Realizable Lists on a Class of Nonnegative Matrices

Enide Andrade*

CIDMA-Center for Research and Development in Mathematics and Applications
Departamento de Matemática, Universidade de Aveiro, 3810-193, Aveiro, Portugal.

Cristina Manzaneda

Departamento de Matemáticas, Facultad de Ciencias. Universidad Católica del Norte.
Av. Angamos 0610 Antofagasta, Chile.

María Robbiano

Departamento de Matemáticas, Facultad de Ciencias. Universidad Católica del Norte.
Av. Angamos 0610 Antofagasta, Chile.

Abstract

A square matrix of order n with $n \geq 2$ is called *permutative matrix* when all its rows are permutations of the first row. In this paper recalling spectral results for partitioned into 2-by-2 symmetric blocks matrices sufficient conditions on a given complex list to be the list of the eigenvalues of a nonnegative permutative matrix are given. In particular, we study NIEP and PNIEP when some complex elements in the lists under consideration have non-zero imaginary part. Realizability regions for nonnegative permutative matrices are obtained. A Guo's realizability-preserving perturbations result is obtained.

Keywords:
permutative matrix; inverse eigenvalue problem; nonnegative matrix; circulant matrix; skew circulant matrix; Guo perturbations
2000 MSC: 15A18, 15A29, 15B99.

*Corresponding author

Email addresses: enide@ua.pt (Enide Andrade), cmanzaneda@ucn.cl (Cristina Manzaneda), mrobbiano@ucn.cl (María Robbiano)

1. Overview of Some Results

The Nonnegative Inverse Eigenvalue Problem (called NIEP), consists on finding necessary and sufficient conditions on a list of n complex numbers

$$\sigma = \{\lambda_1, \lambda_2, \ldots, \lambda_n\} \tag{1}$$

to be the spectrum of an n-by-n entry-wise nonnegative matrix. If there exists an n-by-n nonnegative matrix A with spectrum σ (sometimes denoted by $\sigma(A)$), we will say that σ is realizable and A realizes σ (or, that is a realizing matrix for the list).

The NIEP has a long history since its proposal by Kolmogorov [12] but was first formulated by Suleĭmanova [26] in 1949.

Definition 1. The list σ in (1) is a Suleĭmanova spectrum if the $\lambda's$ are real numbers, $\lambda_1 > 0 \geq \lambda_2 \geq \cdots \geq \lambda_n$ and $\sum_{i=1}^{n} \lambda_i \geq 0$.

The problem has attracted the attention of many authors for more than 50 years. Although some partial results were obtained the NIEP is an open problem for $n \geq 5$. The NIEP for $n \leq 3$ was solved independently by Oliveira [20, Theorem (6.2)] and Loewy and London in [14]. For matrices of order $n = 4$ the problem was solved in [18] and [17]. A recent survey on NIEP can be seen in [9].

Concerning the NIEP there are some immediate necessary conditions on a list of complex numbers $\sigma = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ to be the spectrum of a nonnegative matrix.

1. The list σ is closed under complex conjugation.
2. The Perron eigenvalue $\rho = \max\{|\lambda| : \lambda \in \sigma(A)\}$ lies in σ.
3. $s_k(\sigma) = \sum\limits_{i=1}^{n} \lambda_i^k \geq 0$.
4. $s_k^m(\sigma) \leq n^{m-1} s_{km}(\sigma)$ for $k, m = 1, 2, \ldots$

Theory and applications of nonnegative matrices are blended in the book [1]. Practical problems with applications to Markov chains, queuing networks, economic analysis, or mathematical programming and also inverse eigenvalue problems are presented there. Extensive references are included in each area. See also the book [19] for a general theory on nonnegative matrices.

One of the most promising attempts to solve the NIEP is using constructive methods. For instance, given a class of nonnegative matrices find a large class of spectra realized by these matrices. Companion matrices of polynomials played an important role. Let

$$\begin{aligned} f(x) &= (x-\lambda_1)(x-\lambda_2)\cdots(x-\lambda_n) \\ &= x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_n. \end{aligned}$$

The companion matrix of $f(x)$ is:

$$C(f) = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & 0 & 1 \\ -a_n & -a_{n-1} & \cdots & -a_2 & -a_1 \end{bmatrix}.$$

Note that A is nonnegative if and only if $a_i \leq 0$, for $i = 1, \ldots, n$.

Friedland [5] proved that Suleĭmanova-type spectra are realizable by a companion matrix. Loewy and London, in 1978 [14] showed that the list $(r, a+bi, a-bi)$, with $r \geq 1$, is realizable if and only if it is realizable by the matrix $\alpha I + C$, with $\alpha \geq 0$ and C a nonnegative companion matrix. In [13], T. Laffey and H. Šmigoc established that a list $\sigma = \{\lambda_1, \ldots, \lambda_n\}$, with $\lambda_1 > 0$ and $Re(\lambda_i) \leq 0,\ i = 2, \ldots, n$, is realizable if and only if it is realizable by a matrix of the form $\alpha I + C$, where $\alpha \geq 0$ and C is a companion matrix with trace zero. Rojo and Soto in [22] found sufficient conditions for the realizability of spectra by nonnegative circulant matrices. In [11] the authors studied realizability by nonnegative integral matrices. Leal-Duarte and Johnson in [7] solved the NIEP for the case the realizing matrix is an arbitrary nonnegative diagonal matrix added to a nonnegative matrix whose graph is a tree. G. Soules in [25], construct a symmetric matrix N having specified list of eigenvalues and give conditions on the maximal eigenvalue such that N is nonnegative. A problem proposed by T. Laffey and H. Šmigoc is to find good classes of matrices to study realizability and many authors tried to find good classes of matrices to study this type of problems. Here we will focus on permutative matrices. The summary of the paper is the following: At Section 1 we review some state of the art related with NIEP. At Section 2 we recall some recent results related with permutative matrices. The new results appear at Section 3 where are studied NIEP and PNIEP (we call the problem as PNIEP when the NIEP involves permutative matrices), when some complex elements in the lists under consideration have non-zero imaginary part. It is presented sufficient conditions on a complex list of four elements to be realized by a permutative matrix. The class of circulant and and skew circulant matrices and their properties play an important role at Section 4 and some results are derived. A Guo's realizability-preserving perturbation result is studied in Section 5.

2. A brief History on Permutative Matrices and Recent Results

In this section we recall some useful tools from recent published results that will be used throughout the text. Some auxiliary results from [21], and some recent definitions from the literature are recalled here. The next definition was first presented in [21] and it is the definition of permutative matrix. In fact, this term was given by C. R. Johnson (see footnote at [21]).

Definition 2. *[16] Let $\tau = (\tau_0, \tau_1, \ldots, \tau_{n-1})$ be an n-tuple whose components are permutations in the symmetric group $\mathcal{S}_n$, with $\tau_0 = id$. Let $\mathbf{a} = (a_1, \ldots, a_n) \in \mathbb{C}^n$. Define the row-vector,*

$$\tau_j\left(\mathbf{a}\right) = \left(a_{\tau_j(1)}, \ldots, a_{\tau_j(n)}\right)$$

and consider the matrix

$$\tau\left(\mathbf{a}\right) = \begin{pmatrix} \tau_1\left(\mathbf{a}\right) \\ \tau_2\left(\mathbf{a}\right) \\ \vdots \\ \tau_{n-1}\left(\mathbf{a}\right) \\ \tau_n\left(\mathbf{a}\right) \end{pmatrix}. \tag{2}$$

An n-by-n matrix A is called permutative if $A = \tau\left(\mathbf{a}\right)$ for some n-tuple $\mathbf{a}$.

