a linear combination of unary theta functions.

Remark I.4. Suppose the mock modular form f^+ has shadow $g(\tau) = \sum_{n=1}^{\infty} c_g(n)q^n$ in Definition I.3 (2). Then the non-holomorphic part f^- can be written as a period integral

$$f^-(\tau) = 2^{1-k} i \int_{-\overline{\tau}}^{i\infty} \frac{g^*(z)}{(-i(z+\tau))^{2-k}} dz,$$

where $g^*(\tau) := \overline{g(-\overline{\tau})} = \sum_{n=1}^{\infty} \overline{c_g(n)} q^n$.

I.2 The basics of partitions

Partitions, also called integer partitions, were considered by Leibniz for the first time in a letter to Bernoulli from 1674 [35]. Several decades later, many beautiful partition theorems were proved by Euler [22, 23]. Besides Euler, a number of mathematicians such as Gauss, Hardy, Jacobi, Littlewood, Rademacher, Ramanujan, Schur, Sylvester, and Andrews have contributed to the development of the theory of partitions.

A *partition* of a positive integer n is a non-increasing sequence of positive integers whose sum is n. For instance, there are 5 partitions of 4 :

$$4,\ 3+1,\ 2+2,\ 2+1+1,\ 1+1+1+1.$$

Let $p(n)$ denote the number of partitions of n. For $n = 4$, we have $p(4) = 5$. The generating function of $p(n)$ is given by

$$1 + \sum_{n=1}^{\infty} p(n)q^n = \prod_{k=1}^{\infty} \frac{1}{1-q^k} = \frac{q^{\frac{1}{24}}}{\eta(\tau)},$$

where

$$\eta(\tau) := q^{\frac{1}{24}} \prod_{k=1}^{\infty} \left(1 - q^k\right)$$

is *Dedekind's η-function*, a weight $\frac{1}{2}$ modular form.

I.2.1 Asymptotics

An asymptotic formula for $p(n)$ was first studied by Hardy and Ramanujan [30] in 1918. They showed that

$$p(n) \sim \frac{1}{4\sqrt{3}n} e^{\pi\sqrt{\frac{2n}{3}}}$$

as $n \to \infty$. Hardy and Ramanujan also obtained an asymptotic expansion for $p(n)$ by using so-called Circle Method. This expansion is surprisingly accurate. For example, the first eight terms of the expansion give the correct value of $p(200) = 3972999029388$. Two decades later, Rademacher [38] improved the Hardy and Ramanujan's method to find a convergent series expansion for $p(n)$. That is

$$p(n) = \frac{1}{\pi\sqrt{2}} \sum_{k=1}^{\infty} A_k(n)\sqrt{k} \left[\frac{d}{dx} \frac{\sinh\left(\frac{\pi}{k}\sqrt{\frac{2}{3}\left(x - \frac{1}{24}\right)}\right)}{\sqrt{x - \frac{1}{24}}} \right]_{x=n}.$$

Here, $A_k(n)$ is the Kloosterman sum defined by

$$A_k(n) := \sum_{\substack{h \pmod{k} \\ (h,k)=1}} \omega_{h,k} e^{\frac{-2\pi i n h}{k}},$$

where $\omega_{h,k}$ is a 24th root of unity, and (h, k) denotes the greatest common divisor of h and k.

I.2.2 Congruences

Ramanujan's another remarkable discovery is partition congruences [39, 40]:

$$\begin{aligned} p(5n + 4) &\equiv 0 \pmod{5}, \\ p(7n + 5) &\equiv 0 \pmod{7}, \\ p(11n + 6) &\equiv 0 \pmod{11}, \end{aligned}$$

where n is any nonnegative integer. However, Ramanujan's original proofs for the congruences above are based on q-series identities and do not give a combinatorial understanding of them. In order to provide a combinatorial proof for the Ramanujan's congruences, Dyson [21] defined the *rank* of a partition to be the largest part of the partition minus the number of parts. He also defined $N(m, t, n)$ to be the number of partitions of n with rank congruent to m modulo t and conjectured that

$$N(m, 5, 5n + 4) = \frac{1}{5} p(5n + 4)$$

for $m \in \mathbb{N}_0,\ 0 \le m \le 4$, and

$$N(m, 7, 7n + 5) = \frac{1}{7} p(7n + 5)$$

for $m \in \mathbb{N}_0$, $0 \leq m \leq 6$. Ten years later, this was proved by Atkin and Swinnerton-Dyer [10].

However, a short computation reveals that Dyson's rank does not explain the congruence modulo 11. Naturally, Dyson postulated that there exists another statistic which can divide the partitions of $5n+4$, $7n+5$, and $11n+6$ into 5,7, and 11 groups respectively in equal size, and thus prove the congruences combinatorially. He called it the "crank". In 1988, Andrews and Garvan [4, 26] formulated the definition of the crank and therefore confirmed Dyson's conjecture. For a partition λ, let $o(\lambda)$ denote the number of ones in λ and $\mu(\lambda)$ denote the number of parts larger than $o(\lambda)$. Then the *crank* of λ is defined by

$$\mathrm{crank}(\lambda) := \begin{cases} \text{largest part of } \lambda & \text{if } o(\lambda) = 0, \\ \mu(\lambda) - o(\lambda) & \text{if } o(\lambda) > 0. \end{cases}$$

Remark I.5. Partition rank and crank functions are particularly interesting objects in many aspects. For example, the rank generating function is essentially a mock theta function, i.e., the holomorphic part of a weight $\frac{1}{2}$ harmonic Maass forms, which was proven by Bringmann and Ono [15], while the crank generating function is a modular form, up to a q-power.

I.3 Scope of this thesis

This thesis is cumulative and contains research in various areas related to integer partitions and modular forms, in particular mock theta functions. In the following, we give the motivation of the research projects contained in this thesis as well as a brief description of results.

I.3.1 Radial limits of the universal mock theta function g_3

Although a number of studies had been done on mock theta functions from the modern point of view, for several decades no one had indeed proven that any of Ramanujan's mock theta functions satisfy his own definition (See Definition II.1 below). Here we revisit his original definition and claims.

In his famous 1920 deathbed letter to Hardy [11, p. 220–224], Ramanujan introduced a notion of a mock theta function and offered 17 examples. He then claimed about the behavior of these functions near roots of unity. In particular, for the mock theta function

$$f(q) := 1 + \frac{q}{(1+q)^2} + \frac{q^4}{(1+q)^2\,(1+q^2)^2} + \frac{q^9}{(1+q)^2\,(1+q^2)^2\,(1+q^3)^2} + \cdots ,$$

he claimed that as q approaches a primitive even orde $2k$ root of unity,

$$f(q) - (-1)^k(1-q)\left(1-q^3\right)\left(1-q^5\right)\cdots\left(1-2q+2q^4-\cdots\right) = O(1). \quad \text{(I.3.1)}$$

This claim was proven later by Watson [42]. Moreover, Folsom, Ono, and Rhoades [25] gave a closed formula for (I.3.1). Recently, Griffin, Ono, and Rolen [29] proved that all of Ramanujan's examples satisfy his own definition in an abstract sense. However, this does not give explicit expressions for the radial limits. In order to address this problem, Rhoades [41] asked a question regarding the universal mock theta functions g_2 and g_3, a more general family introduced by Gordon and McIntosh [28]. More precisely, he suggested to find explicit formulas for radial limits of g_2 and g_3, as every mock theta function can be written as specializations of g_2 and g_3 and modular forms.

The work presented in Chapter II is joint work with Steffen Löbrich and confirms Rhoades' question (see Question II.2 below). In fact, Bringmann and Rolen [16] recently found an explicit algorithm to write any specializations of g_2. We treat the case of g_3, generalizing radial limit results for the rank generating function of Folsom, Ono, and Rhoades [25]. Moreover, as a side product, we obtained in Theorem II.5 a generalization of [25, Theorem 1.2] and also non-trivial identities for roots of unity due to the uniqueness of the radial limits (e.g. Corollary II.13 below).

I.3.2 On spt-crank-type functions

Partition-theoretic interpretations of various functions and identities have been intensively studied for many decades. One of the most notable achievements of this area is the Mock Theta Conjectures of Andrews and Garvan [5]. Using the rank of a partition, they showed that proving ten identities for Ramanujan's fifth order mock theta functions is equivalent to the proofs of just two identities. These identities, called Mock Theta Conjectures, were first proved by Hickerson [31]. Recently Andrews, Dixit, and Yee [9] found a partition-theoretic interpretation for Ramanujan's third order mock theta function $\omega(q)$. That is

$$\sum_{n=1}^{\infty} p_\omega(n) q^n = q\omega(q),$$

where $p_\omega(n)$ denotes the number of partitions of a positive integer n such that all odd parts are smaller than twice the smallest part. For example, there are 4 such partitions of 4 :

$$4,\ 2+2,\ 2+1+1,\ 1+1+1+1.$$

Andrews, Dixit, and Yee [9] also studied the smallest parts function for the partitions enumerated by $p_\omega(n)$. The *smallest parts function* $\mathrm{spt}(n)$, introduced by Andrews [7], counts the total number of appearances of the smallest parts in all partitions of n. Likewise, let $\mathrm{spt}_\omega(n)$ denote the total number of smallest parts in the partitions enumerated by $p_\omega(n)$. Recall the partitions counted by $p_\omega(4)$. Here we mark the smallest parts with a circle:

$$4,\ 2+2,\ 2+1+1,\ 1+1+1+1.$$

Therefore, we have $\mathrm{spt}_\omega(4) = 9$. By using q-series identities, Andrews, Dixit, and Yee [9] further showed the congruences

$$\begin{aligned}
\mathrm{spt}_\omega(5n+3) &\equiv \quad \pmod 5,\\
\mathrm{spt}_\omega(10n+7) &\equiv \quad \pmod 5,\\
\mathrm{spt}_\omega(10n+9) &\equiv \quad \pmod 5.
\end{aligned}$$

Chapter III is joint work with Byungchan Kim and concerns the function $\mathrm{spt}_\omega(n)$. We establish the mock modularity of $\mathrm{spt}_\omega(n)$, and thus obtain infinitely many linear congruences for $\mathrm{spt}_\omega(n)$ in Theorem III.4. We also investigate its crank function introduced by Garvan and Jennings-Shaffer [27], which explains the first congruence for $\mathrm{spt}_\omega(n)$ above. We apply Wright's Circle Method [43] to obtain the asymptotic formula for the crank in Theorem III.1 as well as its sign pattern in Theorem III.3. In doing so, we affirm Garvan and Jennings-Shaffer's positivity conjecture on the crank.

I.3.3 Asymptotic behavior of Odd-Even partitions

Besides mock theta functions, another main research area of Ramanujan's work is q-hypergeometric series. Indeed this fact became known thanks to Andrews' discovery of Ramanujan's lost notebook [2, 8]. After this discovery, there have been a number of studies on Ramanujan's identities involving q-hypergeometric series. The *q-hypergeometric series*, also known as *basic hypergeometric series*, are defined for non-negative integers $r,\ s$ by

$${}_r\Phi_s\left[\begin{matrix} a_1, & a_2, & \ldots & a_r \\ b_1, & b_2, & \ldots & b_s\end{matrix};q,z\right] := \sum_{n=0}^{\infty} \frac{(a_1,a_2,\ldots,a_r)_n}{(b_1,b_2,\ldots,b_s)_n}\left((-1)^n q^{\binom{n}{2}}\right)^{1+s-r} z^n,$$

where

$$(a_1, a_2, \ldots, a_\ell)_n = (a_1, a_2, \ldots, a_\ell; q)_n := (a_1)_n\,(a_2)_n \cdots (a_\ell)_n\,,$$

and

$$(a)_n = (a; q)_n := \prod_{k=1}^{n} (1 - aq^{k-1}),$$

for $n \in \mathbb{N}_0 \cup \{\infty\}$.

Andrews [3] looked into a certain q-hypergeometric series

$$\sum_{n=0}^{\infty} \frac{q^{\frac{n(n+1)}{2}}}{(q^2; q^2)_n},$$

which apprears in Ramanujan's identities. In Ramanujan's lost notebook, there are several formulas involving this function, but they are not as simple as the identities with other similar shape of functions (see Section IV.1 for details). Nonetheless, Andrews found out that this function possesses combinatorial information, namely this function is the generating function for odd-even partitions.

In Chapter IV, we analyze odd-even partitions asymptotically. After that we continue by studying their overpartition analogue odd-even overpartitions. Odd-even overpartitions were first consider by Lovejoy [36], and their generating function is given as a mixed mock modular form. The precise definitions and results are given in Section .

II

Radial Limits of the Universal Mock Theta Function g_3

This chapter is based on a manuscript to be published in *Proceedings of the American Mathematical Society* and is joint work with Steffen Löbrich [32].

II.1 Introduction

In his last letter to Hardy, Ramanujan provided 17 examples for what he called a "mock theta function". These are given by q-hypergeometric series that have the same growth behavior as classical modular forms as one radially approaches roots of unity from inside the unit disc. However, he conjectured that one single modular form is not enough to cut out all the poles at roots of unity. More precisely, Ramanujan defined mock theta functions as follows (cf. [AH91], p.63).

Definition II.1 (Ramanujan)**.** A *mock theta function* is a function $F(q)$, defined by a q-series which converges for $|q| < 1$, satisfying the following conditions:

(i) There are infinitely many roots of unity ζ such that $F(q)$ grows exponentially as q approaches ζ radially from inside the unit disc.

(ii) For every root of unity ζ, there exists a weakly holomorphic modular form $M_\zeta(q)$ and a rational number α, such that $F(q) - q^\alpha M_\zeta(q)$ is bounded as q approaches ζ radially from inside the unit disc.

(iii) There does not exist a single weakly holomorphic modular form $M(q)$ and a rational number α, such that for every root of unity ζ, the difference $F(q) - q^\alpha M(q)$ is bounded as q approaches ζ radially from inside the unit disc.

Griffin, Ono, and Rolen [GOR13] proved that all of Ramanujan's examples satisfy Definition II.1. Rhoades [Rho13] asked if one can choose the set

$\{M_\zeta(q)\}_\zeta$ to be finite and can also give explicit expressions for the radial limits in (ii). In particular, he asked this for certain specializations of the universal mock theta functions $g_2(x,q)$ and $g_3(x,q)$, since every mock theta function can be expressed as a linear combination of such specializations, up to a rational q-powers, and modular forms. The problem has been solved for special mock theta functions before: Folsom, Ono, and Rhoades gave a closed expression for the radial limits of Ramanujan's third order mock theta function f (Theorem 1.1 of [FOR13]) as well as a partial answer for Rhoades' question for certain specializations of g_3 (Theorem 1.2 of [FOR13], see also Remark II.7). Moreover, in [BKLMR], the radial limits of Ramanujan's fifth order mock theta functions were computed using bilateral series.

Bringmann and Rolen [BR15] gave a positive answer to Rhoades' question for specializations of the even order universal mock theta function g_2, i.e., functions of the form $F(q) = g_2(\zeta_b^a q^A, q^B)$ for $a, b, A, B \in \mathbb{N}$. Linear combinations of such functions and modular forms comprise all of Ramanujan's mock theta functions up to rational q-powers. They showed that at most four modular forms (including the form identically equal to 0) are needed for such F to keep the radial limits bounded. For this they used a bilateral series identity for g_2 found by Mortenson [Mor16] together with a careful analysis of zeros and poles of g_2 for case-by-case estimates. In this paper, we study the odd order *universal mock theta function* g_3, defined as

$$g_3(x,q) := \sum_{n\geq 1} \frac{q^{n(n-1)}}{(x, q/x; q)_n}$$

(see Section 3 for a definition of the q-Pochhammer symbols $(a_1, \ldots, a_m; q)_n$). Throughout $\zeta_k^h := e^{2\pi i \frac{h}{k}}$. In this case, Rhoades' question for g_3 can be stated as follows:

Question II.2 (Rhoades [Rho13], §3, 3.4.)**.** For every $a, b, A, B \in \mathbb{N}$ with $(a,b) = 1$, can one find for every $k, h \in \mathbb{N}$ with $(k,h) = 1$ a weakly holomorphic modular form $M_{k,h}(q)$ satisfying (ii) of Definition II.1, such that the set $\{M_{k,h}\}_{k,h}$ is finite, and explicitly compute the radial limits

$$Q_{a,b,A,B,h,k} := \lim_{q\to\zeta_k^k} \left(g_3\left(\zeta_b^a q^A, q^B\right) - M_{k,h}(q)\right)?$$

Remark II.3. Note that while for every root of unity ζ_k^h one can choose a lot of modular forms $M_{k,h}(q)$ matching condition (ii) of Definition II.1, Choi, Lim, and Rhoades [CLR16] showed that the expressions $Q_{a,b,A,B,h,k}$ are uniquely determined. Moreover, they proved that for every $a, b, A, B \in \mathbb{N}$, the function

$f : \mathbb{Q} \to \mathbb{C},\ f(\frac{h}{k}) = Q_{a,b,A,B,h,k}$, is a quantum modular form of weight $\frac{1}{2}$ whose cocycles extend to a real analytic function on $\mathbb{R}$, except possibly at one point (for an introduction to quantum modular forms, see [Zag10]). Bringmann and Rolen constructed such quantum modular forms in a different guise in [BR16].

A positive answer to Question II.2 follows from the work of Bringmann and Rolen on g_2 [BR15], since g_3 can be expressed in terms of g_2 (cf. [GM12], eq. (6.1)). However, this relation is quite involved, so finding suitable modular forms and computing the radial limits in the g_3-case can be laborious with this method. We affirm Rhoades' question for most of the choices for the parameters a, b, A, B, h, k directly by constructing the set $\{M_{k,h}\}_{k,h}$ and giving explicit expressions for the radial limits. For this we need an identity analogous to Mortenson's ([Mor16], Corollary 5.2.) as well as a relation between certain g_3-values by Kang ([Kan09], Theorem 1.3). In doing so, we generalize Theorem 1.2 of [FOR13] and find a different proof for it. We also obtain non-trivial identities for roots of unity as a side product. However, for a few special choices of the parameters, we could not find a direct answer.

The paper is organized as follows: In Section 2 we will state our results explicitly. For this we have to distinguish several cases depending on the behavior of g_3 at the root of unity in question. In Section 3 we recall some definitions and identities needed for the proofs, which will be given in Section 4.

Acknowledgements

This paper will be part of the first author's PhD thesis and was supported by the Deutsche Forschungsgemeinschaft (DFG) Grant No. BR 4082/3-1, and partially written in the context of the Cologne Young Researchers in Number Theory Program 2015, supported by University of Cologne postdoc DFG Grant D-72133-G-403-15100101 funded under the Institutional Strategy of the University of Cologne within the German Excellence Initiative. First of all, we would like to thank Kathrin Bringmann and Larry Rolen for suggesting the problem and giving us many helpful instructions and remarks. A good portion of this paper is based on their ideas. Furthermore, we are grateful to Wadim Zudilin for many valuable remarks and a profitable conversation. We would also like to thank Michael Griffin for his advice and Eric Mortenson, Michael Somos, and the referee for useful correspondence.

II.2 Statement of Results

Remark II.4. Throughout this paper we slightly abuse the term “modular form”. To be precise, we mean by a *modular form* a weakly holomorphic modular form (with respect to the τ-variable) of weight $\frac{1}{2}$ for some congruence subgroup and some multiplier, up to multiplication by a rational q-power.

There are several methods to determine the modular forms $M_{k,h}(q)$ and compute the radial limits, depending on the behavior of g_3 at the chosen root of unity. The most straightforward case is when the sum defining g_3 converges absolutely in the radial limit. In this case, we can choose $M_{k,h}(q)$ to be identically zero. If the sum does not converge, some summands can have poles arising from zeros in the denominator. Since there are no zeros in the numerator and the summands have increasing pole order, there is no way for the poles to cancel out (cf. Section 3 in [BR15]). If neither of these cases occur, further analysis is required.

First, we consider the case where a pole occurs in the summands of g_3. Here we need a bilateral series identity for g_3 (see Proposition II.16 below). Let $a, b, A, B, k \in \mathbb{N}$ with $b, k > 0$ and $(a, b) = 1$. For $g_3(\zeta_b^a \zeta_k^{Ah} e^{-At}, \zeta_k^{Bh} e^{-Bt})$ to be well-defined for every $t > 0$, we assume that $B \nmid A$ whenever $b = 1$. Now we set $k' := \frac{k}{(k,B)}$, $B' := \frac{B}{(k,B)}$, and

$$\mathcal{Q}_{a,b,A,B} := \left\{ \frac{h}{k} \in \mathbb{Q} : b|k \text{ and } (k, B) \middle| \left(\frac{ak}{b} + Ah \right) \right\}.$$

As in [BR15], Section 3, this is the set for which some summands of g_3 have poles. All limits in this paper are to be understood as radial limits from within the unit disc. To be precise, we will denote by $\lim_{q\to\zeta} F(q)$ the limit $\lim_{t\to 0^+} F(\zeta e^{-t})$ for ζ a root of unity and F a function on the unit disc. The Jacobi triple product $j(x, q) := (x, q/x, q; q)_\infty$ is defined in Section 3.

