Asymptotic behaviour of nonlocal p-Laplacian reaction-diffusion problems

Tomás Caraballo, Marta Herrera-Cobos and Pedro Marín-Rubio
Dpto. Ecuaciones Diferenciales y Análisis Numérico
Universidad de Sevilla
c/Tarfia s/n, 41012 Sevilla, Spain
E-mail addresses: caraball@us.es, mhc@us.es, pmr@us.es

Abstract

In this paper, we focus on studying the existence of attractors in the phase spaces $L^2(\Omega)$ and $L^p(\Omega)$ (among others) for time-dependent p-Laplacian equations with nonlocal diffusion and nonlinearities of reaction-diffusion type. Firstly, we prove the existence of weak solutions making use of a change of variable which allows us to get rid of the nonlocal operator in the diffusion term. Thereupon, the regularising effect of the equation is shown applying an argument of a posteriori regularity, since under the assumptions made we cannot guarantee the uniqueness of weak solutions. In addition, this argument allows to ensure the existence of an absorbing family in $W_0^{1,p}(\Omega)$. This leads to the existence of the minimal pullback attractors in $L^2(\Omega)$, $L^p(\Omega)$ and some other spaces as $L^{p^*-\epsilon}(\Omega)$. Relationships between these families are also established.

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1 Introduction

Nonlocal problems have become of great interest in many fields by their applications. They play a key role in Medicine [8], in the industry [16, 28] and last but not least, to study the behaviour of a population with accuracy [24]. Within this framework, Chipot and Rodrigues [12] analyse the behaviour of a population of bacteria in a container considering an elliptic nonlocal diffusion equation.

In the parabolic setting, an equation with nonlocal diffusion which has caught the attention of many authors has been

$$u_t - a(l(u))\Delta u = f,$$

where $a \in C(\mathbb{R}; \mathbb{R}_+)$ is bounded from below by a positive constant, and $l \in \mathcal{L}(L^2(\Omega), \mathbb{R})$.

Despite the similarities between the heat equation and the previous one, which looks like a simple perturbation, several difficulties arise in different contexts when nonlinear diffusion appears. For instance, the existence of a Lyapunov structure is only guaranteed under suitable assumptions (see [11] for more details) or for some specific nonlocal operators (cf. [14, 13]).

Regarding the above parabolic equation and its variations, a wide range of results have been published analysing comparison results between the solution of the evolution problem and stationary solutions [10, 11], the existence of global minimizers [14], the convergence of the solution of the evolution problem to a stationary solution [15], existence of pullback attractors [3, 4] or the upper semicontinuous behaviour of attractors [4], amongst others.

When the Laplace operator is replaced by the p-Laplacian, there are only two papers [13, 5] in which close nonlocal problems are analysed. The p-Laplacian appears in a wide range of areas in Physics (see [34, 35, 25, 30] for more details). In addition, we would like to highlight that analysing a p-Laplacian problem involves additional (nontrivial) difficulties compared to the study for the Laplacian (cf. [3, 4, 6]), since, for example, in the result of the existence of solutions, it is necessary to rescale the time in order to use monotonicity arguments for identifying the limit.

In [13] Chipot and Savitska consider

$$\frac{\partial u}{\partial t} - \nabla \cdot a(\|\nabla u\|_p^p)|\nabla u|^{p-2}\nabla u = f,$$

fulfilled with zero Dirichlet boundary conditions. In addition to proving the existence of solutions making use a suitable change of variable (specified below, see (12)), they establish a classification of the critical points of some energy functional. In [5], the existence of the compact global attractor in $L^2(\Omega)$ is analysed when the nonlocal operator is given by $a(l(u))$ instead of $a(\|\nabla u\|_p^p)$ as considered in [13].

In this paper, we study a much more general problem than in [5], since time-dependent terms and nonlinearities of reaction-diffusion type appear here, and the obtained results (see below) are also stronger. Namely, we consider the non-autonomous nonlocal problem for the p-Laplacian

$$\begin{cases} \dfrac{du}{dt} - a(l(u))\Delta_p u = f(u) + h(t) \quad \text{in } \Omega \times (\tau, T), \\ u = 0 \quad \text{on } \partial\Omega \times (\tau, T), \\ u(x,\tau) = u_\tau(x) \quad \text{in } \Omega, \end{cases} \tag{1}$$

where Ω is a bounded open set of $\mathbb{R}^N$, $p \geq 2$,

$$a \in C(\mathbb{R}; [m, \infty)), \tag{2}$$

where $m > 0$. Observe that we do not assume any Lipschitz condition on the function a as in [3, 6]. As a consequence, in the existence result we are not able to guarantee the uniqueness of weak solutions to (1). We also suppose that $l \in (L^2(\Omega))'$, which means that

$$l(u) = l_g(u) = \int_\Omega g(x)u(x)dx \quad \text{for some } g \in L^2(\Omega).$$

Furthermore, $u_\tau \in L^2(\Omega)$, $T > \tau$ and $h \in L^{p'}_{loc}(\mathbb{R}; W^{-1,p'}(\Omega))$, where p' is the conjugate exponent of p. In addition, $f \in C(\mathbb{R})$ and there exist positive constants κ, α_1 and α_2 and $q \geq 2$, such that

$$-\kappa - \alpha_1|s|^q \leq f(s)s \leq \kappa - \alpha_2|s|^q \quad \forall s \in \mathbb{R}. \tag{3}$$

From (3), we deduce that there exists $\beta > 0$ such that

$$|f(s)| \leq \beta(1 + |s|^{q-1}) \quad \forall s \in \mathbb{R}. \tag{4}$$

In this paper, there are two main aims. Firstly, we want to prove the existence (and regularity) of weak solutions to the nonlocal p-Laplacian problem (1). Secondly, we study the asymptotic behaviour of the solutions in a multi-valued setting, as we have mentioned above. We do this in several phase spaces and their corresponding norms, namely in $L^2(\Omega)$, $L^p(\Omega)$ and $L^{p^*-\epsilon}(\Omega)$ among others (see below for more details). Since we are dealing with a non-autonomous problem, there are several approaches to achieve our goal. For instance, one could make use of skew-product flows (cf. [27]), uniform attractors and their kernel sections (cf. [9]) or pullback attractors (see [21]; in relation with random dynamical systems see [17]). In this work, we use the last approach because it allows us to use more general forcing terms and the attractors fulfil an invariance property. Within this framework, there are several choices, we can employ the concept of attraction related to fixed nonempty bounded sets or with a more general class of families called universe $\mathcal{D}$. This class of families is usually made up by sets which vary in time and fulfil a tempered growth condition (see [7, 19] for more details). In addition, it is possible to establish relationships between the attractors defined by the two cited concepts of attraction (cf. [23]).

Although some authors have been interested in proving the existence of pullback attractors for non-autonomous parabolic equations for the p-Laplacian (cf. [1, 29]), as far as we know, in the previous literature there is no study for the dynamical system considered in this paper. To show the existence of these attractors, we need to check the pullback asymptotic compactness. Due to the presence of the nonlocal operator in the diffusion term together with the nonlinearity of the p-Laplacian, it does not seem to be possible to use the energy method applied in [20, 23, 19], in which it is not necessary to consider more regular non-autonomous terms belonging to $L^2_{loc}(\mathbb{R}; L^2(\Omega))$. In this case, we build an absorbing family in $W_0^{1,p}(\Omega)$ and make use of the compact embedding $W_0^{1,p}(\Omega) \subset\subset L^2(\Omega)$. To prove the existence of an absorbing family in $W_0^{1,p}(\Omega)$, in addition to assuming that $h \in L^2_{loc}(\mathbb{R}; L^2(\Omega))$, we suppose that the function $f \in C(\mathbb{R})$ satisfies

$$|f(s)| \leq C(1 + |s|^{\gamma+1}) \quad \forall s \in \mathbb{R}, \tag{5}$$

with γ fulfilling

$$\gamma : \begin{cases} = \dfrac{p}{2} & \text{if } p > N, \\ < \dfrac{N}{2} & \text{if } p = N, \\ = \dfrac{2p + pN - 2N}{2N} & \text{if } p < N. \end{cases}$$

Observe that this estimate has been obtained using interpolation results (cf. [33, Lemma II.4.1, p. 72]) and the regularity of the weak solutions (cf. Remark 20 (iii)). In fact, we also deduce

$$\begin{aligned} \int_\tau^T \int_\Omega |f(u(x,t))|^2 dx dt &\leq 2C^2|\Omega|(T-\tau) + 2C^2 \int_\tau^T \|u(t)\|_{2\gamma+2}^{2\gamma+2} dt \\ &\leq 2C^2|\Omega|(T-\tau) + 2C^2(C_I(N))^{2\theta(\gamma+1)} \|u\|^{2(1-\theta)(\gamma+1)}_{L^\infty(\tau,T;L^2(\Omega))} \|u\|^{2\theta(\gamma+1)}_{L^p(\tau,T;W_0^{1,p}(\Omega))}, \end{aligned} \tag{6}$$

where $\theta \in [0,1]$ is an interpolation exponent and $C_I(N)$ is the constant of the continuous embedding of $W_0^{1,p}(\Omega)$ into the $L^\mathfrak{p}$-spaces. This way, the term $f(u)$, with $u \in L^\infty(\tau, T; L^2(\Omega)) \cap L^p(\tau, T; W_0^{1,p}(\Omega))$, belongs to $L^2(\tau, T; L^2(\Omega))$ and it allows us to prove the regularising effect of the equation (cf. Theorem 4) and the pullback asymptotic compactness (cf. Lemma 12 and Proposition 13).