Ranks of permutative matrices were studied by Hu et al [8]. The authors focus on identifying circumstances under which square permutative matrices are rank deficient.

In [21], dealing with RNIEP, P. Paparella considered cases of realizable spectra when a realizing matrix can be taken to have a specific form, that is, to be a permutative matrix. Paparella raised the question when any realizable (real) list can be realized by such a matrix or a direct sum of permutative matrices. The author showed that for $n \leq 4$ this is always possible. Moreover, it is shown in [21] that if the list σ contains exactly one positive number and is realizable then it can be realized by a permutative matrix, and thus explicit permutative matrices which realize Suleĭmanova spectra were found. The author used a constructive proof. Loewy in [15] showed than in general the answer to the question posed by P. Paparella is no. Loewy [15] resolved this problem in the negative by showing that the list $\sigma = \left(1, \frac{8}{25} + \frac{\sqrt{51}}{50}, \frac{8}{25} - \frac{\sqrt{51}}{50}, -\frac{4}{5}, -\frac{21}{25}\right)$ is realizable but cannot be realized by a permutative matrix or by a direct sum of permutative matrices.

Recently, in [16] the spectra of a class of permutative matrices were studied. In particular, spectral results for matrices partitioned into 2-by-2 symmetric blocks were presented and, using these results sufficient conditions on a given list to be the list of eigenvalues of a nonnegative permutative matrix were obtained and the corresponding permutative matrices were constructed. Throughout the paper and if no misunderstanding arise, the symbol i stands for the complex square root of -1 and the row sub-index of the (i,j)-entry of some considered matrix. Note that the lists considered along the paper are equivalent (up to a permutation of its elements).

3. Complex lists

In this section we study NIEP and PNIEP when some complex elements in the lists under consideration have non-zero imaginary part. The following result gives sufficient conditions on a complex list of four elements to be realized by a permutative matrix.

Theorem 3. *Let* $\sigma = \{\lambda_1, \lambda_2, \lambda_3, \lambda_4\}$ *such that* $\sum_{i=1}^{4} \lambda_i \geq 0$, $\overline{\lambda_3} = \lambda_4$, $\lambda_1 + \lambda_2 \geq 2\mathrm{Re}\lambda_3$, *and* $\lambda_1 - \lambda_2 \geq 2\left|\mathrm{Im}\lambda_3\right|$, *then the nonegative permutative matrix*

$$M = \begin{pmatrix} \frac{\lambda_1+\lambda_2+\lambda_3+\lambda_4}{4} & \frac{\lambda_1+\lambda_2-\lambda_3-\lambda_4}{4} & \frac{\lambda_1-\lambda_2-\lambda_3 i+\lambda_4 i}{4} & \frac{\lambda_1-\lambda_2+\lambda_3 i-\lambda_4 i}{4} \\ \frac{\lambda_1+\lambda_2-\lambda_3-\lambda_4}{4} & \frac{\lambda_1+\lambda_2+\lambda_3+\lambda_4}{4} & \frac{\lambda_1-\lambda_2+\lambda_3 i-\lambda_4 i}{4} & \frac{\lambda_1-\lambda_2-\lambda_3 i+\lambda_4 i}{4} \\ \frac{\lambda_1-\lambda_2+\lambda_3 i-\lambda_4 i}{4} & \frac{\lambda_1-\lambda_2-\lambda_3 i+\lambda_4 i}{4} & \frac{\lambda_1+\lambda_2+\lambda_3+\lambda_4}{4} & \frac{\lambda_1+\lambda_2-\lambda_3-\lambda_4}{4} \\ \frac{\lambda_1-\lambda_2-\lambda_3 i+\lambda_4 i}{4} & \frac{\lambda_1-\lambda_2+\lambda_3 i-\lambda_4 i}{4} & \frac{\lambda_1+\lambda_2-\lambda_3-\lambda_4}{4} & \frac{\lambda_1+\lambda_2+\lambda_3+\lambda_4}{4} \end{pmatrix} \qquad (3)$$

realizes σ.

Proof. Suppose that

$$M=\begin{pmatrix} a & b & c & d\\ b & a & d & c\\ d & c & a & b\\ c & d & b & a \end{pmatrix}.$$

By [16, Theorem 13] the spectrum of M is the union of the spectra of S and C respectively, with

$$S=\begin{pmatrix} a+b & c+d\\ d+c & a+b \end{pmatrix} \text{ and } C=\begin{pmatrix} a-b & c-d\\ d-c & a-b \end{pmatrix}.$$

Thus,

$$\sigma\left(M\right)=\left\{a+b+c+d,a+b-c-d\right\}\cup\left\{a-b+ic-id,a-b-ic+id\right\}.$$

Considering

$$\begin{aligned} \lambda_1 &= a+b+c+d\\ \lambda_2 &= a+b-c-d\\ \lambda_3 &= a-b+ic-id\\ \lambda_4 &= a-b-ic+id \end{aligned}$$

and solving we obtain,

$$\begin{aligned} a &= \frac{\lambda_1+\lambda_2+\lambda_3+\lambda_4}{4}\\ b &= \frac{\lambda_1+\lambda_2-\lambda_3-\lambda_4}{4}\\ c &= \frac{\lambda_1-\lambda_2-\lambda_3 i+\lambda_4 i}{4}\\ d &= \frac{\lambda_1-\lambda_2+\lambda_3 i-\lambda_4 i}{4}. \end{aligned}$$

The conditions in the statement imply that the entries a, b, c and d are nonnegative. Thus, the result follows. □

Example 4. *Since $\sigma=\{8,2,3+2i,3-2i\}$ satisfies the conditions of Theorem 3 we can obtain the nonnegative permutative matrix in (3),*

$$M=\begin{pmatrix} 4 & 1 & \frac{5}{2} & \frac{1}{2}\\ 1 & 4 & \frac{1}{2} & \frac{5}{2}\\ \frac{1}{2} & \frac{5}{2} & 4 & 1\\ \frac{5}{2} & \frac{1}{2} & 1 & 4 \end{pmatrix}. \tag{4}$$

Remark 5. *The previous example have its spectrum outside the region of the complex plane*

$$\Gamma = \{z \in \mathbb{C} : \operatorname{Re} z \leq 0 \text{ and } |\operatorname{Im} z| \leq |\operatorname{Re} z|\},$$

(region presented in [2] for the realizability of some complex spectrum).

Remark 6. *In [4] after describing some previous results the following problem was considered: Find a geometric representation for $(r, a, b) \in \mathbb{R}^3$ such that all lists $\{1, r, a+ib, a-ib\}$ having Perron root 1 in the list, to be a NIEP's solution.*

Towards a response to this problem we present the following result.

