Hyers-Ulam-Rassias Stability of Nonlinear Integral Equations Through the Bielecki Metric[1]

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Abstract: We analyse different kinds of stabilities for classes of nonlinear integral equations of Fredholm and Volterra type. Sufficient conditions are obtained in order to guarantee Hyers-Ulam-Rassias, σ-semi-Hyers-Ulam and Hyers-Ulam stabilities for those integral equations. Finite and infinite intervals are considered as integration domains. Those sufficient conditions are obtained based on the use of fixed point arguments within the framework of the Bielecki metric and its generalizations. The results are illustrated by concrete examples.

Keywords: Hyers-Ulam stability; σ-semi-Hyers-Ulam stability; Hyers-Ulam-Rassias stability; Banach fixed point theorem; Bielecki metric; nonlinear integral equation

1 INTRODUCTION

Hyers-Ulam and Hyers-Ulam-Rassias stabilities are having a great attention recently. In part, this is because of their potential applicability in model situations where although we can not expect to easily obtain the exact solution of the problem we may expect to obtain an approximate solution which should be stable in a certain specific sense. This is being done for a huge number of different type of equations. Among those, we point out functional equations, differential equations and integral equations. This is also connected with the applicability of those equations in different areas of the knowledge like chemical reactions, diffraction theory, elasticity, fluid flow, heat conduction and population dynamic (cf. [1, 2, 3, 4, 5, 6, 7, 9, 8, 10, 12, 13, 11, 14, 15, 16, 19, 20, 21, 22, 23, 24, 25, 26, 29, 28, 27, 30, 31, 32, 33, 34, 35, 37, 17, 39, 40]).

In a sense, we may say that the first results of stability of this type for functional equations were originated from a famous question raised by S. M. Ulam, in 1940, about to discover when a solution of an equation differing "slightly" from a given one must be somehow near to the solution of the given equation. D. H. Hyers obtained a partial answer to the question of S. M. Ulam, for Banach spaces, in the case of the additive Cauchy equation ($f(x+y) = f(x)+f(y)$, cf. [25]), within the framework of (real) Banach spaces. This gave rise to what we now designate as the *Hyers-Ulam stability* of the additive Cauchy equation. Afterwards, Th. M. Rassias (see [36]) introduced new ideas e.g., by proposing to consider unbounded right-hand sides in the involved inequalities, depending on certain functions, introducing therefore the so-called *Hyers-Ulam-Rassias stability*.

In the present work we will analyse Hyers-Ulam-Rassias, σ-semi-Hyers-Ulam and Hyers-Ulam stabilities for the following class of integral equation with two delays

$$y(x) = f\left(x, y(x), y(\alpha(x)), \int_a^b k(x,\tau,y(\tau),y(\beta(\tau)))d\tau\right), \qquad x \in [a,b], \tag{1.1}$$

as well as for the Volterra integral equation

$$y(x) = f\left(x, y(x), y(\alpha(x)), \int_a^x k(x,\tau,y(\tau),y(\beta(\tau)))d\tau\right), \qquad x \in [a,b]. \tag{1.2}$$

Here, for starting, a and b are fixed real numbers, $f : [a,b] \times \mathbb{C} \times \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ and $k : [a,b] \times [a,b] \times \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ are continuous functions, and $\alpha, \beta : [a,b] \to [a,b]$ are continuous delay functions which therefore fulfill $\alpha(\tau) \leq \tau$ and $\beta(\tau) \leq \tau$ for all $\tau \in [a,b]$.

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The formal definition of the above mentioned stabilities are now introduced for the integral equation (1.1). If for each function y satisfying

$$\left|y(x) - f\left(x, y(x), y(\alpha(x)), \int_a^b k(x,\tau,y(\tau),y(\beta(\tau)))d\tau\right)\right| \leq \sigma(x), \qquad x \in [a,b],$$

where σ is a non-negative function, there is a solution y_0 of the integral equation and a constant $C > 0$ independent of y and y_0 such that

$$|y(x) - y_0(x)| \leq C\,\sigma(x),$$

for all $x \in [a,b]$, then we say that the integral equation (1.1) has the *Hyers-Ulam-Rassias stability*.

If for each function y satisfying

$$\left|y(x) - f\left(x, y(x), y(\alpha(x)), \int_a^b k(x,\tau,y(\tau),y(\beta(\tau)))d\tau\right)\right| \leq \theta, \qquad x \in [a,b],$$

where $\theta \geq 0$, there is a solution y_0 of the integral equation and a constant $C > 0$ independent of y and y_0 such that

$$|y(x) - y_0(x)| \leq C\,\theta,$$

for all $x \in [a,b]$, then we say that the integral equation has the *Hyers-Ulam stability*.

We also use a stability in-between the two just mentioned stabilities of Hyers-Ulam-Rassias and Hyers-Ulam, introduced in [11], in the following way:

Definition 1.1 *Let σ a non-decreasing function defined on $[a,b]$. If for each function y satisfying*

$$\left|y(x) - f\left(x, y(x), y(\alpha(x)), \int_a^b k(x,\tau,y(\tau),y(\beta(\tau)))d\tau\right)\right| \leq \theta, \qquad x \in [a,b], \tag{1.3}$$

where $\theta \geq 0$, there is a solution y_0 of the integral equation (1.1) and a constant $C > 0$ independent of y and y_0 such that

$$|y(x) - y_0(x)| \leq C\,\sigma(x), \qquad x \in [a,b] \tag{1.4}$$

then we say that the integral equation (1.1) has the σ-semi-Hyers-Ulam stability.

Typically, the techniques to study the stability of functional equations use a combination of fixed point results with a generalized metric in an appropriate framework. Thus, we shall recall the definition of a generalized metric on a nonempty set X.

Definition 1.2 *A function $d : X \times X \to [0, +\infty]$ is called a generalized metric on X if and only if d satisfies the following three propositions:*

i) $d(x,y) = 0$ if and only if $x = y$;

ii) $d(x,y) = d(y,x)$ for all $x, y \in X$;

iii) $d(x,z) \leq d(x,y) + d(y,z)$ for all $x, y, z \in X$.

We also recall that within the framework of generalized metrics, the well-known *Banach Fixed Point Theorem* also holds true.

Theorem 1.3 *Let (X,d) be a generalized complete metric space and $T : X \to X$ a strictly contractive operator with a Lipschitz constant $L < 1$. If there exists a nonnegative integer k such that $d(T^{k+1}x, T^k x) < \infty$ for some $x \in X$, then the following three propositions hold true:*

i) the sequence $(T^n x)_{n\in\mathbb{N}}$ converges to a fixed point x^* of T;

ii) x^* is the unique fixed point of T in $X^* = \{y \in X : d(T^k x, y) < \infty\}$;

iii) if $y \in X^*$, then

$$d(y, x^*) \leq \frac{1}{1-L}\, d(Ty, y). \tag{1.5}$$

2 HYERS-ULAM-RASSIAS STABILITY IN THE FINITE INTERVAL CASE

The present section is devoted to present sufficient conditions for the Hyers-Ulam-Rassias stability of the integral equation (1.1), where $x \in [a, b]$, for some fixed real numbers a and b.