The structure of the paper is as follows. Section 2 is devoted to studying the existence of solutions and the regularising effect of the equation. To prove the existence result, we perform a change of variable which allows us to deal with a problem with local diffusion. To analyse the regularising effect of the equation, we apply an argument of a posteriori regularity, because under the assumptions made on the functions a and f, we cannot guarantee the uniqueness of weak solutions. Thereupon, in Section 3 we prove our main first result, that is, the existence of pullback attractors in different universes in the phase space $L^2(\Omega)$. To do this, we prove the pullback asymptotic compactness via an absorbing family in $W_0^{1,p}(\Omega)$ and making use of the compactness of the embedding $W_0^{1,p}(\Omega) \subset\subset L^2(\Omega)$. Furthermore, some relationships between these families of pullback attractors are established. To conclude, in Section 4 we show the existence of pullback attractors in more regular spaces, as $L^p(\Omega)$ and in fact any Banach space X such that the embeddings $W_0^{1,p}(\Omega) \subset\subset X$ and $X \subset L^2(\Omega)$ are compact and continuous respectively (cf. Corollary 19), as for instance in the Lebesgue spaces $L^{p^*-\epsilon}(\Omega)$.

Thereupon, the notation utilised along the text is described. We use $(\cdot,\cdot)$ to denote the inner product in $L^2(\Omega)$, $\|\cdot\|_s$ to represent the norm in $L^s(\Omega)$, and $|\cdot|$ for the Lebesgue measure of a subset of $\mathbb{R}^N$. Thanks to the identification of $L^2(\Omega)$ with its dual, the chain of dense and compact embeddings $W_0^{1,p}(\Omega) \subset\subset L^2(\Omega) \subset\subset W^{-1,p'}(\Omega)$ holds. In addition, making use of the Riesz theorem, there exists $\tilde{l} \in L^2(\Omega)$ with $\langle l, u\rangle_{(L^2(\Omega))', L^2(\Omega)} = (\tilde{l}, u)$, and since $(L^2(\Omega))' \equiv L^2(\Omega)$, it holds that $l = \tilde{l}$. However, instead of using (l, u) to denote the inner product between l and u, we employ the traditional notation $l(u)$. The duality product between $L^q(\Omega)$ and $L^{q'}(\Omega)$ elements (where q is the conjugate exponent of q') is also represented by $(\cdot,\cdot)$. By $\langle\cdot,\cdot\rangle$, we denote the duality product between $W_0^{1,p}(\Omega)$ and $W^{-1,p'}(\Omega)$, between $W_0^{1,p}(\Omega) \cap L^q(\Omega)$ and $W^{-1,p'}(\Omega) + L^{q'}(\Omega)$, and between $H^{-s}(\Omega)$ and $H_0^s(\Omega)$. The norm in $W_0^{1,p}(\Omega)$ is represented by $\|\nabla\cdot\|_p$, the $(L^p(\Omega))^N$ norm of the gradient, and the norm in $W^{-1,p'}(\Omega)$, by $\|\cdot\|_*$.

Before starting with the study of the existence of solutions, we would like to recall that the p-Laplacian operator is a one-to-one map from $W_0^{1,p}(\Omega)$ into $W^{-1,p'}(\Omega)$, defined as follows

$$\langle -\Delta_p u, v\rangle = (|\nabla u|^{p-2}\nabla u, \nabla v) \quad \forall u, v \in W_0^{1,p}(\Omega),$$

where for short $(|\nabla u|^{p-2}\nabla u, \nabla v)$ denotes $\sum_{i=1}^N (|\partial_i u|^{p-2}\partial_i u, \partial_i v)$.

Definition 1. *A weak solution to (1) is a function u that belongs to $L^\infty(\tau, T; L^2(\Omega)) \cap L^q(\tau, T; L^q(\Omega))$ $\cap L^p(\tau, T; W_0^{1,p}(\Omega))$ for all $T > \tau$, with $u(\tau) = u_\tau$, such that*

$$\frac{d}{dt}(u(t), v) + a(l(u(t)))\langle -\Delta_p u(t), v\rangle = (f(u(t)), v) + \langle h(t), v\rangle \quad \forall v \in W_0^{1,p}(\Omega) \cap L^q(\Omega), \tag{7}$$

where the above equation is understood in the sense of the distributions.

Remark 2. *If u is a weak solution to* (1)*, taking into account (2), (4) and (7), it holds that $u' \in L^{p'}(\tau, T; W^{-1,p'}(\Omega)) + L^{q'}(\tau, T; L^{q'}(\Omega))$ for any $T > \tau$. Then, $u \in C([\tau, \infty); L^2(\Omega))$ and therefore, the initial datum in* (1) *makes complete sense. Furthermore, it fulfils*

$$\begin{aligned} &\|u(t)\|_2^2 + 2\int_s^t a(l(u(r)))\|\nabla u(r)\|_p^p dr \\ =&\|u(s)\|_2^2 + 2\int_s^t (f(u(r)), u(r))dr + 2\int_s^t \langle h(r), u(r)\rangle dr \end{aligned} \tag{8}$$

for all $\tau \le s \le t$ (for more details see [18, Théorème 2, p. 575], [32, Lemma 3.2, p. 71]).

2 Existence of solution

In this section, we will prove the existence of weak solutions to (1). To that end, we will use the Galerkin approximations and a change of variable which has been already applied for nonlocal problems (cf. [14, 13, 5]). Finally, we pass to the limit by using compactness and monotonicity arguments.

Theorem 3. *Assume that (2)-(3) hold and $h \in L^{p'}_{loc}(\mathbb{R}; W^{-1,p'}(\Omega))$. Then, for each $u_\tau \in L^2(\Omega)$, there exists at least a weak solution to (1).*

Proof. Consider fixed $T > \tau$. We will show the existence result in the interval $(\tau, \widehat{T})$, where $\widehat{T} := \alpha^{-1}(m(T-\tau)) \in (\tau, T]$ and the function α is defined below (see (12)). Repeating the arguments and making use of a concatenation procedure, we can achieve the existence of a solution in (τ, T).

To prove this result, we consider $\{w_j\} \subset H_0^s(\Omega)$, which is a special basis of $L^2(\Omega)$, with $s \geq \max\left\{\frac{N(q-2)}{2q}, \frac{2p+N(p-2)}{2p}\right\}$. This way, $H_0^s(\Omega) \subset W_0^{1,p}(\Omega) \cap L^q(\Omega)$ (see [2, Chapter 1, p. 39]). From now on, we denote by $V_n = span[w_1, \dots, w_n]$. Observe that the set $\bigcup_{n\in\mathbb{N}} V_n$ is dense in $L^2(\Omega)$.

