Invariance under outer inverses

R.E. Hartwig* P. Patrício†

Abstract

We shall use the minus partial order combined with Pierce's decomposition to derive the class of outer inverses for idempotents, units and group invertible elements. Subsequently we show, for matrices over a field $\mathbb{F}$, that the triplet $B\hat{A}C$ is invariant under all choices of outer inverses of A if and only if $B = 0$ or $C = 0$.

Keywords: Invariance, outer inverses, regularity, Pierce decomposition

AMS classification: 15A09, 16E50

1 Introduction

Let R be a ring with 1.

An element a is called *regular* if $aa^-a = a$ for some inner or 1-inverse a^-. The condition for regularity is a *linear* condition, and the set of all inner inverses is given by

$$\{a^{(1)}\} = a^- + (1 - a^-a)R + R(1 - aa^-).$$

An *outer* or *2-inverse* $\hat{a}$ of an element a is such that $\hat{a}a\hat{a} = \hat{a}$. It is a quadratic condition in $\hat{a}$. It is clear that $a\hat{a}a$ will always be regular.

A 1-2 or reflexive inverse of a is denoted by a^+ and satisfies

$$aa^+a = a \text{ and } a^+aa^+ = a^+.$$

*Mathematics Department, N.C.S.U., Raleigh, NC 27695-8205, U.S.A. e-mail: hartwig@unity.ncsu.edu

†CMAT – Centro de Matemática and Departamento de Matemática e Aplicações, Universidade do Minho, 4710-057 Braga, Portugal. e-mail: pedro@math.uminho.pt

Partially supported by FCT- 'Fundação para a Ciência e a Tecnologia', within the project UID-MAT-00013/2013.

Corresponding author.

The set of all outer inverses of an element a will be denoted by T_a or $\{a^{(2)}\}$ and the set of all idempotents will be denoted by E. It is clear that a regular element a admits a^-aa^- as an outer inverse.

Given the **quadratic** nature of the outer inverse condition, the characterization of T_a remains clouded in general. We shall characterize T_a for several special types of elements, such as idempotents and units. We then use these results to nail down T_a for group invertible elements.

We shall also use the concepts of a unit-regular element a, for which there is a unit u in R such that $aua = a$, and (ii) that of a prime ring, for which $aRb = (0)$, forces either $a = 0$ or $b = 0$. It should be noted that a is unit regular if and only if $a = peq$, for some unit p and q and some idempotent e.

2 Classes of outer inverses

We begin by recalling [7] that for any p, a and q, and any $\widehat{(qap)}$

$$p\widehat{(qap)}q \cdot a \cdot p\widehat{(qap)}q = p\widehat{(qap)}q$$

so that it is prudent to define, for a fixed p and q,

Definition 2.1. $S_{p,q} = \{p\widehat{(qap)}q;$ *any* $\widehat{(qap)}\}$

Clearly

$$S_{p,q} = p \cdot T_{qap} \cdot q \subseteq T_a = S_{1,1}. \tag{1}$$

We next turn to the set of all outer inverses of 1. It precisely equal the set of all idempotents E, since $x \cdot 1 \cdot x = x$ if and only if x is idempotent.

Next we recall that for two units p and q

$$T_{paq} = q^{-1}T_a\ p.^{-1} \tag{2}$$

To characterize T_e where e is idempotent, we make use of the minus order as defined in [3] for a regular element a

$$a \leq b \text{ iff } a^-a = a^-b \text{ and } aa^- = ba^-, \text{ for some inner inverse } a^-. \tag{3}$$

The key fact that we need is that given $e^2 = e$ and g regular such that $g \leq e$ then $g^2 = g = ge = eg$. This also tells us that

Lemma 2.1. *The following are equivalent, for $e^2 = e$ and g regular:*

(i) $g \leq e$.

(ii) $g = ge = eg = g^2$.

(iii) $g = e\hat{e}e$ *for some outer inverse* $\hat{e}$.