Theorem II.5. *If $\frac{h}{k} \in \mathcal{Q}_{a,b,A,B}$, then*

$$\lim_{q\to\zeta_h^k} \left(g_3\left(\zeta_b^a q^A, q^B\right) - \frac{\left(q^B; q^B\right)_\infty^2 j\left(q^{\frac{B}{2}}, q^B\right)^2}{\zeta_b^a q^A j\left(\zeta_b^a q^{A+\frac{B}{2}}, q^B\right)^2 j\left(\zeta_b^a q^A, q^B\right)} \right)$$
$$= -\zeta_b^{-a}\zeta_k^{-hA} + \zeta_b^{-2a}\zeta_k^{-2hA} \sum_{n=1}^{k'} \left(\zeta_b^{-a}\zeta_k^{h(B-A)}; \zeta_k^{hB}\right)_{n-1} \left(\zeta_b^a\zeta_k^{hA}; \zeta_k^{hB}\right)_n \zeta_k^{hBn}.$$

Here in the expression $q^{\frac{B}{2}}$ the choice of the possibly implied square root does not matter.

Remark II.6. In fact, more is true: The right hand side, if one replaces ζ_k^h by $\zeta_k^h e^{-t}$, is not just the limit, but an asymptotic expansion of the left hand side as $t \to 0^+$. The same holds for the other theorems below.

Remark II.7. Setting $A = 0$ and $B = 1$, we obtain the same radial limits as Folsom, Ono, and Rhoades in [FOR13], Theorem 1.2. However, the modular form we subtracted on the left hand side is different (see also Remark II.17).

Next, we treat the case where g_3 can be computed directly as an absolutely convergent geometric series. For $x \in \mathbb{R}$, let $\{x\}$ denote the unique number in $[0, 1)$ with $x - \{x\} \in \mathbb{Z}$.

Theorem II.8. *If $\frac{h}{k} \notin \mathcal{Q}_{a,b,A,B}$ and $\{k'(\frac{a}{b} + \frac{Ah}{k})\} \in (\frac{1}{6}, \frac{5}{6})$, then*

$$\lim_{q \to \zeta_h^k} g_3\left(\zeta_b^a q^A, q^B\right) = \frac{1}{1 - \frac{1}{2-2\cos\left(2\pi k'\left(\frac{a}{b}+\frac{Ah}{k}\right)\right)}} \sum_{j=1}^{k'} \frac{\zeta_k^{hBj(j-1)}}{\left(\zeta_b^a \zeta_k^{hA}, \zeta_b^{-a}\zeta_k^{h(B-A)}; \zeta_k^{hB}\right)_j}.$$

Now we need the following beautiful identity of Kang ([Kan09], Theorem 1.3.)

$$g_3(x, q) + g_3\left(\zeta_3 x, q\right) + g_3\left(\zeta_3^2 x, q\right) = \frac{3i\left(q^3; q^3\right)_\infty^3}{(q)_\infty j\left(x^3, q^3\right)}. \tag{II.2.1}$$

If the two shifted values on the left hand side of (II.2.1) converge, we can subtract the modular form on right hand side to obtain a bounded radial limit. This is only possible if k' is not divisible by 3.

Theorem II.9. *Assume that $\frac{h}{k} \notin \mathcal{Q}_{a,b,A,B}$, $\{k'(\frac{a}{b} + \frac{Ah}{k})\} \in [0, \frac{1}{6}) \cup (\frac{5}{6}, 1)$ and $3 \nmid k'$. Then we have*

$$\begin{aligned}
&\lim_{q \to \zeta_h^k} \left(g_3\left(\zeta_b^a q^A, q^B\right) - \frac{3i\left(q^{3B}; q^{3B}\right)_\infty^3}{\left(q^B; q^B\right)_\infty j\left(\zeta_b^{3a} q^{3A}, q^{3B}\right)} \right) \\
&\quad = -\sum_{\ell=1}^{2} \frac{1}{1 - \frac{1}{2-2\cos\left(2\pi k'\left(\frac{a}{b}+\frac{Ah}{k}+\frac{\ell}{3}\right)\right)}} \sum_{j=1}^{k'} \frac{\zeta_k^{hBj(j-1)}}{\left(\zeta_3^\ell \zeta_b^a \zeta_k^{hA}, \zeta_3^{2\ell}\zeta_b^{-a}\zeta_k^{h(B-A)}; \zeta_k^{hB}\right)_j}.
\end{aligned}$$

If $3|k'$ and $g_3(x, q)$ diverges, then also the other two summands in the left hand side of (II.2.1) diverge.

Remark II.10. In trying to treat the case $3|k'$ and $\{k'(\frac{a}{b}+\frac{Ah}{k})\} \notin \{\frac{1}{6},\frac{5}{6}\}$, we first found a naive formula for the radial limits, which does not seem to be true in general due to convergence issues. However, if we additionally require the assumptions of Theorem II.5 to be true, then numerical computations surprisingly suggest the naive formula and the expression in Theorem II.5 to be equal in this overlap case. This leads us to the following conjecture.

Conjecture II.11. Let x, q be roots of unity with $(x, q/x; q)_\infty = 0$ such that q is a primitive root of unity of order $3k$ and x^{3k} is not a primitive sixth order root of unity. Then we have

$$\frac{1}{1-x^{3k}+x^{6k}}\sum_{j=1}^{k}(-1)^j x^{3j-2}q^{-\frac{(3j+1)j}{2}}\left(q\left(1+x^{3k}q^k\right)+x\left(1+x^{3k}q^{2k}\right)\right)$$
$$= -x^{-1}+x^{-2}\sum_{j=1}^{3k}\left(q/x;q\right)_{j-1}(x;q)_j q^j.$$

We challenge the interested reader to find a proof for this identity.

Example II.12. Up to a prefactor, all of Ramanujan's fifth order mock theta functions studied in [BKLMR] can be expressed as the sum of a specialization of g_3 with $\frac{a}{b}\in\frac{1}{5}\mathbb{Z}$ and $B=5$ or 10, and a modular form. In these cases, we have $k'\left(\frac{a}{b}+\frac{Ah}{k}\right)=\frac{Ah}{(k,B)}\in\frac{1}{10}\mathbb{Z}$, so at least one of the above Theorems will apply. However, our expressions for the radial limits are different from those in [BKLMR]. Comparing the results and using the uniqueness of the radial limits yields many curious corollaries. We give one example of such an identity. The fifth order mock theta function $f_0(q)$ can be written as $-2q^2g_3(q^2,q^{10})$ plus a modular form. Here we have $(a,b,A,B)=(0,1,2,10)$ with $\mathcal{Q}_{0,1,2,10} := \left\{\frac{h}{k}\in\mathbb{Q} : (k,10)|2h\right\}$. Let now k be a positive even integer with $5\nmid k$, so that $(k,10)=2|2h$ for every h. Then we have $\frac{h}{k}\in\mathcal{Q}_{0,1,2,10}$ and we can apply Theorem II.5. On the other hand, for even k, we can also apply the first equation of [BKLMR], Theorem 1.1. (a) to compute the radial limit. Therefore we obtain the following corollary.

Corollary II.13. *Let $k\in\mathbb{N}$ with $(k,10)=2$ and ζ a root of unity of order k. Then we have*

$$2-2\zeta^{-2}\sum_{n=1}^{k/2}\left(\zeta^8;\zeta^{10}\right)_{n-1}\left(\zeta^2;\zeta^{10}\right)_n\zeta^{10n} = -2\sum_{n=0}^{k-1}\zeta^{(n+1)(n+2)/2}(-\zeta;\zeta)_n.$$

.

We obtain similar equations for the other cases of [BKLMR], Theorems 1.1. and 1.2. It would be interesting to find a direct explanation for all of these identities.

II.3 Definitions and Useful Formulas

First we fix some notation. For a non-zero complex number q in the open unit disc, $n \in \mathbb{N}_0 \cup \{\infty\}$ and $a \in \mathbb{C}$, the q-Pochhammer symbol is given by

$$(a)_n := (a;q)_n := \prod_{j=0}^{n-1} \left(1 - aq^j\right)$$

and we abbreviate

$$(a_1, \ldots, a_m; q)_n := \prod_{j=1}^{m} (a_j; q)_n \, .$$

We further define the Jacobi triple product as

$$j(x,q) := (x, q/x, q; q)_\infty \, .$$

Now we show that the infinite products we subtracted in Theorems II.5 and II.9 are indeed modular forms.

Lemma II.14. *For every $n \in \mathbb{N}$, ζ a root of unity, and $\alpha \in \mathbb{Q}$, the functions $(q^n)_\infty$ and $j(\zeta q^\alpha, q^n)$ are modular forms in the sense of Remark II.4. In particular, the infinite products on the left hand-sides of Theorems II.5 and II.9 are modular forms in this sense.*

Proof. First we have

$$(q)_\infty = q^{-\frac{1}{24}} \eta(q),$$

where η is the Dedekind eta-function, which is a modular form of weight $\frac{1}{2}$. Furthermore, the well-known Jacobi triple product identity states that

$$j(x,q) = i x^{\frac{1}{2}} q^{-\frac{1}{8}} \vartheta(x,q),$$

where ϑ is the Jacobi theta-function, defined by

$$\vartheta(x,q) := \sum_{n \in \frac{1}{2} + \mathbb{Z}} (-x)^n q^{\frac{n^2}{2}},$$

which is a Jacobi form of weight and index $\frac{1}{2}$. This implies that $\vartheta(\zeta q^\alpha, q)$ is a modular form of weight $\frac{1}{2}$ for every root of unity ζ and $\alpha \in \mathbb{Q}$. Since replacing q by q^n only changes the congruence subgroup, we obtain the statement. $\square$

We next define the Appell-Lerch sum

$$m(x,q,z) := \frac{1}{j(z,q)} \sum_{n\in\mathbb{Z}} \frac{(-z)^n q^{\frac{n(n-1)}{2}}}{1-xzq^{n-1}}.$$

We have the following shifting property in the third argument (see [Zwe02], Propostion 1.4, (7), using a different notation, or [Mor16], Propostion 1.1, (1.4d)):

$$m(x,q,z) - m(x,q,w) = \frac{w(q)_\infty^3 j\left(z/w,q\right) j(xzw,q)}{j(z,q)j(w,q)j(xz,q)j(xw,q)}. \tag{II.3.1}$$

Moreover, we set

$$\widetilde{g}(x,q) := -x \sum_{n\geq 0} \frac{q^{n^2}}{(x)_{n+1}(q/x)_n}.$$

The tail sum for $\widetilde{g}$ is defined as

$$\widetilde{g}_t(x,q) := -x \sum_{n<0} \frac{q^{n^2}}{(x)_{n+1}(q/x)_n}.$$

We want to express $\widetilde{g}_t$ as a sum over positive integers. For this, we use the well-known identity

$$(a)_{-n} = -\frac{q^{\frac{n(n+1)}{2}}}{a^n(q/a)_n}$$

for $a \in \mathbb{C}$ and $n \in \mathbb{N}$, to write

$$\widetilde{g}_t(x,q) = \sum_{n\geq 1} (q/x)_{n-1}(x)_n q^n. \tag{II.3.2}$$

Remark II.15. Note that $R(x,q) = (1-x)(1+xg_3(x,q))$ is Dyson's rank generating function and $U(x,q) = (1-x)^{-1}\widetilde{g}_t(x,q)$ is the generating function for strongly unimodal sequences. These are exactly the functions occurring in [FOR13], Theorem 1.2.

Now we need the following relation between g_3 and $\widetilde{g}$, which can be found in [BFR12], Theorem 3.1.

$$g_3(x,q) = -x^{-1}\left(1 + x^{-1}\widetilde{g}(x,q)\right). \tag{II.3.3}$$

We also require the following identity from the lost notebook, which has already been used in [FOR13] to prove their related Theorem 1.2.

Proposition II.16. *[see [AB09], Entry 3.4.7 or [Mor16], Proposition 1.3] For $a, b \in \mathbb{C}\backslash\{0\}$, we have*

$$\sum_{n=0}^{\infty} \frac{a^{-n-1}b^{-n}}{(-1/a)_{n+1}(-q/b)_n} q^{n^2} + \sum_{n=1}^{\infty}(-aq)_{n-1}(-b)_n q^n = \frac{(-aq)_\infty j(-b,q)}{b(q,-q/b;q)_\infty} m(a/b,q,-b).$$

Plugging in $a = -x^{-1}$ and $b = -x$, we obtain a nice bilateral series identity for $\widetilde{g}$:

$$\widetilde{g}(x,q) + \widetilde{g}_t(x,q) = -\frac{j(x,q)}{x(q)_\infty} m\left(x^{-2},q,x\right). \tag{II.3.4}$$

Remark II.17. Note that (II.3.4) is equivalent to Equation (3) of [Zud15]. Zudilin asked if one can relate the right hand side asymptotically to the Andrews-Garvan crank function $C(x,q)$, in order to obtain a simpler proof of Theorem 1.2. of [FOR13]. However, in our approach, we end up with a modular form different from $C(x,q)$ in Theorem II.5.

We will also need the following functional equation for g_3 (cf. [GM12], eqs. (4.7) and (6.2)):

$$g_3(xq,q) = -x^3 g_3(x,q) - x^2 - x. \tag{II.3.5}$$

Remark II.18. One may also deduce (II.3.4) by applying the heuristic presented in [Mor16] to $\widetilde{g}$ and a functional equation corresponding to (II.3.5).

With the previous results we can express g_3 as a sum of $\widetilde{g}_t$, an Appell-Lerch sum and a Jacobi form.

Lemma II.19. *For every $z \in \mathbb{C}$ with $j(z,q) \neq 0$, we have*

$$g_3(x,q) = -x^{-1} + x^{-2} \sum_{n\geq 1}(q/x)_{n-1}(x)_n q^n + \frac{j(x,q)}{x^3(q)_\infty j(z,q)} \sum_{n\in\mathbb{Z}} \frac{(-z)^n q^{\frac{n(n-1)}{2}}}{1-x^{-2}zq^{n-1}} + \frac{z(q)_\infty j\left(xz^{-1},q\right) j\left(x^{-1}z,\right)}{x^3 j(z,q) j\left(x^{-1},q\right) j\left(x^{-2}z,q\right)}.$$

Proof. Equations (II.3.3) and (II.3.4) imply

$$g_3(x,q) = -x^{-1} + x^{-2}\left(\widetilde{g}_t(x,q) + \frac{j(x,q)}{x(q)_\infty} m\left(x^{-2},q,x\right)\right).$$

Now the statement follows by (II.3.2) and (II.3.1). □

If some summands of g_3 have a pole, then $\widetilde{g}_t$ becomes a finite sum. The idea is to choose an appropriate z, such that the Apell-Lerch sum goes to zero in this case. Now setting $z = x\sqrt{q}$ in Proposition II.19 yields

$$g_3(x,q) = -x^{-1} + x^{-2}\sum_{n\geq 1}(q/x)_{n-1}(x)_n q^n + \frac{j(x,q)}{x^3(q)_\infty j\left(x\sqrt{q},q\right)}\sum_{n\in\mathbb{Z}}\frac{(-x)^n q^{\frac{n^2}{2}}}{1-x^{-1}q^{n-\frac{1}{2}}} + \frac{(q)_\infty^2 j\left(\sqrt{q},q\right)^2}{xj\left(x\sqrt{q},q\right)^2 j(x,q)}. \tag{II.3.6}$$

Since we always have $j\left(\pm x\sqrt{q},q\right)\neq 0$, the choice of the square root $\sqrt{q}$ does not matter.

II.4 Case-by-Case Analysis

II.4.1 Poles in the denominators

First we look at poles of g_3 arising from zeros in the denominators.

Lemma II.20. *The poles in $g_3(\zeta_b^a q^A, q^B)$ arising from zeros in the denominators are exactly at the cusps $\frac{h}{k}\in\mathcal{Q}_{a,b,A,B}$.*

Proof. The statement follows from [BR15], Proposition 3.2, since the summands of the function g_2 examined in [BR15] have exactly the same denominators as the summands of g_3. Moreover, the numerators of the summands of g_3 never vanish. □

Proposition II.21. *For every $\frac{h}{k}\in\mathcal{Q}_{a,b,A,B}$, we have*

$$\lim_{q\to\zeta_h^k}\frac{j\left(\zeta_b^a q^A,q^B\right)}{\left(q^B;q^B\right)_\infty j\left(\zeta_b^a q^{A+\frac{B}{2}},q^B\right)}\sum_{n\in\mathbb{Z}}\frac{(-1)^n\zeta_b^{an}q^{\frac{Bn^2}{2}+An}}{1-\zeta_b^{-a}q^{(B-A)n-\frac{B}{2}}} = 0.$$

The proof is completely analogous to the proof of Lemma 4.2 in [BR15], since $j(\zeta_b^a\zeta_k^{h(A+\frac{B}{2})},\zeta_k^{hB})$ never vanishes if $\frac{h}{k}\in\mathcal{Q}_{a,b,A,B}$.

Proof of Theorem II.5. After using Equation (II.3.6) and Proposition II.21, it remains to show that

$$\left(\zeta_b^{-a}\zeta_k^{h(B-A)};\zeta_k^{hB}\right)_{n-1}\left(\zeta_b^a\zeta_k^{hA};\zeta_k^{hB}\right)_n$$

vanishes for $n > k'$. We have $xq^n = 1$ for $n \in \mathbb{N}$ if and only if

$$\tfrac{ak}{b} + hA + hBn \equiv 0 \pmod{k}.$$

For $\frac{h}{k} \in \mathcal{Q}_{a,b,A,B}$ this equation has a unique solution $\pmod{k'}$. Thus for some $n_0 \in \{1, \dots, k'\}$ we have $xq^{n_0} = 1$ and therefore $(x)_n = 0$ for every $n > n_0$. $\square$

II.4.2 Convergent geometric series

Now we examine the case when the radial limit of g_3 exists as a convergent sum.

