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An elementary proof of the rank-one theorem for BV functions

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Abstract. We provide a simple proof of a result, due to G. Alberti, concerning a rank-one property for the singular part of the derivative of vector-valued functions of bounded variation.

Keywords. Functions of bounded variation, rank-one theorem

In this paper we provide a short, elementary proof of the following result by G. Alberti [1] concerning a rank-one property for the derivative of a function with bounded variation.

Theorem. *Let $\Omega \subset \mathbb{R}^n$ be an open set, $u : \Omega \to \mathbb{R}^m$ a function with bounded variation, and let $D_s u$ be the singular part of Du with respect to the Lebesgue measure $\mathcal{L}^n$. Then $D_s u$ is a rank-one measure, i.e., the (matrix-valued) function $\frac{D_s u}{|D_s u|}(x)$ has rank one for $|D_s u|$-a.e. $x \in \Omega$.*

We recall that a function $u \in L^1(\Omega, \mathbb{R}^m)$ has *bounded variation* in Ω ($u \in \mathrm{BV}(\Omega, \mathbb{R}^m)$) if the derivatives Du of u in the sense of distributions are represented by a (matrix-valued) measure with finite total variation. The measure Du can then be decomposed as the sum $Du = D_a u + D_s u$ of a measure $D_a u$ that is absolutely continuous with respect to $\mathcal{L}^n$, and a measure $D_s u$ that is singular with respect to $\mathcal{L}^n$. The Radon–Nikodym derivative $\frac{D_s u}{|D_s u|}$ of $D_s u$ with respect to its total variation $|D_s u|$ is a $|D_s u|$-measurable map from Ω to $\mathbb{R}^{m\times n}$. The Theorem states that this map takes values in the space of rank-one matrices. See [3] for more details on BV functions.

The Theorem above was conjectured by L. Ambrosio and E. De Giorgi [4]. It was first proved by G. Alberti [1] (see also [2, 5]) by introducing new tools and using sophisticated techniques in geometric measure theory. A new proof follows from the profound PDE result [6], where a rank-one property for maps with bounded deformation is also proved. The Theorem is important for applications to vectorial variational problems (lower semicontinuity, relaxation, approximation and integral representation theorems, etc.) and to systems of PDE.

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On the contrary, our proof of the Theorem is elementary: it stems from well-known geometric properties relating the derivative of a BV function and the *perimeter* of its subgraph. The main new tool is the following lemma, where we denote by $\mathcal{H}^k$ the standard k-dimensional Hausdorff measure and by $\pi : \mathbb{R}^{n+1} \to \mathbb{R}^n$ the canonical projection $\pi(x_1, \dots, x_{n+1}) := (x_1, \dots, x_n)$.

Lemma. *Let Σ_1, Σ_2 be C^1 hypersurfaces in $\mathbb{R}^{n+1}$ with unit normals $\nu_{\Sigma_1}, \nu_{\Sigma_2}$. Then the set*

$$T := \{p \in \Sigma_1 : \exists q \in \Sigma_2 \cap \pi^{-1}(\pi(p)) \text{ with } (\nu_{\Sigma_1}(p))_{n+1} = (\nu_{\Sigma_2}(q))_{n+1} = 0 \text{ and } \nu_{\Sigma_1}(p) \neq \pm\nu_{\Sigma_2}(q)\}$$

is $\mathcal{H}^n$-negligible.

We postpone the proof of the Lemma in order to directly address the proof of the main result.

Proof of the Theorem. Let $u = (u_1, \dots, u_m) \in \mathrm{BV}(\Omega, \mathbb{R}^m)$. It is not restrictive to assume that Ω is bounded. For any $i = 1, \dots, m$ we write $D_s u_i = \sigma_i |D_s u_i|$ for a $|D_s u_i|$-measurable map $\sigma_i : \Omega \to \mathbb{S}^{n-1}$. We also let $E_i := \{(x,t) \in \Omega \times \mathbb{R} : t < u_i(x)\}$ be the subgraph of u_i; it is well known that E_i has finite perimeter in $\Omega \times \mathbb{R}$. Denoting by $\partial^* E_i$ the *reduced boundary* of E_i and by ν_i the measure-theoretic inner normal to E_i, we have (see e.g. [8, Section 4.1.5])

$$|D_s u_i| = \pi_\#(\mathcal{H}^n \llcorner S_i) \quad \text{for } S_i := \{p \in \partial^* E_i : (\nu_i(p))_{n+1} = 0\},$$

where $\pi_\#$ denotes push-forward of measures. The set S_i is n-rectifiable and we can assume that it is contained in the union $\bigcup_{h \in \mathbb{N}} \Sigma_h^i$ of C^1 hypersurfaces Σ_h^i in $\mathbb{R}^{n+1}$.