(iv) $g = exe = g^2$ *for some* x.

We use this in

Theorem 2.1. *If* e *is idempotent then* $T_e = v(x)gw(y)$*, where* $v(x) = 1 + (1-e)xe$*,* $w(y) = 1 + ey(1-e)$*,* $g \leq e$*, and* x *and* y *are arbitrary.*

Proof. Observe that $v(x)$ and $w(y)$ are units for all x and y and that $ev(x) = e = w(y)e$. As such $v(x)gw(y) \cdot e \cdot v(x)gw(y) = vgegw = vgw$. On the other hand, if $t = \hat{e}$ is any outer inverse of e, then we may take $x = y = t$, and select $g = ete$. We then get

$$\begin{aligned} v(t)etew(t) &= [ete + (1-e)te][1 + et(1-e)] \\ &= ete + (1-e)te + et(1-e) + (1-e)t(1-e) \\ &= t, \end{aligned}$$

which is the *Pierce Decomposition* of t. □

The result of Theorem 2.1 is a special case of the following [4]:

Theorem 2.2. *The following are equivalent for two regular elements* a *and* b*:*

(i) $a \leq b$.

(ii) $a = b\hat{b}b$ *for some outer inverse* $\hat{b}$.

(iii) $a = b - b\hat{\hat{b}}b$ *for some outer inverse* $\hat{\hat{b}}$.

In which case $\hat{b}a\hat{b} = \hat{b}$ *and* $\hat{\hat{b}}a = 0 = a\hat{\hat{b}}$.

Proof. (i) implies (ii). Note that $a \leq b$ is equivalent to $a = ba^-a = aa^-b$. By taking $\hat{b} = a^-aa^-$, which is a 2-inverse of b, we have $ba^-aa^-b = aa^-b = a$.

(ii) implies (i). From $a = b\hat{b}b$ it follows $\hat{b}$ is a 1-inverse of a, and we may take $a^- = \hat{b}$. The equalities $a\hat{b} = b\hat{b}$ and $\hat{b}a = \hat{b}b$ show that $a \leq b$.

(iii) implies (ii). It is easy to show that $\hat{b} = b^+ - b^+b\hat{\hat{b}}bb^+$ is indeed a 2-inverse of b, for any choice of a reflexive inverse b^+ of b and any outer inverse $\hat{\hat{b}}$. Furthermore, $b(b^+ - b^+b\hat{\hat{b}}bb^+)b = b - b\hat{\hat{b}}b = a$.

(ii) implies (iii). Since $\hat{\hat{b}} = b^+ - b^+b\hat{b}bb^+$ is a 2-inverse of b, substituting in (iii) we have $b - b(b^+ - b^+b\hat{b}bb^+)b = b - b + b\hat{b}b = a$. The remaining parts are clear. □

In general

$$b \leq a \not\Rightarrow T_b \subseteq T_a,$$

as seen from the example where $b = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and a $= I_2$. If we pick $c = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ then $c \in T_b$ yet $c \notin T_a$.

The idempotent result of Theorem 2.1 parallels the (2×2) matrix case, which uses a Pierce-like decomposition. Indeed if $e = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $\hat{e} = \begin{bmatrix} p & r \\ q & s \end{bmatrix}$ then $p^2 = p$, $q = xp$, and $r = py$, so that

$$\hat{e} = \begin{bmatrix} p & py \\ xp & qr \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ x & 1 \end{bmatrix} \begin{bmatrix} p & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & y \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 \\ x \end{bmatrix} p[1, y].$$

In this x and y are arbitrary and p is an arbitrary idempotent.

We have also seen the importance of outer inverses in the Brown-McCoy transformation [6].

Pierce decomposition appear naturally in the study of outer inverses.