Theorem 7. *Let consider the complex list $\sigma = \{1, r, a+ib, a-ib\}$ with $0 \leq r \leq 1$. If $|a| \leq \frac{1+r}{2}$ and $|b| \leq \frac{1-r}{2}$ then the list σ is a realizable list whose realizing matrix is*

$$M = \begin{pmatrix} \frac{1+r+2a}{4} & \frac{1+r-2a}{4} & \frac{1-r+2b}{4} & \frac{1-r-2b}{4} \\ \frac{1+r-2a}{4} & \frac{1+r+2a}{4} & \frac{1-r-2b}{4} & \frac{1-r+2b}{4} \\ \frac{1-r-2b}{4} & \frac{1-r+2b}{4} & \frac{1+r+2a}{4} & \frac{1+r-2a}{4} \\ \frac{1-r+2b}{4} & \frac{1-r-2b}{4} & \frac{1+r-2a}{4} & \frac{1+r+2a}{4} \end{pmatrix}. \tag{5}$$

Proof. Suppose that $(1, r, a+ib, a-ib) = (\lambda_1, \lambda_2, \lambda_3, \lambda_4) \in \mathbb{C}^4$. Then $\overline{\lambda}_3 = \lambda_4$. From Theorem 3 the conditions:

$$1 + r \geq 2a \qquad \text{and} \qquad 1 - r \geq 2|b|$$

must be fulfilled. Considering the property of the trace we obtain:

$$1 + r + 2a \geq 0.$$

Therefore,

$$|a| \leq \frac{1+r}{2} \qquad \text{and} \qquad |b| \leq \frac{1-r}{2}$$

which is a spectral realization complex rectangular region obtained in terms of r.

$\square$

4. Circulant and skew circulant matrices

The class of circulant matrices and their properties are introduced in [3] and plays an important role here. In [10] it was presented spectral decomposition of four types of real circulant matrices. Among others, right circulants (whose elements topple from right to left) as well as skew right circulants (whose elements change their sign when toppling) are analyzed. The inherent periodicity of circulant matrices means that they are closely related to Fourier analysis and group theory.

Let $s = (s_0, s_1, \ldots, s_{n-1})^T$, $c = (c_0, c_1, \ldots, c_{n-1})^T \in \mathbb{R}^n$ be given.

Definition 8. *[3, 10] A* real right circulant matrix *(or simply,* circulant matrix*), is a matrix of the form*

$$S(s) = \begin{pmatrix} s_0 & s_1 & \ldots & & s_{n-1} \\ s_{n-1} & s_0 & s_1 & & s_{n-2} \\ s_{n-2} & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & s_0 & s_1 \\ s_1 & \ldots & s_{n-2} & s_{n-1} & s_0 \end{pmatrix}$$

where each row is a cyclic shift of the row above to the right.

The matrix $S(s)$ is clearly determined by its first row. Therefore, if no confusion arise, the above circulant matrix is also sometimes denoted by $circ(s_0, s_1, \ldots, s_{n-1})$.

Definition 9. *[10] A* real skew right circulant matrix *or simply a* skew circulant matrix *is a matrix of the form*

$$C(c) = \begin{pmatrix} c_0 & c_1 & \ldots & & c_{n-1} \\ -c_{n-1} & c_0 & c_1 & & c_{n-2} \\ -c_{n-2} & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & c_0 & c_1 \\ -c_1 & \ldots & -c_{n-2} & -c_{n-1} & c_0 \end{pmatrix}$$

If no confusion arise, the above skew circulant matrix is sometimes also denoted by $skwcirc(c_0, \ldots, c_{n-1})$.

The next concepts can be seen in [10]. Define the orthogonal (anti-diagonal unit) matrix $J_m \in \mathbb{R}^{m\times m}$ as

$$J_m := \begin{pmatrix} 0 & 0 & \dots & 1 \\ \vdots & & 1 & 0 \\ 0 & \iddots & \vdots & \vdots \\ 1 & \dots & 0 & 0 \end{pmatrix}.$$

The matrix

$$\Gamma_n := \left(\begin{array}{c|c} 1 & \dots \\ \hline \vdots & J_{n-1} \end{array}\right)$$

is an orthogonal cyclic shift matrix (and a left circulant matrix).

It follows that,

$$\Gamma_n = FF^T = F^2,$$

where the entries of the unitary discrete Fourier transform (DFT) matrix $F = (f_{pq})$ are given by

$$f_{pq} := \frac{1}{\sqrt{n}}\omega^{pq},\ p = 0, 1, \dots, n-1,\ q = 0, 1, \dots, n-1,$$

where

$$\omega = \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}.$$

For the orthogonal matrix

$$\Xi_n = \left(\begin{array}{c|c} 1 & \dots \\ \hline \vdots & -J_{n-1} \end{array}\right)$$

it is straightforward to verify that

$$\Xi_n = GG^T,$$

where $G = (g_{pq})$ with

$$g_{pq} = \frac{1}{\sqrt{n}}\omega^{p\left(q+\frac{1}{2}\right)},\quad p = 0, 1, \dots, n-1,\ q = 0, 1, \dots, n-1,$$

is strongly related to the DFT matrix, i.e.,

$$G = diag\left(1, \iota, \ldots, \iota^{n-1}\right) F$$

with

$$\iota = \omega^{\frac{1}{2}} = \cos\frac{\pi}{n} + i\sin\frac{\pi}{n}.$$

Therefore, G is also unitary.

Remark 10. *Let*

$$S = circ\,(s_0, s_1, \ldots, s_{n-1}) := (s_{ij})$$

and

$$C = skwcirc\,(c_0, c_1, \ldots, c_{n-1}) := (c_{ij}),$$

then

$$s_{ij} = \begin{cases} s_{j-i} & 1 \le i \le j \le n \\ s_{n-i+j} & 1 \le j < i \le n \end{cases}$$

and

$$c_{ij} = \begin{cases} c_{j-i} & j \ge i \\ -c_{n-i+j} & i > j. \end{cases} \tag{6}$$

Theorem 11. *Consider*

$$J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Let

$$S = circ\,(s_0, \ldots, s_{n-1}) \tag{7}$$

and

$$\pm\gamma C = skwcirc\,(\pm\gamma c_0, \pm\gamma c_1, \ldots, \pm\gamma c_{n-1}) \tag{8}$$

be a circulant and a skew circulant matrix, respectively. If

$$|c_\ell| \le s_\ell, \quad \textit{for } \ell = 0, \ldots, n-1 \tag{9}$$

then, the matrix M obtained in [16, Remark 15] from S and C is a nonnegative permutative matrix which takes the form

$$N_{\pm}\gamma := \begin{pmatrix} N_0 & N_1 & \dots & & & N_{n-1} \\ JN_{n-1} & N_0 & N_1 & \dots & \dots & N_{n-2} \\ JN_{n-2} & JN_{n-1} & N_0 & \dots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & & \\ JN_2 & \dots & & & & \\ JN_1 & \dots & & & JN_{n-1} & N_0 \end{pmatrix} \tag{10}$$

where, for $j = 0, 1, \dots, n-1$

$$N_j = \begin{pmatrix} \frac{s_j \pm \gamma c_j}{2} & \frac{s_j \mp \gamma c_j}{2} \\ \frac{s_j \mp \gamma c_j}{2} & \frac{s_j \pm \gamma c_j}{2} \end{pmatrix}.$$

Proof. This result is a direct consequence of the condition in the statement and of the construction of the matrix M in [16, Remark 15]. □

To illustrate the above fact we give the following example.