For every integer $n \geq 1$, the function $u_n(t; \tau, u_\tau) = \sum_{j=1}^n \varphi_{nj}(t) w_j$ (for short, $u_n(t)$) denotes a (local) solution to

$$\begin{cases} \dfrac{d}{dt}(u_n(t), w_j) + a(l(u_n(t)))\langle -\Delta_p u_n(t), w_j\rangle = (f(u_n(t)), w_j) + \langle h(t), w_j\rangle & \text{a.e. } t \in (\tau, T), \\ (u_n(\tau), w_j) = (u_\tau, w_j), & j = 1, \dots, n. \end{cases} \tag{9}$$

Indeed, it is defined in a maximal interval (τ, t_n) with $t_n < T$ and after the following a priori estimates, one can see that $t_n = T$. Multiplying by $\varphi_{nj}(t)$ in (9) and summing from $j = 1$ to n, making use of (2), we have

$$\frac{1}{2}\frac{d}{dt}\|u_n(t)\|_2^2 + m\int_\Omega \|\nabla u_n(t)\|_p^p dx \leq (f(u_n(t)), u_n(t)) + \langle h(t), u_n(t)\rangle \quad \text{a.e. } t \in (\tau, t_n). \tag{10}$$

From (3) and the Young inequality, we deduce

$$(f(u_n(t)), u_n(t)) \leq \kappa|\Omega| - \alpha_2\|u_n(t)\|_q^q,$$
$$\langle h(t), u_n(t)\rangle \leq \frac{1}{p'}\left(\frac{2}{mp}\right)^{p'/p}\|h(t)\|_*^{p'} + \frac{m}{2}\|\nabla u_n(t)\|_p^p.$$

Then, taking this into account, from (10) we obtain

$$\frac{1}{2}\frac{d}{dt}\|u_n(t)\|_2^2 + \frac{m}{2}\|\nabla u_n(t)\|_p^p + \alpha_2\|u_n(t)\|_q^q \leq \kappa|\Omega| + \frac{1}{p'}\left(\frac{2}{mp}\right)^{p'/p}\|h(t)\|_*^{p'} \quad \text{a.e. } t \in (\tau, t_n).$$

Therefore, the sequence $\{u_n\}$ can be considered in the whole interval (τ, T) and it is actually bounded in $L^\infty(\tau, T; L^2(\Omega)) \cap L^p(\tau, T; W_0^{1,p}(\Omega)) \cap L^q(\tau, T; L^q(\Omega))$.

From this, we deduce that $\{-\Delta_p u_n\}$ is bounded in $L^{p'}(\tau, T; W^{-1,p'}(\Omega))$. Furthermore, making use of (4), it fulfils that $\{f(u_n)\}$ is bounded in $L^{q'}(\tau, T; L^{q'}(\Omega))$.

As a consequence, the sequence $\{u_n'\}$ is bounded in $L^{p'}(\tau, T; H^{-s}(\Omega))$. To prove that, we need to consider $P_n : H^{-s}(\Omega) \ni f \mapsto P_n f := \sum_{j=1}^n \langle f, w_j\rangle w_j \in V_n$, which is the continuous extension of the projector P_n defined as $P_n : L^2(\Omega) \ni f \mapsto P_n f := \sum_{j=1}^n (f, w_j) w_j \in V_n$. Moreover, the sequences $\{f(u_n)/a(l(u_n))\}$, $\{(f(u_n)u_n)/a(l(u_n))\}$ and $\{u_n/a(l(u_n))\}$ are bounded in $L^{q'}(\tau, T; L^{q'}(\Omega))$, $L^1(\tau, T; L^1(\Omega))$ and $L^p(\tau, T; W_0^{1,p}(\Omega))$, respectively, thanks to the fact that $f \in C(\mathbb{R})$, $l \in L^2(\Omega)$ and $a \in C(\mathbb{R}; \mathbb{R}_+)$ fulfilling (2).

Therefore, using the Aubin-Lions lemma, we deduce that there exist a subsequence of $\{u_n\}$ (relabeled the same), $\xi \in L^{p'}(\tau, T; W^{-1,p'}(\Omega))$, $u \in L^\infty(\tau, T; L^2(\Omega)) \cap L^p(\tau, T; W_0^{1,p}(\Omega)) \cap L^q(\tau, T; L^q(\Omega))$

with $u' \in L^{p'}(\tau,T;H^{-s}(\Omega))$, such that

$$
\begin{cases}
u_n \overset{*}{\rightharpoonup} u \quad \text{weakly-star in } L^\infty(\tau,T;L^2(\Omega)),\\
u_n \rightharpoonup u \quad \text{weakly in } L^p(\tau,T;W_0^{1,p}(\Omega)),\\
u_n \rightharpoonup u \quad \text{weakly in } L^q(\tau,T;L^q(\Omega)),\\
u_n \to u \quad \text{strongly in } L^p(\tau,T;L^p(\Omega)),\\
-\Delta_p u_n \rightharpoonup \xi \quad \text{weakly in } L^{p'}(\tau,T;W^{-1,p'}(\Omega)),\\
u_n' \rightharpoonup u' \quad \text{weakly in } L^{p'}(\tau,T;H^{-s}(\Omega)),\\
\dfrac{f(u_n)}{a(l(u_n))} \rightharpoonup \dfrac{f(u)}{a(l(u))} \quad \text{weakly in } L^{q'}(\tau,T;L^{q'}(\Omega)),\\
\dfrac{f(u_n)u_n}{a(l(u_n))} \rightharpoonup \dfrac{f(u)u}{a(l(u))} \quad \text{weakly in } L^1(\tau,T;L^1(\Omega)),\\
\dfrac{u_n}{a(l(u_n))} \rightharpoonup \dfrac{u}{a(l(u))} \quad \text{weakly in } L^p(\tau,T;W_0^{1,p}(\Omega)),
\end{cases}
\tag{11}
$$

where the last three convergences have been proved applying [22, Lemme 1.3, p. 12].

The next aim is to check that $\xi = -\Delta_p u$. To do this we will use the same idea from [13] (see also [14, 5]). We rescale the time in the following way

$$
\alpha(t) = \int_\tau^t a(l(u(s)))ds. \tag{12}
$$

Then, making use of (12), solving problem (1) is reduced to dealing with

$$
\begin{cases}
v_s(\alpha(t)) - \Delta_p v(\alpha(t)) = \dfrac{f(v(\alpha(t)))}{a(l(v(\alpha(t))))} + \dfrac{h(t)}{a(l(v(\alpha(t))))} \quad \text{in } \Omega\times(\tau,T),\\
v = 0 \quad \text{on } \partial\Omega\times(\tau,T),\\
v(x,\alpha(\tau)) = u_\tau(x) \quad \text{in } \Omega,
\end{cases}
$$

where $u(x,t) = v(x,\alpha(t))$. Then, $u_t(x,t) = v_s(x,\alpha(t))\alpha'(t) = v_s(x,\alpha(t))a(l(u(t)))$.

Observe that the previous problem becomes

$$
\begin{cases}
v_t - \Delta_p v = \dfrac{f(v)}{a(l(v))} + \dfrac{h(\alpha^{-1}(t))}{a(l(v))} \quad \text{in } \Omega\times(0,\alpha(T)),\\
v = 0 \quad \text{on } \partial\Omega\times(0,\alpha(T)),\\
v(x,0) = u_\tau(x) \quad \text{in } \Omega.
\end{cases}
\tag{13}
$$

To deal with this problem rigorously, we consider the Galerkin approximation problems associated to (13) and

$$
\alpha_n(t) := \int_0^t a(l(u_n(s)))ds.
$$

Namely, $v_n(t) = \sum_{j=1}^n \widetilde{\varphi}_{nj}(t)w_j$, which is set such that $v_n(x,\alpha_n(t)) := u_n(x,t)$, solves

$$
\begin{cases}
\dfrac{d}{dt}(v_n(t),w_j) + (|\nabla v_n(t)|^{p-2}\nabla v_n(t),\nabla w_j) = \dfrac{\langle f(v_n),w_j\rangle}{a(l(v_n(t)))} + \dfrac{\langle h(\alpha_n^{-1}(t)),w_j\rangle}{a(l(v_n(t)))} \quad \text{a.e. } t\in(0,\alpha_n(T)),\\
(v_n(0),w_j) = (u_\tau,w_j), \qquad j = 1,\dots,n.
\end{cases}
\tag{14}
$$

Now, consider $\varphi \in \mathcal{D}(0, m(T-\tau))$ and $w \in V_n$. Then, making use of (14), we obtain

$$-\int_0^{m(T-\tau)} (v_n(t), w)\varphi'(t)dt + \int_0^{m(T-\tau)} (|\nabla v_n(t)|^{p-2}\nabla v_n(t), \nabla w)\varphi(t)dt$$
$$= \int_0^{m(T-\tau)} \left(\frac{f(v_n(t))}{a(l(v_n(t)))}, w\right)\varphi(t)dt + \int_0^{m(T-\tau)} \frac{\langle h(\alpha_n^{-1}(t)), w\rangle}{a(l(v_n(t)))}\varphi(t)dt,$$

since $0 < m(T-\tau) \le \alpha_n(T)$.