Lemma 2.2. *Given p, q, not necessarily idempotent,*

1. $a = qap + qa(1-p) + (1-q)ap + (1-q)a(1-p)$.
2. $(\gamma) : a = qap + (1-q)a(1-p)$ *if and only if* $(\alpha) : qa(1-p) + (1-q)ap = 0$.
3. *If* $q = ax$, $p = xa$ *then* (α) *holds if and only if* $axa(1-xa) + (1-ax)axa = 0$ *if and only if* $(\beta) : 2(axaxa - axa) = 0$. *In the case* $char(R) \neq 2$, *these are equivalent to* $a(xax - x)a = 0$.
4. $(\beta) \Rightarrow (\alpha) \Rightarrow (\gamma)$.

Proof. (4). If $x = \hat{a}$, $q = a\hat{a}$ and $p = \hat{a}a$ then $a\hat{a}a\hat{a}a = a\hat{a}a$. □

Corollary 2.1. *If $a = peq$, with p, q units and e idempotent, then*

$$T_a = q^{-1}T_e p^{-1} = q^{-1}[1 + (1-e)xe]e\hat{e}e[1 + ey(1-e)]p^{-1},$$

with x, y arbitrary.

Corollary 2.2. *If u is a unit then the set of* all *outer inverses of u is given by $T_u = Eu^{-1} = u^{-1}E$.*

Proof. It is clear that $(eu^{-1})u(eu^{-1}) = eu^{-1}$, for any $e \in E$. Conversely, if $xux = x$, then $(xu)1(xu) = xu$, so that $xu = e$ must be idempotent. Consequently, $x = eu^{-1}$ and the set of all outer inverses of u becomes $T_u = Eu^{-1}$. The rest follows by symmetry. □

Corollary 2.3. *If $a = ue$, where e is idempotent and u is a unit, then $T_a = T_e u^{-1}$.*

Proof. Clearly $\hat{e}u^{-1}(ue)\hat{e}u^{-1} = \hat{e}u^{-1}$. Conversely, if $x(ue)x = x$, then $(xu)e(xu) = xu$ and hence $xu \in T_e$. This means that $x \in T_e u^{-1}$. □

Corollary 2.4. *If a has a group inverse $a^\#$ then*

$$T_a = [1 + (1 - aa^\#)xaa^\#]g[1 + aa^\# y(1 - aa^\#)](a^\# + 1 - aa^\#),$$

where $g \leq aa^\#$ and g is regular.

Proof. $a = (a + 1 - aa^\#)(aa^\#) = ue$, where u is a unit and e is idempotent. □

Remark. Since $g = aa^\# g aa^\#$ the product simplifies considerably.

Next we give an example of a nilpotent element.

Proposition 2.1. *If $N = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ then $T_N = \begin{bmatrix} x \\ 1 \end{bmatrix} e[1, y]$ where e is idempotent and x and y are arbitrary.*

Proof. Equating $\begin{bmatrix} a & c \\ b & d \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$ gives $ab = a$, $b^2 = b$, $c = ad$ and $d = bd$, from which the result follows. □

Remark. This is a special case of the full-rank-factorization result. Indeed, if $a = bc$, with $b^- b = 1 = cc^-$ then $c^- e b^- \in T_a$ for all idempotents e. Conversely, if $xbcx = x$, then $cxb = e$ is idempotent. This may be solved to give

$$x = c^- e b^- + (1 - c^- c)R + R(1 - bb^-).$$

But not all of these solutions are outer inverses of a. The solution set reduces again to a quadratic equation!

The question of containment for sets of outer inverses can be extended to unit regular elements. Inded we have the following.

Theorem 2.3. *If a and b are unit regular elements in a prime ring, then the following are equivalent:*
(i) $T_b \subseteq T_a$ (ii) $b = a$,   (iii) $T_b = T_a$.