We are interested in considering the Bielecki metric. Namely, taking a positive constant $p > 0$, we will be using the space $C_p([a, b])$ of continuous functions $u : [a, b] \to \mathbb{C}$ endowed with the Bielecki metric

$$d_p(u, v) = \sup_{x \in [a,b]} \frac{|u(x) - v(x)|}{e^{p(x-a)}}. \tag{2.6}$$

Anyway, in a more global sense, we will also consider the space $C([a, b])$ of continuous functions on $[a, b]$, endowed with a generalization of the Bielecki metric

$$d(u, v) = \sup_{x \in [a,b]} \frac{|u(x) - v(x)|}{\sigma(x)}, \tag{2.7}$$

where σ is a non-decreasing continuous function $\sigma : [a, b] \to (0, \infty)$. We recall that $(C_p([a, b]), d_p)$ and $(C([a, b]), d)$ are complete metric spaces (cf., [18], [38]).

Theorem 2.1 *Let $\alpha, \beta : [a, b] \to [a, b]$ be continuous delay functions with $\alpha(t) \leq t$ and $\beta(t) \leq t$ for all $t \in [a, b]$ and $\sigma : [a, b] \to (0, \infty)$ a non-decreasing function. In addition, suppose that there is $\eta \in \mathbb{R}$ such that*

$$\int_a^b \sigma(\tau) d\tau \leq \eta \sigma(x),$$

for all $x \in [a, b]$. Moreover, suppose that $f : [a, b] \times \mathbb{C} \times \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ is a continuous function satisfying the Lipschitz condition

$$|f(x, u(x), u(\alpha(x)), g(x)) - f(x, v(x), v(\alpha(x)), h(x))| \leq M \left(|u(x) - v(x)| + |u(\alpha(x)) - v(\alpha(x))| + |g(x) - h(x)|\right) \tag{2.8}$$

with $M > 0$ and the kernel $k : [a, b] \times [a, b] \times \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ is a continuous kernel function satisfying the Lipschitz condition

$$|k(x, t, u(t), u(\beta(t))) - k(x, t, v(t), v(\beta(t)))| \leq L|u(\beta(t)) - v(\beta(t))| \tag{2.9}$$

with $L > 0$.

If $y \in C([a, b])$ is such that

$$\left| y(x) - f\left(x, y(x), y(\alpha(x)), \int_a^b k(x, \tau, y(\tau), y(\beta(\tau)))d\tau\right)\right| \leq \sigma(x), \qquad x \in [a, b], \tag{2.10}$$

and $M(2 + L\eta) < 1$, then there is a unique function $y_0 \in C([a, b])$ such that

$$y_0(x) = f\left(x, y_0(x), y_0(\alpha(x)), \int_a^b k(x, \tau, y_0(\tau), y_0(\beta(\tau)))d\tau\right)$$

and

$$|y(x) - y_0(x)| \leq \frac{1}{1 - M(2 + L\eta)} \sigma(x) \tag{2.11}$$

for all $x \in [a, b]$.

This means that under the above conditions, the integral equation (1.1) has the Hyers-Ulam-Rassias stability.

Proof. We will consider the operator $T : C([a, b]) \to C([a, b])$, defined by

$$(Tu)(x) = f\left(x, u(x), u(\alpha(x)), \int_a^b k(x, \tau, u(\tau), u(\beta(\tau)))d\tau\right),$$

for all $x \in [a,b]$ and $u \in C([a,b])$.

Under the present conditions, we will deduce that the operator T is strictly contractive with respect to the metric (2.7). Indeed, for all $u, v \in C([a,b])$, we have,

$$
\begin{aligned}
& d(Tu, Tv) \\
= & \sup_{x\in[a,b]} \frac{|(Tu)(x) - (Tv)(x)|}{\sigma(x)} \\
= & \sup_{x\in[a,b]} \frac{1}{\sigma(x)} \left| f\left(x, u(x), u(\alpha(x)), \int_a^b k(x,\tau,u(\tau),u(\beta(\tau)))d\tau\right) - f\left(x, v(x), v(\alpha(x)), \int_a^b k(x,\tau,y(\tau),y(\beta(\tau)))d\tau\right)\right| \\
\leq & M \sup_{x\in[a,b]} \frac{1}{\sigma(x)} \left\{ |u(x)-v(x)| + |u(\alpha(x)) - v(\alpha(x))| + \left| \int_a^b k(x,\tau,u(\tau),u(\beta(\tau)))d\tau - \int_a^b k(x,\tau,v(\tau),v(\beta(\tau)))d\tau \right| \right\} \\
\leq & M \sup_{x\in[a,b]} \frac{1}{\sigma(x)} \left\{ |u(x)-v(x)| + |u(\alpha(x)) - v(\alpha(x))| + \int_a^b |k(x,\tau,u(\tau),u(\beta(\tau))) - k(x,\tau,v(\tau),v(\beta(\tau)))| \, d\tau \right\} \\
\leq & M \sup_{x\in[a,b]} \frac{1}{\sigma(x)} \left\{ |u(x)-v(x)| + |u(\alpha(x)) - v(\alpha(x))| + L\int_a^b |u(\beta(\tau)) - v(\beta(\tau))| \, d\tau \right\} \\
\leq & M \left\{ 2\sup_{x\in[a,b]} \frac{|u(x)-v(x)|}{\sigma(x)} + L \sup_{x\in[a,b]} \frac{1}{\sigma(x)} \int_a^b |u(\beta(\tau)) - v(\beta(\tau))| \, d\tau \right\} \\
= & M \left\{ 2\sup_{x\in[a,b]} \frac{|u(x)-v(x)|}{\sigma(x)} + L \sup_{x\in[a,b]} \frac{1}{\sigma(x)} \int_a^b \sigma(\tau)\frac{|u(\beta(\tau)) - v(\beta(\tau))|}{\sigma(\tau)} d\tau \right\} \\
\leq & M \left\{ 2\sup_{x\in[a,b]} \frac{|u(x)-v(x)|}{\sigma(x)} + L \sup_{\tau\in[a,b]} \frac{|u(\tau)-v(\tau)|}{\sigma(\tau)} \sup_{x\in[a,b]} \frac{1}{\sigma(x)} \int_a^b \sigma(\tau) d\tau \right\} \\
\leq & M \left\{ 2d(u,v) + Ld(u,v) \sup_{x\in[a,b]} \frac{\eta\sigma(x)}{\sigma(x)} \right\} \\
= & M\left(2 + L\eta\right) d(u,v).
\end{aligned}
$$

Due to the fact that $M(2 + L\eta) < 1$ it follows that T is strictly contractive. Thus, we can apply the above mentioned Banach Fixed Point Theorem, which ensures that we have the Hyers-Ulam-Rassias stability for the integral equation (1.1). Additionally, we can apply again the Banach Fixed Point Theorem, which guarantees us that

$$d(y, y_0) \leq \frac{1}{1 - M(2 + L\eta)} d(Ty, y). \tag{2.12}$$

From the definition of the metric d and by (2.10) follows that

$$\sup_{x\in[a,b]} \frac{|y(x) - y_0(x)|}{\sigma(x)} \leq \frac{1}{1 - M(2 + L\eta)} \tag{2.13}$$

and consequently (2.11) holds. □

Corollary 2.2 *Let $\alpha, \beta : [a,b] \to [a,b]$ be continuous delay functions with $\alpha(t) \leq t$ and $\beta(t) \leq t$ for all $t \in [a,b]$. Moreover, suppose that $f : [a,b] \times \mathbb{C} \times \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ is a continuous function satisfying the Lipschitz condition (2.8), with $M > 0$, and the kernel $k : [a,b] \times [a,b] \times \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ is a continuous kernel function satisfying the Lipschitz condition (2.9), with $L > 0$.*

If $y \in C_p([a,b])$ is such that

$$\left| y(x) - f\left(x, y(x), y(\alpha(x)), \int_a^b k(x,\tau,y(\tau),y(\beta(\tau)))d\tau\right)\right| \leq e^{p(x-a)}, \qquad x \in [a,b], \tag{2.14}$$

and $M\left(2+\frac{L}{p}\left(e^{p(b-a)}-1\right)\right)<1$, *then there is a unique function* $y_0 \in C_p([a,b])$ *such that*

$$y_0(x) = f\left(x, y_0(x), y_0(\alpha(x)), \int_a^b k(x,\tau,y_0(\tau),y_0(\beta(\tau)))d\tau\right)$$

and

$$|y(x)-y_0(x)| \leq \frac{p}{p-2pM-ML\left(e^{p(b-a)}-1\right)}\, e^{p(x-a)} \tag{2.15}$$

for all $x \in [a,b]$.