Proof of Theorem II.8. We show that the sum $g_3(\zeta_b^a \zeta_k^{hA}, \zeta_k^{hB})$ converges absolutely in the above case by rearranging terms

$$\begin{aligned}
&g_3\left(\zeta_b^a \zeta_k^{hA}, \zeta_k^{hB}\right)\\
&\quad= \sum_{n\geq 1} \frac{\zeta_k^{hBn(n-1)}}{\left(\zeta_b^a\zeta_k^{hA}, \zeta_b^{-a}\zeta_k^{h(B-A)}; \zeta_k^{hB}\right)_n}\\
&\quad= \sum_{m\geq 0}\sum_{j=1}^{k'} \frac{\zeta_k^{hB(mk'+j)(mk'+j-1)}}{\left(\zeta_b^a\zeta_k^{hA}, \zeta_b^{-a}\zeta_k^{h(B-A)}; \zeta_k^{hB}\right)_{mk'+j}}\\
&\quad= \sum_{m\geq 0}\sum_{j=1}^{k'} \frac{\zeta_k^{hBj(j-1)}}{\left(\zeta_b^a\zeta_k^{hA}, \zeta_b^{-a}\zeta_k^{h(B-A)}; \zeta_k^{hB}\right)_{k'}^m \left(\zeta_b^a\zeta_k^{hA}, \zeta_b^{-a}\zeta_k^{h(B-A)}; \zeta_k^{hB}\right)_j}\\
&\quad= \sum_{j=1}^{k'} \frac{\zeta_k^{hBj(j-1)}}{\left(\zeta_b^a\zeta_k^{hA}, \zeta_b^{-a}\zeta_k^{h(B-A)}; \zeta_k^{hB}\right)_j} \sum_{m\geq 0} \frac{1}{\left(\zeta_b^a\zeta_k^{hA}, \zeta_b^{-a}\zeta_k^{h(B-A)}; \zeta_k^{hB}\right)_{k'}^m}.
\end{aligned}$$

The second sum is a geometric series that converges absolutely if and only if

$$\left|\left(\zeta_b^a\zeta_k^{hA}, \zeta_b^{-a}\zeta_k^{h(B-A)}; \zeta_k^{hB}\right)_{k'}\right| > 1.$$

Now consider the expression $(x, q/x; q)_{k'}$ for q a primitive k'-th root of unity. We use the factorization

$$1 - x^{k'} = \prod_{j=0}^{k'-1} \left(1 - xq^j\right)$$

to obtain

$$\begin{aligned}(x, q/x; q)_{k'} &= \prod_{j=0}^{k'-1} \left(1 - xq^j\right)\left(1 - x^{-1}q^{j+1}\right)\\ &= \prod_{j=0}^{k'-1} \left(1 - xq^j\right) \prod_{j=0}^{k'-1} \left(1 - x^{-1}q^j\right)\\ &= \left(1 - x^{k'}\right)\left(1 - x^{-k'}\right) = 2 - 2\,\mathrm{Re}\left(x^{k'}\right).\end{aligned}$$

Setting $x = \zeta_b^a \zeta_k^{hA}$, we get

$$\mathrm{Re}\left(x^{k'}\right) = \cos\left(2\pi k'\left(\tfrac{a}{b} + \tfrac{Ah}{k}\right)\right)$$

and thus

$$\left(\zeta_b^a \zeta_k^{hA}, \zeta_b^{-a}\zeta_k^{h(B-A)}; \zeta_k^{hB}\right)_{k'} = 2 - 2\cos\left(2\pi k'\left(\tfrac{a}{b} + \tfrac{Ah}{k}\right)\right).$$

Therefore the series converges absolutely if and only if

$$\cos\left(2\pi k'\left(\tfrac{a}{b} + \tfrac{Ah}{k}\right)\right) < \tfrac{1}{2},$$

or equivalently

$$\left\{k'\left(\tfrac{a}{b} + \tfrac{Ah}{k}\right)\right\} \in \left(\tfrac{1}{6}, \tfrac{5}{6}\right).$$

By Abel's Theorem, the limit of the series equals the radial limit of g_3 from within the unit disc. □

II.4.3 Shifting by third roots of unity

Proof of Theorem II.9. We use Kang's identity (II.2.1). Note that under the assumptions of Theorem II.9, if $\frac{a}{b} \in \{\frac{1}{3}, \frac{2}{3}\}$, we always have $A \neq 0$ and $B \nmid A$, so that all three summands of (II.2.1) are well-defined for $|x|, |q| < 1$. First we show that $\frac{h}{k} \notin \mathcal{Q}_{3a+b,3b,A,B} \cup \mathcal{Q}_{3a+2b,3b,A,B}$, so that the summands of $g_3(\zeta_3\zeta_b^a\zeta_k^{hA}, \zeta_k^{hB})$ and $g_3(\zeta_3^2\zeta_b^a\zeta_k^{hA}, \zeta_k^{hB})$ have non-vanishing denominators. Here we abuse notation and write $\mathcal{Q}_{a,b,A,B}$ for $\mathcal{Q}_{\frac{a}{(a,b)},\frac{b}{(a,b)},A,B}$ if $(a,b) > 1$. Suppose that

$$(B, k) \;\Big|\; \tfrac{ak}{b} + Ah + \tfrac{\ell k}{3}$$

for some $\ell \in \{1, 2\}$. This means that $k'(\frac{a}{b} + \frac{Ah}{k} + \frac{\ell}{3})$ is an integer. Since k' is not divisible by 3, it follows that $\{k'(\frac{a}{b} + \frac{Ah}{k})\} \in \{\frac{1}{3}, \frac{2}{3}\}$, which contradicts our

assumption.
Since we also have

$$\left\{k'\left(\tfrac{a}{b}+\tfrac{Ah}{k}+\tfrac{1}{3}\right)\right\},\left\{k'\left(\tfrac{a}{b}+\tfrac{Ah}{k}+\tfrac{2}{3}\right)\right\}\in\left(\tfrac{1}{6},\tfrac{5}{6}\right),$$

we see by Theorem II.8 that $g_3(\zeta_3\zeta_b^a\zeta_k^{hA},\zeta_k^{hB})$ and $g_3(\zeta_3^2\zeta_b^a\zeta_k^{hA},\zeta_k^{hB})$ are absolutely convergent sums. Together with (II.2.1) the statement follows. □

II.4.4 Future directions and discussion

Apart from the case where $g_3(x,q)$ diverges and k' is divisible by 3, we are left with the case where $\frac{h}{k}\notin\mathcal{Q}_{a,b,A,B}$ and $\{k'(\frac{a}{b}+\frac{Ah}{k})\}\in\{\frac{1}{6},\frac{5}{6}\}$. Again we set $x=\zeta_b^a\zeta_k^{hA}$ and $q=\zeta_k^{hB}$. Then x is a $6k'$-th root of unity of order $\pm1 \pmod 6$, say $x=\zeta_{6k'}^{\pm1+6\ell}=\zeta_{6k'}^{\pm1}\zeta_{k'}^{\ell}$, $\ell\in\mathbb{Z}$. Notice that since $(k',hB')=1$, we can always find an integer n satisfying $hB'n\equiv-\ell \pmod{k'}$, meaning that

$$q^n=\zeta_{k'}^{hB'n}=\zeta_{k'}^{-\ell}.$$

By iteratively applying (II.3.5), we can reduce this to the cases $x=\zeta_{6k'}^{\pm1}$. Also, by easy computation, we can obtain

$$g_3\left(x^{-1},q\right)=g_3(xq,q)=-x^3g_3(x,q)-x^2-x.$$

Because of this identity, it is enough to consider the case $x=\zeta_{6k'}$. Now we obtain

$$q=\zeta_{k'}^{hB'}=x^{6hB'}.$$

Assume that $hB'\equiv1 \pmod{k'}$, i.e. $q=\zeta_{k'}=x^6$. In this case, we can use the following mock theta "conjecture" (see [GM12], eq. (7.3)):

$$g_3\left(x,x^6\right)=-\frac{1}{2x}+\frac{x}{2}g_3\left(x^3,x^6\right)+\frac{\left(x^2;x^2\right)_\infty^4}{2x(x;x)_\infty^2\left(x^6;x^6\right)_\infty}.$$

First, we can apply Theorem II.8 to $g_3(x^3,x^6)$, because $g_3(x^3,x^6)=g_3(\zeta_{2k'},\zeta_{k'})$ and $k'\cdot\frac{1}{2k'}=\frac{1}{2}\in(\frac{1}{6},\frac{5}{6})$. Thus we have

$$g_3\left(\zeta_{2k'},\zeta_{k'}\right)=\frac{4}{3}\sum_{j=1}^{k'}\frac{\zeta_{k'}^{j(j-1)}}{\left(\zeta_{2k'};\zeta_{k'}\right)_j^2}.$$

Next, we look at the remaining modular term. It can be easily checked that

$$\frac{\left(x^2;x^2\right)_\infty^4}{(x;x)_\infty^2\left(x^6;x^6\right)_\infty}=(-x;x)_\infty^4(x;x)_\infty\left(x,x^2,x^3,x^4,x^5;x^6\right)_\infty,$$

which vanishes for $x = \zeta_{6k'}$.

Combining all of the above, we obtain

$$\lim_{q \to \zeta_k^h} g_3\left(\zeta_b^a q^A, q^B\right) = -\frac{1}{2\zeta_{6k'}} + \frac{2\zeta_{6k'}}{3} \sum_{j=1}^{k'} \frac{\zeta_{k'}^{j(j-1)}}{(\zeta_{2k'};\zeta_{k'})_j^2}$$

if $\frac{h}{k} \notin \mathcal{Q}_{a,b,A,B}$, $k'(\frac{a}{b} + \frac{Ah}{k}) = \frac{1}{6}$, and $hB' \equiv 1 \pmod{k'}$.

If $hB' \not\equiv 1 \pmod{k'}$, one can use the relation [GM12], eq. (6.1) between g_2 and g_3 and the results for g_2 in [BR15] to compute the modular forms to be subtracted and the radial limits of g_3. Unfortunately, this method is quite laborious and not straightforward. It would be interesting to find direct approach to these cases.

Bibliography

[AB09] G. E. Andrews and B. C. Berndt, *Ramanujan's lost notebook. Part II*, Springer, New York, 2009.

[AH91] G. E. Andrews and D. Hickerson, *Ramanujan's "lost" notebook. VII. The sixth order mock theta functions*, Adv. Math. **89** (1991), no.1, 60–105.

[BKLMR] J. Bajpai, S. Kimport, J. Liang, D. Ma, and J. Ricci, *Bilateral series and Ramanujan's radial limits*, Proc. Amer. Math. Soc. **143** (2015), no. 2, 479–492.

[BFR12] K. Bringmann, A. Folsom, and R. C. Rhoades, *partial theta functions and mock modular forms as q-hypergeometric series*, Ramanujan J. **29** (2012), no. 1-3, 295–310.

[BR15] K. Bringmann and L. Rolen, *Radial limits of mock theta functions*, Res. Math. Sci. **2** (2015), Art. 17,18.

[BR16] K. Bringmann and L. Rolen, *Half-integral weight Eichler integrals and quantum modular forms*, J. Number Theory **161** (2016), 240–254.

[CLR16] D. Choi, S. Lim, and R. C. Rhoades, *Mock modular forms and quantum modular forms*, Proc. Amer. Math. Soc. **144** (2016), no.6 2337–2349.

[FOR13] A. Folsom, K. Ono, and R. C. Rhoades, *Mock theta functions and quantum modular forms*, Forum Math. Pi **1** (2013), e2, 27.

[GM12] B. Gordon and R. J. McIntosh, *A survey of classical mock theta functions*, Partitions, q-series, and modular forms, Dev. Math., vol. 23, Springer, New York, 2012, pp. 95–144.

[GOR13] M. Griffin, K. Ono, and L. Rolen, *Ramanujan's mock theta functions*, Proc. Natl. Acad. Sci. USA **110** (2013), no. 15, 5765–5768.

[Kan09] S.-Y. Kang, *Mock Jacobi forms in basic hypergeometric series*, Compos. Math. **145** (2009), no. 3, 553–565.

[Mor16] E. Mortenson. *Ramanujan's radial limits and mixed mock modular bilateral q-hypergeometric series*, Proc. Edinb. Math. Soc., **59** (3), (2016), 787-799.

[Rho13] R. C. Rhoades, *On Ramanujan's definition of mock theta function*, Proc. Natl. Acad. Sci. USA **110** (2013), no. 10, 7592–7594.

[Zag10] D. Zagier, *Quantum modular forms*, Quanta of maths, 2010, pp.659–675.

[Zud15] W. Zudilin, *On Three theorems of Folsom, Ono and Rhoades*, Proc. Amer. Math. Soc. **143** (2015), no. 4, 1471–1476.

[Zwe02] S. Zwegers, *Mock theta functions*, 2002, Ph.D. Thesis, Universiteit Utrecht.

III

On spt-crank-type functions

This chapter is based on a manuscript to be published in *The Ramanujan Journal* and is joint work with Byungchan Kim [34].

III.1 Introduction

Since Andrews [And08] introduced the *spt-function*, which counts the total number of appearance of the smallest part in each integer partition of n, there have been numerous studies on spt function and its variants. For example, see [ACK13, AGL13, FO08, Gar10, Gar11, Rol16] to name a few. In particular, Andrews proves striking congruences:

$$\begin{aligned} \mathrm{spt}(5n+4) &\equiv 0 \pmod{5}, \\ \mathrm{spt}(7n+5) &\equiv 0 \pmod{7}, \\ \mathrm{spt}(13n+6) &\equiv 0 \pmod{13}. \end{aligned}$$

Motivated from Dyson's rank [Dys44] and Andrews and Garvan's crank [AG88] which explain Ramanujan's famous partition congruences, Andrews, Garvan, and Liang [AGL13] introduced an spt-crank which explains the modulo 5 and modulo 7 congruences of the spt function.

More recently, Andrews, Dixit, and Yee [ADY15] introduced a new spt function $\mathrm{spt}_\omega(n)$. The partition function $p_\omega(n)$ is defined to be the number of partitions of n such that all odd parts are smaller than twice the smallest part. A new spt-type function $\mathrm{spt}_\omega(n)$ is defined by the total number of appearances of the smallest part in each partition enumerated by $p_\omega(n)$. Recall that Ramanujan's third order mock theta function is

$$\omega(q) := \sum_{n=0}^{\infty} \frac{q^{2n^2+2n}}{(q;q^2)_{n+1}^2}.$$

The subscript ω is used in p_ω because its generating function is essentially $\omega(q)$ [ADY15, Theorem 3.1] :

$$\sum_{n\geq 1} p_\omega(n)q^n = \sum_{n\geq 1} \frac{q^n}{(1-q^n)\left(q^{n+1};q\right)_n \left(q^{2n+2};q^2\right)_\infty} = q\omega(q).$$

As usual, $(a)_n := (a;q)_n := \prod_{k=1}^n (1-aq^{k-1})$ for $n \in \mathbb{N}_0 \cup \{\infty\}$. In particular, Andrews, Dixit, and Yee proved the congruence

$$\mathrm{spt}_\omega(5n+3) \equiv 0 \pmod 5.$$

On the other hand, motivated by Andrews, Garvan and Liang [AGL13], Garvan and Jennings-Shaffer [GJS16] introduced many spt-like functions with corresponding spt-crank-type functions. Garvan and Jennings-Shaffer first find a proper spt-crank-type function which dissects appropriately, and later define corresponding spt-type functions which have congruences inherited from beautiful dissections of crank functions. To introduce new spt-crank-type functions, they investigated Bailey pairs in Slater's list. In particular, they defined $N_{C_1}(m,n)$ by

$$\sum_{\substack{n\geq 0 \\ m\in\mathbb{Z}}} N_{C_1}(m,n)z^m q^n := \frac{(q;q^2)_\infty (q)_\infty}{(z)_\infty (z^{-1})_\infty} \sum_{n=1}^\infty \frac{q^n (z)_n (z^{-1})_n}{(q;q^2)_n (q)_n},$$

and showed that

$$N_{C_1}(0,5,5n+3) = N_{C_1}(1,5,5n+3) = \cdots = N_{C_1}(4,5,5n+3),$$

where

$$N_{C_1}(j,5,n) := \sum_{\substack{m\in\mathbb{Z} \\ m\equiv j \pmod 5}} N_{C_1}(m,n).$$

This clearly implies that

$$\mathrm{spt}_{C_1}(5n+3) \equiv 0 \pmod 5,$$

where $\mathrm{spt}_{C_1}(n) := \sum_{m\in\mathbb{Z}} N_{C_1}(m,n)$. Actually, the generating function for $\mathrm{spt}_{C_1}(n)$ is identical with that of $\mathrm{spt}_\omega(n)$, and thus we see that $\mathrm{spt}_{C_1}(n) = \mathrm{spt}_\omega(n)$ and $N_{C_1}(m,n)$ can be regarded as a crank function for $\mathrm{spt}_\omega(n)$.

In this paper, we investigate arithmetic properties of $\mathrm{spt}_\omega(n)$ and its crank function $N_{C_1}(m,n)$. The main result is an asymptotic formula for $N_{C_1}(m,n)$ which we derive by using Wright's Circle Method. Typically, Wright's Circle

Method is employed only when the generating function has just one dominant pole. In our case, due to the presence of the factor $\frac{1}{(q^2;q^2)_\infty}$ in the generating function (see Section 2 for details), there are two dominant poles, namely $q = \pm 1$.

Theorem III.1. *As $n \to \infty$,*

$$N_{C_1}(m,n) \sim \frac{\log(2)}{4\pi\sqrt{n}} e^{\frac{\pi\sqrt{n}}{\sqrt{3}}}.$$

The following corollary is an immediate result from Theorem III.1 and confirms Garvan and Jennings-Shaffer's positivity conjecture on $N_{C_1}(m,n)$ asymptotically.

Corollary III.2. *For a fixed integer m,*

$$N_{C_1}(m,n) > 0,$$

for large enough integers n.

Bringmann and the second author [BK16] proved that various unimodal ranks satisfy inequalities of the form $u(m,n) > u(m+1,n)$ for large enough integers n. For the spt-crank $N_{C_1}(m,n)$, the situation is slightly different.

Theorem III.3. *For a fixed nonnegative integer m,*

$$(-1)^{m+n+1}(N_{C_1}(m,n) - N_{C_1}(m+1,n)) > 0,$$

for large enough integers n.

We also prove a congruence of $\mathrm{spt}_\omega(n)$ via the mock modularity of its generating function.

Theorem III.4. *Suppose that $p \geq 5$ is a prime, and j, m and n are positive integers with $\left(\frac{n}{p}\right) = -1$. If m is sufficiently large, then there are infinitely many primes $Q \equiv -1 \pmod{576p^j}$ satisfying*

$$\mathrm{spt}_\omega\left(\frac{Q^3p^mn+1}{12}\right) \equiv 0 \pmod{p^j}.$$

The rest of paper is organized as follows. In Section 2, we derive the generating functions for $N_{C_1}(m,n)$ and its companion $N_{C_5}(m,n)$. In Section 3, we define an auxiliary function and investigate its asymptotic behavior near and away from dominant poles. These estimate will play important roles in Section 4, where we employ Wright's Circle Method to prove Theorem III.1. In Section 5, we prove Theorem III.3. We conclude the paper with the proof of Theorem III.4 in Section 6.

Acknowledgements

This paper will be a part of the first author's PhD thesis. The authors thank Kathrin Bringmann, Michael Woodbury, and the referee for valuable comments on an earlier version of this paper.

III.2 Generating functions and combinatorics

From [GJS16, Proposition 5.1], we know that

$$\begin{aligned} S_{C_1}(z,q) :&= \sum_{\substack{n\geq 0\\ m\in\mathbb{Z}}} N_{C_1}(m,n) z^m q^n \\ &= \frac{1}{(1-z)\left(1-z^{-1}\right)} \left(R\left(z,q^2\right) - \left(q;q^2\right)_\infty C(z,q)\right), \end{aligned}$$

where $R(z,q)$ and $C(z,q)$ are the generating functions for ordinary partition ranks and cranks. From the Lambert series expansion of $C(z,q)$ and $R(z,q)$, we obtain that

$$\begin{aligned} &S_{C_1}(z,q) \\ &= \frac{1}{(q^2;q^2)_\infty} \sum_{n=1}^{\infty} (-1)^{n-1} \left(\frac{q^{n(n+1)/2}\left(1+q^n\right)}{\left(1-zq^n\right)\left(1-q^n/z\right)} - \frac{q^{3n^2+n}\left(1+q^{2n}\right)}{\left(1-zq^{2n}\right)\left(1-q^{2n}/z\right)} \right). \end{aligned}$$

By noting that

$$\frac{1+q^n}{(1-zq^n)(1-q^n/z)} = \frac{1}{1-q^n}\left(\frac{1}{1-zq^n} + \frac{q^n/z}{1-q^n/z}\right),$$

we deduce that

$$\begin{aligned} S_{C_1,m}(q) :&= \sum_{n\geq 0} N_{C_1}(m,n) q^n \\ &= \frac{1}{(q^2;q^2)_\infty} \sum_{n\geq 1} (-1)^{n-1} \left(\frac{q^{n(n+1)/2+|m|n}}{1-q^n} - \frac{q^{3n^2+n+2|m|n}}{1-q^{2n}} \right). \end{aligned} \tag{III.2.1}$$

Garvan and Jennings-Shaffer [GJS16] also conjectured the positivity of $N_{C_5}(m,n)$, which is a crank to explain the congruence $\mathrm{spt}_{C_5}(5n+3) \equiv 0 \pmod 5$. They showed that $\mathrm{spt}_{C_5}(n) = \sum_{m\in\mathbb{Z}} N_{C_5}(m,n) = \mathrm{spt}_\omega(n) - \mathrm{spt}(n/2)$, where $\mathrm{spt}(n/2) = 0$ for odd n. As the shape of the generating function is similar to $N_{C_1}(m,n)$, we also investigate this function. From [GJS16], and after some manipulations, we find that

$$\begin{aligned}
S_{C_5,m} &:= \sum_{n\geq 0} N_{C_5}(m,n)q^n \\
&= \frac{1}{(q^2;q^2)_\infty} \sum_{n\geq 1}(-1)^{n-1}\left(\frac{q^{n(n+1)/2+|m|n}}{1-q^n} - \frac{q^{n^2+n+2|m|n}}{1-q^{2n}}\right).
\end{aligned} \tag{III.2.2}$$

III.3 An auxiliary function

The generating functions in the previous section suggest defining

$$h_{A,B}(q) := \sum_{n\geq 1}(-1)^n \frac{q^{An^2+Bn}}{1-q^n},$$

where $2A \in \mathbb{N}$ and $2B \in \mathbb{Z}$ with $A+B$ is a positive integer. The key step to prove Theorem III.1 is estimating its asymptotic behaviors near dominant poles, namely $q=1$ and $q=-1$, and away from them.

We start with investigating $h_{A,B}(q)$ near $q=1$. Throughout the paper, we set $q = e^{2\pi i z}$ with $z = x+iy$.