Proof. Suppose $a = peq$ and $b = rgs$, where e and g are idempotent and p, q, r and s are units. Then $T_a = q^{-1}T_e p^{-1}$ and $T_b = s^{-1}T_g r^{-1}$. Hence $T_b \subseteq T_a$ if and only if

$(qs^{-1})T_g(r^{-1}p) \subseteq T_e$. Because an arbitrary element in T_g has the form $v_g h w_g$, where $h \le g$, $v_g = 1 + (1-g)xg$, $w_g = 1 + gy(1-g)$, we see that

$$T_b \subseteq T_a \text{ if and only if } (qs^{-1})(v_g h w_g)(r^{-1}p).e.(qs^{-1})(v_g h w_g)(r^{-1}p) = qs^{-1}(v_g h w_g)(r^{-1}p). \tag{4}$$

Since q, s, v_g, r, p and w_g are units, and $h = gh = hg$, the latter reduces to

$$hgw_g(r^{-1}p)e(qs^{-1})v_g gh = h \tag{5}$$

whch can be simplified to

$$h[1 + y(1-g)](r^{-1}as^{-1})[1 + (1-g)x]h = h, \quad for \quad all \quad h \le g. \tag{6}$$

Since the x and y are arbitrary and h is any element below g, we may select them suitably. Selecting $x = 0 = y$ and $h = g$, we arrive at the first necessary condition

$$(hr^{-1})a(s^{-1}h) = h, \tag{7}$$

which gives $g(r^{-1}as^{-1})g = g$. Consequently $(rgr^{-1})a(s^{-1}gs) = rgs = b$. Next we consider the inner inverse $b^- = s^{-1}gr^{-1}$, and obtain

$$bb^- = rgr^{-1} \quad and \quad b^-b = s^{-1}gs. \tag{8}$$

We thus arrive at our first necessary condition

$$b = bb^-ab^-b. \tag{9}$$

Next we set $x = 0$ in (6). This gives

$$hy(1-g)r^{-1}as^{-1}h = 0. \tag{10}$$

Since y is arbitrary and R is prime this ensures that

$$(1-g)r^{-1}as^{-1}h = 0.$$

Left and right multiplication by r and s respectively gives our second necessary condition

$$(1 - bb^-)ab^-b = 0. \tag{11}$$

By symmetry (i.e. setting $y = 0$) we obtain the third condition

$$hr^{-1}as^{-1}(1-g)xh = 0 \tag{12}$$

which turns into

$$bb^-a(1-b^-b) = 0. \tag{13}$$

Substituting (7), (10), and (12) into (6) we are left with

$$hy(1-g)r^{-1}as^{-1}(1-g)xh = 0. \tag{14}$$

Again since R is prime and selecting $h \neq 0$ we obtain

$$(1-g)r^{-1}as^{-1}(1-g) = 0. \tag{15}$$

This collapses to

$$(1-bb^-)a(1-b^-b) = 0. \tag{16}$$

Lastly, combining equations (9), (11), (13) and (16), yields

$$a = ab^-b = bb^-a = bb^-ab^-b = b. \tag{17}$$

□

3 Triplet invariance over $\mathbb{F}$

In the theory of linear models [1] the invariance of the matrix triplet BA^-C under all choices of inner inverses A^- is essential. It is shown there that this invariance occurs if and only if the range-row-space conditions $B = XA$ and $C = AY$, for some X and Y, are satisfied.

In a later paper [2], it was shown that over a prime regular ring, the invariance of the triplet BA^+C under any choice of reflexive inverse A^+ is also equivalent to the range-row-space conditions $B = XA$ and $C = AY$.

Here we shall examine the invariance of the triplet $b\hat{a}c$ under all choices of outer inverses $\hat{a}$. In particular we shall show that $B\hat{A}C$ is invariant over a field $\mathbb{F}$, under all choices of outer inverses $\hat{A}$, if and only if either $B = 0$, or $C = 0$.

We now need the following definition:

Definition 3.1. *Given an idempotent e, R is an e-*prime ring *if $bgc = 0$, $\forall\, g \leq e$, implies $b = 0$ or $c = 0$.*

Note that if $f = u^{-1}eu$ and R is e-prime then R is f-prime. Indeed, if $bgc = 0$ for all idempotents $g \leq f$ then $bu^{-1}(ugu^{-1})uc = 0$ for all idempotents $g \leq f$, ie for all $ugu^{-1} \leq e$. Since R is e-prime then this implies $bu^{-1} = 0$ or $uc = 0$, that is $b = 0$ or $c = 0$.