This means that under the above conditions, the integral equation (1.1) has the Hyers-Ulam-Rassias stability.

Proof. This is a direct consequence of the previous theorem when considering $\sigma(x) = e^{p(x-a)}$ and realizing that

$$\int_a^b e^{p(\tau-a)}d\tau \leq \eta e^{p(x-a)}, \qquad x \in [a,b],$$

for any $\eta \geq (e^{p(b-a)}-1)/p$. □

3 σ-SEMI-HYERS-ULAM AND HYERS-ULAM STABILITIES IN THE FINITE INTERVAL CASE

The present section is devoted to present sufficient conditions for the σ-semi-Hyers-Ulam and Hyers-Ulam stabilities of the integral equation (1.1) (and where we continue to use both the metrics (2.6) and (2.7)).

Theorem 3.1 *Let* $\sigma(x) = e^{p(x-a)}$ *(a non-decreasing function defined on* $[a,b]$ *with* $p>0$*),* $\alpha,\beta : [a,b] \to [a,b]$ *be continuous delay functions (with* $\alpha(t) \leq t$ *and* $\beta(t) \leq t$*, for all* $t \in [a,b]$*). Moreover, suppose that* $f : [a,b]\times\mathbb{C}\times\mathbb{C}\times\mathbb{C} \to \mathbb{C}$ *is a continuous function satisfying the Lipschitz condition*

$$|f(x,u(x),u(\alpha(x)),g(x)) - f(x,v(x),v(\alpha(x)),h(x))| \leq M\left(|u(x)-v(x)|+|u(\alpha(x))-v(\alpha(x))|+|g(x)-h(x)|\right) \tag{3.16}$$

with $M>0$ *and the kernel* $k : [a,b]\times[a,b]\times\mathbb{C}\times\mathbb{C} \to \mathbb{C}$ *is a continuous kernel function satisfying the Lipschitz condition*

$$|k(x,t,u(t),u(\beta(t))) - k(x,t,v(t),v(\beta(t)))| \leq L|u(\beta(t))-v(\beta(t))| \tag{3.17}$$

with $L>0$.

If $y \in C_p([a,b])$ *is such that*

$$\left| y(x) - f\left(x, y(x), y(\alpha(x)), \int_a^b k(x,\tau,y(\tau),y(\beta(\tau)))d\tau\right)\right| \leq \theta, \qquad x \in [a,b], \tag{3.18}$$

where $\theta>0$ *and* $M\left(2+\frac{L}{p}\left(e^{p(b-a)}-1\right)\right)<1$, *then there is a unique function* $y_0 \in C_p([a,b])$ *such that*

$$y_0(x) = f\left(x, y_0(x), y_0(\alpha(x)), \int_a^b k(x,\tau,y_0(\tau),y_0(\beta(\tau)))d\tau\right)$$

and

$$|y(x)-y_0(x)| \leq \frac{p\theta}{p-2pM-ML\left(e^{p(b-a)}-1\right)}\, e^{p(x-a)} \tag{3.19}$$

for all $x \in [a,b]$.

This means that under the above conditions, the integral equation (1.1) has the σ*-semi-Hyers-Ulam stability.*

Proof. We will consider the operator $T : C_p([a, b]) \to C_p([a, b])$, defined by

$$(Tu)(x) = f\left(x, u(x), u(\alpha(x)), \int_a^b k(x, \tau, u(\tau), u(\beta(\tau)))d\tau\right),$$

for all $x \in [a, b]$ and $u \in C_p([a, b])$.

By the same procedure as above, we have that T is strictly contractive with respect to the metric (2.6) due to the fact that $M\left(2 + \frac{L}{p}\left(e^{p(b-a)} - 1\right)\right) < 1$. Thus, we can again apply the Banach Fixed Point Theorem, which ensures that we have the σ-semi-Hyers-Ulam stability for the integral equation (1.1) with (3.19) being obtained by the definition of the metric d_p, using (1.5) and (3.18). □

Still having in mind that $e^{p(\cdot - a)}$ is a positive non-decreasing function, and considering an obvious upper bound in (3.19), we directly obtain from the last result the following Hyers-Ulam stability of the integral equation (1.1).

Corollary 3.2 *Let $\alpha, \beta : [a, b] \to [a, b]$ be continuous delay functions and suppose that $f : [a, b] \times \mathbb{C} \times \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ is a continuous function satisfying the Lipschitz condition (3.16), with $M > 0$, and the kernel $k : [a, b] \times [a, b] \times \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ is a continuous kernel function satisfying the Lipschitz condition (3.17), with $L > 0$.*

If $y \in C_p([a, b])$ is such that (3.18), where $\theta > 0$ and $M\left(2 + \frac{L}{p}\left(e^{p(b-a)} - 1\right)\right) < 1$, then there is a unique function $y_0 \in C_p([a, b])$ such that

$$y_0(x) = f\left(x, y_0(x), y_0(\alpha(x)), \int_a^b k(x, \tau, y_0(\tau), y_0(\beta(\tau)))d\tau\right)$$

and

$$|y(x) - y_0(x)| \leq \frac{pe^{p(b-a)}}{p - 2pM - ML\left(e^{p(b-a)} - 1\right)}\,\theta \tag{3.20}$$

for all $x \in [a, b]$.

This means that under the above conditions, the integral equation (1.1) has the Hyers-Ulam stability.

Let us now turn to the generalized metric (2.7).

Theorem 3.3 *Let $\alpha, \beta : [a, b] \to [a, b]$ be continuous delay functions and $\sigma : [a, b] \to (0, \infty)$ a non-decreasing function. In addition, suppose that there is $\eta \in \mathbb{R}$ such that*

$$\int_a^b \sigma(\tau)d\tau \leq \eta\sigma(x), \tag{3.21}$$

for all $x \in [a, b]$. Moreover, suppose that $f : [a, b] \times \mathbb{C} \times \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ is a continuous function satisfying the Lipschitz condition

$$|f(x, u(x), u(\alpha(x)), g(x)) - f(x, v(x), v(\alpha(x)), h(x))| \leq M\left(|u(x) - v(x)| + |u(\alpha(x)) - v(\alpha(x))| + |g(x) - h(x)|\right) \tag{3.22}$$

with $M > 0$ and the kernel $k : [a, b] \times [a, b] \times \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ is a continuous kernel function satisfying the Lipschitz condition

$$|k(x, t, u(t), u(\beta(t))) - k(x, t, v(t), v(\beta(t)))| \leq L|u(\beta(t)) - v(\beta(t))| \tag{3.23}$$

with $L > 0$.