Example 12. *Let*

$$S = \begin{pmatrix} 2 & 4 & 0 & 2 \\ 2 & 2 & 4 & 0 \\ 0 & 2 & 2 & 4 \\ 4 & 0 & 2 & 2 \end{pmatrix} \quad \text{and} \quad C = \begin{pmatrix} -1 & 1 & 0 & 1 \\ -1 & -1 & 1 & 0 \\ 0 & -1 & -1 & 1 \\ -1 & 0 & -1 & -1 \end{pmatrix}$$

be a circulant and a skew circulant matrix whose spectra, respectively, are $\{8, -4, 2+2i, 2-2i\}$ y $\{-1+i\sqrt{2}, -1+i\sqrt{2}, -1-i\sqrt{2}, -1-i\sqrt{2}\}$. Both matrices satisfy (9). Thus, the matrix in (10) becomes

$$M = \begin{pmatrix} \frac{1}{2} & \frac{3}{2} & \frac{5}{2} & \frac{3}{2} & 0 & 0 & \frac{3}{2} & \frac{1}{2} \\ \frac{3}{2} & \frac{1}{2} & \frac{3}{2} & \frac{5}{2} & 0 & 0 & \frac{1}{2} & \frac{3}{2} \\ \frac{1}{2} & \frac{3}{2} & \frac{1}{2} & \frac{3}{2} & \frac{5}{2} & \frac{3}{2} & 0 & 0 \\ \frac{3}{2} & \frac{1}{2} & \frac{3}{2} & \frac{1}{2} & \frac{3}{2} & \frac{5}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{3}{2} & \frac{1}{2} & \frac{3}{2} & \frac{5}{2} & \frac{3}{2} \\ 0 & 0 & \frac{3}{2} & \frac{1}{2} & \frac{3}{2} & \frac{1}{2} & \frac{3}{2} & \frac{5}{2} \\ \frac{3}{2} & \frac{5}{2} & 0 & 0 & \frac{1}{2} & \frac{3}{2} & \frac{1}{2} & \frac{3}{2} \\ \frac{5}{2} & \frac{3}{2} & 0 & 0 & \frac{3}{2} & \frac{1}{2} & \frac{3}{2} & \frac{1}{2} \end{pmatrix}$$

and is a nonnegative permutative matrix with complex spectrum

$$\{8, -4, 2+2i, 2-2i, -1+i\sqrt{2}, -1+i\sqrt{2}, -1-i\sqrt{2}, -1-i\sqrt{2}\}.$$

The following results characterize the circulant and skew circulant spectra.

Theorem 13. *[10] Let* $s = (s_0, \ldots, s_{n-1})$ *and* $S(s) = circ(s_0, \ldots, s_{n-1})$. *Then*

$$S(s) = F^* \Lambda(s) F,$$

with

$$\Lambda(s) = diag(\lambda_0(s), \lambda_1(s), \ldots, \lambda_{n-1}(s))$$

and

$$\lambda_k(s) = \sum_{j=0}^{n-1} s_j \omega^{kj}, \quad k = 0, 1, \ldots, n-1.$$

Theorem 14. *[10] Let* $c = (c_0, \ldots, c_{n-1})$ *and* $C(c) = skwcirc(c_0, \ldots, c_{n-1})$. *Then* $C(c) = C^* \Delta(c) C$, *where*

$$\Delta(c) = diag(\mu_0(c), \mu_1(c), \ldots, \mu_{n-1}(c))$$

and

$$\mu_k(c) = \sum_{j=0}^{n-1} c_j \omega^{\left(k+\frac{1}{2}\right)j}, \quad k = 0, 1, \ldots, n-1.$$

Taking into account that F and G are unitary matrices the next corollary follows.

Corollary 15. *Let* s *and* c *as defined in Theorem 13 and Theorem 14, respectively. Let*

$$v := v(s) = (\lambda_0(s), \lambda_1(s), \ldots, \lambda_{n-1}(s))^T$$

and

$$u := u(c) = (\mu_0(c), \mu_1(c), \ldots, \mu_{n-1}(c))^T.$$

1. *where* $v = \sqrt{n} F s$, *and*

$$s_k = \frac{1}{n} \sum_{j=0}^{n-1} \lambda_j \omega^{-kj}, \quad k = 0, 1, \ldots, n-1.$$

2. *where* $u = \sqrt{n} Gc$, *and*

$$c_k = \frac{1}{n} \sum_{j=0}^{n-1} \mu_j \omega^{-\left(k+\frac{1}{2}\right)j}, \quad k = 0, 1, \ldots, n-1. \tag{11}$$

The following results, deal with conjugate symmetry within the spectrum of $S(s)$ and $C(c)$.

Theorem 16. *[10] Let* $s = (s_0, \ldots, s_{n-1})$ *and* $c = (c_0, \ldots, c_{n-1})$ *be two n-tuples with real entries. Then*

1. $\lambda_{n-k}(s) = \overline{\lambda_k(s)}$, *for* $k = 1, 2, \ldots, n-1$ *and* $\lambda_0(s) = \sum_{j=1}^{n-1} s_j$.
2. $\mu_{n-1-k}(c) = \overline{\mu_k(c)}$, *for* $k = 0, 1, \ldots, n-1$.

Theorem 17. *Let* $\Lambda = \{\lambda_0, \lambda_1, \ldots, \lambda_{n-1}\}$ *and* $\Upsilon = \{\mu_0, \mu_1, \ldots, \mu_{n-1}\}$ *be the sets such that*

1. $\lambda_0 = \rho = \max\{|\lambda_j| : j = 1, 2, \ldots, n\}$ *and*
2. $\lambda_{n-k} = \overline{\lambda_k}$, *for* $k = 1, 2, \ldots, n-1$
3. $\mu_{n-1-k} = \overline{\mu_k}$, *for* $k = 0, 1, \ldots, n-1$,

and consider the sets

$$\mathcal{P} = \left\{ \alpha \in \mathcal{S}_n \colon \alpha = \begin{pmatrix} 0 & 1 & 2 & \cdots & \cdots & n-1 \\ 0 & \ell_1 & \ell_2 & \cdots & \cdots & \ell_{n-1} \end{pmatrix};\ \lambda_{n-\ell_k} = \overline{\lambda}_{\ell_k}, k = 1, 2, \ldots, n-1 \right\}. \tag{12}$$

and

$$\mathcal{Q} = \left\{ \beta \in \mathcal{S}_n \colon \beta = \begin{pmatrix} 0 & 1 & 2 & \cdots & \cdots & n-1 \\ \ell_0 & \ell_1 & \ell_2 & \cdots & \cdots & \ell_{n-1} \end{pmatrix};\ \mu_{n-1-\ell_k} = \overline{\mu}_{\ell_k}, k = 0, 1, \ldots, n-1 \right\}. \tag{13}$$