Then, taking limit when $n \to \infty$, from (11), we deduce

$$v'(t) + \widehat{\xi}(t) = \frac{f(v(t))}{a(l(v(t)))} + \frac{h(\alpha^{-1}(t))}{a(l(v(t)))} \quad \text{a.e. } t \in (0, m(T-\tau)), \tag{15}$$

where

$$\widehat{\xi}(x, \alpha(t)) = \xi(x, t) \quad \text{a.e. } t \in (\tau, \alpha^{-1}(m(T-\tau))). \tag{16}$$

Observe that in order to pass to the limit with the non-autonomous term, we have taken into account that

$$\int_0^{m(T-\tau)} \frac{\langle h(\alpha_n^{-1}(t)), w\rangle}{a(l(v_n(t)))}\varphi(t)dt = \int_0^{\alpha_n(T)} \frac{\langle h(\alpha_n^{-1}(t)), w\rangle}{a(l(v_n(t)))}\varphi(t)dt,$$
$$\to \int_0^{\alpha(T)} \frac{\langle h(\alpha^{-1}(t)), w\rangle}{a(l(v(t)))}\varphi(t)dt,$$
$$= \int_0^{m(T-\tau)} \frac{\langle h(\alpha^{-1}(t)), w\rangle}{a(l(v(t)))}\varphi(t)dt.$$

To prove that $\widehat{\xi} = -\Delta_p v$, we first check

$$\liminf_{n\to\infty} \|v_n(m(T-\tau))\|_2 \ge \|v(m(T-\tau))\|_2. \tag{17}$$

On the one hand, considering (15) in $(0, m(T-\tau))$, we have that

$$\int_0^{m(T-\tau)} \langle\widehat{\xi}(t), w\rangle dt$$
$$= \int_0^{m(T-\tau)} \left(\frac{f(v(t))}{a(l(v(t)))}, w\right)dt + \int_0^{m(T-\tau)} \frac{\langle h(\alpha^{-1}(t)), w\rangle}{a(l(v(t)))}dt + (v(0), w) - (v(m(T-\tau)), w)$$

for all $w \in V_n$.

On the other hand, from (14), we deduce

$$\int_0^{m(T-\tau)} \langle -\Delta_p v_n(t), w\rangle dt$$
$$= \int_0^{m(T-\tau)} \left(\frac{f(v_n(t))}{a(l(v_n(t)))}, w\right)dt + \int_0^{m(T-\tau)} \frac{\langle h(\alpha_n^{-1}(t)), w\rangle}{a(l(v_n(t)))}dt + (v_n(0), w) - (v_n(m(T-\tau)), w)$$

for all $w \in V_n$. Then, taking limit when $n \to \infty$ in the above expression, using (11) and the fact that $v(0) = u_\tau$, (17) holds.

Now, we are ready to prove $\widehat{\xi} = -\Delta_p v$. From (14), making use of (11) and (17), we have

$$\limsup_{n\to\infty} \int_0^{m(T-\tau)} \|\nabla v_n(t)\|_p^p dt$$
$$\leq \int_0^{m(T-\tau)} \left(\frac{f(v(t))}{a(l(v(t)))}, v(t) \right) dt + \int_0^{m(T-\tau)} \frac{\langle h(\alpha^{-1}(t)), v(t)\rangle}{a(l(v(t)))} dt + \frac{\|u_\tau\|_2^2}{2} - \frac{\|v(m(T-\tau))\|_2^2}{2}. \quad (18)$$

Consider $w \in L^p(0, m(T-\tau); W_0^{1,p}(\Omega))$. Then, thanks to the inequality

$$\int_0^{m(T-\tau)} (|\nabla v_n(t)|^{p-2}\nabla v_n(t) - |\nabla w(t)|^{p-2}\nabla w(t), \nabla(v_n(t) - w(t)))dt \geq 0,$$

taking limit superior and bearing in mind (11), (16) and (18), we have

$$\int_0^{m(T-\tau)} \left(\frac{f(v(t))}{a(l(v(t)))}, v(t) \right) dt + \int_0^{m(T-\tau))} \frac{\langle h(\alpha^{-1}(t)), v(t)\rangle}{a(l(v(t)))} dt + \frac{\|u_\tau\|_2^2}{2} - \frac{\|v(m(T-\tau))\|_2^2}{2}$$
$$- \int_0^{m(T-\tau)} \langle \widehat{\xi}(t), w(t)\rangle dt - \int_0^{m(T-\tau)} (|\nabla w(t)|^{p-2}\nabla w(t), \nabla(v(t) - w(t)))dt \geq 0. \quad (19)$$

Then, combining (15) and (19), we obtain

$$\int_0^{m(T-\tau)} \langle \widehat{\xi}(t) + \nabla \cdot |\nabla w(t)|^{p-2}\nabla w(t), v(t) - w(t)\rangle dt \geq 0,$$

for all $w \in L^p(0, m(T-\tau); W_0^{1,p}(\Omega))$.

Now, considering $w = v - \delta z$ with $\delta > 0$ and $z \in L^p(0, m(T-\tau); W_0^{1,p}(\Omega))$, we have

$$\int_0^{m(T-\tau)} \langle \widehat{\xi}(t) + \nabla \cdot |\nabla(v(t) - \delta z(t))|^{p-2}\nabla(v(t) - \delta z(t)), z(t)\rangle dt \geq 0.$$

As a consequence, the equality $\widehat{\xi}(x,t) = -\Delta_p v(x,t)$ a.e. $t \in (0, m(T-\tau))$ holds. Then, using (16) and the fact that $u(x,t) = v(x, \alpha(t))$ for all $t \in [\tau, \widehat{T}]$, we deduce that $\xi(x,t) = -\Delta_p u(x,t)$ a.e. $t \in (\tau, \widehat{T})$. Finally, to obtain a global-in-time solution just simply reproduce (and concatenate) the previous arguments in intervals of the form $[k\widehat{T}, (k+1)\widehat{T}]$ with $k \in \mathbb{N}$. □

Theorem 4. *Under the assumptions of Theorem 3, if f also fulfils (5) and $h \in L^2_{loc}(\mathbb{R}; L^2(\Omega))$, then any weak solution u to (1) belongs to $C_w((\tau, T]; W_0^{1,p}(\Omega))$. Moreover, if the initial condition $u_\tau \in W_0^{1,p}(\Omega)$, then $u \in C_w([\tau, T]; W_0^{1,p}(\Omega))$.*

Proof. Consider $T > \tau$ and a solution $u(\cdot; \tau, u_\tau)$ to (1), for short denoted by u. Observe that problem

$$(P_u) \begin{cases} \dfrac{\partial y}{\partial t} - a(l(u))\Delta_p y = f(y) + h(t) & \text{in } \Omega \times (\tau, T), \\ y = 0 \quad \text{on } \partial\Omega \times (\tau, T), \\ y(x, \tau) = u_\tau(x) \quad \text{in } \Omega, \end{cases}$$

possesses a unique solution because of the monotonicity of the p-Laplacian (cf. [22, Chapitre II]). Therefore, more regular (a posteriori) estimates as well as using the Galerkin approximations make complete sense. Furthermore, by the uniqueness of solution to (P_u) and the fact that u is a solution to (1), it follows that $y = u$.

Then, we consider the Galerkin formulation associated to problem (P_u)

$$\begin{cases} \dfrac{d}{dt}(\hat{u}_n(t), w_j) + a(l(u))(|\nabla \hat{u}_n(t)|^{p-2}\nabla \hat{u}_n(t), \nabla w_j) = (f(\hat{u}_n(t)), w_j) + (h(t), w_j) & \text{a.e. } t \in (\tau, T), \\ (\hat{u}_n(\tau), w_j) = (u_\tau, w_j), & j = 1, \ldots, n, \end{cases} \tag{20}$$

with $\hat{u}_n(t;\tau,u_\tau) = \sum_{j=1}^n \widehat{\varphi}_{nj}(t) w_j$, which is denoted by $\hat{u}_n(t)$ in what follows.