If $y \in C([a, b])$ is such that

$$\left|y(x) - f\left(x, y(x), y(\alpha(x)), \int_a^b k(x, \tau, y(\tau), y(\beta(\tau)))d\tau\right)\right| \leq \theta, \qquad x \in [a, b], \tag{3.24}$$

where $\theta > 0$ and $M\,(2 + L\eta) < 1$, then there is a unique function $y_0 \in C([a,b])$ such that

$$y_0(x) = f\left(x, y_0(x), y_0(\alpha(x)), \int_a^b k(x,\tau,y_0(\tau),y_0(\beta(\tau)))d\tau\right)$$

and

$$|y(x) - y_0(x)| \le \frac{\theta}{(1 - M\,(2 + L\eta))\,\sigma(a)}\,\sigma(x) \tag{3.25}$$

for all $x \in [a,b]$.

This means that under the above conditions, the integral equation (1.1) has the σ-semi-Hyers-Ulam stability.

Proof. We will consider the operator $T : C([a,b]) \to C([a,b])$, defined by

$$(Tu)\,(x) = f\left(x, u(x), u(\alpha(x)), \int_a^b k(x,\tau,u(\tau),u(\beta(\tau)))d\tau\right),$$

for all $x \in [a,b]$ and $u \in C([a,b])$.

By the same procedure as above we derive that T is strictly contractive with respect to the metric (2.7) (due to the fact that $M\,(2 + L\eta) < 1$). Thus, we can again apply the Banach Fixed Point Theorem, which ensures that we have the σ-semi-Hyers-Ulam stability for the integral equation (1.1), with (3.25) being obtained by the definition of the metric d, using (1.5) and (3.24). □

Now, having in mind that σ is a positive non-decreasing function, and considering an upper bound in (3.25), we directly obtain from the last result the following Hyers-Ulam stability of the integral equation (1.1).

Corollary 3.4 *Let $\alpha, \beta : [a,b] \to [a,b]$ be continuous delay functions and $\sigma : [a,b] \to (0,\infty)$ a non-decreasing function. In addition, suppose that there is $\eta \in \mathbb{R}$ such that (3.21) holds (for all $x \in [a,b]$). Moreover, suppose that $f : [a,b] \times \mathbb{C} \times \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ is a continuous function satisfying the Lipschitz condition (3.22) (with $M > 0$) and the kernel $k : [a,b] \times [a,b] \times \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ is a continuous kernel function satisfying the Lipschitz condition (3.23) (for $L > 0$).*

If $y \in C([a,b])$ is such that (3.24) holds (where $\theta > 0$) and $M\,(2 + L\eta) < 1$, then there is a unique function $y_0 \in C([a,b])$ such that

$$y_0(x) = f\left(x, y_0(x), y_0(\alpha(x)), \int_a^b k(x,\tau,y_0(\tau),y_0(\beta(\tau)))d\tau\right)$$

and

$$|y(x) - y_0(x)| \le \frac{\sigma(b)}{(1 - M\,(2 + L\eta))\,\sigma(a)}\,\theta \tag{3.26}$$

for all $x \in [a,b]$

This means that under the above conditions, the integral equation (1.1) has the Hyers-Ulam stability.

4 HYERS-ULAM-RASSIAS STABILITY FOR VOLTERRA INTEGRAL EQUATION IN THE FINITE INTERVAL CASE

In this section, we will derive sufficient conditions for the Hyers-Ulam-Rassias stability of the Volterra integral equation (1.2). We will continue to use the metrics (2.6) and (2.7).

Thus, we will now be dealing with the integral equation

$$y(x) = f\left(x, y(x), y(\alpha(x)), \int_a^x k(x,\tau,y(\tau),y(\beta(\tau)))d\tau\right), \qquad x \in [a,b],$$

where, for starting, a and b are fixed real numbers, $f : [a,b] \times \mathbb{C} \times \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ and $k : [a,b] \times [a,b] \times \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ are continuous functions, and $\alpha, \beta : [a,b] \to [a,b]$ are continuous delay functions which therefore fulfill $\alpha(\tau) \leq \tau$ and $\beta(\tau) \leq \tau$, for all $\tau \in [a,b]$.

Theorem 4.1 *Let $\alpha, \beta : [a,b] \to [a,b]$ be continuous delay functions and $\sigma : [a,b] \to (0,\infty)$ a non-decreasing function. In addition, suppose that there is $\eta \in \mathbb{R}$ such that*

$$\int_a^x \sigma(\tau) d\tau \leq \eta \sigma(x),$$

for all $x \in [a,b]$. Moreover, suppose that $f : [a,b] \times \mathbb{C} \times \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ is a continuous function satisfying the Lipschitz condition

$$|f(x,u(x),u(\alpha(x)),g(x)) - f(x,v(x),v(\alpha(x)),h(x))| \leq M\left(|u(x)-v(x)| + |u(\alpha(x)) - v(\alpha(x))| + |g(x)-h(x)|\right) \quad (4.27)$$

with $M > 0$ and the kernel $k : [a,b] \times [a,b] \times \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ is a continuous kernel function satisfying the Lipschitz condition

$$|k(x,t,u(t),u(\beta(t))) - k(x,t,v(t),v(\beta(t)))| \leq L|u(\beta(t)) - v(\beta(t))| \quad (4.28)$$

with $L > 0$.

If $y \in C([a,b])$ is such that

$$\left| y(x) - f\left(x, y(x), y(\alpha(x)), \int_a^x k(x,\tau,y(\tau),y(\beta(\tau)))d\tau\right)\right| \leq \sigma(x), \qquad x \in [a,b], \quad (4.29)$$

and $M(2 + L\eta) < 1$, then there is a unique function $y_0 \in C([a,b])$ such that

$$y_0(x) = f\left(x, y_0(x), y_0(\alpha(x)), \int_a^x k(x,\tau,y_0(\tau),y_0(\beta(\tau)))d\tau\right)$$

and

$$|y(x) - y_0(x)| \leq \frac{1}{1 - M(2+L\eta)} \sigma(x) \quad (4.30)$$

for all $x \in [a,b]$.

This means that under the above conditions, the Volterra integral equation (1.2) has the Hyers-Ulam-Rassias stability.

Proof. We consider the operator $T : C([a,b]) \to C([a,b])$, defined by

$$(Tu)(x) = f\left(x, u(x), u(\alpha(x)), \int_a^x k(x,\tau,u(\tau),u(\beta(\tau)))d\tau\right),$$

for all $x \in [a,b]$ and $u \in C([a,b])$ and, under the present conditions, similarly with what was done in the proof of Theorem 2.1, we are able to deduce that the operator T is strictly contractive with respect to the metric (2.7). Besides this, from the definition of the metric d and by (4.29), it follows

$$\sup_{x\in[a,b]} \frac{|y(x) - y_0(x)|}{\sigma(x)} \leq \frac{1}{1 - M(2+L\eta)} \quad (4.31)$$

and consequently (4.30) holds. □

Just as a particular case of the last result, we obtain the next corollary.