Let $0 \leq \gamma \leq 1$. *A sufficient condition for the lists* $\Lambda \cup (\pm\gamma)\Upsilon$ *to be realized by a permutative matrix* $M_{\pm\gamma}$ *as in [16, Remark 15] is*

$$\lambda_0 \geq \min_{\alpha \in \mathcal{P}} \max_{0 \leq k \leq 2m} \left\{ \begin{array}{c} -2 \sum_{j=1}^{m} \mathrm{Re}\lambda_{\alpha(\mathrm{j})} \cos \frac{2\mathrm{kj}\pi}{2\mathrm{m}+1} - \\ -2 \sum_{j=1}^{m} \mathrm{Im}\lambda_{\alpha(\mathrm{j})} \sin \frac{2\mathrm{kj}\pi}{2\mathrm{m}+1} \end{array} \right. \tag{14}$$

whenever $n = 2m - 1$

$$\lambda_0 \geq \min_{\alpha\in\mathcal{P}} \max_{0\leq k\leq 2m+1} \begin{cases} -2\sum_{j=1}^{m-1} \mathrm{Re}\lambda_{\alpha(\mathrm{j})} \cos\frac{2\mathrm{kj}\pi}{\mathrm{m+1}} - (-1)^{\mathrm{k}}\lambda_{\mathrm{m}} - \\ \qquad -2\sum_{j=1}^{m-1} \mathrm{Im}\lambda_{\alpha(\mathrm{j})} \sin\frac{2\mathrm{kj}\pi}{\mathrm{m+1}} \end{cases} \tag{15}$$

whenever $n = 2m-2$, *and there exists* $(\alpha, \beta) \in \mathcal{P}\times\mathcal{Q}$, *for all* $k = 0, 1, \ldots, n-1$ *such that the following inequalities hold*

$$\sum_{j=0}^{n-1} \lambda_{\alpha(j)}\omega^{-kj} \geq \left|\sum_{j=0}^{n-1} \mu_{\beta(j)}\omega^{-\left(k+\frac{1}{2}\right)j}\right|. \tag{16}$$

Proof. A set $\Lambda = \{\lambda_0, \lambda_1, \ldots, \lambda_{n-1}\}$ satisfying the conditons 1. and 2. in the statement is called *even conjugate* and by Theorem 4 in [23] a necessary and sufficient condition for Λ to be the spectrum of a real circulant matrix are given by (14) and (15). On the other hand, let us consider the set of permutations (13). For $\beta \in \mathcal{Q}$ the skew circulant matrix whose first row is given by

$$c_\beta = \frac{1}{\sqrt{n}} G^*\beta\,(u)\,, \tag{17}$$

(where $\beta\,(u)$ is considered as in Definition 2) is a real skew circulant matrix and all the components of the vector $u = (\mu_0, \mu_1, \ldots, \mu_{n-1})$ belongs to its spectrum. The conditions in (16) reflect the conditions in 9 as to obtain a nonnegative matrix with the shape in (10). □

From the above facts the following definition can be established.

Definition 18. *Given the sets* $\Upsilon = \{\mu_0, \mu_1, \ldots, \mu_{n-1}\}$ *and* $\Lambda = \{\rho, \lambda_1, \lambda_2, \ldots, \lambda_n\}$ *we say that* Υ *(resp.* Λ*) is skew circulant (resp. circulant) spectrum if the condition* 3. *(resp.* 1 *and* 2.*) in Theorem 17 holds. Note that, if* $\Lambda = \{\rho, \lambda_1, \lambda_2, \ldots, \lambda_n\}$ *is a circulant spectrum, then* $\Lambda \setminus \{\rho\}$ *is a skew circulant spectrum.*

Corollary 19. *Let* $\Lambda = \{1, \lambda_1, \ldots, \lambda_{n-1}\}$ *and* $\Upsilon = \{\mu_0, \mu_1, \ldots, \mu_{n-1}\}$ *be a skew circulant and a circulant spectrum, respectively, with*

$$1 = \max\{|\lambda_j| : j = 1, 2, \ldots, n\}$$

and consider the sets $\mathcal{P}$ and $\mathcal{Q}$ as in (12) and (13), respectively. Let $0 \leq \gamma \leq 1$. A sufficient condition for $\Lambda \cup (\pm\gamma)\,\Upsilon$ to be the spectrum of a permutative nonnegative matrix is:

$$1 \geq \min_{\alpha \in \mathcal{P}} \max_{0 \leq k \leq 2m} \left\{ -2\sum_{j=1}^{m} \mathrm{Re}\lambda_{\alpha(\mathrm{j})} \cos \tfrac{2\mathrm{kj}\pi}{2\mathrm{m}+1} - 2\sum_{\mathrm{j}=1}^{\mathrm{m}} \mathrm{Im}\lambda_{\alpha(\mathrm{j})} \sin \tfrac{2\mathrm{kj}\pi}{2\mathrm{m}+1} \right\}$$

whenever $n = 2m - 1$ and

$$1 \geq \min_{\alpha \in \mathcal{P}} \max_{0 \leq k \leq 2m+1} \left\{ -2\sum_{j=1}^{m-1} \mathrm{Re}\lambda_{\alpha(\mathrm{j})} \cos \tfrac{2\mathrm{kj}\pi}{\mathrm{m}+1} - (-1)^{\mathrm{k}}\,\lambda_{\mathrm{m}} - 2\sum_{\mathrm{j}=1}^{\mathrm{m}-1} \mathrm{Im}\lambda_{\alpha(\mathrm{j})} \sin \tfrac{2\mathrm{kj}\pi}{\mathrm{m}+1} \right\}$$

whenever $n = 2m - 2$.
Moreover, there exists $(\alpha, \beta) \in \mathcal{P} \times \mathcal{Q}$, for all $k = 0, 1, \ldots, n-1$, such that the following inequalities hold

$$\left| \sum_{j=0}^{n-1} \mu_{\beta(j)} \omega^{-\left(k+\frac{1}{2}\right)j} \right| \leq 1 + \sum_{j=1}^{n-1} \lambda_{\alpha(j)} \omega^{-kj} \quad .$$

Thus,

$$\bigcup_{(\alpha,\beta)\in\mathcal{P}\times\mathcal{Q}} \{ \left(\lambda_{\alpha(1)}, \ldots, \lambda_{\alpha(n-1)}, \mu_{\beta(0)}, \ldots, \mu_{\beta(n-1)}\right) : \left| \sum_{j=0}^{n-1} \mu_{\beta(j)} \omega^{-\left(k+\frac{1}{2}\right)j} \right| \leq 1 + \sum_{j=1}^{n-1} \lambda_{\alpha(j)} \omega^{-kj} \}$$

is a complex permutative realizability region for

$$\{1, \lambda_1, \ldots, \lambda_{n-1}, \mu_0, \mu_1, \ldots, \mu_{n-1}\}$$

in terms of $\{\lambda_1 \ldots, \lambda_{n-1}\}$. In particular for

$$r = \min_{\alpha \in \mathcal{P}} \left\{ \sum_{j=1}^{n-1} \lambda_{\alpha(j)} \omega^{-kj} : k = 0, 1, \ldots, n-1 \right\}$$

the set

$$\bigcup_{\beta \in \mathcal{Q}} \{ \left(\mu_{\beta(0)}, \ldots, \mu_{\beta(n-1)}\right) : \left| \sum_{j=0}^{n-1} \mu_{\beta(j)} \omega^{-\left(k+\frac{1}{2}\right)j} \right| \leq 1 + r, \quad k = 0, 1, \ldots, n-1 \}$$

is also a complex permutative realizability region in terms of $\{\lambda_1 \ldots, \lambda_{n-1}\}$.