Multiplying (20) by $\widehat{\varphi}_{nj}(t)$, summing from $j=1$ until n and making use of (2), we have

$$\frac{1}{2}\frac{d}{dt}\|\hat{u}_n(t)\|_2^2 + m\|\nabla \hat{u}_n(t)\|_p^p \le (f(\hat{u}_n(t)), \hat{u}_n(t)) + (h(t), \hat{u}_n(t)) \quad \text{a.e. } t \in (\tau, T).$$

Applying the Young inequality and (3) in the previous inequality, analogously as in Theorem 3, we obtain

$$\frac{d}{dt}\|\hat{u}_n(t)\|_2^2 + m\|\nabla \hat{u}_n(t)\|_p^p + 2\alpha_2\|\hat{u}_n(t)\|_q^q \le 2\kappa|\Omega| + \frac{2}{p'}\left(\frac{2}{mp}\right)^{p'/p}\|h(t)\|_*^{p'} \quad \text{a.e. } t \in (\tau, T).$$

Integrating between τ and T, we deduce the analogous uniform estimates for $\hat{u}_n$ (as in Theorem 3) and in particular

$$m\int_\tau^T \|\nabla \hat{u}_n(t)\|_p^p dt \le \|u_\tau\|_2^2 + 2\kappa|\Omega|(T-\tau) + \frac{2}{p'}\left(\frac{2}{mp}\right)^{p'/p}\|h\|_{L^{p'}(\tau,T;W^{-1,p'}(\Omega))}^{p'}. \tag{21}$$

Now, multiplying (20) by $\widehat{\varphi}'_{nj}(t)/a(l(u(t)))$ and summing from $j=1$ to n, we have

$$\frac{\|\hat{u}'_n(t)\|_2^2}{a(l(u(t)))} + \frac{1}{p}\frac{d}{dt}\|\nabla \hat{u}_n(t)\|_p^p = \frac{(f(\hat{u}_n(t)), \hat{u}'_n(t))}{a(l(u(t)))} + \frac{(h(t), \hat{u}'_n(t))}{a(l(u(t)))} \quad \text{a.e. } t \in (\tau, T).$$

Then, the Cauchy inequality and (2) imply

$$\frac{1}{p}\frac{d}{dt}\|\nabla \hat{u}_n(t)\|_p^p \le \frac{\|f(\hat{u}_n(t))\|_2^2}{2m} + \frac{\|h(t)\|_2^2}{2m} \quad \text{a.e. } t \in (\tau, T).$$

Now, integrating between s and t, with $\tau < s \le t \le T$, and bearing in mind that $f(\hat{u}_n) \in L^2(\tau, T; L^2(\Omega))$ thanks to the regularity of $\hat{u}_n$ (recall after (5) we have (6)), we obtain

$$\|\nabla \hat{u}_n(t)\|_p^p \le \|\nabla \hat{u}_n(s)\|_p^p + \frac{p}{2m}\int_\tau^T \|f(\hat{u}_n(t))\|_2^2 dt + \frac{p}{2m}\int_\tau^T \|h(t)\|_2^2 dt.$$

Then, integrating w.r.t. s between τ and t, and using (6), we deduce

$$\begin{aligned} &\|\nabla \hat{u}_n(t)\|_p^p \\ \le &\frac{1}{\varepsilon}\int_\tau^T \|\nabla \hat{u}_n(s)\|_p^p ds + \frac{C^2 p(T-\tau)}{\varepsilon m}\left[|\Omega| + (C_I(N))^{2\theta(\gamma+1)}\|\hat{u}_n\|_{L^\infty(\tau,T;L^2(\Omega))}^{2(1-\theta)(\gamma+1)}\|\hat{u}_n\|_{L^p(\tau,T;W_0^{1,p}(\Omega))}^{2\theta(\gamma+1)}\right] \\ &+ \frac{p(T-\tau)}{2\varepsilon m}\int_\tau^T \|h(t)\|_2^2 dt, \end{aligned}$$

for all $t \in [\tau+\varepsilon, T]$ with $\varepsilon \in (0, T-\tau)$. From this, taking into account (21), we deduce that the sequence $\{\hat{u}_n\}$ is bounded in $L^\infty(\tau+\varepsilon, T; W_0^{1,p}(\Omega))$. By the uniqueness of solution to (P_u), the whole sequence

$$\hat{u}_n \overset{*}{\rightharpoonup} u \quad \text{weakly-star in } L^\infty(\tau+\varepsilon, T; W_0^{1,p}(\Omega)).$$

In addition, since $u \in C([\tau, T]; L^2(\Omega))$, it holds that $u \in C_w((\tau, T]; W_0^{1,p}(\Omega))$ (cf. [31, Theorem 2.1, p. 544] or [32, Lemma 3.3, p. 74]).

The case in which the initial datum u_τ belongs to $W_0^{1,p}(\Omega)$ allows to simplify the above estimates in a standard way and the solution u belongs in fact to $C_w([\tau, T]; W_0^{1,p}(\Omega))$. □

3 Existence of minimal pullback attractors in $L^2(\Omega)$

The main goal of this section is to study the asymptotic behaviour of the solutions to (1) analysing the existence of the minimal pullback attractors in the$L^2(\Omega)$ norm for different universes. A brief recall of the main definitions and abstract results required in order to achieve our aim in a set-valued framework can be found, e.g., in [4, Section 3].

The most difficult part about proving the existence of these families is to check the pullback asymptotic compactness. To do this, we build a pullback absorbing family in $W_0^{1,p}(\Omega)$ and make use of the compact embedding $W_0^{1,p}(\Omega) \subset\subset L^2(\Omega)$. Once the existence of the minimal pullback attractors is established, some relationships between these families are analysed.

In what follows, we denote by $\Phi(\tau, u_\tau)$ the set of solutions to (1) in $[\tau, \infty)$ with initial datum $u_\tau \in L^2(\Omega)$.

Now, thanks to Theorem 3 we can define a multi-valued map $U : \mathbb{R}^2_d \times L^2(\Omega) \to \mathcal{P}(L^2(\Omega))$ (e.g., cf. [4, Definition 2]) as

$$U(t,\tau)u_\tau = \{u(t) : u \in \Phi(\tau, u_\tau)\}, \quad \tau \le t, \quad u_\tau \in L^2(\Omega), \tag{22}$$

where $\mathbb{R}^2_d = \{(t,s) \in \mathbb{R}^2 : t \ge s\}$.

Lemma 5. *Assume that (2)-(3) hold and $h \in L^{p'}_{loc}(\mathbb{R}; W^{-1,p'}(\Omega))$. Then, the multi-valued map U defined in (22) is a strict multi-valued process in $L^2(\Omega)$.*

Now, to study more properties of the multi-valued process U, we need the following result.

Lemma 6. *Under the assumptions of Lemma 5, given a convergent sequence of initial data $\{u^n_\tau\} \subset L^2(\Omega)$, i.e. $u^n_\tau \to u_\tau$ strongly in $L^2(\Omega)$, it fulfils that for any sequence $\{u^n\}$, where $u^n \in \Phi(\tau, u^n_\tau)$ for all n, there exist a subsequence of $\{u^n\}$ (relabeled the same) and $u \in \Phi(\tau, u_\tau)$, such that*

$$u^n(s) \to u(s) \quad \textit{strongly in } L^2(\Omega) \quad \forall s \ge \tau. \tag{23}$$

Proof. Consider fixed $\tau > T$. Arguing similarly to Theorem 3, making use of the energy equality, (2), (3) and the Young inequality we can deduce

$$\frac{d}{dt}\|u^n(t)\|_2^2 + m\|\nabla u^n(t)\|_p^p + 2\alpha_2\|u^n(t)\|_q^q \le 2\kappa|\Omega| + \frac{2}{p'}\left(\frac{2}{mp}\right)^{p'/p}\|h(t)\|_*^{p'} \quad \text{a.e. } t \in (\tau, T).$$

Thus, $\{u^n\}$ is bounded in $L^\infty(\tau, T; L^2(\Omega)) \cap L^p(\tau, T; W_0^{1,p}(\Omega)) \cap L^q(\tau, T; L^q(\Omega))$. Bearing this in mind, together with the facts that $a \in C(\mathbb{R}; \mathbb{R}_+)$ and $l \in L^2(\Omega)$, we deduce that there exists a positive constant $M_{C_\infty} > 0$ such that

$$a(l(u^n(t))) \le M_{C_\infty} \quad \forall t \in [\tau, T] \quad \forall n \in \mathbb{N}.$$

Then, the sequence $\{-a(l(u^n))\Delta_p u^n\}$ is bounded in $L^{p'}(\tau, T; W^{-1,p'}(\Omega))$. In addition, $\{f(u^n)\}$ is bounded in $L^{q'}(\tau, T; L^{q'}(\Omega))$ thanks to (4) and the boundedness of $\{u^n\}$ in $L^q(\tau, T; L^q(\Omega))$.