Corollary 4.2 *Let $\alpha, \beta : [a,b] \to [a,b]$ be continuous delay functions and $\sigma : [a,b] \to (0,\infty)$ a non-decreasing function. Moreover, suppose that $f : [a,b] \times \mathbb{C} \times \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ is a continuous function satisfying the Lipschitz condition*

$$|f(x,u(x),u(\alpha(x)),g(x)) - f(x,v(x),v(\alpha(x)),h(x))| \leq M\left(|u(x)-v(x)| + |u(\alpha(x)) - v(\alpha(x))| + |g(x)-h(x)|\right) \quad (4.32)$$

with $M > 0$ and the kernel $k : [a,b] \times [a,b] \times \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ is a continuous kernel function satisfying the Lipschitz condition

$$|k(x,t,u(t),u(\beta(t))) - k(x,t,v(t),v(\beta(t)))| \leq L|u(\beta(t)) - v(\beta(t))| \tag{4.33}$$

with $L > 0$.

If $y \in C_p([a,b])$ is such that

$$\left| y(x) - f\left(x, y(x), y(\alpha(x)), \int_a^x k(x,\tau,y(\tau),y(\beta(\tau)))d\tau\right)\right| \leq e^{p(x-a)}, \qquad x \in [a,b], \tag{4.34}$$

and $M\left(2 + \frac{L}{p}\left(e^{p(b-a)} - 1\right)\right) < 1$, then there is a unique function $y_0 \in C_p([a,b])$ such that

$$y_0(x) = f\left(x, y_0(x), y_0(\alpha(x)), \int_a^x k(x,\tau,y_0(\tau),y_0(\beta(\tau)))d\tau\right)$$

and

$$|y(x) - y_0(x)| \leq \frac{p}{p - 2pM - ML\left(e^{p(b-a)} - 1\right)} e^{p(x-a)} \tag{4.35}$$

for all $x \in [a,b]$.

This means that under the above conditions, the Volterra integral equation (1.2) has the Hyers-Ulam-Rassias stability.

5 σ-SEMI-HYERS-ULAM AND HYERS-ULAM STABILITY FOR VOLTERRA INTEGRAL EQUATION IN THE FINITE INTERVAL CASE

The present section is devoted to present sufficient conditions for the σ-semi-Hyers-Ulam and Hyers-Ulam stabilities of the Volterra integral equation (1.2). We will continue to use the metric (2.6) on the first two results and the metric (2.7) on the others two.

Theorem 5.1 *Let $\sigma(x) = e^{p(x-a)}$ (a non-decreasing function defined on $[a,b]$ with $p > 0$), $\alpha, \beta : [a,b] \to [a,b]$ be continuous delay functions with $\alpha(t) \leq t$ and $\beta(t) \leq t$, for all $t \in [a,b]$. Moreover, suppose that $f : [a,b] \times \mathbb{C} \times \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ is a continuous function satisfying the Lipschitz condition*

$$|f(x,u(x),u(\alpha(x)),g(x)) - f(x,v(x),v(\alpha(x)),h(x))| \leq M\left(|u(x) - v(x)| + |u(\alpha(x)) - v(\alpha(x))| + |g(x) - h(x)|\right) \tag{5.36}$$

with $M > 0$ and the kernel $k : [a,b] \times [a,b] \times \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ is a continuous kernel function satisfying the Lipschitz condition

$$|k(x,t,u(t),u(\beta(t))) - k(x,t,v(t),v(\beta(t)))| \leq L|u(\beta(t)) - v(\beta(t))| \tag{5.37}$$

with $L > 0$.

If $y \in C_p([a,b])$ is such that

$$\left| y(x) - f\left(x, y(x), y(\alpha(x)), \int_a^x k(x,\tau,y(\tau),y(\beta(\tau)))d\tau\right)\right| \leq \theta, \qquad x \in [a,b], \tag{5.38}$$

where $\theta > 0$ and $M\left(2 + \frac{L}{p}\left(e^{p(b-a)} - 1\right)\right) < 1$, then there is a unique function $y_0 \in C_p([a,b])$ such that

$$y_0(x) = f\left(x, y_0(x), y_0(\alpha(x)), \int_a^x k(x,\tau,y_0(\tau),y_0(\beta(\tau)))d\tau\right)$$

and

$$|y(x) - y_0(x)| \leq \frac{p\theta}{p - 2pM - ML\left(e^{p(b-a)} - 1\right)} e^{p(x-a)} \tag{5.39}$$

for all $x \in [a,b]$.

This means that under the above conditions, the Volterra integral equation (1.2) has the σ-semi-Hyers-Ulam stability.

Proof. We will consider the operator $T : C_p([a,b]) \to C_p([a,b])$, defined by

$$(Tu)(x) = f\left(x, u(x), u(\alpha(x)), \int_a^x k(x,\tau,u(\tau),u(\beta(\tau)))d\tau\right),$$

for all $x \in [a,b]$ and $u \in C_p([a,b])$. Similarly as above, we have T strictly contractive with respect to the metric (2.6) due to the fact that $M\left(2 + \frac{L}{p}\left(e^{p(b-a)} - 1\right)\right) < 1$. Thus, we can once again apply the Banach Fixed Point Theorem, which ensures that we have the σ-semi-Hyers-Ulam stability for the Volterra integral equation (1.2) with (5.39) being obtained by using the definition of the metric d_p, (1.5) and (5.38). □

Corollary 5.2 *Let $\alpha, \beta : [a,b] \to [a,b]$ be continuous delay functions. Moreover, suppose that $f : [a,b] \times \mathbb{C} \times \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ is a continuous function satisfying the Lipschitz condition*

$$|f(x,u(x),u(\alpha(x)),g(x)) - f(x,v(x),v(\alpha(x)),h(x))| \leq M\left(|u(x)-v(x)| + |u(\alpha(x)) - v(\alpha(x))| + |g(x)-h(x)|\right) \quad (5.40)$$

with $M > 0$ and the kernel $k : [a,b] \times [a,b] \times \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ is a continuous kernel function satisfying the Lipschitz condition

$$|k(x,t,u(t),u(\beta(t))) - k(x,t,v(t),v(\beta(t)))| \leq L|u(\beta(t)) - v(\beta(t))| \quad (5.41)$$

with $L > 0$.

If $y \in C_p([a,b])$ is such that

$$\left|y(x) - f\left(x, y(x), y(\alpha(x)), \int_a^x k(x,\tau,y(\tau),y(\beta(\tau)))d\tau\right)\right| \leq \theta, \qquad x \in [a,b], \quad (5.42)$$

where $\theta > 0$ and $M\left(2 + \frac{L}{p}\left(e^{p(b-a)} - 1\right)\right) < 1$, then there is a unique function $y_0 \in C_p([a,b])$ such that

$$y_0(x) = f\left(x, y_0(x), y_0(\alpha(x)), \int_a^x k(x,\tau,y_0(\tau),y_0(\beta(\tau)))d\tau\right)$$

and

$$|y(x) - y_0(x)| \leq \frac{pe^{p(b-a)}}{p - 2pM - ML\left(e^{p(b-a)} - 1\right)}\theta \quad (5.43)$$

for all $x \in [a,b]$.

This means that under the above conditions, the Volterra integral equation (1.2) has the Hyers-Ulam stability.

Theorem 5.3 *Let $\alpha, \beta : [a,b] \to [a,b]$ be continuous delay functions, and $\sigma : [a,b] \to (0,\infty)$ a non-decreasing function. In addition, suppose that there is $\eta \in \mathbb{R}$ such that*

$$\int_a^x \sigma(\tau)d\tau \leq \eta\sigma(x),$$

for all $x \in [a,b]$. Moreover, suppose that $f : [a,b] \times \mathbb{C} \times \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ is a continuous function satisfying the Lipschitz condition

$$|f(x,u(x),u(\alpha(x)),g(x)) - f(x,v(x),v(\alpha(x)),h(x))| \leq M\left(|u(x)-v(x)| + |u(\alpha(x)) - v(\alpha(x))| + |g(x)-h(x)|\right) \quad (5.44)$$

with $M > 0$ and the kernel $k : [a,b] \times [a,b] \times \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ is a continuous kernel function satisfying the Lipschitz condition

$$|k(x,t,u(t),u(\beta(t))) - k(x,t,v(t),v(\beta(t)))| \leq L|u(\beta(t)) - v(\beta(t))| \quad (5.45)$$

with $L > 0$.