Remark 20. *By the trace property, for all $(\alpha, \beta) \in \mathcal{P} \times \mathcal{Q}$ the constant diagonal elements of the matrices $C = skwcirc(c_\beta^T)$ and $S = circ(s_\alpha^T)$ obtained from the vectors c_β in (17) and*

$$s_\alpha = \frac{1}{\sqrt{n}} F^* \alpha\left(v\right), \tag{18}$$

(where $\alpha\left(v\right)$ is considered as in Definition 2) coincide, respectively.

The proof of the following result is similar to the proof of Theorem 24 in [16].

Theorem 21. *Let $S = (s_{ij})$ be a nonnegative matrix of order $n+1$ and consider the $C = skwcirc\left(c_0, c_1, \ldots, c_{n-1}\right) := (c_{ij})$ whose spectra (counted with their multiplicities) are $\{\lambda_0, \lambda_1, \ldots, \lambda_n\}$ and $\{\mu_0, \mu_1, \ldots, \mu_{n-1}\}$, respectively. Moreover, suppose that $|c_{ij}| \leq s_{ij}$, $1 \leq i, j \leq n$. Then the nonnegative matrix*

$$M = \begin{pmatrix} \frac{s_{11} \pm \gamma c_{11}}{2} & \frac{s_{11} \mp \gamma c_{11}}{2} & \cdots & \cdots & \frac{s_{1n} \pm \gamma c_{1n}}{2} & \frac{s_{1n} \mp \gamma c_{1n}}{2} & s_{1,n+1} \\ \frac{s_{11} \mp \gamma c_{11}}{2} & \frac{s_{11} \pm \gamma c_{11}}{2} & \cdots & \cdots & \frac{s_{1n} \mp \gamma c_{1n}}{2} & \frac{s_{1n} \pm \gamma c_{1n}}{2} & s_{1,n+1} \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \vdots \\ \frac{s_{n1} \mp \gamma c_{n1}}{2} & \frac{s_{n1} \pm \gamma c_{n1}}{2} & \cdots & \cdots & \frac{s_{nn} \pm \gamma c_{nn}}{2} & \frac{s_{nn} \mp \gamma c_{nn}}{2} & s_{n,n+1} \\ \frac{s_{n1} \pm \gamma c_{n1}}{2} & \frac{s_{n1} \mp \gamma c_{n1}}{2} & \cdots & \cdots & \frac{s_{nn} \mp \gamma c_{nn}}{2} & \frac{s_{nn} \pm \gamma c_{11}}{2} & s_{n,n+1} \\ s_{n+1,1}^1 & s_{n+1,1}^2 & \cdots & \cdots & s_{n+1,n}^1 & s_{n+1,n}^2 & s_{n+1,n+1} \end{pmatrix}, \tag{19}$$

where

$$s_{n+1,i}^1 + s_{n+1,i}^2 = s_{n+1,i} \qquad 1 \leq i \leq n$$

realizes the list

$$\{\lambda_0, \lambda_1, \ldots, \lambda_n, \pm\gamma\mu_0, \pm\gamma\mu_1, \ldots, \pm\gamma\mu_{n-1}\}.$$

The next problem can be formulated:

Problem 22. *Given the skew circulant spectra $\Upsilon = \{\mu_0, \mu_1, \ldots, \mu_{n-1}\}$, and $\{\lambda_1, \ldots, \lambda_n\}$ under which conditions on ρ does there exist a realizable circulant spectrum $\Lambda = \{\rho, \lambda_1, \ldots, \lambda_n\}$ such that $\Lambda \cup \pm\gamma\Upsilon$ is realizable, for all $\gamma \in [0, 1]$?*

In order to give an answer to this problem, we need to recall the following facts:

1. From formula (11) at item 2. in Corollary 15 and Theorem 17, the following inequalities can be easily obtained

$$|c_\ell| \leq \max_{0\leq k\leq n-1} \frac{1}{n}\left|\sum_{j=0}^{n-1} \mu_j \omega^{-\left(k+\frac{1}{2}\right)j}\right| \quad , \quad \ell = 0, 1, \ldots, n-1.$$

2. Let

$$\chi = \max_{0\leq k\leq n-1} \frac{1}{n}\left|\sum_{j=0}^{n-1} \mu_j \omega^{-\left(k+\frac{1}{2}\right)j}\right|.$$

3. Eqs. (14) and (15) give necessary and sufficient conditions for

$$\widetilde{\Lambda} = \{\rho - (n+1)\,\chi, \lambda_1, \lambda_2, \ldots, \lambda_n\}$$

to be the spectrum of a nonnegative circulant matrix $B = (b_{ij})$. Furthermore, by Perron Frobenius Theory (see [1]), B is an irreducible nonnegative matrix and the positive $(n+1)$ -dimensional eigenvector $\mathbf{e} = (1, \ldots, 1)^T$ is associated to the eigenvalue $\rho - (n+1)\,\chi$ of B.
4. By Brauer Theorem (see [24]) the matrix $R = B + \chi \mathbf{e}\mathbf{e}^T$ has spectrum

$$\widetilde{\Lambda} \setminus \{\rho - (n+1)\,\chi\} \cup \left\{\rho - (n+1)\,\chi + \chi \mathbf{e}^T\mathbf{e} = \rho\right\} = \Lambda.$$

5. Moreover, for the (i,j)-entry of $R := (r_{ij})$ we have $r_{ij} = b_{ij} + \chi \geq \chi \geq |c_k|\,, \quad k = 0, 1, \ldots, n-1.$
6. By Theorem 21 a nonnegative matrix of order $2n+1$ of the form of M in (19) can be constructed from the matrices R and $skwcirc\,(c_0, c_1, \ldots, c_{n-1})$.

In consequence, the following result can be proved.

Proposition 23. *Let $\Upsilon = \{\mu_0, \mu_1, \ldots, \mu_{n-1}\}$, and $\{\lambda_1, \ldots, \lambda_n\}$ be skew circulant spectra. If there exists a nonnegative circulant matrix with spectrum*

$$\widetilde{\Lambda} = \{\rho - (n+1)\,\chi, \lambda_1, \lambda_2, \ldots, \lambda_n\},$$

where χ was defined in previous item 2 then, there exists a nonnegative matrix with spectrum $\Lambda \cup \pm\gamma\Upsilon$ where $\gamma \in [0,1]$ and

$$\Lambda = \{\rho, \lambda_1, \lambda_2, \ldots, \lambda_n\}.$$

In particular, if $\lambda_\ell = 0, \quad \ell = 1, \ldots, n-1$ the matrix M in (19) obtained from $C := skwcirc(c_0, \ldots, c_{n-1})$, where c_j for $j = 1 \ldots n-1$ are defined as in Corollary 15 and the rank one matrix

$$S = circ(\frac{\rho}{n+1}, \ldots, \frac{\rho}{n+1})$$

is permutative.

The following example shows that the condition in the above proposition can be weakened.