Lemma III.5. *For $|x| \leq y$, as $y \to 0^+$,*

$$h_{A,B}(q) = \frac{\log(2)}{2\pi i z} + O(1).$$

Proof. From the Mittag-Leffler partial fraction decomposition ([BM14, eq. (3.1)] with corrected signs), for $w \in \mathbb{C}$ we find that

$$\frac{e^{\pi i w}}{1-e^{2\pi i w}} = \frac{1}{-2\pi i w} + \frac{1}{-2\pi i}\sum_{k\geq 1}(-1)^k\left(\frac{1}{w-k}+\frac{1}{w+k}\right). \tag{III.3.1}$$

Using this, we rewrite $h_{A,B}(q)$ as

$$\begin{aligned}
h_{A,B}(q) &= \sum_{n\geq 1}(-1)^n \frac{q^{An^2+Bn}}{1-q^n} = \sum_{n\geq 1}(-1)^n q^{An^2+\left(B-\frac{1}{2}\right)n}\frac{q^{\frac{n}{2}}}{1-q^n} \\
&= \sum_{n\geq 1}(-1)^n q^{An^2+\left(B-\frac{1}{2}\right)n}\left(\frac{1}{-2\pi i n z} + \frac{1}{-2\pi i}\sum_{k\geq 1}(-1)^k\left(\frac{1}{nz-k}+\frac{1}{nz+k}\right)\right) \\
&= -\frac{1}{2\pi i z}\sum_{n\geq 1}(-1)^n n^{-1} q^{An^2+\left(B-\frac{1}{2}\right)n}
\end{aligned}$$

$$- \frac{1}{2\pi i} \sum_{n\geq 1} (-1)^n q^{An^2+\left(B-\frac{1}{2}\right)n} \sum_{k\geq 1} (-1)^k \left(\frac{1}{nz-k} + \frac{1}{nz+k} \right).$$

We first note that

$$\frac{1}{nz-k} + \frac{1}{nz+k} = \frac{2nz}{n^2z^2-k^2} = 2nz \left(\frac{1}{n^2z^2-k^2} + \frac{1}{k^2} - \frac{1}{k^2} \right),$$

and for $|x| \leq y$

$$\sum_{k\geq 1} \left| \frac{1}{n^2z^2-k^2} + \frac{1}{k^2} \right| = \sum_{k\geq 1} \left| \frac{n^2z^2}{k^2\left(n^2z^2-k^2\right)} \right| \leq \sum_{k\geq 1} \frac{2n^2y^2}{k^4} \leq 3n^2y^2.$$

Therefore, it follows that

$$h_{A,B}(q) = -\frac{1}{2\pi i z} f_{1,2A,2B-1}(z) - \frac{z\pi}{12i} f_{-1,2A,2B-1}(z) \\ - \frac{z}{\pi i} \sum_{n\geq 1} (-1)^n n q^{An^2+\left(B-\frac{1}{2}\right)n} S_n(z),$$

where $|S_n(z)| \leq 3n^2y^2$ and $f_{j,a,b}(z) := \sum_{n=1}^{\infty} (-1)^n n^{-j} q^{\frac{an^2+bn}{2}}$. For nonnegative integers j, the asymptotic behavior of $f_{j,a,b}(z)$ is well-known (See [BK16, KKS14]). By adopting the same technique in [BK16] using Zagier's asymptotic expansion [Zag06], we find that $f_{-1,a,b}(z) = \frac{1}{4} + O(y)$ for $|x| \leq y$ and $y \to 0^+$. Note also that

$$\left| \sum_{n\geq 1} (-1)^n n q^{An^2+\left(B-\frac{1}{2}\right)n} S_n(z) \right| \leq 3y^2 \sum_{n\geq 1} n^3 e^{-2\pi y\left(An^2+\left(B-\frac{1}{2}\right)n\right)} \ll 1.$$

Since $f_{1,a,b}(z) = -\log(2) + O(y)$ for $|x| \leq y$ and $y \to 0^+$, the proof is complete. □

Now we turn to investigating $h_{A,B}(z)$ near $q = -1$.

Lemma III.6. *For* $\left|x - \frac{1}{2}\right| \leq y$ *and a half-integer* B*, as* $y \to 0^+$*,*

$$h_{A,B}(q) = \frac{\log(2)}{4\pi i\tau} + O\left(1\right).$$

Proof. By setting $\tau := z - \frac{1}{2} = x - \frac{1}{2} + iy$ and $Q := e^{2\pi i\tau} = -q$, we derive that

$$h_{A,B}(q) = h_{A,B}(-Q) = \sum_{n\geq 1} (-1)^n \frac{(-Q)^{An^2+Bn}}{1-(-Q)^n} = \sum_{n\geq 1} \frac{(-1)^{(An+B+1)n} Q^{An^2+Bn}}{1-(-1)^nQ^n}$$

$$= \sum_{n\geq 1} \frac{(-1)^n Q^{4An^2+2Bn}}{1-Q^{2n}} + (-1)^{A+B} \sum_{n\geq 1} \frac{(-1)^n Q^{4An^2-(4A-2B)n+A-B}}{1+Q^{2n-1}},$$

provided B is a half-integer. The first sum can be estimated using Lemma III.5, and thus we need only deal with the second sum.

By setting $w = (2n-1)\tau + \frac{1}{2}$ in (III.3.1), we find that

$$\begin{aligned}
\frac{e^{\pi i(2n-1)\tau} i}{1+e^{2\pi i(2n-1)\tau}} &= \frac{1}{-2\pi i} \frac{1}{(2n-1)\tau+1/2} \\
&\quad + \sum_{k=1}^{\infty} \frac{(-1)^k}{-2\pi i} \left(\frac{1}{(2n-1)\tau - k + 1/2} + \frac{1}{(2n-1)\tau + k + 1/2} \right) \\
&= \sum_{k=1}^{\infty} \frac{(-1)^k}{-2\pi i} \left(\frac{1}{(2n-1)\tau - k + 1/2} - \frac{1}{(2n-1)\tau + k - 1/2} \right) \\
&= \sum_{k=1}^{\infty} \frac{(-1)^k}{-\pi i} \left(\frac{1}{(4n-2)\tau - (2k-1)} - \frac{1}{(4n-2)\tau + 2k - 1} \right) \\
&= \sum_{k=1}^{\infty} \frac{(-1)^k}{-\pi i} \frac{4k-2}{(4n-2)^2\tau^2 - (2k-1)^2}.
\end{aligned}$$

Applying this, we have

$$\begin{aligned}
&\sum_{n\geq 1} \frac{(-1)^n Q^{4An^2-(4A-2B)n+A-B}}{1+Q^{2n-1}} \\
&= \sum_{n\geq 1} (-1)^n Q^{4An^2-(4A-2B+1)n+A-B+\frac{1}{2}} \frac{Q^{n-\frac{1}{2}}}{1+Q^{2n-1}} \\
&= \frac{1}{\pi} \sum_{n\geq 1} (-1)^n Q^{4An^2-(4A-2B+1)n+A-B+\frac{1}{2}} \sum_{k=1}^{\infty} (-1)^k \frac{4k-2}{(4n-2)^2\tau^2 - (2k-1)^2}.
\end{aligned}$$

By decomposing

$$\begin{aligned}
&\frac{1}{(4n-2)^2\tau^2 - (2k-1)^2} \\
&\qquad = \left(\frac{1}{(4n-2)^2\tau^2 - (2k-1)^2} + \frac{1}{(2k-1)^2} - \frac{1}{(2k-1)^2} \right)
\end{aligned}$$

and noting that

$$\left|\frac{1}{(4n-2)^2\tau^2-(2k-1)^2}+\frac{1}{(2k-1)^2}\right|$$
$$=\left|\frac{(4n-2)^2\tau^2}{(2k-1)^2\left((4n-2)^2\tau^2-(2k-1)^2\right)}\right|\leq\frac{2(4n-2)^2y^2}{(2k-1)^4},$$

we conclude that the whole sum is bounded. □

Since the term $\frac{1}{(q^2;q^2)_\infty}$ in (III.2.2) is exponentially small away from the dominant poles, we do not need a sharp bound for $h_{A,B}(z)$ in this region.

Lemma III.7. *For $y>0$ with $y\leq|x|\leq\frac{1}{2}-y$,*

$$|h_{A,B}(q)|\ll y^{-\frac{3}{2}}.$$

Proof. A direct calculation reveals that

$$|h_{A,B}(q)|\leq\frac{1}{1-|q|}\sum_{n\geq1}|q|^{An^2+Bn}\ll\frac{1}{y}y^{-\frac{1}{2}}.$$

□

III.4 Proof of Theorem III.1

In this section, we use Wright's Circle Method to complete the proof of Theorem III.1. Before beginning the proof, we first investigate $\frac{1}{(q^2;q^2)_\infty}$ near and away from $q=\pm1$. Recall that, from the modular inversion formula for Dedekind's eta-function ([Kob84, p. 121, Proposition 14]),

$$(q;q)_\infty=\frac{1}{\sqrt{-iz}}e^{-\frac{\pi iz}{12}-\frac{\pi i}{12z}}\left(1+O\left(e^{-\frac{2\pi i}{z}}\right)\right).\tag{III.4.1}$$

Therefore, we find that

$$\begin{aligned}\frac{1}{(q^2;q^2)_\infty}&=\sqrt{-2iz}\;e^{\frac{\pi i}{24z}}+O\left(y^{\frac{3}{2}}e^{\frac{\pi}{24}\mathrm{Im}\left(\frac{-1}{z}\right)}\right)\qquad\text{for } |x|<y,\\ \frac{1}{(Q^2;Q^2)_\infty}&=\sqrt{-2i\tau}\;e^{\frac{\pi i}{24\tau}}+O\left(y^{\frac{3}{2}}e^{\frac{\pi}{24}\mathrm{Im}\left(\frac{-1}{\tau}\right)}\right)\qquad\text{for } \left|x-\frac{1}{2}\right|<y.\end{aligned}\tag{III.4.2}$$

Now we consider the behavior away from the dominant poles, i.e., in a range of $y\leq|x|\leq\frac{1}{2}-y$. Note that

$$\mathrm{Log}\left(\frac{1}{(q^2;q^2)_\infty}\right)=-\sum_{n=1}^{\infty}\mathrm{Log}\left(1-q^{2n}\right)=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{q^{2nm}}{m}=\sum_{m=1}^{\infty}\frac{q^{2m}}{m\left(1-q^{2m}\right)}.$$

Thus,

$$\begin{aligned}\left|\operatorname{Log}\left(\frac{1}{(q^2;q^2)_\infty}\right)\right| &\leq \sum_{m=1}^{\infty}\frac{|q|^{2m}}{m\,|1-q^{2m}|}\\ &\leq \sum_{m=1}^{\infty}\frac{|q|^{2m}}{m\,(1-|q|^{2m})}+\frac{|q|^2}{|1-q^2|}-\frac{|q|^2}{1-|q|^2}\\ &= \operatorname{Log}\left(\frac{1}{(|q|^2;|q|^2)_\infty}\right)-|q|^2\left(\frac{1}{1-|q|^2}-\frac{1}{|1-q^2|}\right).\end{aligned} \tag{III.4.3}$$

Plugging $z\mapsto 2iy$ into (IV.4.3), we find that

$$\operatorname{Log}\left(\frac{1}{(|q|^2;|q|^2)_\infty}\right)=\frac{\pi}{24y}+\frac{1}{2}\operatorname{Log}(2y)+O(y).$$

To estimate the other term in (III.4.3), first note that if $y\leq|x|\leq\frac{1}{4}$, then $\cos(4\pi y)\geq\cos(4\pi x)$. On the other hand, if $\frac{1}{4}\leq|x|\leq\frac{1}{2}-y$, then $\cos(4\pi x)\leq\cos(2\pi-4\pi y)=\cos(4\pi y)$. Thus, for all $y\leq|x|\leq\frac{1}{2}-y$, $\cos(4\pi x)\leq\cos(4\pi y)$. Therefore,

$$\left|1-q^2\right|^2=1-2e^{-4\pi y}\cos(4\pi x)+e^{-8\pi y}\geq 1-2e^{-4\pi y}\cos(4\pi y)+e^{-8\pi y}.$$

From the Taylor expansion, we conclude that $|1-q^2|\leq 4\sqrt{2}\pi y+O\left(y^2\right)$. Since $1-|q|^2=1-e^{-4\pi y}=4\pi y+O\left(y^2\right)$, we arrive at

$$\left|\frac{1}{(q^2;q^2)_\infty}\right|\ll\sqrt{2y}\exp\left[\frac{1}{y}\left(\frac{\pi}{24}-\frac{1}{4\pi}\left(1-\frac{1}{\sqrt{2}}\right)\right)\right]. \tag{III.4.4}$$

Now we are ready to prove Theorem III.1.

Proof of Theorem III.1. First, rewrite (III.2.1) in terms of $h_{A,B}(q)$ as follows:

$$S_{C_1,m}(q)=\sum_{n\geq 0}N_{C_1}(m,n)q^n=\frac{-1}{(q^2;q^2)_\infty}\left(h_{\frac{1}{2},\frac{1+2|m|}{2}}(q)-h_{\frac{3}{2},\frac{1+|m|}{2}}\left(q^2\right)\right).$$

By Cauchy's Theorem, we see for $y=\frac{1}{4\sqrt{3n}}$ that

$$\begin{aligned}N_{C_1}(m,n)&=\frac{1}{2\pi i}\int_{\mathcal{C}}\frac{S_{C_1,m}(q)}{q^{n+1}}dq=\int_{-\frac{1}{2}}^{\frac{1}{2}}S_{C_1,m}\left(e^{2\pi ix-\frac{\pi}{2\sqrt{3n}}}\right)e^{-2\pi inx+\frac{\pi\sqrt{n}}{2\sqrt{3}}}dx\\ &=\int_{|x|\leq y}S_{C_1,m}\left(e^{2\pi ix-\frac{\pi}{2\sqrt{3n}}}\right)e^{-2\pi inx+\frac{\pi\sqrt{n}}{2\sqrt{3}}}dx\end{aligned}$$

$$+\int_{y\le|x|\le\frac{1}{2}-y} S_{C_1,m}\left(e^{2\pi ix-\frac{\pi}{2\sqrt{3n}}}\right)e^{-2\pi inx+\frac{\pi\sqrt{n}}{2\sqrt{3}}}dx$$
$$+\int_{\left|x-\frac{1}{2}\right|\le y} S_{C_1,m}\left(e^{2\pi ix-\frac{\pi}{2\sqrt{3n}}}\right)e^{-2\pi inx+\frac{\pi\sqrt{n}}{2\sqrt{3}}}dx$$
$$=: \mathcal{I}_1+\mathcal{I}_2+\mathcal{I}_3,$$

where $\mathcal{C}=\{|q|=e^{-\frac{\pi}{2\sqrt{3n}}}\}$. In this case, the integral $\mathcal{I}_1$ contributes the main term and the integrals $\mathcal{I}_2$ and $\mathcal{I}_3$ are absorbed in the error term.

To evaluate $\mathcal{I}_1$ we first introduce a function $P_s(u)$ which is defined by Wright [Wri34]. For fixed $M>0$ and $u\in\mathbb{R}^+$

$$P_s(u):=\frac{1}{2\pi i}\int_{1-Mi}^{1+Mi} v^s e^{u\left(v+\frac{1}{v}\right)}dv.$$

This functions is rewritten in terms of the I-Bessel function up to an error term.

Lemma III.8 ([Wri34])**.** *As $n\to\infty$*

$$P_s(u)=I_{-s-1}(2u)+O\left(e^u\right),$$

where I_ℓ denotes the usual the I-Bessel function of order ℓ.

From Lemma III.5 and (III.4.2), we find that for $|x|\le\frac{1}{4\sqrt{3n}}=y$ as $n\to\infty$

$$S_{C_1,m}(q)=\frac{e^{\frac{\pi i}{24z}}\log(2)}{2\pi\sqrt{-2iz}}+O\left(n^{-\frac{1}{4}}e^{\frac{\pi}{24}\mathrm{Im}\left(-\frac{1}{z}\right)}\right).$$

Thus the integral $\mathcal{I}_1$ becomes

$$\int_{|x|\le\frac{1}{4\sqrt{3n}}}\left(\frac{e^{\frac{\pi i}{24z}}\log(2)}{2\pi\sqrt{-2iz}}+O\left(n^{-\frac{1}{4}}e^{\frac{\pi}{24}\mathrm{Im}\left(-\frac{1}{z}\right)}\right)\right)e^{-2\pi inx+\frac{\pi\sqrt{n}}{2\sqrt{3}}}dx.$$

By making the change of variables $v=1-i4\sqrt{3n}x$, we arrive at

$$\int_{1-i}^{1+i}\frac{1}{i4\sqrt{3n}}\left(\frac{e^{\frac{\pi\sqrt{n}}{2\sqrt{3}v}}(3n)^{\frac{1}{4}}\log(2)}{\pi\sqrt{2v}}+O\left(n^{-\frac{1}{4}}e^{\frac{\pi\sqrt{n}v}{2\sqrt{3}}}\right)\right)e^{\frac{\pi\sqrt{n}v}{2\sqrt{3}}}dv$$
$$=\frac{(3n)^{-\frac{1}{4}}\log(2)}{2\sqrt{2}}P_{-\frac{1}{2}}\left(\frac{\pi\sqrt{n}}{2\sqrt{3}}\right)+O\left(n^{-\frac{3}{4}}e^{\frac{\pi\sqrt{n}}{\sqrt{3}}}\right)$$
$$=\frac{(3n)^{-\frac{1}{4}}\log(2)}{2\sqrt{2}}I_{-\frac{3}{2}}\left(\frac{\pi\sqrt{n}}{\sqrt{3}}\right)+O\left(n^{-\frac{3}{4}}e^{\frac{\pi\sqrt{n}}{\sqrt{3}}}\right)$$

$$= \frac{\log(2)}{4\pi} n^{-\frac{1}{2}} e^{\frac{\pi\sqrt{n}}{\sqrt{3}}} + O\left(n^{-\frac{3}{4}} e^{\frac{\pi\sqrt{n}}{\sqrt{3}}}\right),$$

where we use the asymptotic formula for the I-Bessel function [AAR01, 4.12.7]

$$I_\ell(x) = \frac{e^x}{\sqrt{2\pi x}} + O\left(\frac{e^x}{x^{\frac{3}{2}}}\right).$$

Now we consider the integral $\mathcal{I}_2$. From the Lemma III.7 and (IV.4.6), for $\frac{1}{4\sqrt{3n}} \le |x| \le \frac{1}{2} - \frac{1}{4\sqrt{3n}}$, we have

$$S_{C_1,m}(q) \ll n^{\frac{1}{2}} \exp\left(\frac{\pi\sqrt{n}}{2\sqrt{3}} - \frac{\sqrt{3n}}{\pi}\left(1 - \frac{1}{\sqrt{2}}\right)\right),$$

as $n \to \infty$. Hence, we have

$$\mathcal{I}_2 \ll n^{\frac{1}{2}} \exp\left(\frac{\pi\sqrt{n}}{\sqrt{3}} - \frac{\sqrt{3n}}{\pi}\left(1 - \frac{1}{\sqrt{2}}\right)\right).$$

Finally, to estimate $\mathcal{I}_3$ first we shift $x \mapsto \widetilde{x} + \frac{1}{2}$ (thus $\tau = \widetilde{x} + i\frac{\pi}{4\sqrt{3n}}$), and rewrite $\mathcal{I}_3$ as follows:

$$\begin{aligned}\mathcal{I}_3 &= \int_{\left|x-\frac{1}{2}\right| \le y} S_{C_1,m}\left(e^{2\pi i z}\right) e^{-2\pi i n x + \frac{\pi\sqrt{n}}{2\sqrt{3}}} dx \\ &= (-1)^n \int_{|\widetilde{x}| \le y} S_{C_1,m}\left(e^{2\pi i \tau}\right) e^{-2\pi i n \widetilde{x} + \frac{\pi\sqrt{n}}{2\sqrt{3}}} d\widetilde{x}.\end{aligned}$$

As before, from Lemma III.6 and (III.4.2), we find that for $\left|x - \frac{1}{2}\right| = |\widetilde{x}| \le \frac{1}{4\sqrt{3n}}$ as $n \to \infty$

$$S_{C_1,m}\left(e^{2\pi i \tau}\right) = O\left(y^{\frac{1}{2}} e^{\frac{\pi}{24}\mathrm{Im}\left(\frac{-1}{\tau}\right)}\right).$$

Thus, we arrive at

$$\mathcal{I}_3 \ll \int_{|\widetilde{x}| \le \frac{1}{4\sqrt{3n}}} n^{-\frac{1}{4}} e^{\frac{\pi\sqrt{n}}{\sqrt{3}}} d\widetilde{x} \ll n^{-\frac{3}{4}} e^{\frac{\pi\sqrt{n}}{\sqrt{3}}},$$

which finishes the proof of Theorem III.1. □

By noting that

$$S_{C_5}(q) = \sum_{n \ge 0} N_{C_5}(m,n) q^n = \frac{-1}{(q^2;q^2)_\infty}\left(h_{\frac{1}{2},\frac{1+2|m|}{2}}(q) - h_{\frac{1}{2},\frac{1+2|m|}{2}}(q^2)\right),$$

we can deduce the following asymptotic formula by proceeding in the same way as before.