If $y \in C([a,b])$ is such that

$$\left|y(x) - f\left(x, y(x), y(\alpha(x)), \int_a^x k(x,\tau,y(\tau),y(\beta(\tau)))d\tau\right)\right| \leq \theta, \qquad x \in [a,b], \quad (5.46)$$

where $\theta > 0$ *and* $M(2 + L\eta) < 1$, *then there is a unique function* $y_0 \in C([a,b])$ *such that*

$$y_0(x) = f\left(x, y_0(x), y_0(\alpha(x)), \int_a^x k(x,\tau,y_0(\tau),y_0(\beta(\tau)))d\tau\right)$$

and

$$|y(x) - y_0(x)| \leq \frac{\theta}{(1 - M(2 + L\eta))\,\sigma(a)}\,\sigma(x) \tag{5.47}$$

for all $x \in [a,b]$.

This means that under the above conditions, the Volterra integral equation (1.2) has the σ*-semi-Hyers-Ulam stability.*

Proof. Considering the operator $T : C([a,b]) \to C([a,b])$, defined by

$$(Tu)(x) = f\left(x, u(x), u(\alpha(x)), \int_a^x k(x,\tau,u(\tau),u(\beta(\tau)))d\tau\right),$$

for all $x \in [a,b]$ and $u \in C([a,b])$, we conclude that T is strictly contractive with respect to the metric (2.7) due to the fact that $M(2 + L\eta) < 1$. Thus, we can again apply the Banach Fixed Point Theorem, which ensures that we have the σ-semi-Hyers-Ulam stability for the Volterra integral equation (1.2) with (5.47) being obtained by the definition of the metric d and using (1.5) and (5.46). □

Corollary 5.4 *Let* $\alpha, \beta : [a,b] \to [a,b]$ *be continuous delay functions and* $\sigma : [a,b] \to (0,\infty)$ *a non-decresing function. In addition, suppose that there is* $\eta \in \mathbb{R}$ *such that*

$$\int_a^x \sigma(\tau)d\tau \leq \eta\sigma(x),$$

for all $x \in [a,b]$. *Moreover, suppose that* $f : [a,b] \times \mathbb{C} \times \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ *is a continuous function satisfying the Lipschitz condition*

$$|f(x,u(x),u(\alpha(x)),g(x)) - f(x,v(x),v(\alpha(x)),h(x))| \leq M(|u(x) - v(x)| + |u(\alpha(x)) - v(\alpha(x))| + |g(x) - h(x)|) \tag{5.48}$$

with $M > 0$ *and the kernel* $k : [a,b] \times [a,b] \times \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ *is a continuous kernel function satisfying the Lipschitz condition*

$$|k(x,t,u(t),u(\beta(t))) - k(x,t,v(t),v(\beta(t)))| \leq L|u(\beta(t)) - v(\beta(t))| \tag{5.49}$$

with $L > 0$.

If $y \in C([a,b])$ *is such that*

$$\left|y(x) - f\left(x, y(x), y(\alpha(x)), \int_a^x k(x,\tau,y(\tau),y(\beta(\tau)))d\tau\right)\right| \leq \theta, \qquad x \in [a,b], \tag{5.50}$$

where $\theta > 0$ *and* $M(2 + L\eta) < 1$, *then there is a unique function* $y_0 \in C([a,b])$ *such that*

$$y_0(x) = f\left(x, y_0(x), y_0(\alpha(x)), \int_a^x k(x,\tau,y_0(\tau),y_0(\beta(\tau)))d\tau\right)$$

and

$$|y(x) - y_0(x)| \leq \frac{\sigma(b)}{(1 - M(2 + L\eta))\,\sigma(a)}\,\theta \tag{5.51}$$

for all $x \in [a,b]$.

This means that under the above conditions, the Volterra integral equation (1.2) has the Hyers-Ulam stability.

6 HYERS-ULAM-RASSIAS STABILITY IN THE INFINITE INTERVAL CASE

In this section, we will analyse the Hyers-Ulam-Rassias stability of the Volterra integral equation but when considering infinite intervals. This means that instead of considering, as before in (1.2), a finite interval $[a,b]$ (with $a,b \in \mathbb{R}$), we will now consider e.g. corresponding intervals $[a,\infty)$, for some fixed $a \in \mathbb{R}$.

Thus, we will now be dealing with the integral equation

$$y(x) = f\left(x, y(x), y(\alpha(x)), \int_a^x k(x,\tau,y(\tau),y(\beta(\tau)))d\tau\right), \qquad x \in [a,\infty), \tag{6.52}$$

where a is a fixed real number, $f : [a,\infty) \times \mathbb{C} \times \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ and $k : [a,\infty) \times [a,\infty) \times \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ are continuous functions, and $\alpha, \beta : [a,\infty) \to [a,\infty)$ are continuous delay functions which therefore fulfill $\alpha(\tau) \leq \tau$ and $\beta(\tau) \leq \tau$, for all $\tau \in [a,\infty)$. Here, our strategy will be based on a recurrence procedure due to the already obtained result for the corresponding finite interval case.

Let us consider a fixed non-decreasing continuous function $\sigma : [a,\infty) \to (\varepsilon,\omega)$, for some $\varepsilon, \omega > 0$ and the space $C^b([a,\infty))$ of bounded continuous functions endowed with the metric

$$d^b(u,v) = \sup_{x\in[a,\infty)} \frac{|u(x)-v(x)|}{\sigma(x)}. \tag{6.53}$$

Theorem 6.1 *Let $\alpha, \beta : [a,\infty) \to [a,\infty)$ be continuous delay functions and $\sigma : [a,\infty) \to (\varepsilon,\omega)$, for some $\varepsilon,\omega > 0$, a non-decreasing function. In addition, suppose that there is $\eta \in \mathbb{R}$ such that*

$$\int_a^x \sigma(\tau)d\tau \leq \eta\sigma(x),$$

for all $x \in [a,\infty)$. Moreover, suppose that $f : [a,\infty) \times \mathbb{C} \times \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ is a continuous function satisfying the Lipschitz condition

$$|f(x,u(x),u(\alpha(x)),g(x)) - f(x,v(x),v(\alpha(x)),h(x))| \leq M\left(|u(x)-v(x)| + |u(\alpha(x))-v(\alpha(x))| + |g(x)-h(x)|\right) \tag{6.54}$$

with $M > 0$ and the kernel $k : [a,\infty)\times[a,\infty)\times\mathbb{C}\times\mathbb{C} \to \mathbb{C}$ is a continuous kernel function so that $\int_a^x k(x,\tau,z(\tau),z(\beta(\tau)))d\tau$ is a bounded continuous function for any bounded continuous function z. In addition, suppose that k satisfying the Lipschitz condition

$$|k(x,t,u(t),u(\beta(t))) - k(x,t,v(t),v(\beta(t)))| \leq L|u(\beta(t)) - v(\beta(t))| \tag{6.55}$$

with $L > 0$.