Example 24. *Let $\Upsilon = \left\{7, \frac{5+i\sqrt{3}}{2}, \frac{5-i\sqrt{3}}{2}\right\}$, $\{1, 2+5i, 2-5i\}$ be two skew circulant spectra. The matrix*

$$C = \begin{pmatrix} 4 & -2 & 1 \\ -1 & 4 & -2 \\ 2 & -1 & 4 \end{pmatrix}.$$

is a skew circulant matrix with spectrum Υ, then the value of χ is $max\{4, 2, 1\} = 4$. For the condition in Proposition 23 we need at least

$$0 \leq \rho - (n+1)\chi = \rho - 4 \cdot 4 = \rho - 16.$$

Nevertheless the set

$$\Omega = \left\{15, 1, 7, 2+5i, 2-5i, \frac{5+i\sqrt{3}}{2}, \frac{5-i\sqrt{3}}{2}\right\}$$

can be partitioned into the circulant spectrum $\Lambda = \{15, 1, 2+5i, 2-5i\}$ and the skew circulant spectrum Υ. The former is realized by the circulant matrix S where

$$S = \begin{pmatrix} 5 & 6 & 3 & 1 \\ 1 & 5 & 6 & 3 \\ 3 & 1 & 5 & 6 \\ 6 & 3 & 1 & 5 \end{pmatrix}.$$

So, the union Ω is realized by the nonnegative matrix

$$M = \begin{pmatrix} \frac{9}{2} & \frac{1}{2} & 2 & 4 & 2 & 1 & 1 \\ \frac{1}{2} & \frac{9}{2} & 4 & 2 & 1 & 2 & 1 \\ 0 & 1 & \frac{9}{2} & \frac{1}{2} & 2 & 4 & 3 \\ 1 & 0 & \frac{1}{2} & \frac{9}{2} & 4 & 2 & 3 \\ \frac{5}{2} & \frac{1}{2} & 0 & 1 & \frac{9}{2} & \frac{1}{2} & 6 \\ \frac{1}{2} & \frac{5}{2} & 1 & 0 & \frac{1}{2} & \frac{9}{2} & 6 \\ 3 & 3 & 3 & 0 & 1 & 0 & 5 \end{pmatrix}.$$

The next definition generalizes the definitions of circulant and skew ciculant matrices.

Definition 25. *Let C be a matrix of order n. Then $C = (c_{ij})$ is an absolutely circulant matrix if the absolute value matrix of C, $|C| := (|c_{ij}|)$ is circulant, being the diagonal elements either non positive or all non negative. If C is an absolutely circulant matrix with first row $(c_0, c_1, \ldots, c_{n-1})$ we write,*

$$C = \mathit{abscirc}\,(c_0, c_1, \ldots, c_{n-1})\,.$$

Lemma 26. *Let $C = \mathit{abscirc}\,(c_0, c_1, \ldots, c_{n-1}) := (c_{ij})$. Then*

$$c_{ij} = \begin{cases} \pm c_{j-i} & j \geq i \\ \pm c_{n-i+j} & i > j. \end{cases} \tag{20}$$

Proof. Is a direct consequence of Definition 25. □

The example below gives an absolutely circulant matrix which is neither circulant nor skew circulant.

Example 27. *Let us consider*

$$C = \begin{pmatrix} 1 & -2 & 3 \\ 3 & 1 & 2 \\ -2 & -3 & 1 \end{pmatrix}.$$

It is clear that C is an absolutely circulant matrix which is neither circulant nor skew circulant. In fact, $|C| = \mathit{circ}(1, 2, 3)$.

Proposition 28. *Let $S = \mathit{circ}(s_0, s_1, \ldots, s_{n-1})$ and $C = \mathit{abscirc}\,(c_0, c_1, \ldots, c_{n-1})$ be a circulant and an absolutely circulant matrix, respectively such that*

$$|c_\ell| \leq s_\ell,\ \ \ell = 0, \ldots, n-1.$$

Then the matrix M in [16, Remark 15] is a nonnegative permutative matrix.

Proof. This proof is constructive and it is an analogous proof to the one presented at Theorem 11. □

From the above definition and properties we now formulate the following question:

Problem 29. *Could it be obtained a spectral characterization for an absolutely circulant matrix in terms of matrices related to discrete Fourier transform (DFT)?*

5. Guo Perturbations

In this section it is introduced a Guo's realizability- preserving perturbation result. The reader should refer to Guo in 1997 [6, Theorem 2.1,Theorem 3.1]. In [6] the author proposed the following question:

For a list $\sigma = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ that is symmetrically realizable (which means that σ is the spectrum of a symmetric nonnegative matrix), and $t > 0$, whether (or not) the perturbed list $\sigma_t = \{\lambda_1 + t, \lambda_2 \pm t, \lambda_3, \ldots, \lambda_n\}$ is also symmetrically realizable?

In [23] the authors gave an affirmative answer to this question for the special case when the realizing matrix is circulant. Next we use the above mentioned result to prove the following.

Theorem 30. *Let $n = 2m + 2$ and consider the n-tuples*

$$v = v(s) = \left(\lambda_0, \lambda_1, \lambda_2, \ldots, \lambda_m, \lambda_{m+1}, \overline{\lambda}_m, \ldots, \overline{\lambda}_2, \overline{\lambda}_1\right)^T$$

and

$$u = u(c) = (\mu_0, \mu_1, , \ldots, \overline{\mu}_1, \overline{\mu}_0)^T$$

where $s = (s_0, s_1, \ldots, s_{n-2}, s_{n-1})$ and $c = (c_0, c_1, \ldots, c_{n-2}, c_{n-1})$ are such that the equations in Corollary 15 and (9) hold. Let t_0 be a positive real number. Then, there exists a nonnegative permutative matrix M realizing the list $v(s, t_0) \cup u(c)$, where

$$v(s, t_0) = \left(\lambda_0 + t_0, \lambda_1, \lambda_2, \ldots, \lambda_m, \lambda_{m+1} \pm t_0, \overline{\lambda}_m, \ldots, \overline{\lambda}_2, \overline{\lambda}_1\right)^T$$

Proof. Let $s = (s_1, \ldots, s_n)^T$ and $c = (c_1, \ldots, c_n)^T$. By Corollary 15

$$s = \frac{1}{\sqrt{n}} F^* v(s)$$

and

$$c = \frac{1}{\sqrt{n}} G^* u(c),$$

respectively. In consequence,

$$c = \frac{1}{\sqrt{n}} F^* D^* u(c),$$

where

$$D^* = diag(1, \bar{\iota}, \ldots, \bar{\iota}^{n-1}).$$

By the condition in (9), the following holds

$$s + c = \frac{1}{\sqrt{n}} F^*(v(s) + D^* u(c)) \geq 0$$

and

$$s - c = \frac{1}{\sqrt{n}} F^*(v(s) - D^* u(c)) \geq 0.$$

Note that $v(s, t_0)$ is a circulant spectrum, where the first row of the associated circulant matrix (that is, a matrix whose spectrum is $v(s, t_0)$) is given by

$$s(t_0) \quad = \quad \frac{1}{\sqrt{n}} F^* v(s, t_0) \tag{21}$$

and

$$c \quad = \quad \frac{1}{\sqrt{n}} F^* D^* u(c). \tag{22}$$

Now we prove that the condition in (9) still holds. Let $\mathbf{e}_1$ and $\mathbf{e}_{m+2}$ be the first and the $(m+2)$-nd canonical vectors of $\mathbb{C}^n$. Adding the expressions in (21) and in (22), and afterwards taking the difference between the previous expressions we obtain