Proposition III.9. *As* $n \to \infty$,

$$N_{C_5}(m,n) \sim \frac{\log(2)}{4\pi\sqrt{n}} e^{\frac{\pi\sqrt{n}}{\sqrt{3}}}.$$

III.5 Proof of Theorem III.3

In this section, we study crank differences and prove their sign pattern. From (III.2.1), one easily sees that for a nonnegative integer m

$$\begin{aligned} SD_{C_1,m}(q) &:= S_{C_1,m}(q) - S_{C_1,m+1}(q) \\ &= \frac{1}{(q^2;q^2)_\infty} \sum_{n\geq 1} (-1)^{n-1} \left(q^{\frac{n(n+1)}{2}+mn} - q^{n(3n+1)+2mn} \right). \end{aligned}$$

In [KKS15, Theorem 1.1], the asymptotic behavior of $f_{0,a,b}$ is effectively given. It implies that for $|x| < y$ as $y \to 0^+$,

$$f_{0,a,b}(z) = -\frac{1}{2} + \frac{b}{8}(-2\pi i z) + O(y^2).$$

From the above, we easily see that for $|x| < y$ as $y \to 0^+$,

$$SD_{C_1,m}(q) = -\frac{(1+2m)\pi\sqrt{2i}}{4} z^{\frac{3}{2}} e^{\frac{\pi i}{24 z}} + O\left(y^{\frac{5}{2}} e^{\frac{\pi}{24}\mathrm{Im}\left(\frac{-1}{z}\right)} \right).$$

For asymptotic behavior near $q = -1$, we set $\tau = z - \frac{1}{2} = x - \frac{1}{2} + iy$ and $Q = e^{2\pi i \tau} = -q$ as before. Then, we obtain that

$$\begin{aligned} &SD_{C_1,m}(q) \\ &= \frac{1}{(Q^2;Q^2)_\infty} \left(-f_{0,4,2(2m+1)}(\tau) + (-1)^m Q^{-m} f_{0,4,2(2m-1)}(\tau) + f_{0,6,2(2m+1)}(\tau) \right) \\ &= \left(\sqrt{-2i\tau} e^{\frac{\pi i}{24\tau}} + O\left(y^{\frac{3}{2}} e^{\frac{\pi}{24}\mathrm{Im}\left(\frac{-1}{\tau}\right)} \right) \right) \left(-\frac{(-1)^m}{2} + \frac{(-1)^m}{2}\pi i \tau + O\left(y^2\right) \right) \\ &= \frac{(-1)^{m+1}}{2} \sqrt{-2i\tau} e^{\frac{\pi i}{24\tau}} + O\left(y^{\frac{3}{2}} e^{\frac{\pi}{24}\mathrm{Im}\left(\frac{-1}{\tau}\right)} \right), \end{aligned} \tag{III.5.1}$$

as $y \to 0^+$ for $|x - \frac{1}{2}| \leq y$. The rest of proof is almost identical to that of Theorem III.1 except that $\mathcal{I}_3$ contributes the main term this time. Since the other estimates are similar, we only give a brief explanation for the main term. In this case, $\mathcal{I}_3$ becomes

$$\mathcal{I}_3 = \int_{\left|x-\frac{1}{2}\right|\leq y} SD_{C_1,m}\left(e^{2\pi i z}\right) e^{-2\pi i n x + \frac{\pi\sqrt{3n}}{6}} dx$$

$$= (-1)^n \int_{|\widetilde{x}|\leq y} SD_{C_1,m}\left(e^{2\pi i\tau}\right) e^{-2\pi i n\widetilde{x}+\frac{\pi\sqrt{3n}}{6}} d\widetilde{x}.$$

As before, by setting $v = 1 - i4\sqrt{3n}\widetilde{x}$ and the equation (III.5.1), we find that

$$\mathcal{I}_3 \sim \frac{(-1)^{n+m+1}\pi(3n)^{-\frac{3}{4}}}{4\sqrt{2}} P_{\frac{1}{2}}\left(\frac{\pi\sqrt{n}}{2\sqrt{3}}\right) \sim \frac{(-1)^{n+m+1}}{8\sqrt{3}n} e^{\frac{\pi\sqrt{n}}{\sqrt{3}}},$$

which completes the proof of Theorem III.3.

Again, we can also think about the differences for $N_{C_5}(m,n)$. From (III.2.2) it follows that

$$\begin{aligned} SD_{C_5,m} :&= S_{C_5,m}(q) - S_{C_5,m+1}(q) \\ &= \frac{1}{(q^2;q^2)_\infty} \sum_{n\geq 1} (-1)^{n-1}\left(q^{\frac{n(n+1)}{2}+|m|n} - q^{n(n+1)+2|m|n}\right). \end{aligned}$$

By proceeding the same way, we can also conclude that for a fixed nonnegative integer m,

$$(-1)^{n+m+1}(N_{C_5}(m,n) - N_{C_5}(m+1,n)) > 0$$

holds for large enough integers n.

III.6 Modularity and congruences

In this section, we show that the generating function for $\mathrm{spt}_\omega(n)$ is a part of the holomorphic part of a certain weight $\frac{3}{2}$ harmonic weak Maass form to prove the Theorem III.4.

We first note that from [ADY15, Lemma 6.1]

$$\begin{aligned} S_\omega(z) :&= \sum_{n\geq 1} \mathrm{spt}_\omega(n) q^n = \sum_{n\geq 1} \frac{q^n}{(1-q^n)^2 (q^{n+1};q)_n (q^{2n+2};q^2)_\infty} \\ &= \frac{1}{(q^2;q^2)_\infty} \sum_{n\geq 1} \frac{nq^n}{1-q^n} + \frac{1}{(q^2;q^2)_\infty} \sum_{n\geq 1} \frac{(-1)^n (1+q^{2n}) q^{n(3n+1)}}{(1-q^{2n})^2}. \end{aligned}$$

We also note that

$$E_2(z) := 1 - 24\sum_{n=1}^{\infty} \sigma_1(n) q^n = 1 - 24 \sum_{n\geq 1} \frac{nq^n}{1-q^n},$$

as usual the Eisenstein series with $\sigma_1(n) := \sum_{d|n} d$, and

$$R_2(z) := \sum_{n\geq 0} \frac{1}{2} N_2(n) q^n$$

$$= \frac{-1}{(q;q)_\infty} \sum_{n\geq 1} \frac{(-1)^n(1+q^n)q^{\frac{n(3n+1)}{2}}}{(1-q^n)^2}, \qquad \text{[And08, eq. (3.4)]},$$

with $N_j(n) := \sum_{m\in\mathbb{Z}} m^j N(m,n)$ the moments of the rank. By using these, we rewrite $S_\omega(z)$ as follows:

$$S_\omega(z) = \frac{q^{\frac{1}{12}}(1-E_2(z))}{24\eta(2z)} - R_2(2z), \tag{III.6.1}$$

where $\eta(z) := q^{\frac{1}{24}} \prod_{n=1}^\infty (1-q^n)$ is Dedekind's η-function. To state more define

$$\mathcal{N}(z) := \frac{i}{4\sqrt{2}\pi} \int_{-\bar{z}}^{i\infty} \frac{\eta(24z)}{(-i(\tau+z))^{\frac{3}{2}}} d\tau,$$
$$\mathcal{M}(z) := q^{-1}S_\omega(12z) - \frac{E_2(24z)}{24\eta(24z)} - \mathcal{N}(z).$$

Lemma III.10. *The function $\mathcal{M}(z)$ is a harmonic weak Maass form of weight $\frac{3}{2}$ on $\Gamma_0(576)$ with Nebentypus character $\chi_{12}(\cdot) := \left(\frac{12}{\cdot}\right)$.*

It is an immediate result from [Bri08, Theorem 1.1] and the basic properties of $E_2(z)$ and $\eta(z)$. Here $\mathcal{N}(z)$ is the non-holomorphic part and supported on finitely many square classes. Hence, the holomorphic part $q^{-1}S_\omega(12z) - \frac{E_2(24z)}{24\eta(24z)}$ is a mock modular form of weight $\frac{3}{2}$ and becomes an weakly holomorphic modular form with appropriate arithmetic progressions which make $\mathcal{N}(z)$ vanish. Theorem III.4 follows from [Tre06, Theorem 1.1] together with the fact that $E_2(z)$ is a p-adic modular form for any prime p.

Bibliography

[And08] G. E. Andrews, *The number of smallest parts in the partitions of n*, J. reine Angew. Math., **624** (2008), 133–142.

[AG88] G. E. Andrews and F. G. Garvan, *Dyson's crank of a partition*, Bull. Amer. Math. Soc. (N.S.) **18** (1988), 167–171.

[AAR01] G. E. Andrews, R. Askey, and R. Roy, Special Functions, Encyclopedia of Mathematics and its Applications, Cambridge University Press, 2001.

[ACK13] G. E. Andrews, S.H. Chan, B. Kim, *The odd moments of ranks and cranks*, J. Combin. Theory Ser. A **120** (2013), no. 1, 77–91.

[ADY15] G. E. Andrews, A. Dixit, and A. J. Yee, *Partitions associated with the Ramanujan/Watson mock theta functions $\omega(q)$, $\nu(q)$ and $\phi(q)$*, Research in Number Theory **1** Issue 1 (2015), 1–25.

[AGL13] G. E. Andrews, F. G. Garvan, and J. Liang, *Combinatorial interpretations of congruences for the spt-function*, Ramanujan J. **29** (2013), 321–338.

[Bri08] K. Bringmann, *On the explicit construction of higher deformations of partition statistics*, Duke Math. **144** (2008), 195–233.

[BK16] K. Bringmann and B. Kim, *On the asymptotic behavior of unimodal rank generating functions*, J. Math. Anal. Appl. **435** (2016), 627–645.

[BM14] K. Bringmann and K. Mahlburg, *Asymptotic inequalities for positive crank and rank moments*, Trans. Amer. Math. Soc. **366** (2014), 1073–1094.

[Dys44] F. Dyson, *Some guesses in the theory of partitions*, Eureka (Cambridge) **8** (1944), 10–15.

[FO08] A. Folsom and K. Ono, *The spt-function of Andrews*, Proc. Natl. Acad. Sci. USA **105** (2008), no. 51, 20152–20156.

[Gar10] F.G. Garvan, *Congruences for Andrews' smallest parts partition function and new congruences for Dyson's rank*, Int. J. Number Theory **6** (2010), no. 2, 281–309.

[Gar11] F. G. Garvan, *Higher order spt-functions*, Adv. Math. **228** (2011), no. 1, 241–265.

[GJS16] F. Garvan and C. Jennings-Shaffer, *Exotic Bailey-Slater SPT-function II: Hecke-Rogers-type double sums and Bailey pairs from group A, C, E*, Adv. Math. **299** (2016), 605–639.

[KKS14] B. Kim, E. Kim, and J. Seo, *On the number of even and odd strings along overpartitions of* n, Arch. Math. (Basel) **102** (2014), 357–368.

[KKS15] B. Kim, E. Kim, and J. Seo, *Asymptotics for* q*-expansions involving partial theta functions*, Discrete Math. **338** (2015), 180–189.

[Kob84] N. Koblitz, Introdunction to elliptic curves and modular forms, Graduate Texts in Mathematics **97**, Springer-Verlag, NewYork, 1984.

[Rol16] J. M. Z. Rolon, *Asymptotics of higher order spt-functions for overpartitions*, Ann. Comb. **20** (2016), no. 1, 177–191.

[Tre06] S. Treneer, *Congruences for the coefficients of weakly holomorphic modular forms*, Proc. London Math. Soc. **93** (2006), 304–324.

[Wri34] E. Wright, *Asymptotic partition formulae II. Weighted partitions*, Prod. London Math. Soc. (2) **36** (1934), 117–141.

[Zag06] D. Zagier, *The Mellin transform and other useful analytic techniques*, Appendix to E. Zeidler, Quantum Field Theory I: Basics in Mathematics and Physics. A Bridge Between Mathematicians and Physicists Springer-Verlag, Berlin-Heidelberg-New York (2006), 305–323

IV

Asymptotic behavior of Odd-Even partitions

This chapter is based on a manuscript to be published in *The Electronic Journal of Combinatorics* [33].

IV.1 Introduction and Statement of results

Andrews [And84] considered a certain family of functions and noticed a mysterious phenomenon. More precisely, Andrews investigated q-series identities involving hypergeometric functions, for example in particular ([And84], [And86, eq. (4.10) and (4.12)] and [And88, p. 19 and 104])

$$1+\sum_{n=1}^{\infty}\frac{q^n}{(1-q)(1-q^2)\cdots(1-q^n)}=\prod_{n=1}^{\infty}\frac{1}{(1-q^n)},$$

$$1+\sum_{n=1}^{\infty}\frac{q^{\frac{n(n+1)}{2}}}{(1-q)(1-q^2)\cdots(1-q^n)}=\prod_{n=1}^{\infty}(1+q^n),$$

$$1+\sum_{n=1}^{\infty}\frac{q^{n^2}}{(1-q)(1-q^2)\cdots(1-q^n)}=\prod_{n=1}^{\infty}\frac{1}{(1-q^{5n-4})(1-q^{5n-1})},$$

$$1+\sum_{n=1}^{\infty}\frac{q^n}{(1-q^2)(1-q^4)\cdots(1-q^{2n})}=\prod_{n=1}^{\infty}\frac{1}{(1-q^{2n-1})},$$

$$1+\sum_{n=1}^{\infty}\frac{q^{n^2}}{(1-q^2)(1-q^4)\cdots(1-q^{2n})}=\prod_{n=1}^{\infty}\left(1+q^{2n-1}\right),$$

$$1+\sum_{n=1}^{\infty}\frac{q^{\frac{n(n+1)}{2}}}{(1-q^2)(1-q^4)\cdots(1-q^{2n})}=?, \tag{IV.1.1}$$

$$1+\sum_{n=1}^{\infty}\frac{q^{n^2}}{(1-q^4)(1-q^8)\cdots(1-q^{4n})}=\prod_{n=1}^{\infty}\frac{1}{(1+q^{2n})(1-q^{5n-4})(1-q^{5n-1})}, \tag{IV.1.2}$$

$$1+\sum_{n=1}^{\infty}\frac{q^{n(n+2)}}{(1-q^4)(1-q^8)\cdots(1-q^{4n})}=\prod_{n=1}^{\infty}\frac{1}{(1+q^{2n})(1-q^{5n-3})(1-q^{5n-2})}. \tag{IV.1.3}$$

While the others can be nicely written in terms of infinite product (so that it turns out that they are modular forms up to q powers), Andrews did not find any such shape of identities for (IV.1.1). Moreover, Zagier [Zag07, Table 1] determined that (IV.1.1) is not modular. Nonetheless, Andrews [And84] provided a combinatorial interpretation for this function, namely *odd-even partitions*.

Recall that a *partition* of positive integer n is a nowhere increasing sequence of positive integers whose sum is n. Define a partition function $OE(n)$ by the number of partitions of n in which the parts alternate in parity starting with the smallest part odd. In other words, $OE(n)$ counts the number of *odd-even partitions of n*. For instance, there are no odd-even partitions of 2 and the odd-even partitions of 3 are 3 and 2+1. Therefore $OE(2)=0$ and $OE(3)=2$. By Andrews' proof a generating function (in Eulerian form) for the odd-even partitions is given by

$$\mathscr{O}(q):=1+\sum_{n=1}^{\infty}OE(n)q^n=\sum_{m=0}^{\infty}\frac{q^{\frac{m(m+1)}{2}}}{\left(q^2;q^2\right)_m}, \tag{IV.1.4}$$

which is (IV.1.1). Here the q-Pochhammer symbol or q-shifted factorial is defined as $(a)_n:=(a;q)_n:=\prod_{j=1}^{n}(1-aq^{j-1})$ for $n\in\mathbb{N}_0\cup\{\infty\}$.

In this paper we investigate the asymptotic behavior of $OE(n)$. In order to study the asymptotic behavior of the coefficients of a series, one can either use the Circle Method [HR18, Vau81, Wri34] or apply Ingham's Tauberian Theorem [Ing41]. Since $\mathscr{O}(q)$ has a pole at every root of unity and it is not easy to find the bounds for $\mathscr{O}(q)$ at every root of unity, it is difficult to use the Circle Method in our case. Moreover, as $OE(n)$ is not monotonically increasing, we cannot directly apply Ingham's Tauberian Theorem to our case either (see Section 2 for more details). Thus, we need to slightly modify our function so that we can apply Ingham's Tauberian Theorem.

Theorem IV.1. *We have*

$$OE(n)\sim\frac{1}{2\sqrt{5}n^{\frac{3}{4}}}e^{\pi\sqrt{\frac{n}{5}}}$$

as $n\to\infty$.

We also investigate the asymptotics of *odd-even overpartitions*, studied by Lovejoy [Lov08]. Recall that an *overpartition* of positive interger n is a partition of n in which the first occurrence (equivalently, the final occurrence) of a number may be overlined. An *odd-even overpartition* is an overpartition with the smallest part odd and such that the difference between successive parts is odd if the smaller is nonoverlined and even otherwise. For example, there are no odd-even overpartitions of 2, the odd-even overpartitions of 3 are $\overline{3}$, 3, $\overline{2}+1$, and $2+1$, and the odd-even partitions of 4 are $\overline{3}+\overline{1}$ and $3+\overline{1}$. Notice that if all parts are non-overlined, then we have the odd-even partitions. We denote $\overline{OE}(n)$ by the number of odd-even overpartitions of n and define $\overline{OE}(0) := 1$. The generating function is given in [Lov08]

$$\overline{\mathscr{O}}(q) := \sum_{n=0}^{\infty} \overline{OE}(n)q^n = \sum_{m=0}^{\infty} \frac{(-1)_m q^{\frac{m(m+1)}{2}}}{(q^2;q^2)_m} = (-q)_\infty f(q),$$

where

$$f(q) := \sum_{n=0}^{\infty} \frac{q^{n^2}}{(-q)_n^2} \tag{IV.1.5}$$

is one of Ramanujan's third order mock theta functions. These functions appeared in Ramanujan's deathbed letter to Hardy and are now known as the holomorphic parts of weight $\frac{1}{2}$ *harmonic Maass forms* (see [Zwe02]). We remark that the generating function for the odd-even overpartitions is a *mixed mock modular form*, i.e., the product of a modular form and a mock theta function. From this fact, we can apply Wright's Circle Method [Wri34] to obtain the asymptotic formula for $\overline{OE}(n)$.

Theorem IV.2. *We have*

$$\overline{OE}(n) \sim \frac{1}{3^{\frac{5}{4}} n^{\frac{3}{4}}} e^{\pi\sqrt{\frac{n}{3}}}$$

as $n \to \infty$.

This paper is organized as follows. In Section 2 we study some basic properties of odd-even partitions and introduce an auxiliary theorem which play important roles to prove Theorem IV.1. The proof is given in Section 3. We conclude the paper with the proof of Theorem IV.2 in Section 4.

Acknowledgement

These results are part of the author's PhD thesis supervised by Kathrin Bringmann. The author thanks her for suggesting this problem and valuable advice, Don Zagier and Byungchan Kim for insightful comments and for providing the numerical results to the main theorems, and Jeremy Lovejoy for offering the idea to consider odd-even overpartitions which expanded the scope of this paper. The author also thanks Steffen Löbrich and Michael Woodbury for their support and fruitful conversation regarding this topic, and Jaebum Sohn and the referee for a careful reading of this paper and many helpful comments.