Suppose $b\hat{a}c$ is invariant under $\widehat{(\cdot)}$. Since 0 is an outer inverse of a, this means that $b\hat{a}c = 0$ for **all** $\hat{a}$. If we assume that $a = peq$, for some units p and q and some idempotent e, then we arrive at

$$b\hat{a}c = bq^{-1}v(x)gw(y)p^{-1}c = 0,$$

for all x, y and $g \leq e$, where $v(x) = 1 + (1-g)xg$, $w(y) = 1 + gy(1-g)$.

Since R is e-prime we may conclude that either $bq^{-1}v(x) = 0$ or $w(y)p^{-1}c = 0$. That is, either $b = 0$ or $c = 0$.

It is clear from the normal form that a square matrix over $\mathbb{F}$ can be written as PEQ, where P and Q are invertible and $E = \text{diag}(I_r, 0)$. So let us next show that $\mathbb{F}_{n\times n}$ is an I-prime ring.

We begin with

Lemma 3.1. *$\mathbb{F}_{n\times n}$ is I-prime. That is, if $RES = 0$ for all choices of $E^2 = E$, either $R = 0$ or $S = 0$.*

Proof. Setting $R = \begin{bmatrix} \mathbf{r}_1 & \cdots & \mathbf{r}_n \end{bmatrix}$ and $S = \begin{bmatrix} \mathbf{s}_1^T \\ \vdots \\ \mathbf{s}_n^T \end{bmatrix}$ and taking $E = E_{kk}$ we see that $\mathbf{r}_k\mathbf{s}_k^T = 0$ for all $k = 1, \ldots, n$. Since we are over a field this ensures that either $\mathbf{r}_k = \mathbf{0}$ or $\mathbf{s}_k = 0$. If $R \neq 0$, we may rearrange the columns such that $R = \left[\begin{array}{ccc|ccc} \mathbf{r}_1 & \cdots & \mathbf{r}_u & \mathbf{0} & \cdots & \mathbf{0} \end{array}\right]$ with $\mathbf{r}_i \neq \mathbf{0}$, $i = 1, \ldots, u$ and $\mathbf{s}_j^T = \mathbf{0}^T$ for $j = 1, \ldots, u$.

Next we select $E = \mathbf{e}_1\mathbf{e}_1^T + \mathbf{e}_1\mathbf{e}_{u+v}^T$, which yields $RES = \mathbf{r}_1\mathbf{s}_{u+v}^T = 0$ and thus forces $\mathbf{s}_{u+v}^T = \mathbf{0}^T$ for $v = 1, \ldots, n-u$. Thus $S = 0$. Likewise if $S \neq 0$ then $R = 0$. □

Theorem 3.1. *$B\hat{A}C$ is 2-invariant if and only if $B = 0$ or $C = 0$.*

Proof. Let $A = PEQ$, where $E = \text{diag}(I_r, 0)$. Now if $G \leq E$ then G has the form $G = \text{diag}(H, 0)$, where $H^2 = H$ is $r \times r$. As such

$$B\hat{A}C = (BQ^{-1}V)G(WP^{-1}C) = RGS = R\,\text{diag}(H, 0)S = 0$$

where $V = I + (I-E)XE$, $W = I + EY(I-E)$, for all X, Y and for all $r \times r$ idempotent H, using Theorem 2.1. Partitioning we get

$$\begin{bmatrix} R_1 & R_2 \end{bmatrix} \begin{bmatrix} H & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} S_1 \\ S_2 \end{bmatrix} = 0$$

and

$$R_1HS_1 = 0.$$

where R_1 and S_1^T are $n \times r$.