As a consequence, the sequence $\{(u^n)'\}$ is bounded in $L^{p'}(\tau, T; W^{-1,p'}(\Omega)) + L^{q'}(\tau, T; L^{q'}(\Omega))$. Now, applying the Aubin-Lions lemma, there exist a subsequence of $\{u^n\}$ (relabeled the same) and $u \in L^\infty(\tau, T; L^2(\Omega)) \cap L^p(\tau, T; W_0^{1,p}(\Omega)) \cap L^q(\tau, T; L^q(\Omega))$ with $u' \in L^{p'}(\tau, T; W^{-1,p'}(\Omega)) + L^{q'}(\tau, T; L^{q'}(\Omega))$

such that

$$
\begin{cases}
u^n \overset{*}{\rightharpoonup} u \quad \text{weakly-star in } L^\infty(\tau,T;L^2(\Omega)),\\
u^n \rightharpoonup u \quad \text{weakly in } L^p(\tau,T;W_0^{1,p}(\Omega)),\\
u^n \rightharpoonup u \quad \text{weakly in } L^q(\tau,T;L^q(\Omega)),\\
u^n \to u \quad \text{strongly in } L^p(\tau,T;L^p(\Omega)),\\
u^n(t) \to u(t) \quad \text{strongly in } L^p(\Omega) \quad \text{a.e. } t\in(\tau,T),\\
(u^n)' \rightharpoonup u' \quad \text{weakly in } L^{p'}(\tau,T;W^{-1,p'}(\Omega)) + L^{q'}(\tau,T;L^{q'}(\Omega)),\\
f(u^n) \rightharpoonup f(u) \quad \text{weakly in } L^{q'}(\tau,T;L^{q'}(\Omega)),\\
-a(l(u^n))\Delta_p u^n \rightharpoonup -a(l(u))\Delta_p u \quad \text{weakly in } L^{p'}(\tau,T;W^{-1,p'}(\Omega)),
\end{cases}
\tag{24}
$$

where the limit of the sequence $\{f(u^n)\}$ has been obtained using [22, Lemme 1.3, p. 12] and the last one, the limit of the sequence $\{-a(l(u^n))\Delta_p u^n\}$, has been obtained applying the change of variable (12) used to prove the existence of solution in Theorem 3.

Now we are ready to prove (23). Observe that

$$
u^n(s) \rightharpoonup u(s) \quad \text{weakly in } L^2(\Omega) \quad \forall s\in[\tau,T], \tag{25}
$$

just simply applying the Ascoli-Arzelà theorem and the fact that the sequence $\{u^n\}$ is bounded in $C([\tau,T];L^2(\Omega))$. Namely, since the sequence $\{u^n\}$ is equicontinuous in $W^{-1,p'}(\Omega)+L^{q'}(\Omega)$ on $[\tau,T]$, bounded in $C([\tau,T];L^2(\Omega))$ and the embedding $L^2(\Omega)\subset\subset W^{-1,p'}(\Omega)+L^{q'}(\Omega)$ is compact, by the Ascoli-Arzelà theorem, it fulfils

$$
u^n \to u \quad \text{strongly in } C([\tau,T];W^{-1,p'}(\Omega)+L^{q'}(\Omega)).
$$

From this, bearing in mind the boundedness of $\{u^n\}$ in $C([\tau,T];L^2(\Omega))$, (25) holds.

Now, to prove (23), we only need to check

$$
\limsup_{n\to\infty}\|u^n(s)\|_2 \le \|u(s)\|_2 \quad \forall s\in[\tau,T]. \tag{26}
$$

To do that, we use an energy method which relies on the continuity of the solutions (see [20, 23, 19] for more details).

Observe that from the energy equality (8), we deduce

$$
\|z(s)\|_2^2 \le \|z(r)\|_2^2 + 2\kappa|\Omega|(s-r) + \frac{2}{p'}\left(\frac{1}{mp}\right)^{p'/p}\int_r^s \|h(\xi)\|_*^{p'}d\xi \quad \forall \tau\le r\le s\le T,
$$

where z is replaced by u or any u^n.

Now, we define the following continuous and non-increasing functions on $[\tau,T]$

$$
J_n(s) = \|u^n(s)\|_2^2 - 2\kappa|\Omega|(s-\tau) - \frac{2}{p'}\left(\frac{1}{mp}\right)^{p'/p}\int_\tau^s \|h(\xi)\|_*^{p'}d\xi,
$$
$$
J(s) = \|u(s)\|_2^2 - 2\kappa|\Omega|(s-\tau) - \frac{2}{p'}\left(\frac{1}{mp}\right)^{p'/p}\int_\tau^s \|h(\xi)\|_*^{p'}d\xi.
$$

Observe that from (24), the continuity of the functional J on $[\tau,T]$ and the non-increasing character of the function J_n on $[\tau,T]$, we deduce

$$
J_n(s) \to J(s) \quad \forall s\in(\tau,T).
$$

As a consequence, (26) holds. A diagonal argument in increasing intervals yield (23) for all $s\ge\tau$. □

Now we are ready to prove that the multi-valued process U is upper-semicontinuous with closed values.

Proposition 7. *Under the assumptions of Lemma 5, the multi-valued process U is upper-semicontinuous with closed values.*

Proof. To prove that the multi-valued process U is upper-semicontinuous, we argue by contradiction. Suppose that there exist $(t,\tau) \in \mathbb{R}^2_d$, a neighbourhood $\mathcal{N}(U(t,\tau)u_\tau)$ and a sequence $\{y_n\}$ which fulfils that each $y_n \in U(t,\tau)u^n_\tau$, where $u^n_\tau \to u_\tau$ strongly in $L^2(\Omega)$, but for all $n \in \mathbb{N}$ $y_n \notin \mathcal{N}(U(t,\tau)u_\tau)$.

Since each $y_n \in U(t,\tau)u^n_\tau$, there exists $u^n \in \Phi(\tau, u^n_\tau)$ such that $y_n = u^n(t)$. Now, applying Lemma 6, we deduce that there exists a subsequence of $\{u^n(t)\}$ (relabeled the same) which converges to a function $u(t) \in U(t,\tau)u_\tau$. This is contradictory because $y_n \notin \mathcal{N}(U(t,\tau)u_\tau)$ for all n.

Finally, Lemma 6 implies again that U has closed values. □

Now, to define a suitable tempered universe we introduce the following estimate.

Proposition 8. *Under the assumptions of Lemma 5, given $u_\tau \in L^2(\Omega)$, for any $\mu \in (0, 2m)$, any solution to (1) fulfils*

$$\|u(t)\|_2^2 \le e^{-\mu(t-\tau)}\|u_\tau\|_2^2 + C_1(\mu) + C_2(\mu)e^{-\mu t}\int_\tau^t e^{\mu s}\|h(s)\|_*^{p'}\,ds \tag{27}$$

for all $t \ge \tau$, where C_I is the constant of the continuous embedding $W_0^{1,p}(\Omega) \subset L^2(\Omega)$, and $C_1(\mu)$ and $C_2(\mu)$ are positive constants depending on μ.

Proof. Bearing in mind the energy equality (8), (2) and (3), we obtain

$$\frac{d}{dt}\|u(t)\|_2^2 + 2m\|\nabla u(t)\|_p^p + 2\alpha_2\|u(t)\|_q^q \le 2\kappa|\Omega| + 2\langle h(t), u^n(t)\rangle \quad \text{a.e. } t \ge \tau.$$

Now, adding $\mu\|u(t)\|_2^2$ in both sides in the above expression, using the Young inequality suitably in order to cancel the $2m\|\nabla u(t)\|_p^p$ term and making use of

$$\|u(t)\|_2^2 \le \frac{p-2}{p}\left(\frac{2C_I^p}{p}\right)^{2/(p-2)} + \|\nabla u(t)\|_p^p,$$

after multiplying by $e^{\mu t}$ we deduce

$$\frac{d}{dt}(e^{\mu t}\|u(t)\|_2^2) \le \mu C_1(\mu)e^{\mu t} + C_2(\mu)e^{\mu t}\|h(t)\|_*^{p'} \quad \text{a.e. } t \ge \tau,$$

where

$$C_1(\mu) = \frac{2\kappa|\Omega|}{\mu} + \frac{(p-2)}{p}\left(\frac{2C_I^p}{p}\right)^{2/(p-2)} \quad \text{and} \quad C_2(\mu) = \frac{1}{p'}\left(\frac{2^p}{2mp-\mu p}\right)^{p'/p}.$$

Then, integrating between τ and t we obtain (27). □

Now, we are ready to define an adequate tempered universe in $\mathcal{P}(L^2(\Omega))$.