$$\begin{aligned}
s(t_0) + c &= \frac{1}{\sqrt{n}} F^*(v(s, t_0) + D^* u(c)) \\
&= \frac{1}{\sqrt{n}} F^*(v(s) + D^* u(c) + t_0 \mathbf{e}_1 \pm t_0 \mathbf{e}_{m+2}) \\
&= \frac{1}{\sqrt{n}} F^*(v(s) + D^* u(c)) + t_0 \frac{1}{\sqrt{n}} F^* \mathbf{e}_1 \pm t_0 \frac{1}{\sqrt{n}} F^* \mathbf{e}_{m+2} \\
&\geq t_0 \frac{1}{\sqrt{n}} F^* \mathbf{e}_1 \pm t_0 \frac{1}{\sqrt{n}} F^* \mathbf{e}_{m+2} \\
&\geq 0,
\end{aligned}$$

and

$$
\begin{aligned}
s(t_0) - c &= \frac{1}{\sqrt{n}} F^*(v(s,t_0) - D^*u(c)) \\
&= \frac{1}{\sqrt{n}} F^*(v(s) - D^*u(c) + t_0\mathbf{e}_1 \pm t_0\mathbf{e}_{m+2}) \\
&= \frac{1}{\sqrt{n}} F^*(v(s) - D^*u(c)) + t_0\frac{1}{\sqrt{n}} F^*\mathbf{e}_1 \pm t_0\frac{1}{\sqrt{n}} F^*\mathbf{e}_{m+2} \\
&\geq t_0\frac{1}{\sqrt{n}} F^*\mathbf{e}_1 \pm t_0\frac{1}{\sqrt{n}} F^*\mathbf{e}_{m+2} \\
&\geq 0.
\end{aligned}
$$

The last inequalities are due to [23, Theorem 7], where is it shown that, if $\beta \geq 0$ then

$$\beta\frac{1}{\sqrt{n}} F^*(\mathbf{e}_1) \pm \beta\frac{1}{\sqrt{n}} F^*(\mathbf{e}_{m+2}) \geq 0.$$

Finally, the matrix N as in (10) obtained from the matrices $S(t_0) := circ(s(t_0))$ and $C := skwcirc(c)$ is the matrix M required in the statement. □

Acknowledgments. Enide Andrade was supported in part by the Portuguese Foundation for Science and Technology (FCT-Fundação para a Ciência e a Tecnologia), through CIDMA - Center for Research and Development in Mathematics and Applications, within project UID/MAT/04106/2013. M. Robbiano was partially supported by project VRIDT UCN 170403003.
The authors would like to thank the referee for his/her carefully reading and his/her suggestions/comments that improved the text of this work.

[1] A. Berman, R. J. Plemmons, Nonnegative Matrices in the Mathematical Sciences, Series: Classics in Applied Mathematics, SIAM, 1994.

[2] A. Borobia, J. Moro, R. L. Soto, Negativity compensation in the nonnegative inverse eigenvalue problem, Lin. Algebra Appl. 393 (2004): 73−89.

[3] P.J. Davis, Circulant matrices, John Wiley and Sons, New York, Chichester, Brisbane, Toronto (1979).

[4] P. D. Egleston, T. D. Lenker, S. K. Narayan, The nonnegative inverse eigenvalue problem, Lin. Algebra Appl. 379 (2004): 475490

[5] Sh. Friedland, On an inverse problem for nonnegative and eventually nonnegative matrices, Israel T. Math. 29 (1978): 1, 43-60.

[6] W. Guo, Eigenvalues of nonnegative matrices, Lin. Algebra Appl. 266 (1997): 261-270.

[7] A. Leal- Duarte, C.R. Johnson, Resolution of the symmetric nonnegative inverse eigenvalue problem for matrices subordinate to a bipartite graph, Positivity 8 (2004): 209-2013.

[8] X. Hu, C. R. Johnson, C. E. Davis, and Y. Zhang, Ranks of permutative matrices, Spec. Matrices, 4, (2016): 233-246.

[9] C. R. Johnson, C. Marijuán, P. Paparella, M. Pisonero, The NIEP, https://arxiv.org/pdf/1703.10992.pdf (2017).

[10] H. Karner, J. Schneid, C. W. Ueberhuber, Spectral decomposition of real circulant matrices. Lin. Algebra Appl. 367 (2003): 301-311.

[11] Ki Hang Kim, S. N. Ormes, W. F. Roush, The spectra of nonnegative integer matrices via formal power series, J. Amer. Math. Soc. 13, 4 (2000): 773-806 (electronic).

[12] A. N. Kolmogorov, Markov chains with a countable number of possible states, Bull. Moskow Gosu-darstvennogo Univ. Mat. Meh., 1(3)(1937): 1-16.

[13] T. Laffey, H. Šmigoc, Nonnegative realization of spectra having negative real parts, Lin. Algebra Appl. 416, 1 (2006): 148-159.

[14] R. Loewy, D. London, A note on an inverse problem for nonnegative matrices, Linear and Multilinear Algebra 6 (1978/79): 83-90.

[15] R. Loewy. A note on the real nonnegative inverse eigenvalue problem, Electron. J. Linear Algebra, 31 (2016): 765-773.

[16] C. Manzaneda, E. Andrade, M. Robbiano. Realizable lists via the spectra of structured matrices, Lin. Algebra Appl. 534 (2017): 51-72.

[17] J. Torre Mayo, M. R. Abril Raymund, E. Alarcia Estvez, C. Marijuán, M. Pisonero, The nonnegative inverse problema from the coefficients of the characteristic polynomial EBL digraphs, Lin. Algebra Appl. 426, (2007): 729-773.

[18] M. E. Meehan, Some results on matrix spectra, Phd thesis, National University of Ireland, Dublin, 1998.

[19] H. Minc, Non-negative Matrices, John Wiley and Sons, New York, 1988.

[20] G. N. Oliveira, Sobre matrizes estocásticas e duplamente estocásticas, PhD Thesis, (1968), Coimbra.

[21] P. Paparella, Realizing Suleimanova-type Spectra via Permutative Matrices, Electron. Journal of Linear Algebra, Volume 31, (2016): 306-312.

[22] O. Rojo, H. Soto, Some results on symmetric circulant matrices and on symmetric centrosymmetric matrices, Lin. Algebra Appl., 392, (2004): 211-233.

[23] O. Rojo, R. L. Soto. Guo perturbations for symmetric nonnegative circulant matrices, Lin. Algebra Appl. 431 (2009): 594-607.

[24] O. Rojo, R. L. Soto. Applications of a Brauer Theorem in the nonnegative inverse eigenvalue problem, Lin. Algebra Appl. 416 (2007): 1-18.

[25] G. W. Soules, Constructing symmetric nonnegative matrices, Linear and Multilinear Algebra 13-3 (1983): 241-251.

[26] H. R. Suleĭmanova, Stochastic matrices with real characteristic numbers, Doklady, Akad. Nuk SSSR (N. S.) 66 (1949): 343-345.