IV.2 Preliminaries

IV.2.1 Basic properties of odd-even partitions

First we look into the first few values of the odd-even partition function $OE(n)$:

n	relevant partitions of n	$OE(n)$
1	1	1
2	—	0
3	3, 1+2	2
4	—	0
5	5, 1+4	2
6	1+2+3	1
7	7, 1+6, 3+4	3
8	1+2+5	1
$\vdots$	$\vdots$	$\vdots$

From these values, we see that $OE(n)$ is not monotonically increasing. Nevertheless, $OE(n) \leq OE(n+2)$ holds for every n due to the fact that we can always make an odd-even partition of $n+2$ from the one of n by adding 2 to the largest part. Thus, $OE(n)$ is monotonically increasing for even (odd resp.) n. This suggests that the appropriate approach to understand the asymptotic behavior of $OE(n)$ is to split the power series of $OE(n)$ into two parts, one with even n and the other with odd n, as follows:

$$\begin{aligned}\mathscr{O}(q) = \sum_{n=0}^{\infty} OE(n)q^n &= \sum_{n=0}^{\infty} OE(2n)q^{2n} + \sum_{n=0}^{\infty} OE(2n+1)q^{2n+1} \\ &=: \mathscr{O}_e(q) + \mathscr{O}_o(q).\end{aligned}$$

Here, for convenience we define $OE(0) := 1$. We further split the q hypergeometric series in (IV.1.4) accordingly by considering the parity of powers of q for each summand. Since the q-Pochhammer symbol $(q^2; q^2)_m$ in the denominator always produces even powers of q, the parity of powers of q depends only on

$\frac{m(m+1)}{2}$. Note that $\frac{m(m+1)}{2}$ is even iff $m \equiv 0, 3 \pmod 4$ and odd iff $m \equiv 1, 2 \pmod 4$. Hence

$$\mathscr{O}_e(q) = \sum_{\substack{m \geq 0 \\ m \equiv 0,3 \pmod 4}} \frac{q^{\frac{m(m+1)}{2}}}{(q^2; q^2)_m}, \qquad \mathscr{O}_o(q) = \sum_{\substack{m \geq 0 \\ m \equiv 1,2 \pmod 4}} \frac{q^{\frac{m(m+1)}{2}}}{(q^2; q^2)_m}.$$

IV.2.2 Ingham's Tauberian Theorem

From the asymptotic behavior of a power series, Ingham's Tauberian Theorem [Ing41] gives an asymptotic formula for its coefficients.

Theorem (Ingham [Ing41])**.** *Let $f(q) = \sum_{n\geq 0} a(n)q^n$ be a power series with weakly increasing nonnegative coefficients and radius of convergence equal to 1. If there are constants $A > 0$, $\lambda, \alpha \in \mathbb{R}$ such that*

$$f\left(e^{-\varepsilon}\right) \sim \lambda \varepsilon^{\alpha} e^{\frac{A}{\varepsilon}}$$

as $\varepsilon \to 0^+$, then

$$a(n) \sim \frac{\lambda}{2\sqrt{\pi}} \frac{A^{\frac{\alpha}{2}+\frac{1}{4}}}{n^{\frac{\alpha}{2}+\frac{3}{4}}} e^{2\sqrt{An}}$$

as $n \to \infty$.

IV.3 Proof of Theorem IV.1

IV.3.1 Asymptotics for the generating functions

In this section we investigate the asymptotic behavior of the functions $\mathscr{O}_e(q)$ and $\mathscr{O}_o(q)$. Throughout the section we set $q = e^{-t}$. In order to get the asymptotic formulas for these functions, we exploit the second proof of [Zag07, Proposition 5]. The idea of the proof is based on the asymptotics of the individual terms in the series. We first study the asymptotic behavior of the summand and then sum up the asymptotics. We denote the m-th term in the series (IV.1.4) by

$$f_m = f_m(q) := \frac{q^{\frac{m(m+1)}{2}}}{(q^2; q^2)_m}.$$

The sequence $(f_m)_{m\in\mathbb{N}}$ is unimodal, meaning that f_m increases until f_m reaches a maximum value and then decreases. More precisely, for $0 < |q| < 1$ the ratio

$$\frac{f_m}{f_{m-1}} = \frac{q^m}{1 - q^{2m}} \tag{IV.3.1}$$

goes to ∞ as $m \to 0$, decreases as m grows, and tends to 0 as $m \to \infty$. To determine when f_m takes the maximum value, we check when the ratio (IV.3.1) becomes 1. This ratio is equal to 1 exactly for q^{2m} the unique root of the equation $Q^{\frac{1}{2}} + Q = 1$ in the interval $(0, 1)$, namely $Q := \frac{3-\sqrt{5}}{2}$. In other words, f_m approaches the maximum value when q^{2m} is close to Q and m near $\frac{\mathrm{Log}(Q)}{2\,\mathrm{Log}(q)}$. We further note that

$$\frac{\mathrm{Log}(Q)}{2\,\mathrm{Log}(q)} \to \infty, \quad q^{2m} \to Q \qquad \text{as} \quad q \to 1^-.$$

Thus, the main contribution occurs when the terms are of the form $q^{2m} = Qq^{-2\nu}$ (or $q^m = Q^{\frac{1}{2}} q^{-\nu}$) with $\nu \in \nu_0 + \mathbb{Z}$ satisfying $\nu = o(m)$ and ν_0 denotes the fractional part of $\frac{\mathrm{Log}(Q)}{2\,\mathrm{Log}(q)}$. In this setting, we evaluate the size of f_m. For this, we use the asymptotic expansion from Zagier [Zag07, p. 53]. Here the *dilogarithm function* $\mathrm{Li}_2(z)$ is defined, for $|z| < 1$, by

$$\mathrm{Li}_2(z) := \sum_{n=1}^{\infty} \frac{z^n}{n^2}.$$

Lemma IV.3. *Let $A, B \in \mathbb{R}$ and $A > 0$. For the unique root $R \in (0, 1)$ of the equation $R + R^A = 1$ and $q = e^{-t}$ with $q^n = Rq^{-\nu}$, $\nu = o(n)$ as $n \to \infty$, we have*

$$\begin{aligned}
&\mathrm{Log}\left(\frac{q^{\frac{1}{2}An^2+Bn}}{(q)_n}\right) \\
&= \left(\frac{\pi^2}{6} - \mathrm{Li}_2(R) - \frac{1}{2}\mathrm{Log}(R)\,\mathrm{Log}(1-R)\right) t^{-1} \\
&\quad - \frac{1}{2}\mathrm{Log}\left(\frac{2\pi}{t}\right) + \mathrm{Log}\left(\frac{R^B}{\sqrt{1-R}}\right) \\
&\quad - \left(\frac{A+R-AR}{2(1-R)}\nu^2 - \left(B + \frac{R}{2(1-R)}\right)\nu + \frac{1+R}{24(1-R)}\right) t + O\left(t^2\right),
\end{aligned}$$

as $t \to 0^+$.

Remark IV.4. In fact, Zagier obtained the asymptotic expansion with arbitrary many terms. Since we only use the first few terms in this paper, we do not need to consider the complete expansion.

We set $q \mapsto q^2$, $A \mapsto \frac{1}{2}$, and $B \mapsto \frac{1}{4}$ in Lemma IV.3. Thus, R becomes Q and we have, recalling that $Q^{\frac{1}{2}} + Q = 1$ and $Q = \frac{3-\sqrt{5}}{2}$,

$$\begin{aligned}\mathrm{Log}\left(\frac{q^{\frac{m(m+1)}{2}}}{(q^2;q^2)_m}\right) &= \left(\frac{\pi^2}{6} - \mathrm{Li}_2(Q) - \left(\frac{1}{2}\mathrm{Log}(Q)\right)^2\right)\frac{1}{2t} \\ &\quad - \frac{1}{2}\mathrm{Log}\left(\frac{\pi}{t}\right) - \frac{\sqrt{5}}{2}\left(\nu^2 - \nu + \frac{1}{6}\right)t + O\left(t^2\right). \quad \text{(IV.3.2)}\end{aligned}$$

Furthermore, we use the special value of the dilogarithm from [Zag07, Section I.1]

$$\mathrm{Li}_2(Q) = \frac{\pi^2}{15} - \left(\mathrm{Log}\left(\frac{1+\sqrt{5}}{2}\right)\right)^2 \qquad \text{(IV.3.3)}$$

and note that

$$\left(\frac{1}{2}\mathrm{Log}\left(Q\right)\right)^2 = \left(\mathrm{Log}(1-Q)\right)^2 = \left(\mathrm{Log}\left((1-Q)^{-1}\right)\right)^2 = \left(\mathrm{Log}\left(\frac{1+\sqrt{5}}{2}\right)\right)^2. \qquad \text{(IV.3.4)}$$

Combining (IV.3.2), (IV.3.3), and (IV.3.4) gives

$$\begin{aligned}\mathrm{Log}\left(\frac{q^{\frac{m(m+1)}{2}}}{(q^2;q^2)_m}\right) &= \frac{\pi^2}{20t} - \frac{1}{2}\mathrm{Log}\left(\frac{\pi}{t}\right) - \frac{\sqrt{5}}{2}\left(\nu^2 - \nu + \frac{1}{6}\right)t + O\left(t^2\right) \\ &= \mathrm{Log}\left(\varphi(\nu)\right) + O\left(t^2\right),\end{aligned} \qquad \text{(IV.3.5)}$$

where

$$\varphi(\nu) := \sqrt{\frac{t}{\pi}}\exp\left[\frac{\pi^2}{20t} - \frac{\sqrt{5}}{2}\left(\nu^2 - \nu + \frac{1}{6}\right)t\right].$$

We additionally define for $j \in \{0,1,2,3\}$

$$\mathcal{S}_j := \sum_{m \equiv j \pmod 4} \frac{q^{\frac{m(m+1)}{2}}}{(q^2;q^2)_m}, \qquad \text{(IV.3.6)}$$

so that we can write

$$\mathscr{O}_e(q) = \mathcal{S}_0 + \mathcal{S}_3, \qquad \mathscr{O}_o(q) = \mathcal{S}_1 + \mathcal{S}_2. \qquad \text{(IV.3.7)}$$

Theorem IV.5. *We have*

$$\mathscr{O}_e\left(e^{-t}\right) \sim \mathscr{O}_o\left(e^{-t}\right) \sim \frac{1}{\sqrt{2\sqrt{5}}}e^{\frac{\pi^2}{20t}}$$

as $t \to 0^+$.

Proof. Using (IV.3.5), we can also rewrite $\mathcal{S}_j$ in terms of $\varphi(\nu)$ as

$$\mathcal{S}_j = \left(1 + O\left(t^2\right)\right) \sum_{\nu \equiv \nu_0 + j \pmod 4} \varphi(\nu). \tag{IV.3.8}$$

To estimate $\mathcal{S}_j$, we begin by rewriting the sum in ν on the right-hand side of (IV.3.8) as

$$\begin{aligned}\sum_{n\in\mathbb{Z}} \varphi(4n + \nu_0 + j) &= \sum_{n \in \frac{1}{2}+\mathbb{Z}} \varphi(4n+\alpha) \\ &= \sqrt{\frac{t}{\pi}} e^{\frac{\pi^2}{20t}} \sum_{n\in\frac{1}{2}+\mathbb{Z}} e^{-\frac{\sqrt{5}}{2}\left((4n+\alpha)^2-(4n+\alpha)+\frac{1}{6}\right)t} \\ &= \sqrt{\frac{t}{\pi}} e^{\frac{\pi^2}{20t}-\frac{\sqrt{5}}{2}\left(\alpha^2-\alpha+\frac{1}{6}\right)t} \vartheta\left(\frac{\sqrt{5}\left(2\alpha-1\right)ti}{\pi} - \frac{1}{2}; \frac{8\sqrt{5}ti}{\pi}\right),\end{aligned} \tag{IV.3.9}$$

where $\alpha := 2 + \nu_0 + j$ and the Jacobi Theta function is given for $z \in \mathbb{C}$ and $\tau \in \mathbb{H}$ by

$$\vartheta\left(z;\tau\right) := \sum_{n\in\frac{1}{2}+\mathbb{Z}} e^{\pi i n^2\tau + 2\pi i n\left(z+\frac{1}{2}\right)}.$$

The modular inversion formula for the Jacobi theta function [Zwe02, Proposition 1.3 (7)] implies that for $a, b \in \mathbb{C}$ with $\mathrm{Re}(a) > 0$

$$\begin{aligned}\vartheta\left(\frac{bti}{\pi} - \frac{1}{2}; \frac{ati}{\pi}\right) &= i\sqrt{\frac{\pi}{at}} e^{-\frac{\pi^2}{at}\left(\frac{bti}{\pi}-\frac{1}{2}\right)^2} \vartheta\left(\frac{b}{a} + \frac{\pi i}{2at}; \frac{\pi i}{at}\right) \\ &= \sqrt{\frac{\pi}{at}} \sum_{n\in\mathbb{Z}} (-1)^n e^{-\frac{\pi^2}{at}\left(n-\frac{bti}{\pi}\right)^2}.\end{aligned}$$

Plugging in $a \mapsto 8\sqrt{5}$ and $b \mapsto \sqrt{5}(2\alpha - 1)$ and simplifying the summation yields that

$$\begin{aligned}&\vartheta\left(\frac{\sqrt{5}\left(2\alpha-1\right)ti}{\pi} - \frac{1}{2}; \frac{8\sqrt{5}ti}{\pi}\right) = \sqrt{\frac{\pi}{8\sqrt{5}t}} \sum_{n\in\mathbb{Z}} (-1)^n e^{-\frac{\pi^2}{8\sqrt{5}t}\left(n-\frac{\sqrt{5}(2\alpha-1)ti}{\pi}\right)^2} \\ &= \sqrt{\frac{\pi}{8\sqrt{5}t}} e^{\frac{\sqrt{5}(2\alpha-1)^2t}{32}} \left(1 + O\left(\sum_{n\in\mathbb{Z}\setminus\{0\}} e^{-\frac{\pi^2 n^2}{8\sqrt{5}t}}\right)\right) = \sqrt{\frac{\pi}{8\sqrt{5}t}} \left(1 + O\left(t\right)\right).\end{aligned} \tag{IV.3.10}$$

The last equality comes directly from the fact that, as $t \to 0^+$,

$$e^{\frac{\sqrt{5}(2\alpha-1)^2 t}{32}} = 1 + O(t),$$

and

$$\sum_{n\in\mathbb{Z}\setminus\{0\}} e^{-\frac{\pi^2 n^2}{8\sqrt{5}t}} \ll e^{-\frac{\pi^2}{8\sqrt{5}t}}.$$

From (IV.3.8), (IV.3.9), and (IV.3.10), we have for any $j \in \{0, 1, 2, 3\}$

$$\mathcal{S}_j \sim \frac{1}{2\sqrt{2\sqrt{5}}} e^{\frac{\pi^2}{20t} - \frac{\sqrt{5}}{2}\left(\alpha^2-\alpha+\frac{1}{6}\right)t} \sim \frac{1}{2\sqrt{2\sqrt{5}}} e^{\frac{\pi^2}{20t}}$$

as $t \to 0^+$. Recalling (IV.3.7), we obtain the desired result.

□

Moreover, since $\mathscr{O}(q) = \mathscr{O}_e(q) + \mathscr{O}_o(q)$, we have following Corollary.

Corollary IV.6. *We have*

$$\mathscr{O}\left(e^{-t}\right) \sim \sqrt{\frac{2}{\sqrt{5}}} e^{\frac{\pi^2}{20t}}$$

as $t \to 0^+$.

Remark IV.7. One can directly estimate the series $\mathscr{O}(q)$ by using the Constant Term Method, inserting an additional variable to identify the series as the constant term of the product of more familiar number-theoretic functions in a new variable. (See [Zag07, First proof of Proposition 5] for more details.)

Remark IV.8. McIntosh [McI95] derived the complete asymptotic expansion of the more general q-series

$$\sum_{n=0}^{\infty} \frac{a^n q^{bn^2+cn}}{(q)_n}$$

in full detail using elementary methods (Euler-Maclaurin sum formula). The asymptotic of $\mathscr{O}(q)$ is the case $c = \frac{1}{4}$ and $q \mapsto q^2$ (hence $t \mapsto 2t$) in the following formula [McI95, p. 134]

$$\begin{aligned}&\operatorname{Log}\left(\sum_{n=0}^{\infty} \frac{q^{\frac{n^2}{4}+cn}}{(q)_n}\right)\\ &=\frac{\pi^2}{10t} + 2c \operatorname{Log}\left(\frac{\sqrt{5}-1}{2}\right) - \frac{1}{2}\operatorname{Log}\left(\frac{5-\sqrt{5}}{4}\right) + \left(\frac{4c-1}{40} + \frac{c(2c-1)}{10}\sqrt{5}\right) t\end{aligned}$$

$$- c(2c-1)\left(\frac{1}{25}+\frac{4c-1}{150}\sqrt{5}\right)t^2 + c(2c-1)\left(\frac{4c-1}{250}+\frac{2c^2-c+3}{750}\sqrt{5}\right)t^3$$
$$- c(2c-1)\left(\frac{2c^2-c+13}{3750}-\frac{(4c-1)\left(12c^2-6c-31\right)}{45000}\sqrt{5}\right)t^4 + O\left(t^5\right),$$

as $t \to 0^+$.

IV.3.2 Applying Ingham's Tauberian Theorem

Now we are ready to apply Ingham's Tauberian Theorem to the functions $\mathscr{O}_e\left(e^{-t}\right)$ and $\mathscr{O}_o\left(e^{-t}\right)$. We first deal with the even case. Setting $a(n) := OE(2n)$ and replacing q by q^2 in Theorem IV.2.2 determines the constants

$$\lambda = \frac{1}{\sqrt{2\sqrt{5}}}, \qquad \alpha = 0, \qquad A = \frac{\pi^2}{10}.$$

We remark that since $OE(n)$ does not satisfy weakly increasing property with $n = 0$, we only consider when $n \geq 1$. Thus, we have

$$OE(2n) \sim \frac{1}{2\sqrt{5}(2n)^{\frac{3}{4}}}e^{2\pi\sqrt{\frac{n}{10}}}.$$

By letting $n \mapsto \frac{n}{2}$, we obtain the desired asymptotic formula for $OE(n)$ with even n, namely

$$OE(n) \sim \frac{1}{2\sqrt{5}n^{\frac{3}{4}}}e^{\pi\sqrt{\frac{n}{5}}}. \tag{IV.3.11}$$

For odd n, we rewrite the series as

$$\mathscr{O}_o(q) = \sum_{n=0}^{\infty} OE(2n+1)q^{2n+1} = q\sum_{n=0}^{\infty} OE(2n+1)q^{2n}.$$

Since by Theorem IV.5

$$\mathscr{O}_o\left(e^{-t}\right) = e^{-t}\sum_{n=0}^{\infty} OE(2n+1)e^{-2tn} \sim \frac{1}{\sqrt{2\sqrt{5}}}e^{\frac{\pi^2}{20t}},$$

we have

$$\sum_{n=0}^{\infty} OE(2n+1)e^{-2tn} \sim \frac{1}{\sqrt{2\sqrt{5}}}e^{\frac{\pi^2}{20t}}.$$

Similar to the case of even n, setting $a(n) := OE(2n+1)$ and replacing q by q^2 yields

$$OE(2n+1) \sim \frac{1}{2\sqrt{5}(2n)^{\frac{3}{4}}} e^{2\pi\sqrt{\frac{n}{10}}}.$$

As before we let $n \mapsto \frac{n}{2}$ and thus we have for even n

$$OE(n+1) \sim \frac{1}{2\sqrt{5}n^{\frac{3}{4}}} e^{\pi\sqrt{\frac{n}{5}}}. \tag{IV.3.12}$$

Finally from (IV.3.11) and (IV.3.12) we get the desired asymptotic formula for $OE(n)$, for every n,

$$OE(n) \sim \frac{1}{2\sqrt{5}n^{\frac{3}{4}}} e^{\pi\sqrt{\frac{n}{5}}}$$

as $n \to \infty$.

IV.4 Proof of Theorem IV.2

We follow the same method of the proof of Theorem 4 in [BDLM15]. The strategy is to estimate the generating function near and away from a dominant pole, and then apply Wright's Circle Method. Although the method of proof is not new, because we are dealing with a different function, the result does not follow directly from the statement of Theorem 4 in [BDLM15], and thus we include its proof here. However, the proof is basically the same. Throughout this section we let $q = e^{2\pi i\tau}$ with $\tau = x + iy$, where $x, y \in \mathbb{R}$.

IV.4.1 Asymptotics of $\overline{\mathscr{O}}(q)$

Using the Watson's identity for Ramanujan's third order mock theta function $f(q)$ [Wat36]

$$f(q) = \frac{2}{(q)_\infty} \sum_{n\in\mathbb{Z}} \frac{(-1)^n q^{\frac{n(3n+1)}{2}}}{1+q^n},$$

we rewrite $\overline{\mathscr{O}}(q)$ as

$$\overline{\mathscr{O}}(q) = \frac{2(-q)_\infty}{(q)_\infty} \sum_{n\in\mathbb{Z}} \frac{(-1)^n q^{\frac{n(3n+1)}{2}}}{1+q^n}. \tag{IV.4.1}$$

From this expression we can see that $\overline{\mathscr{O}}(q)$ has a dominant pole at $q = 1$.