If $y \in C^b([a,\infty))$ is such that

$$\left|y(x) - f\left(x, y(x), y(\alpha(x)), \int_a^x k(x,\tau,y(\tau),y(\beta(\tau)))d\tau\right)\right| \leq \sigma(x), \qquad x \in [a,\infty), \tag{6.56}$$

and $M(2+L\eta) < 1$, then there is a unique function $y_0 \in C^b([a,b])$ such that

$$y_0(x) = f\left(x, y_0(x), y_0(\alpha(x)), \int_a^x k(x,\tau,y_0(\tau),y_0(\beta(\tau)))d\tau\right) \tag{6.57}$$

and

$$|y(x) - y_0(x)| \leq \frac{1}{1-M(2+L\eta)}\,\sigma(x) \tag{6.58}$$

for all $x \in [a,\infty)$.

This means that under the above conditions, the Volterra integral equation (6.52) has the Hyers-Ulam-Rassias stability.

Proof. For any $n \in \mathbb{N}$, we will define $I_n = [a, a+n]$. By Theorem 4.1, there exists a unique bounded continuous function $y_{0,n} : I_n \to \mathbb{C}$ such that

$$y_{0,n}(x) = f\left(x, y_{0,n}(x), y_{0,n}(\alpha(x)), \int_a^x k(x,\tau, y_{0,n}(\tau), y_{0,n}(\beta(\tau)))d\tau\right) \tag{6.59}$$

and

$$|y(x) - y_{0,n}(x)| \leq \frac{1}{1 - M(2+L\eta)}\,\sigma(x) \tag{6.60}$$

for all $x \in I_n$. The uniqueness of $y_{0,n}$ implies that if $x \in I_n$ then

$$y_{0,n}(x) = y_{0,n+1}(x) = y_{0,n+2}(x) = \cdots . \tag{6.61}$$

For any $x \in [a,\infty)$, let us define $n(x) \in \mathbb{N}$ as $n(x) = \min\{n \in \mathbb{N} \mid x \in I_n\}$. We also define a function $y_0 : [a,\infty) \to \mathbb{C}$ by

$$y_0(x) = y_{0,n(x)}(x). \tag{6.62}$$

For any $x_1 \in [a,\infty)$, let $n_1 = n(x_1)$. Then $x_1 \in Int\, I_{n_1+1}$ and there exists an $\epsilon > 0$ such that $y_0(x) = y_{0,n_1+1}(x)$ for all $x \in (x_1 - \epsilon, x_1 + \epsilon)$. By Theorem 4.1, y_{0,n_1+1} is continuous at x_1, and so it is y_0.

Now, we will prove that y_0 satisfies

$$y_0(x) = f\left(x, y_0(x), y_0(\alpha(x)), \int_a^x k(x,\tau, y_0(\tau), y_0(\beta(\tau)))d\tau\right)$$

and

$$|y(x) - y_0(x)| \leq \frac{1}{1 - M(2+L\eta)}\,\sigma(x) \tag{6.63}$$

for all $x \in [a,\infty)$. For an arbitrary $x \in [a,\infty)$ we chose $n(x)$ such that $x \in I_{n(x)}$. By (6.59) and (6.62), we have

$$\begin{aligned} y_0(x) = y_{0,n(x)}(x) &= f\left(x, y_{0,n(x)}(x), y_{0,n(x)}(\alpha(x)), \int_a^x k(x,\tau, y_{0,n(x)}(\tau), y_{0,n(x)}(\beta(\tau)))d\tau\right) \\ &= f\left(x, y_0(x), y_0(\alpha(x)), \int_a^x k(x,\tau, y_0(\tau), y_0(\beta(\tau)))d\tau\right). \end{aligned} \tag{6.64}$$

Note that $n(\tau) \leq n(x)$, for any $\tau \in I_{n(x)}$, and it follows from (6.61) that $y_0(\tau) = y_{0,n(\tau)}(\tau) = y_{0,n(x)}(\tau)$, so, the last equality in (6.64) holds true.

From (6.62) and (6.60), we have that for all $x \in [a,\infty)$ it holds

$$|y(x) - y_0(x)| = \left|y(x) - y_{0,n(x)}(x)\right| \leq \frac{1}{1 - M(2+L\eta)}\,\sigma(x)$$

which is (6.58). Finally, we will prove the uniqueness of y_0. Let us consider another bounded continuous function y_1 which satisfies (6.57) and (6.58), for all $x \in [a,\infty)$. By the uniqueness of the solution on $I_{n(x)}$ for any $n(x) \in \mathbb{N}$ we have that $y_{0|I_{n(x)}} = y_{0,n(x)}$ and $y_{1|I_{n(x)}}$ satisfies (6.57) and (6.58) for all $x \in I_{n(x)}$, so

$$y_0(x) = y_0|_{I_{n(x)}}(x) = y_1|_{I_{n(x)}}(x) = y_1(x).$$

□

Remark 6.2 With the necessary adaptations, Theorem 6.1 also holds true for infinite intervals $(-\infty, b]$, with $b \in \mathbb{R}$, as well as for $(-\infty,\infty)$.

7 EXAMPLES

In this section we will present some examples to illustrate that the conditions of the above results are possible to attain.

For continuous functions $y : [0,1] \to \mathbb{R}$, let us start by considering the integral equation

$$y(x) = \frac{x^5}{60} - \frac{x^2}{5} + x + \frac{1}{5}y(\alpha(x)) + \frac{1}{3}\int_0^x ((\tau - x)y(\beta(\tau)))\, d\tau, \quad x \in [0,1]\,, \tag{7.65}$$

as well as the non-decreasing continuous function $\sigma : [0,1] \to (0,\infty)$ defined by $\sigma(x) = 0.0083\,x + 0.0005$ and the continuous delay functions $\alpha : [0,1] \to [0,1]$ given by $\alpha(x) = x^2$ and $\beta : [0,1] \to [0,1]$ defined by $\beta(x) = x^3$.

We have all the conditions of Theorem 4.1 being satisfied. In fact, such α and β are continuous functions, and obviously $\alpha(x) \le x$ and $\beta(x) \le x$. Moreover, for $\eta = 0.52841$ we have that $\sigma : [0,1] \to (0,\infty)$ defined by $\sigma(x) = 0.0083\,x + 0.0005$ is a continuous function fulfilling

$$\int_0^x 0.0083\,\tau + 0.0005\, d\tau \le \eta\,(0.0083\,x + 0.0005) = 0.52841\,\sigma(x), \quad x \in [0,1]; \tag{7.66}$$

$f : [0,1] \times \mathbb{C} \times \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ defined by $f(x, y(x), y(\alpha(x)), g(x)) = \frac{x^5}{60} - \frac{x^2}{5} + x + \frac{1}{5}y(\alpha(x)) + \frac{1}{3}g(x)$ is a continuous function which fulfills

$$|f(x,u(x),u(\alpha(x)),g(x)) - f(x,v(x),v(\alpha(x)),h(x))| \le \frac{1}{3}\left(|u(x)-v(x)| + |u(\alpha(x)) - v(\alpha(x))| + |g(x)-h(x)|\right), \tag{7.67}$$

for all $x \in [0,1]$ (and so the previous constant M is here taking the value 1/3); the kernel $k : [0,1] \times [0,1] \times \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ defined by $k(x,\tau,y(\tau),y(\beta(\tau))) = (\tau - x)y(\beta(\tau))$ is a continuous function which fulfils the condition

$$|k(x,\tau,u(\tau),u(\beta(\tau))) - k(x,\tau,v(\tau),v(\beta(\tau)))| \le |u\,(\beta(\tau)) - v\,(\beta(\tau))|\,, \quad \tau \in [0,x], \quad x \in [0,1] \tag{7.68}$$

(where we may identify 1 as the constant previously denoted by L). Thus, $M(2 + L\eta) = 252841/300000 < 1$.