By [8, Section 4.1.5], the Lemma above and the well-known properties of rectifiable sets, the following properties hold for $\mathcal{H}^n$-a.e. $p \in S_i$:

$$\nu_{\partial^* E_i}(p) = (\sigma_i(\pi(p)), 0); \tag{1}$$

$$\text{if } p \in \Sigma_h^i, \text{ then } \nu_i(p) = \pm\nu_{\Sigma_h^i}(p); \tag{2}$$

$$\text{if } p \in \Sigma_h^i \text{ and } q \in S_j \cap \Sigma_k^j \cap \pi^{-1}(\pi(p)), \text{ then } \nu_{\Sigma_h^i}(p) = \pm\nu_{\Sigma_k^j}(q). \tag{3}$$

Up to modifying S_i on an $\mathcal{H}^n$-negligible set and σ_i on a $|D_s u_i|$-negligible set, we can assume that (1)–(3) hold everywhere on S_i and $\sigma_i = 0$ on $\Omega \setminus \pi(S_i)$.

Since $D_s u = (\sigma_1 |D_s u_1|, \dots, \sigma_m |D_s u_m|)$ and $|D_s u|$ is concentrated on the union $\pi(S_1) \cup \dots \cup \pi(S_m)$, it is enough to prove that the matrix-valued function $(\sigma_1, \dots, \sigma_m)$ has rank 1 on $\pi(S_1) \cup \dots \cup \pi(S_m)$. This will follow if we prove the implication

$$i, j \in \{1, \dots, m\},\ i \neq j,\ x \in \pi(S_i) \implies \sigma_j(x) \in \{0, \sigma_i(x), -\sigma_i(x)\}.$$

If i, j, x are as above and $x \notin \pi(S_j)$, then $\sigma_j(x) = 0$. Otherwise, $x \in \pi(S_i) \cap \pi(S_j)$, i.e., there exist $p \in S_i$ and $h \in \mathbb{N}$ such that $\pi(p) = x$ and $\sigma_i(x) = \pm\nu_{\Sigma_h^i}(p)$ and there exist $q \in S_j$ and $k \in \mathbb{N}$ such that $\pi(q) = x$ and $\sigma_j(x) = \pm\nu_{\Sigma_k^j}(p)$. By (3) we obtain $\sigma_j(x) = \pm\sigma_i(x)$, as wished. □

Proof of the Lemma. Consider the C^1 hypersurfaces $\widetilde{\Sigma}_1, \widetilde{\Sigma}_2$ in $\mathbb{R}^{n+2}$ defined by

$$\widetilde{\Sigma}_1 := \{(x,t,s) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R} : (x,t) \in \Sigma_1,\ s \in \mathbb{R}\},$$
$$\widetilde{\Sigma}_2 := \{(x,t,s) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R} : (x,s) \in \Sigma_2,\ t \in \mathbb{R}\},$$

and let

$$\widetilde{T} := \{(x,t,s) \in \widetilde{\Sigma}_1 \cap \widetilde{\Sigma}_2 : (\nu_{\widetilde{\Sigma}_1}(x,t,s))_{n+1} = (\nu_{\widetilde{\Sigma}_2}(x,t,s))_{n+2} = 0, \\ \nu_{\widetilde{\Sigma}_1}(x,t,s) \neq \nu_{\widetilde{\Sigma}_2}(x,t,s)\}.$$

Clearly, $\widetilde{T}$ is contained in the n-dimensional C^1 submanifold $R := \{(x,t,s) \in \widetilde{\Sigma}_1 \cap \widetilde{\Sigma}_2 : \nu_{\widetilde{\Sigma}_1}(x,t,s) \neq \nu_{\widetilde{\Sigma}_2}(x,t,s)\}$ and $\widetilde{\pi}(\widetilde{T}) = T$, where $\widetilde{\pi}$ denotes the projection $\widetilde{\pi}(x,t,s) = (x,t)$. Given $(x,t,s) \in \widetilde{T}$ it can be easily checked that the vector $(0,\dots,0,1)$ is tangent to R and belongs to the kernel of $d\widetilde{\pi} : T_{(x,t,s)}R \to T_{(x,t)}\Sigma_1$. It follows that $d\widetilde{\pi}$ is not surjective at points of $\widetilde{T}$ and, by the area formula, we deduce that $\mathcal{H}^n(T) = \mathcal{H}^n(\widetilde{\pi}(\widetilde{T})) = 0$, as desired. □

We finally mention that the proof above can be adapted to prove a rank-one theorem for BV functions in sub-Riemannian Heisenberg groups, as well as in a more general class of Carnot groups [7].

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