Theorem IV.9. *Let $M > 0$ be fixed.*

(i) *For $|x| \le My$, as $y \to 0^+$*

$$\overline{\mathscr{O}}(q) = \frac{2\sqrt{2}}{3} e^{\frac{\pi i}{24\tau}} + O\left(y e^{\frac{\pi}{24} \operatorname{Im}\left(\frac{-1}{\tau}\right)} \right).$$

(ii) *For $My < |x| \le 1/2$, as $y \to 0^+$*

$$\overline{\mathscr{O}}(q) \ll \frac{1}{y\sqrt{2}} \exp\left[\frac{1}{y} \left(\frac{\pi}{8} - \frac{1}{\pi} \left(1 - \frac{1}{\sqrt{1+M^2}} \right) \right) \right].$$

Remark IV.10. One can find $M > \sqrt{\left(\frac{12}{12-\pi^2}\right)^2 - 1} = 5.543\ldots$, so that the bound in the part (ii) is indeed an error term.

Proof. (i) To estimate the function $\overline{\mathscr{O}}(q)$ near $q = 1$, we first examine $f(q)$. By Taylor's Theorem, we have

$$f(q) = f(1) + O(|\tau|),$$

and from (IV.1.5) we see that

$$f(1) = \sum_{n=0}^{\infty} \frac{1}{4^n} = \frac{4}{3}.$$

Thus, we have for $|x| \le My$

$$f(q) = \frac{4}{3} + O(y) \tag{IV.4.2}$$

as $y \to 0^+$.

Now we turn to the infinite product $(-q)_\infty$ in front of $f(q)$. Recall that, from the modular inversion formula for Dedekind's eta-function ([Kob84, p. 121, Proposition 14]),

$$(q)_\infty = \frac{1}{\sqrt{-i\tau}} e^{-\frac{\pi i \tau}{12} - \frac{\pi i}{12\tau}} \left(1 + O\left(e^{-\frac{2\pi i}{\tau}} \right) \right). \tag{IV.4.3}$$

Therefore, we find that

$$(-q)_\infty = \frac{(q^2;q^2)_\infty}{(q)_\infty} = \frac{1}{\sqrt{2}} e^{\frac{\pi i}{24\tau}} + O\left(y e^{\frac{\pi}{24} \operatorname{Im}\left(\frac{-1}{\tau}\right)} \right). \tag{IV.4.4}$$

Combining (IV.4.2) and (IV.4.4) gives the proof of the part (i).

(ii) In order to examine $\overline{\mathscr{O}}(q)$ away from $q = 1$, we first consider the expression in (IV.4.1). Note that

$$\sum_{n\in\mathbb{Z}} \frac{(-1)^n q^{\frac{n(3n+1)}{2}}}{1+q^n} = \frac{1}{2} + 2\sum_{n\geq 1} \frac{(-1)^n q^{\frac{n(3n+1)}{2}}}{1+q^n}$$

and that, for $My < |x| \leq \frac{1}{2}$,

$$\left|\sum_{n\geq 1} \frac{(-1)^n q^{\frac{n(3n+1)}{2}}}{1+q^n}\right| \leq \frac{1}{1-|q|}\sum_{n\geq 1} |q|^{\frac{n(3n+1)}{2}} \ll \frac{1}{y}\cdot y^{-\frac{1}{2}} = y^{-\frac{3}{2}}.$$

This implies

$$\left|\sum_{n\in\mathbb{Z}} \frac{(-1)^n q^{\frac{n(3n+1)}{2}}}{1+q^n}\right| \ll y^{-\frac{3}{2}}. \tag{IV.4.5}$$

Now it remains to bound the infinity product

$$\frac{(-q)_\infty}{(q)_\infty} = \frac{(q^2;q^2)_\infty}{(q)_\infty^2}.$$

We write this as

$$\begin{aligned}
&\mathrm{Log}\left(\frac{(q^2;q^2)_\infty}{(q)_\infty^2}\right) \\
&\quad= \sum_{n\geq 1}\big(\mathrm{Log}\left(1-q^{2n}\right) - 2\,\mathrm{Log}\left(1-q^n\right)\big) = \sum_{n\geq 1}\sum_{m\geq 1}\frac{2q^{nm}}{m} - \sum_{n\geq 1}\sum_{m\geq 1}\frac{q^{2nm}}{m} \\
&\quad= \sum_{m\geq 1}\left(\frac{2q^m}{m\left(1-q^m\right)} - \frac{q^{2m}}{m\left(1-q^{2m}\right)}\right) = \sum_{m\geq 1}\frac{2q^{2m-1}}{(2m-1)\left(1-q^{2m-1}\right)}.
\end{aligned}$$

Thus,

$$\begin{aligned}
\left|\mathrm{Log}\left(\frac{(q^2;q^2)_\infty}{(q)_\infty^2}\right)\right| &\leq \sum_{m\geq 1}\frac{2|q|^{2m-1}}{(2m-1)\left|1-q^{2m-1}\right|} \\
&\leq \sum_{m\geq 1}\frac{2|q|^{2m-1}}{(2m-1)\left(1-|q|^{2m-1}\right)} + \frac{2|q|}{|1-q|} - \frac{2|q|}{1-|q|} \\
&= \mathrm{Log}\left(\frac{(|q|^2;|q|^2)_\infty}{(|q|)_\infty^2}\right) - 2|q|\left(\frac{1}{1-|q|} - \frac{1}{|1-q|}\right).
\end{aligned}$$

From (IV.4.3), we have

$$\frac{(|q|^2;|q|^2)_\infty}{(|q|)_\infty^2} = \sqrt{\frac{y}{2}} e^{\frac{\pi}{8y}} \left(1 + O\left(e^{-\frac{\pi}{y}}\right)\right).$$

To evaluate the remaining term, we note that for $My < |x| \leq \frac{1}{2}$, $\cos(\pi My) > \cos(\pi x)$. Therefore,

$$|1-q|^2 = 1 - 2e^{-2\pi y}\cos(2\pi x) + e^{-4\pi y} > 1 - 2e^{-2\pi y}\cos(2\pi My) + e^{-4\pi y}.$$

Using the Taylor expansion around $y = 0$, we conclude that

$$|1-q| > 2\pi y\sqrt{1+M^2} + O\left(y^2\right).$$

Since $1 - |q| = 2\pi y + O\left(y^2\right)$, we arrive at

$$\left|\frac{(q^2;q^2)_\infty}{(q)_\infty^2}\right| \ll \sqrt{\frac{y}{2}} \exp\left[\frac{1}{y}\left(\frac{\pi}{8} - \frac{1}{\pi}\left(1 - \frac{1}{\sqrt{1+M^2}}\right)\right)\right]. \tag{IV.4.6}$$

Plugging (IV.4.5) and (IV.4.6) into (IV.4.1) yields the part (ii). □

Theorem IV.9 (ii) and Remark IV.10 then immediately reveal the following.

Corollary IV.11. *For $My < |x| \leq \frac{1}{2}$ with $M > \sqrt{\left(\frac{12}{12-\pi^2}\right)^2 - 1}$, there exists $\varepsilon > 0$ such that, as $y \to 0^+$,*

$$\overline{\mathscr{O}}(q) \ll \frac{1}{y\sqrt{2}} e^{\frac{\pi}{24}\left(\mathrm{Im}\left(\frac{-1}{\tau}\right) - \varepsilon\right)}.$$

IV.4.2 Wright's Circle Method

In this section we complete the proof of Theorem IV.2 by applying Wright's Circle Method. By Cauchy's Theorem, we see, for $y = \frac{1}{4\sqrt{3n}}$, that

$$\begin{aligned}
\overline{OE}(n) &= \frac{1}{2\pi i}\int_{\mathcal{C}} \frac{\overline{\mathscr{O}}(q)}{q^{n+1}} dq = \int_{-\frac{1}{2}}^{\frac{1}{2}} \overline{\mathscr{O}}\left(e^{2\pi i x - \frac{\pi}{2\sqrt{3n}}}\right) e^{-2\pi i n x + \frac{\pi\sqrt{n}}{2\sqrt{3}}} dx \\
&= \int_{|x|\leq My} \overline{\mathscr{O}}\left(e^{2\pi i x - \frac{\pi}{2\sqrt{3n}}}\right) e^{-2\pi i n x + \frac{\pi\sqrt{n}}{2\sqrt{3}}} dx \\
&\quad + \int_{My<|x|\leq\frac{1}{2}} \overline{\mathscr{O}}\left(e^{2\pi i x - \frac{\pi}{2\sqrt{3n}}}\right) e^{-2\pi i n x + \frac{\pi\sqrt{n}}{2\sqrt{3}}} dx =: \mathcal{I}_1 + \mathcal{I}_2,
\end{aligned}$$

where $\mathcal{C} = \{|q| = e^{-\frac{\pi}{2\sqrt{3n}}}\}$. In fact, the integral $\mathcal{I}_1$ contributes the main term as the integral $\mathcal{I}_2$ is an error term.

In order to evaluate $\mathcal{I}_1$, we introduce a function $P_s(u)$, defined by Wright [Wri34], for fixed $M > 0$ and $u \in \mathbb{R}^+$

$$P_s(u) := \frac{1}{2\pi i} \int_{1-Mi}^{1+Mi} v^s e^{u\left(v+\frac{1}{v}\right)} dv.$$

This function is rewritten in terms of the I-Bessel function up to an error term.

Lemma IV.12 ([Wri34]). *As $n \to \infty$, we have*

$$P_s(u) = I_{-s-1}(2u) + O\left(e^u\right),$$

where I_ℓ denotes the usual the I-Bessel function of order ℓ.

Using Theorem IV.9 (i), we write the integral $\mathcal{I}_1$ as

$$\mathcal{I}_1 = \int_{|x| \le \frac{M}{4\sqrt{3n}}} \left(\frac{2\sqrt{2}}{3} e^{\frac{\pi i}{24\tau}} + O\left(n^{-\frac{1}{2}} e^{\frac{\pi\sqrt{n}}{2\sqrt{3}}}\right)\right) e^{-2\pi i n x + \frac{\pi\sqrt{n}}{2\sqrt{3}}} dx.$$

By making the change of variables $v = 1 - i4\sqrt{3n}x$, we arrive at

$$\begin{aligned}
\mathcal{I}_1 &= \int_{1-Mi}^{1+Mi} \frac{-i}{4\sqrt{3n}} \left(\frac{2\sqrt{2}}{3} e^{\frac{\pi\sqrt{n}}{2\sqrt{3}v}} + O\left(n^{-\frac{1}{2}} e^{\frac{\pi\sqrt{n}}{2\sqrt{3}}}\right)\right) e^{\frac{\pi\sqrt{n}v}{2\sqrt{3}}} dv \\
&= \frac{\pi\sqrt{2}}{3\sqrt{3n}} P_0\left(\frac{\pi\sqrt{n}}{2\sqrt{3}}\right) + O\left(n^{-\frac{3}{2}} e^{\frac{\pi\sqrt{n}}{\sqrt{3}}}\right) \\
&= \frac{\pi\sqrt{2}}{3\sqrt{3n}} I_{-1}\left(\frac{\pi\sqrt{n}}{\sqrt{3}}\right) + O\left(n^{-\frac{3}{2}} e^{\frac{\pi\sqrt{n}}{\sqrt{3}}}\right) \\
&= \frac{1}{3^{\frac{5}{4}} n^{\frac{3}{4}}} e^{\frac{\pi\sqrt{n}}{\sqrt{3}}} + O\left(n^{-\frac{3}{2}} e^{\frac{\pi\sqrt{n}}{\sqrt{3}}}\right),
\end{aligned} \tag{IV.4.7}$$

where we use the asymptotic formula for the I-Bessel function [AAR01, 4.12.7]

$$I_\ell(x) = \frac{e^x}{\sqrt{2\pi x}} + O\left(\frac{e^x}{x^{\frac{3}{2}}}\right).$$

Now we turn to the integral $\mathcal{I}_2$. From the Corollarly IV.11, we have for $My < |x| \le \frac{1}{2}$

$$\mathcal{I}_2 \ll \int_{My < |x| \le \frac{1}{2}} 2\sqrt{6n} e^{\frac{1}{y}\left(\frac{\pi}{24} - \varepsilon\right)} e^{\frac{\pi\sqrt{n}}{2\sqrt{3}}} dx \ll n^{\frac{1}{2}} e^{\frac{\pi\sqrt{n}}{\sqrt{3}}(1-\varepsilon)},$$

which together with (IV.4.7) completes the proof.

IV.5 Concluding remarks

The referee has kindly pointed out that in Section 3.1 by the unimodality of the sequence $(f_m)_{m\in\mathbb{N}}$ all of the $\mathcal{S}_j$ have the same asymptotic expansion since any differences in the $\mathcal{S}_j$ are exponentially small compared to other terms in their asymptotic expansions. Therefore the asymptotic expansion of each $\mathcal{S}_j$ is equal to the asymptotic expansion of the full q-series multiplied by $\frac{1}{4}$. In this case it is not necessary to use theta functions for the proof. We refer the interested reader to [McI95] for details.

The referee has also suggested a interesting discussion. In [McI95] McIntosh obtained

$$\sum_{n=0}^{\infty} \frac{q^{\frac{n(n+1)}{4}}}{(q)_n} = \sqrt{\frac{2}{\sqrt{5}}}\exp\left\{\frac{\pi^2}{10}t^{-1} - \frac{\sqrt{5}}{80}t + \frac{1}{200}t^2 - \frac{23\sqrt{5}}{48000}t^3 + \frac{103}{240000}t^4 + O\left(t^5\right)\right\}$$

and

$$\sum_{n=0}^{\infty} \frac{q^{n\left(n+\frac{1}{2}\right)}}{(q)_n} = \sqrt{\frac{1}{\sqrt{5}}}\exp\left\{\frac{\pi^2}{15}t^{-1} + \left(\frac{1}{48} + \frac{\sqrt{5}}{80}\right)t + \frac{1}{200}t^2 + \frac{23\sqrt{5}}{48000}t^3 + \frac{103}{240000}t^4 + O\left(t^5\right)\right\},$$

as $t \to 0^+$. These expansions appear to agree up to sign from the t^2 term onward. If we replace q by q^4 in the first series, then in some sense it is in the middle of the identities (IV.1.2) and (IV.1.3). Watson [Wat36] discussed the second series, denoted by $G_{1/2}(q)$, in terms of Ramanujan's concept of 'closed' and 'unclosed' asymptotic expansion. This appears to be somewhat related to the concept of 'modular' and 'nonmodular'.

Bibliography

[And84] G. E. Andrews, *Ramanujan's "lost" notebook. IV. Stacks and alternating parity in partitions*, Adv. in Math. **53** (1984), no. 1, 55–74.

[And86] G. E. Andrews, q-Series: Their Development and Application in Analysis, Number Theory, Combinatorics, Physics, and Computer Algebra, CBMS conference series **66**, Amer. Math. Soc., Providence, Rhode Island, 1986.

[And88] G. E. Andrews, The Theory of partitions, Cambridge University Press, Cambridge, 1988.

[AAR01] G. E. Andrews, R. Askey, and R. Roy, Special Functions, Encyclopedia of Mathematics and its Applications, Cambridge University Press, 2001.

[BDLM15] K. Bringmann, J. Dousse, J. Lovejoy, and K. Mahlburg, *Overpartitions with restricted odd differences*, Electron. J. Combin. (3) **22** (2015).

[HR18] G. H. Hardy and S. Ramanujan. *Asymptotic formulae in combinatory analysis*, Proc. London Math. Soc. (2) **17** (1918), 75–115.

[Ing41] A. Ingham, *A Tauberian theorem for partitions*, Ann. of Math. **42** (1941), 1075–1090.

[Kob84] N. Koblitz, Introdunction to elliptic curves and modular forms, Graduate Texts in Mathematics **97**, Springer-Verlag, NewYork, 1984.

[Lov08] J. Lovejoy, *Rank and conjugation for the Frobenius presentation of an overpartition*, Ann. Comb. **12** (2008), 101–113.

[McI95] R. J. McIntosh, *Some asymptotic formulae for q-hypergeometric series*, J. London Math. Soc. (2) **51** (1995), 120–136.

[Vau81] R.C. Vaughan, The Hardy-Littlewood method, Second edition, Cambridge tracts in Mathematics **125**, Cambridge University Press, 1981.

[Wat36] G.N. Watson, *The final problem: an account of the mock theta functions*, J. London Math. Soc. **11** (1936), 55–80.

[Wri34] E. Wright, *Asymptotic partition formulae II. Weighted partitions*, Prod. London Math. Soc. (2) **36** (1934), 117–141.

[Zag07] D. Zagier, The Dilogarithm function. Frontiers in Number Theory, Physics, and Geometry II, 3–65, Springer, Berlin, 2007.

[Zwe02] S. Zwegers, Mock theta functions, Ph.D. Thesis, Universiteit Utrecht, 2002.

V

Summary and Discussion

In this chapter, we conclude the thesis with a brief overview of the studies presented in this thesis and some suggestions for future research.

Radial limits of the universal mock theta function g_3

In Chapter II we gave explicit formulas for specializations of g_3 by applying similar arguments from [16]. To be more precise, we determined associated modular forms $M_{k,h}(q)$ and obtained explicit expressions for the radial limits $Q_{a,b,A,B,h,k}$ such that (recall Rhoades' question II.2)

$$Q_{a,b,A,B,h,k} := \lim_{q \to \zeta_k^k} \left(g_3\left(\zeta_b^a q^A, q^B\right) - M_{k,h}(q) \right).$$

For this we used case-by-case analysis based on the behavior of g_3 at each root of unity. The case where a pole occurs in the summands of g_3 is described in Theorem II.5, while the case the sum defining g_3 converges absolutely in the radial limit is stated in Theorem II.8. The other cases are discussed in Section II.4.4 as well as in Theorem II.9 and Conjecture II.11, as further analysis is required.

Here we recall quantum modular forms introduced by Zagier [46]. A *quantum modular form of weight k*, $k \in \frac{1}{2}\mathbb{Z}$, is a complex-valued function f on $\mathbb{Q}$, such that for all $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$, a congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$, and for multiplier systems $\varepsilon(\gamma)$ the function $h_\gamma(x) : \mathbb{Q} \setminus \gamma^{-1}\{i\infty\} \to \mathbb{C}$, defined by

$$h_\gamma(x) := f(x) - \varepsilon^{-1}(\gamma)(cx+d)^{-k} f\left(\frac{ax+b}{cx+d}\right),$$

has a suitable property of continuity or analyticity in $\mathbb{R}$. The connection between quantum modular forms and mock theta functions was considered in [19] from a formal point of view. Later, Bringmann and Rolen [16] gave a very large and explicit family of quantum modular forms by solving Rhoades' question on the universal mock theta function g_2. Our result further gives another large and explicit family of quantum modular forms.

Since our methods are restricted to Ramanujan's mock theta functions, which all have weight $\frac{1}{2}$, it would be worthwhile to study radial limits of weight $\frac{3}{2}$ mock theta functions. An interesting function might be the rank generating function for overpartitions. Recall that an *overpartition* is a partition where the first occurrence of a part may be overlined. The rank of an overpartition is defined in [20] to be one less than the largest part minus the number of overlined parts less than the largest part. We let $\overline{N}(m,n)$ denote the number of overpartitions of n with rank m. Lovejoy [36] then showed that the rank generating function for overpartitions is given by

$$\mathcal{O}(w;q) := 1 + \sum_{n=1}^{\infty} \overline{N}(m,n) w^m q^n = \sum_{n=0}^{\infty} \frac{(-1)_n q^{\frac{n(n+1)}{2}}}{(wq, q/w)_n}.$$

Moreover, Bringmann and Lovejoy [12] showed that $\mathcal{O}(-1;q)$ is a weight $\frac{3}{2}$ mock theta function, while $\mathcal{O}(w;q)$ is a mock theta function of weight $\frac{1}{2}$ if w is a primitive root of unity and $w \neq \pm 1$. We notice that the function $\mathcal{O}(w;q)$ is surprisingly similar to the universal mock theta function g_2 defined by

$$g_2(x,q) := \sum_{n=0}^{\infty} \frac{(-q)_n q^{\frac{n(n+1)}{2}}}{(x, q/x)_{n+1}}.$$

On spt-crank-type functions

In Chapter III we constructed mock modularity of the new spt function $\mathrm{spt}_\omega(n)$ by which we were able to establish infinitely many linear congruences of $\mathrm{spt}_\omega(n)$ in Theorem III.4. Applying Wright's Circle Method, we also obtained the asymptotic formula for its crank function $N_{C_1}(m,n)$ in Theorem III.1 and deduced the sign pattern of the crank in Theorem III.3. As an application, we proved Garvan and Jennings-Shaffer's positivity conjecture on the crank asymptotically in Corollary III.2.