If we choose $y(x) = 101\,x/100$, it follows

$$\left|y(x) - f\left(x, y(x), y(\alpha(x)), \int_a^x k(x,\tau,y(\tau),y(\beta(\tau)))d\tau\right)\right| = \left|\frac{x^5}{6000} - \frac{x^2}{500} + \frac{x}{100}\right| \le 0.0083\,x + 0.0005 = \sigma(x), \tag{7.69}$$

for all $x \in [0,1]$.

Therefore, from Theorem 4.1, we have the Hyers-Ulam-Rassias stability of the integral equation (7.65). In particular, having in mind the exact solution $y_0(x) = x$ of (7.65), it follows that

$$|y(x) - y_0(x)| = \left|\frac{101}{100}x - x\right| \le \frac{1}{1 - M(2 + L\eta)}\,\sigma(x) = \frac{300000}{47159}\,\sigma(x), \quad x \in [0,1]\,. \tag{7.70}$$

Still within this last example associated with the integral equation (7.65), and using the same η, M and L (and so still having $M(2 + L\eta) = 252841/300000 < 1$), if we choose $y(x) = 101\,x/100$, it follows

$$\left|y(x) - f\left(x, y(x), y(\alpha(x)), \int_a^x k(x,\tau,y(\tau),y(\beta(\tau)))d\tau\right)\right| = \left|\frac{x^5}{6000} - \frac{x^2}{500} + \frac{x}{100}\right| \le \theta := \frac{11}{1250}, \quad x \in [0,1]\,. \tag{7.71}$$

From Theorem 5.3, we have the σ-semi-Hyers-Ulam stability of the integral equation (7.65), and from Corollary 5.4 we have the consequent Hyers-Ulam stability. In particular, having in mind the exact solution $y_0(x) = x$ of (7.65), it follows that (5.47) and (5.51) are

$$|y(x) - y_0(x)| = \left|\frac{101}{100}x - x\right| \le \frac{\theta}{(1 - M\,(2 + L\eta))\,\sigma(0)}\,\sigma(x) = \frac{5280000}{47159}\,\sigma(x), \quad x \in [0,1]\,, \tag{7.72}$$

and

$$|y(x) - y_0(x)| = \left|\frac{101}{100}x - x\right| \le \frac{\sigma(1)}{(1 - M\,(2 + L\eta))\,\sigma(0)}\,\theta = \frac{46464}{47159} \approx 0.98527, \quad x \in [0,1]\,, \tag{7.73}$$

respectively.

Let us now turn to a second example in which the σ-semi-Hyers-Ulam and the Hyers-Ulam stabilities are illustrated.

For continuous functions $y : [0,1] \to \mathbb{R}$, let us start be considering the integral equation

$$y(x) = \left(\frac{1}{8}x + \frac{1}{4}\right) e^{x/2} + \frac{1}{4}y(\alpha(x)) - \frac{3}{16}e^{x/2} \int_0^x ((\tau - x)y(\beta(\tau)))\, d\tau, \quad x \in [0,1]\,, \tag{7.74}$$

as well as the non-decreasing continuous function $\sigma : [0,1] \to (0,\infty)$ defined by $\sigma(x) = 0.007\,x + 0.006$ and the continuous delay functions $\alpha : [0,1] \to [0,1]$ given by $\alpha(x) = x$ and $\beta : [0,1] \to [0,1/2]$ given by $\beta(x) = x/2$.

We have all the conditions of Theorem 5.3 being satisfied. In fact, such $\alpha : [0,1] \to [0,1]$ defined by $\alpha(x) = x$ and $\beta : [0,1] \to [0,1/2]$ defined by $\beta(x) = x/2$ are continuous functions, and obviously $\alpha(x) \le x$ and $\beta(x) \le x$. Moreover, for $\eta = 0.74$ we have that $\sigma : [0,1] \to (0,\infty)$ defined by $\sigma(x) = 0.007\,x + 0.006$ is a continuous function fulfilling

$$\int_0^x 0.007\,\tau + 0.006\, d\tau \le \eta\,(0.007\,x + 0.006) = 0.74\,\sigma(x), \quad x \in [0,1]; \tag{7.75}$$

$f : [0,1] \times \mathbb{C} \times \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ defined by $f(x, y(x), y(\alpha(x)), g(x)) = \left(\frac{1}{8}x + \frac{1}{4}\right) e^{x/2} + \frac{1}{4}y(\alpha(x)) - \frac{3}{16}e^{x/2}g(x)$ is a continuous function which fulfills

$$|f(x,u(x),u(\alpha(x)),g(x)) - f(x,v(x),v(\alpha(x)),h(x))| \le \frac{3\sqrt{e}}{16}\left(|u(x) - v(x)| + |u(\alpha(x)) - v(\alpha(x))| + |g(x) - h(x)|\right), \tag{7.76}$$

for all $x \in [0,1]$ (and so the previous constant M is here taking the value $3\sqrt{e}/16$); the kernel $k : [0,1] \times [0,1] \times \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ defined by $k(x,\tau,y(\tau),y(\beta(\tau))) = (\tau - x)y(\beta(\tau))$ is a continuous function which fulfils the condition

$$|k(x,\tau,u(\tau),u(\beta(\tau))) - k(x,\tau,v(\tau),v(\beta(\tau)))| \le |u\,(\beta(\tau)) - v\,(\beta(\tau))|\,, \quad \tau \in [0,x], \quad x \in [0,1] \tag{7.77}$$

(where we may identify 1 as the constant previously denoted by L). Thus, $M(2 + L\eta) = 411\sqrt{e}/800 < 1$.

If we choose $y(x) = 100\,e^x/99$, it follows

$$\left|y(x) - f\left(x, y(x), y(\alpha(x)), \int_a^x k(x,\tau,y(\tau),y(\beta(\tau)))d\tau\right)\right| = \left|\left(\frac{201}{792}x + \frac{402}{792}\right)e^{x/2}\right| \le \theta := \frac{67\sqrt{e}}{88}, \quad x \in [0,1]\,. \tag{7.78}$$

Therefore, from Theorem 5.3, we have the σ-semi-Hyers-Ulam stability of the integral equation (7.74). In particular, having in mind the exact solution $y_0(x) = e^x$ of (7.74), it follows that

$$|y(x) - y_0(x)| = \left|\frac{100}{99}e^x - e^x\right| \le \frac{\theta}{(1 - M(2 + L\eta))\,\sigma(0)}\,\sigma(x) \approx 1367.677\,\sigma(x), \quad x \in [0,1]\,. \tag{7.79}$$

From Corollary 5.4 we have the consequent Hyers-Ulam stability. In particular, having in mind the exact solution $y_0(x) = e^x$ of (7.74), it follows that (5.51) is

$$|y(x) - y_0(x)| = \left|\frac{100}{99}e^x - e^x\right| \le \frac{\sigma(1)}{(1 - M(2 + L\eta))\sigma(0)}\,\theta \approx 17.7789, \quad x \in [0,1]\,. \tag{7.80}$$

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