Definition 9. *For each $\mu > 0$, $\mathcal{D}_\mu^{L^2}$ denotes the class of all families of nonempty subsets $\widehat{D} = \{D(t) : t \in \mathbb{R}\} \subset \mathcal{P}(L^2(\Omega))$ such that*

$$\lim_{\tau \to -\infty}\left(e^{\mu\tau}\sup_{v \in D(\tau)}\|v\|_2^2\right) = 0.$$

Remark 10. *Observe that* $\mathcal{D}_F^{L^2} \subset \mathcal{D}_\mu^{L^2}$ *and the universe* $\mathcal{D}_\mu^{L^2}$ *is inclusion-closed.*

After Proposition 8 and Definition 9, it is possible to ensure the existence of a pullback $\mathcal{D}_\mu^{L^2}$-absorbing family under a suitable weighted assumption on h. Namely, for some $\mu \in (0, 2m)$

$$\int_{-\infty}^{0} e^{\mu s} \|h(s)\|_*^{p'} ds < \infty. \tag{28}$$

Corollary 11. *Under the assumptions of Lemma 5, if for some* $\mu \in (0, 2m)$ *the function* h *fulfils (28), then the family* $\widehat{D}_0 = \{D_0(t) : t \in \mathbb{R}\}$ *defined by* $D_0(t) = \overline{B}_{L^2}(0, R_{L^2}(t))$*, the closed ball in* $L^2(\Omega)$ *of center zero and radius* $R_{L^2}(t)$*, where*

$$(R_{L^2}(t))^2 = 1 + C_1(\mu) + e^{-\mu t} C_2(\mu) \int_{-\infty}^{t} e^{\mu s} \|h(s)\|_*^{p'} ds,$$

is pullback $\mathcal{D}_\mu^{L^2}$*-absorbing for the multi-valued process* U*. In addition,* $\widehat{D}_0 \in \mathcal{D}_\mu^{L^2}$*.*

Now, to prove the existence of pullback attractors, it suffices to check the pullback $\widehat{D}_0$-asymptotic compactness. To that end, let us firstly establish some useful estimates. To do this, rather than (28), we assume the stronger assumption

$$\int_{-\infty}^{0} e^{\mu s} \|h(s)\|_2^2 ds < \infty \tag{29}$$

for some $\mu \in (0, 2m)$.

Lemma 12. *Under the assumptions of Lemma 5, if (5) holds and* $h \in L^2_{loc}(\mathbb{R}; L^2(\Omega))$ *fulfils (29) for some* $\mu \in (0, 2m)$*, then for any* $t \in \mathbb{R}$ *and* $\widehat{D} \in \mathcal{D}_\mu^{L^2}$*, there exists* $\tau_1(\widehat{D}, t) < t - 2$ *such that for any* $\tau \le \tau_1(\widehat{D}, t)$*,* $u_\tau \in D(\tau)$ *and* $u \in \Phi(\tau, u_\tau)$*, it holds*

$$\begin{cases} \|u(r; \tau, u_\tau)\|_2^2 \le \rho_1(t) \quad \forall r \in [t-2, t], \\[2mm] \displaystyle\int_{r-1}^{r} \|\nabla u(\xi; \tau, u_\tau)\|_p^p d\xi \le \rho_2(t) \quad \forall r \in [t-1, t], \\[2mm] \|\nabla u(r; \tau, u_\tau)\|_p^p \le \rho_3(t) \quad \forall r \in [t-1, t], \end{cases} \tag{30}$$

where

$$\rho_1(t) = 1 + \frac{2\kappa|\Omega|}{\mu} + \frac{p-2}{p} \left(\frac{2C_I^p}{p} \right)^{2/(p-2)} + \left(\frac{2^p}{2mp - \mu p} \right)^{p'/p} \frac{e^{-\mu(t-2)}}{p'} \int_{-\infty}^{t} e^{\mu \xi} \|h(\xi)\|_*^{p'} d\xi,$$

$$\rho_2(t) = \frac{\rho_1(t)}{m} + \frac{2\kappa|\Omega|}{m} + \frac{1}{mp'} \left(\frac{2^p}{mp} \right)^{p'/p} \max_{r \in [t-1,t]} \int_{r-1}^{r} \|h(\xi)\|_*^{p'} d\xi,$$

$$\rho_3(t) = \rho_2(t) + \frac{p}{2m} \max_{r \in [t-1,t]} \int_{r-1}^{r} \|h(\xi)\|_2^2 d\xi + \frac{C^2 p}{m} \left\{ |\Omega| + (\rho_1(t))^{(1-\theta)(\gamma+1)} [(C_I(N))^2 \rho_2(t)]^{\theta(\gamma+1)} \right\}.$$

Proof. Consider fixed t and $\widehat{D} \in \mathcal{D}_\mu^{L^2}$. Analogously as in Corollary 11, from the energy equality (8), we deduce that there exists $\tau_1(\widehat{D}, t) < t - 2$ such that

$$\|u(r; \tau, u_\tau)\|_2^2 \le \rho_1(t) \quad \forall r \in [t-2, t] \quad \forall u_\tau \in D(\tau) \quad \forall \tau \le \tau_1(\widehat{D}, t),$$

where $\rho_1(t)$ is given in the statement.

On the other hand, from (8) in $[r-1,r]$, making use of (2), we have

$$\|u(r)\|_2^2 + 2m\int_{r-1}^{r} \|\nabla u(\xi)\|_p^p d\xi \le \|u(r-1)\|_2^2 + 2\int_{r}^{r-1} (f(u(\xi)), u(\xi))d\xi + 2\int_{r-1}^{r} \langle h(\xi), u(\xi)\rangle d\xi.$$

Since $h \in L^2_{loc}(\mathbb{R}; L^2(\Omega)) \subset L^q_{loc}(\mathbb{R}; L^q(\Omega))$,

$$(f(u(\xi)), u(\xi)) \le \kappa|\Omega| - \alpha_2\|u(\xi)\|_q^q,$$
$$2\langle h(\xi), u(\xi)\rangle \le \frac{1}{p'}\left(\frac{2^p}{mp}\right)^{p'/p} \|h(\xi)\|_*^{p'} + m\|\nabla u(\xi)\|_p^p,$$

we obtain

$$\|u(r)\|_2^2 + m\int_{r-1}^{r} \|\nabla u(\xi)\|_p^p d\xi + 2\alpha_2\int_{r-1}^{r} \|u(\xi)\|_q^q d\xi \le \|u(r-1)\|_2^2 + 2\kappa|\Omega| + \frac{1}{p'}\left(\frac{2^p}{mp}\right)^{p'/p}\int_{r-1}^{r} \|h(\xi)\|_*^{p'} d\xi.$$

Therefore,

$$\int_{r-1}^{r} \|\nabla u(\xi;\tau,u_\tau)\|_p^p d\xi \le \rho_2(t) \quad \forall r \in [t-1,t] \quad \forall u_\tau \in D(\tau) \quad \tau \le \tau_1(\widehat{D}, t),$$

where $\rho_2(t)$ is given in the statement.

Finally, to prove the last estimate of (30), we argue as in Theorem 4, making use of an a posteriori regularity argument combined with the Galerkin approximations for the problem (P_u) and compactness arguments. Namely, multiplying (20) by $\widehat{\varphi}'_{nj}(\xi)/a(l(u(\xi)))$, summing from $j=1$ to n, and using the Cauchy inequality and (2), we have

$$\frac{1}{p}\frac{d}{d\xi}\|\nabla \hat{u}_n(\xi)\|_p^p \le \frac{\|f(\hat{u}_n(\xi))\|_2^2}{2m} + \frac{\|h(\xi)\|_2^2}{2m} \quad \text{a.e. } \xi > \tau.$$

Now, integrating between r and s, with $\tau < r-1 \le s \le r$,

$$\|\nabla \hat{u}_n(r)\|_p^p \le \|\nabla \hat{u}_n(s)\|_p^p + \frac{p}{2m}\int_{r-1}^{r} \|f(\hat{u}_n(\xi))\|_2^2 d\xi + \frac{p}{2m}\int_{r-1}^{r} \|h(\xi)\|_2^2 d\xi.$$