We next consider two arbitrary $r \times r$ submatrices R_I and S_J from R_1 and S_1, respectively. Then

$$R_IHS_J = 0, \ \forall\ H = H^2. \tag{18}$$

Hence if some $S_J \neq 0$, then *all* R_I vanish and conversely, using Lemma 3.1. In other words, either $R_1 = 0$ or $S_1 = 0$.

If $R_1 = 0$ then R is of the form $R = BQ^{-1}V = \begin{bmatrix} 0 & \alpha \\ 0 & \beta \end{bmatrix}$.

V is of the form $I + (I - E)XE = \begin{bmatrix} I & 0 \\ x & I \end{bmatrix}$. Since X is arbitrary, then x is arbitrary.

Denote $BQ^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. We then have $BQ^{-1}V = \begin{bmatrix} a + bx & b \\ c + dx & d \end{bmatrix} = R = \begin{bmatrix} 0 & \alpha \\ 0 & \beta \end{bmatrix}$, from which α and β are constant, and $\alpha = b, \beta = d$. Also, $a + bx = 0$ for all x. Taking $x = 0$ we have $a = 0$. So, $bx = 0$ for all possible choices of x, which gives $b = 0$ by taking successively $x = \begin{bmatrix} \mathbf{0} & \cdots & \mathbf{e}_i & \cdots & \mathbf{0} \end{bmatrix}$ for $i = 1, \ldots, n - r$. Similarly from $c + dx = 0$ we obtain $c = 0$ and $d = 0$. So, $BQ = 0$ and therefore $B = 0$.

The converse is clear. □

4 Remarks and questions

1. The element $a\hat{a}a$ is always regular, with a reflexive inverse of $\hat{a}$, and the set of regular elements is given by $\{b\hat{b}b;\ \text{all } b\}$.

2. If $\hat{a}$ is a outer inverse of a, then so is $\hat{a} + (1 - \hat{a}a)(a - a\hat{a}a)^{\wedge}(1 - a\hat{a})$.

3. When A is $m \times n$ over $\mathbb{F}$, we know that $A = R \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} S = RES$, for some invertible R and S. As such $\hat{A} = S^{-1}\hat{E}R^{-1}$ is $n \times m$. The class of all outer inverses of E can be found by direct computation as $\begin{bmatrix} I \\ \alpha \end{bmatrix} H[I, \beta]$, where $H^2 = H$ and α, β are arbitrary.

4. The expression of $u^{-1}E$ in Corollary 2.2 is very similar to the theory of cosets.

5. Can we select p and q such that it is easier to classify $\widehat{(qap)}$?

6. Can we generalize Theorem 3.1 to $R_{n\times n}$ where R is a unit regular ring?

7. Theorem 3.1 shows that over $\mathbb{F}$ there are basically *not* enough outer inverses for the triple to be trivially invariant, i.e., $B = 0$ or $C = 0$.

8. What are some of the other classes of outer inverses that can be characterized?

Acknowledgment

The authors thank an anonymous referee for his/her valuable corrections.

References

[1] F. Graybill; *Introduction to Matrices with Applications to Statistics*, Wadsworth, Belmont, CA 1969, p.124.

[2] R.E. Hartwig; 1-2 Inverses and the Invariance of $BA^{+}C$, *Lin. Algebra App.* 11 (1975), 27–275.

[3] R.E. Hartwig; How to order regular elements, *Math. Jap.* 25 (1980), 1–13.

[4] R.E. Hartwig, J. Luh; Decomposition of an orthogonally complete atomic unit regular ring, 1979, *unpublished*.

[5] R.E Hartwig, P. Patrício, R. Puystjens; Diagonalizing triangular matrices via orthogonal Pierce decompositions. *Linear Algebra Appl.* 401 (2005), 381–391.

[6] R.E. Hartwig, P. Patricio, Two by two units, *Electron. J. Linear Algebra* 27 (2014), 489–503.

[7] C.R. Rao, S. K. Mitra; *Generalized inverse of matrices and its applications.* John Wiley & Sons, Inc., New York-London-Sydney, 1971.