The modularity (or mock modularity) of partition-theoretic functions has many important consequences, for example asymptotics and congruences. In particular, the partition rank function $N(m,n)$ has a rich history as mentioned in Section I.2.2 above. Recall the rank of a partition is defined to be the largest part of the partition minus the number of parts and let $N(m,n)$ denote the number of partitions of n with rank equal to m. Then we have the generating function

$$R(w;q) := 1 + \sum_{m\in\mathbb{Z}}\sum_{n=1}^{\infty} N(m,n)w^m q^n = 1 + \sum_{n=1}^{\infty} \frac{q^{n^2}}{(wq,q/w)_n}.$$

In particualr,

$$R(1;q) = 1 + \sum_{n=1}^{\infty} p(n)q^n = \frac{q^{\frac{1}{24}}}{\eta(\tau)},$$
$$R(-1;q) = 1 + \sum_{n=1}^{\infty} \frac{q^{n^2}}{(-q)_n^2} = f(q),$$

where $f(q)$ is a famous Ramanujan's third order mock theta function. Furthermore, Bringmann and Ono [15] related this rank generating function $R(w;q)$ to harmonic Maass forms.

We recall (III.6.1), the generating function for $\mathrm{spt}_\omega(n)$ given by

$$S_\omega(\tau) = \sum_{n\geq 1} \mathrm{spt}_\omega(n)q^n = \frac{q^{\frac{1}{12}}\left(1-E_2(\tau)\right)}{24\eta(2\tau)} - R_2(2\tau).$$

Here $R_k(z)$ is the generating function of $\eta_k(n)$, a symmetrized k-th moment of the rank, defined in [6] by

$$\eta_k(n) = \sum_{m\in\mathbb{Z}} \binom{m+\lfloor\frac{k-1}{2}\rfloor}{k} N(m,n).$$

Andrews [7] showed that

$$\mathrm{spt}(n) = np(n) - \eta_2(n).$$

In another paper [6], he proved more generally that the function $\eta_k(n)$ enumerates so-called k-marked Durfee symbols. Before explaining k-marked Durfee symbols, we first recall that the largest square of nodes in the Ferrers graph is called the *Durfee square*. To each partition Andrews associated a *Durfee symbol* based on its Ferrers graph and Durfee square. The Durfee symbol consists of two rows and a subscript. The top row indicates the columns to the right of the Durfee square, the bottom row indicates the rows below the Durfee square, and the subscript denotes the side length of the Durfee square. An example of the Durfee symbol for a partition $4+4+3+2+1$ of 14 is illustrated below:

$$\begin{array}{ccc|c} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \\ \hline \cdot & \cdot & & \\ \cdot & & & \end{array} \qquad \longrightarrow \qquad \begin{pmatrix} 2 & \\ 2 & 1 \end{pmatrix}_3 .$$

Using k copies of integers denoted by $\{1_1, 2_1, 3_1 \ldots\}$, $\{1_2, 2_2, 3_2, \ldots\}, \ldots,$ and $\{1_k, 2_k, 3_k, \ldots\}$, Andrews more generally defined the k-marked Durfee symbols. Returning to the Dyson's rank, Andrews also introduced the j-th rank for the j-th copy of integers $\{1_j, 2_j, 3_j, \ldots,\}$ and deduced the rank generating function of k-marked Durfee symbols. Let $\mathcal{D}(m_1, m_2, \ldots, m_k; n)$ denote the number of k-marked Durfee symbols arising from partitions of n with j-th rank equal to m_j. Then we have

$$\begin{aligned} R_k(x_1, x_2, \ldots, x_k; q) &:= \sum_{m_1,\ldots,m_k \in \mathbb{Z}} \sum_{n=0}^{\infty} \mathcal{D}(m_1, m_2, \ldots, m_k; n) x_1^{m_1} x_2^{m_2} \cdots x_k^{m_k} q^n \\ = \sum_{\substack{m_1>0 \\ m_2, m_3, \ldots, m_k \geq 0}} & \frac{q^{(m_1+\cdots m_k)^2 + (m_1+m_2+\cdots+m_{k-1}) + (m_1+\cdots+m_{k-2}) + \cdots + m_1}}{(x_1 q, x_1^{-1} q)_{m_1} (x_2 q^{m_1}, x_2^{-1} q^{m_1})_{m_2+1}} \\ & \times \frac{1}{(x_3 q^{m_1+m_2}, x_3^{-1} q^{m_1+m_2})_{m_3+1} \cdots (x_k q^{m_1+\cdots+m_{k-1}}, x_k^{-1} q^{m_1+\cdots+m_{k-1}})_{m_k+1}}. \end{aligned}$$

As this combinatorial series have numerous interesting properties, for example when $k=1$ one recovers original Dyson's rank, there have been many studies on its modularity [13, 14, 24]. In upcoming work with Amanda Folsom, Susie Kimport and Holly Swisher, we establish quantum modularity of this series.

Asymptotic behavior of Odd-Even partitions

Using various methods, we examined the asymptotic behaviors of odd-even partitions and overpartitions in Chapter IV. As the odd-even partition function $OE(n)$ is not monotonically increasing, we first split the generating function for $OE(n)$ into two series depending on residue classes. This allowed us to apply Ingham's Tauberian Theorem. We then estimated each series. For that, we mimicked the second proof of [44, Proposition 5]. The strategy is to examine the asymptotic behavior of the summand first and then sum up the asymptotics.

In order to analyze odd-even overpartitions, we used Wright's Circle Method. To be more precise, using the Watson's identity for Ramanujan's mock theta function $f(q)$, we rewrote the generating function for odd-even overpartitions as in (IV.4.1). From that we could easily see that the generating function has only one dominant pole, namely $q = 1$, and thus we applied Wright's Circle Method to study the Fourier coefficients of the series, the odd-even overpartition function $\overline{OE}(n)$.

Here we have two remarks regarding the proof of Theorem IV.5. First, one can estimate $\mathcal{S}_j$, defined in (IV.3.6), without using the Jacobi Theta function and its modularity properties. More precisely, one can directly apply the Poisson summation formula to (IV.3.8), which yields the same results.

Furthermore, by simple observation one can rewrite the functions $\mathscr{O}_e(q)$ and $\mathscr{O}_o(q)$ in terms of $\mathscr{O}(q)$ without using residue classes,

$$\mathscr{O}_e(q) = \sum_{n=0} OE(2n)q^{2n} = \frac{\mathscr{O}(q) + \mathscr{O}(-q)}{2},$$
$$\mathscr{O}_o(q) = \sum_{n=0} OE(2n+1)q^{2n} = \frac{\mathscr{O}(q) - \mathscr{O}(-q)}{2}.$$

For the asymptotics of $\mathscr{O}(q)$, we can directly use the results of McIntosh [37] or Zagier [44, Proposision 5] as mentioned in Remarks IV.7 and IV.8 above. Now we turn to $\mathscr{O}(-q)$,

$$\mathscr{O}(-q) = \sum_{n=0}^{\infty} \frac{(-q)^{\frac{n(n+1)}{2}}}{((-q)^2;(-q)^2)_n} = \sum_{n=0}^{\infty} \frac{(-1)^n q^{\frac{n(n+1)}{2}}}{(q^2;q^2)_n}.$$

The only difference from $\mathscr{O}(q)$ is the term $(-1)^n$ in the numerator of the summand. Due to this additional factor, the sequence of the summands is not unimodal anymore. Therefore, we can not apply the same method as what we used for $\mathscr{O}(q)$. Nevertherless, since we know the asymptotics of $\mathscr{O}_e(q)$ and $\mathscr{O}_o(q)$ from Theorem IV.5, and $\mathscr{O}(-q)$ is just the sum of $\mathscr{O}_e(q)$ and $\mathscr{O}_o(q)$, we can deduce the asymptotic formulas for $\mathscr{O}(-q)$. In this case, we have to obtain more main terms than in Theorem IV.5. Thus, for $q = e^{-t}$ we have

$$\sum_{n=0}^{\infty} \frac{(-1)^n q^{\frac{n(n+1)}{2}}}{(q^2;q^2)_n} \sim -\frac{3t}{4}\sqrt{\frac{\sqrt{5}}{2}}e^{\frac{\pi^2}{20t}}$$

as $t \to 0^+$.

It would be interesting to study the asymptotics behavior of q-hypergeometric series of the form

$$\sum_{n=0}^{\infty} \frac{q^{An^2+Bn}}{(q)_n},$$

where $A, B \in \mathbb{R}$ and $A > 0$, as $q \to -1$. More generally, an interesting but challenging problem might be to find the asymptotics of this q-hypergeometric series as q approaches any root of unity.

Bibliography

[1] G. E. Andrews, The Theory of Partitions. Addison-Wesley, Reading, Mass. (1976).

[2] G. E. Andrews, *An introduction to Ramanujan's "lost" notebook*, Amer. Math. Monthly **86** (1979), 89–108.

[3] G. E. Andrews, *Ramanujan's "lost" notebook. IV. Stacks and alternating parity in partitions*, Adv. in Math. **53** (1984), no. 1, 55–74.

[4] G. E. Andrews and F. G. Garvan, *Dyson's crank of a partition*, Bull. Amer. Math. Soc. (N.S.) **18** (1988), no. 2, 167–171.

[5] G. E. Andrews and F. G. Garvan, *Ramanujan's "Lost" Notebook VI: The Mock Theta Conjectures*, Adv. in Math. **73** (1989), no.2, 242–255.

[6] G. E. Andrews *Partitions, Durfee symbols, and the Atkin-Garvan moments of ranks*, Invent. Math. **169** (2007), 37–73.

[7] G. E. Andrews, *The number of smallest parts in the partitions of* n, J. reine Angew. Math., **624** (2008), 133–142.

[8] G. E. Andrews, The Discovery of Ramanujan's Lost Notebook, in *The Legacy of Srinivasa Ramanujan: Proceedings of an International Conference in Celebration of the 125th Anniversary of Ramanujan's Birth: University of Delhi, 17–22 December 2012*, B. C. Berndt and D. Prasad, eds., Ramanujan Mathe- matical Society, Mysore, 2013, pp. 77–88.

[9] G. E. Andrews, A. Dixit, and A. J. Yee, *Partitions associated with the Ramanujan/Watson mock theta functions* $\omega(q)$, $\nu(q)$ *and* $\phi(q)$, Research in Number Theory, **1** Issue 1 (2015), 1–25.

[10] A. L. Atkin and P. Swinnerton-Dyer, *Some properties of partitions*, Proc. London Math. Soc. (3) **4** (1954), 84–106.

[11] B. C. Berndt and R. A. Rankin, *Ramanujan: Letters and Commentary*, History of Mathematics **9** (1995), Amer. Math. Soc., Providence, RI.

[12] K. Bringmann and J. Lovejoy, *Dyson's rank, overpartitions, and weak Maass forms*, Int. Maht. Res. Not., (2007), rnm063.

[13] K. Bringmann, *On the explicit construction of higher deformations of partition statistics*, Duke Math. **144** (2008), 195–233.

[14] K. Bringmann, F. G. Garvan, and K. Mahlburg, *Partition statistics and quasiharmonic Maass forms*, In. Math. Res. Notices 2009, 63–97.

[15] K. Bringmann and K. Ono, *Dyson's ranks and Maass forms*, Ann. of Math. **171** (2010), 419–449.

[16] K. Bringmann and L. Rolen, *Radial limits of mock theta functions*, Res. Math. Sci. **2** (2015), Art. 17,18.

[17] K. Bringmann, A. Folsom, K. Ono, and L. Rolen, Harmonic Maass forms and mock modular forms: theory and applications, American Mathematical Society Colloquium Publications. American Mathematical Society, Providence, RI, to appear.

[18] J. H. Bruinier and J. Funke, *On two geometric theta lifts*, Duke Math. J. **125** (2004), no. 1, 45–90.

[19] D. Choi, S. Lim, and R. C. Rhoades, *Mock modular forms and quantum modular forms*, Proc. Amer. Math. Soc. **144** (2016), no.6 2337–2349.

[20] S. Corteel and J. Lovejoy, *Overpartitions*, Trans. Amer. Math. Soc. **356** (2004), 1623–1635.

[21] F. J. Dyson, *Some guesses in the theory of partitions*, Eureka (1944), no. 8, 10–15.

[22] L. Euler, *Observationes analyticae variae de combinationibus*, Commentarii academiae scientiarum Petropolitanae 13 (1751), 64–93.

[23] L. Eulero, *De partitione numerorum.* Novi Commentarii Academiae scientiarum Petropolitanae 3, (1753), 125–169.

[24] A. Folsom and S. Kimport, *Mock modular forms and singular combinatorial series*, Acta Arith. **159** (2013), 257–297.

[25] A. Folsom, K. Ono, and R. C. Rhoades, *Mock theta functions and quantum modular forms*, Forum Math. Pi **1** (2013), e2, 27.

[26] F. G. Garvan, *New combinatorial interpretations of Ramanujan's partition congruences mod 5, 7 and 11*, Trans. Amer. Math. Soc. **305** (1988), no. 1, 47–77.

[27] F. G. Garvan and C. Jennings-Shaffer, *Exotic Bailey-Slater SPT-function II: Hecke-Rogers-type double sums and Bailey pairs from group A, C, E*, Adv. Math. **299** (2016), 605–639.

[28] B. Gordon and R. J. McIntosh, *A survey of classical mock theta functions*, Partitions, q-series, and modular forms, Dev. Math., vol. 23, Springer, New York, 2012, pp. 95–144.

[29] M. Griffin, K. Ono, and L. Rolen, *Ramanujan's mock theta functions*, Proc. Natl. Acad. Sci. USA **110** (2013), no. 15, 5765–5768.

[30] G. H. Hardy and S. Ramanujan, *Asymptotic formulae in combinatory analysis*, Proc. London Math. Soc. **17** (1918), 75–115.

[31] *A proof of the mock theta conjectures*, Invent. Math. **94** (1988), no. 3, 639–660.

[32] M.-J. Jang and S. Löbrich, *Radial limits of universal mock theta function g_3*, Proc. Amer. Math. Soc., **145** (2017), 925-935.

[33] M.-J. Jang, *Asymptotic behavior of Odd-Even partitions*, Electron. J. Combin. (3) **24** (2017) #P3.62.

[34] M.-J. Jang and B. Kim, *On spt-crank-type functions*, Ramanujan J. **45** Issue 1 (2018), 211–225.

[35] G. W. Leibniz, *Specimen de divulsionibus aequationum ad problemata indefinita in numeris rationalibus solvenda. 2. September 1674*, in: G. W. Leibniz, *Sämtliche Schriften und Briefe. Siebente Reihe: Mathematische Schriften, Bd. 1: 1672–1676. Geometrie – Zahlentheorie – Algebra (1. Teil)*, Akademie-Verlag, Berlin, 1990, 740–755.

[36] J. Lovejoy, *Rank and conjugation for the Frobenius representation of an overpartition*, Ann. Comb. **9** (2005), 321–331.

[37] R. J. McIntosh, *Some asymptotic formulae for q-hypergeometric series*, J. London Math. Soc. (2) **51** (1995), 120–136.

[38] H. Rademacher, *On the partition function* $p(n)$, Proc. London Math. Soc. **s2-43** (1938), 241–254.

[39] S. Ramanujan, *Some properties of* $p(n)$*, the number of partitions of* n, Proc. Cambridge Philos. Soc. **19** (1919), 207–210.

[40] S. Ramanujan, *Congruence properties of partitions*, Math. Z. **9** (1921), no. 1-2, 147–153.

[41] R. C. Rhoades, *On Ramanujan's definition of mock theta function*, Proc. Natl. Acad. Sci. USA **110** (2013), no. 10, 7592–7594.

[42] G. N. Watson, *The final problem: An account of the mock theta functions*, J. London Math. Soc. (2) **2** (1936), 55–80.

[43] E. Wright, *Asymptotic partition formulae II. Weighted partitions*, Prod. London Math. Soc. (2) **36** (1934), 117–141.

[44] D. Zagier, The Dilogarithm function. Frontiers in Number Theory, Physics, and Geometry II, 3–65, Springer, Berlin, 2007.

[45] D. Zagier, *Ramanujan's mock theta functions and their applications (after Zwegers and Ono- Bringmann)*, AstÃľrisque (2009), no. 326, Exp. No. 986, vii–viii, 143–164 (2010), SÃľminaire Bourbaki. Vol. 2007/2008.

[46] D. Zagier, *Quantum modular forms, in Quanta of Maths: Conference in Honor of Alain Connes*, Clay Math. Proc., **11** (Amer. Math. Soc., Prov, 2010), 659–675.

[47] S. Zwegers, Mock theta functions, Ph.D. Thesis, Universiteit Utrecht, 2002.

Declaration

I hereby declare that the article *Radial limits of the universal mock theta function g_3* [32] was jointly written with Steffen Löbrich and my share of work amounted to 50%. The article *On spt-crank-type functions* [32] was jointly written with Prof. Dr. Byungchan Kim and my share of work amounted to 50%. The articles *Asymptotic behavior of Odd-Even partitions* [33] is entirely my own work.

Köln, April 13, 2018

Min-Joo Jang

Erklärung

Ich versichere, dass ich die von mir vorgelegte Dissertation selbstständig angefertigt, die benutzten Quellen und Hilfsmittel vollständig angegeben und die Stellen der Arbeit – einschließlich Tabellen, Karten und Abbildungen –, die anderen Werken entnommen sind, in jedem Einzelfall als Entlehnung kenntlich gemacht habe; dass diese Dissertation noch keiner anderen Fakultät oder Universit ät zur Prüfung vorgelegen hat; dass sie – abgesehen von unten angegebenen Teilpublikationen – noch nicht verößentlicht worden ist, sowie, dass ich eine solche Verößentlichung vor Abschluss des Promotionsverfahrens nicht vornehmen werde. Die Bestimmungen der Promotionsordnung sind mir bekannt. Die von mir vorgelegte Dissertation ist von Prof. Dr. Kathrin Bringmann betreut worden.

Köln, April 13, 2018

Min-Joo Jang

Curriculum Vitae

Personal Information

Name	**Min-Joo Jang**
Date of Birth	**Dec. 4, 1987**
Nationality	**Republic of Korea**

Contact Information

Email	**min-joo.jang@uni-koeln.de**
Address	**Mathematisches Institut, Universtät Köln** **Weyertal 86-90, 50931 Köln, Germany**
Telephone	**+49 (0)0221 470 6802**
Homepage	**http:// www.mi.uni-koeln.de/∼jminjoo/**

Education

Mar. 2018	**PhD Course**, Cologne University Advisor: Kathrin Bringmann
Feb. 2015	**Master in Mathematics**, Yonsei University Advisor: Jaebum Sohn
Feb. 2012	**Bachelor in Mathematics**, Yonsei University

Professional Experience

Winter 2017	**Co-adviser for Bachelor Thesis** Student: Stefan Thullen, Supervisor: Kathrin Bringmann
Summer 2017	**Co-organizer for lecture**, *Mathematik für Lehramtsstudierende II*, Lecturer : Stephan Ehlen, Cologne University

Winter 2016-17	**Co-organizer for lecture**, *Mathematik für Lehramtsstudierende I*, Lecturer : Stephan Ehlen, Cologne University
2016-17	**Co-adviser for Master Thesis** Student: Marcel Kraft, Supervisor: Kathrin Bringmann
2015-17	**Research Assistant**, Cologne University
2012-14	**Teaching Assistant**, Yonsei University Course : Number Theory, Calculus & Vector Analysis, Advanced Calculus

Research Interests

Number Theory and Combinatorics involving partitions and modular forms

Papers and preprints

1. *Radial limits of the universal mock theta function g_3* (with Steffen Löbrich)
   Proceedings of the American Mathematical Society, **145** (2017), 925–935.
2. *On spt-crank-type functions* (with Byungchan Kim)
   The Ramanujan Journal, **45** Issue 1 (2018), 211–225.
3. *Asymptotic behavior of Odd-Even partitions*
   The Electronic Journal of Combinatorics, **22** (3) (2017) #P3.62.
4. *Identities for overpartitions with even smallest parts* (with Jeremy Lovejoy)
   International Journal of Number Theory, accepted for publication.
5. *Quantum modular forms and singular combinatorial series with distinct roots of unity* (with Amanda Folsom, Susie Kimport, and Holly Swisher)
   Submitted for publication

Talks

Dec. 2017	**Quantum modular forms and singular combinatorial series** Trends in Modular Forms Conference, NIMS (South Korea)
Oct. 2017	**Identities for overpartitions with even smallest parts**

Partitions/Combinatorics Seminar, Penn State Uni. (U.S.)

Mar. 2016 **On spt-crank-type functions**
The 2016 Gainesville International Number Thoery Conference, Uni. of Florida. (U.S.)

Sep. 2015 **Radial Limits of a Universal Mock Theta Function**
Student Seminar, Hongkong Uni. (Hongkong)

Sep. 2015 **Radial Limits of a Universal Mock Theta Function**
Discoverers avant le PhD (DISCO), MPIM Bonn (Germany)

Nov. 2014 **Asymptotic formula for certain distinct partitions**
Yonsei-Keio Workshop, Keio Uni. (Japan)