Then, taking into account (6) and integrating w.r.t. s between $r-1$ and r, we have

$$\begin{aligned}
&\|\nabla \hat{u}_n(r)\|_p^p \\
&\le \int_{r-1}^{r} \|\nabla \hat{u}_n(s)\|_p^p ds + \frac{C^2 p}{m}\left[|\Omega| + (C_I(N))^{2\theta(\gamma+1)}\|\hat{u}_n\|^{2(1-\theta)(\gamma+1)}_{L^\infty(r-1,r;L^2(\Omega))}\|\hat{u}_n\|^{2\theta(\gamma+1)}_{L^p(r-1,r;W_0^{1,p}(\Omega))}\right] \\
&\quad + \frac{p}{2m}\int_{r-1}^{r} \|h(\xi)\|_2^2 d\xi.
\end{aligned}$$

Therefore, as the sequence $\{\hat{u}_n\}$ also fulfils the first two inequalities appearing in (30), we deduce

$$\|\nabla \hat{u}_n(r)\|_p^p \le \rho_2(t) + \frac{C^2 p}{m}\left\{|\Omega| + (\rho_1(t))^{(1-\theta)(\gamma+1)}[(C_I(N))^2\rho_2(t)]^{\theta(\gamma+1)}\right\} + \frac{p}{2m}\int_{r-1}^{r} \|h(\xi)\|_2^2 d\xi,$$

for all $r \in [t-1,t]$, $\tau \le \tau_1(\widehat{D}, t)$ and $u_\tau \in D(\tau)$.

From this, taking into account that $\hat{u}_n \overset{*}{\rightharpoonup} u$ weakly-star in $L^\infty(t-1,t;W_0^{1,p}(\Omega))$ (cf. Theorem 4) and the fact that $u \in C([t-1,t];L^2(\Omega))$, the last estimate in (30) holds. □

After the previous result we have a better absorbing property, namely a family whose time sections are bounded in $W_0^{1,p}(\Omega)$ (by $\rho_3(t)$). Then the following result follows immediately, taking into account the compact embedding $W_0^{1,p}(\Omega) \subset\subset L^2(\Omega)$.

Proposition 13. *Under the assumptions and notation of Lemma 12, the multi-valued process U possesses a pullback $\mathcal{D}_\mu^{L^2}$-absorbing family, namely $\{\overline{B}_{W_0^{1,p}}(0, \rho_3(t)^{1/p}) : t \in \mathbb{R}\}$. In particular U is pullback $\mathcal{D}_\mu^{L^2}$-asymptotically compact.*

From the above results, we deduce the existence of the minimal pullback attractors in $L^2(\Omega)$, which is the main result of this section.

Theorem 14. *Assume (2)-(3) and (5) hold and $h \in L^2_{loc}(\mathbb{R}; L^2(\Omega))$ satisfies condition (29) for some $\mu \in (0, 2m)$. Then, there exist the minimal pullback $\mathcal{D}_F^{L^2}$-attractor $\mathcal{A}_{\mathcal{D}_F^{L^2}} = \{\mathcal{A}_{\mathcal{D}_F^{L^2}}(t) : t \in \mathbb{R}\}$ and the minimal pullback $\mathcal{D}_\mu^{L^2}$-attractor $\mathcal{A}_{\mathcal{D}_\mu^{L^2}} = \{\mathcal{A}_{\mathcal{D}_\mu^{L^2}}(t) : t \in \mathbb{R}\}$ for the process $U : \mathbb{R}^2_d \times L^2(\Omega) \to \mathcal{P}(L^2(\Omega))$. The minimal pullback $\mathcal{D}_\mu^{L^2}$-attractor $\mathcal{A}_{\mathcal{D}_\mu^{L^2}}$ belongs to $\mathcal{D}_\mu^{L^2}$ and the following relationships hold*

$$\mathcal{A}_{\mathcal{D}_F^{L^2}}(t) \subset \mathcal{A}_{\mathcal{D}_\mu^{L^2}}(t) \subset D_0(t) = \overline{B}_{L^2}(0, R_{L^2}(t)) \quad \forall t \in \mathbb{R}. \tag{31}$$

In addition, if the function h fulfils

$$\sup_{s \le 0} \left(e^{-\mu s} \int_{-\infty}^{s} e^{\mu\theta} \|h(\theta)\|_*^{p'} d\theta \right) < \infty, \tag{32}$$

then $\mathcal{A}_{\mathcal{D}_F^{L^2}}(t) = \mathcal{A}_{\mathcal{D}_\mu^{L^2}}(t)$ for all $t \in \mathbb{R}$.

Proof. The existence of $\mathcal{A}_{\mathcal{D}_\mu^{L^2}}$ and $\mathcal{A}_{\mathcal{D}_F^{L^2}}$, together with the first relationship established in (31) are a consequence of [4, Corollary 1], since the strict multi-valued process U is upper-semicontinuous with closed values (cf. Proposition 7), there is an adequate relationship between the universes (cf. Remark 10), there exists a pullback $\mathcal{D}_\mu^{L^2}$-absorbing family (cf. Corollary 11) and the multi-valued process U is pullback $\mathcal{D}_\mu^{L^2}$-asymptotically compact (cf. Proposition 13).

The second relationship established in (31) between the family $\mathcal{A}_{\mathcal{D}_\mu^{L^2}}$ and $\widehat{D}_0$ together with the fact that $\mathcal{A}_{\mathcal{D}_\mu^{L^2}}$ belongs to $\mathcal{D}_\mu^{L^2}$ are a direct consequence of [4, Theorem 2], Remark 10 and Corollary 11.

Finally, under the stronger assumption (32), it fulfils that the set $\cup_{t \le T} R_{L^2}(t)$ is bounded for each $T \in \mathbb{R}$, where the expression of $R_{L^2}(t)$ has been provided in Corollary 11. Therefore, from [4, Corollary 2], we deduce that both attractors, $\mathcal{A}_{\mathcal{D}_F^{L^2}}$ and $\mathcal{A}_{\mathcal{D}_\mu^{L^2}}$, coincide. □

4 Attraction in more regular spaces

In the spirit of the bi-space attractor theory (e.g. cf. [2, 34, 35]) we expect here better results concerning the pullback attractors obtained in the previous section. Indeed, in the stronger setting established in Theorem 14, we will first ensure the existence of the minimal pullback attractor in $L^p(\Omega)$ (as usual in the literature related to the p-Laplacian operator), since the sections of the absorbing family belongs to $W_0^{1,p}(\Omega)$. Actually, we do this just for the sake of simplicity in the exposition, since the main result is stronger, as shown in Corollary 19.

Consider the class Φ_p of weak solutions to (1) where the initial datum belongs to $L^p(\Omega)$. It is immediate to deduce that $\Phi_p \equiv \Phi|_{\mathbb{R} \times L^p(\Omega)}$. Thanks to the regularity property of the problem (cf. Theorem 4) we have that the (possibly multi-valued) solution operator $U_p : \mathbb{R}^2_d \times L^p(\Omega) \to \mathcal{P}(L^p(\Omega))$, given analogously to (22), is well-defined. Indeed, since the phase-spaces will be explicited in the results, we keep the usual notation U for both processes, the one in $L^2(\Omega)$ and $L^p(\Omega)$.

Now we introduce the universes which involve more regularity.

Definition 15. $\mathcal{D}_\mu^{L^2,L^p}$ *denotes the class of all families of nonempty subsets* $\widehat{D}_{L^p} = \{D(t) \cap L^p(\Omega) : t \in \mathbb{R}\}$, *where* $\widehat{D} = \{D(t) : t \in \mathbb{R}\} \in \mathcal{D}_\mu^{L^2}$.

The upper-semicontinuity of U in $L^p(\Omega)$ follows from the next result.

Proposition 16. *Assume (2)-(3) and (5) hold, and* $h \in L^2_{loc}(\mathbb{R}; L^2(\Omega))$. *Then, the multi-valued process* U *is upper-semicontinuous in* $L^p(\Omega)$ *with closed values.*

Proof. We proceed analogously to Proposition 7, using the additional regularity result obtained in Theorem 4. Namely, given $(t,\tau) \in \mathbb{R}^2_d$, suppose by contradiction that $U(t,\tau)$ is not upper-semicontinuous in $L^p(\Omega)$. Then, there exist u_τ and $\{u^n_\tau\}$ in $L^p(\Omega)$ with $u^n_\tau \to u_\tau$ (in $L^p(\Omega)$) and $y_n = u^n(t) \in U(t,\tau)u^n_\tau$ such that $y_n \notin \mathcal{N}(U(t,\tau)u_\tau)$ for all n.

It is already known from Lemma 6 that there exist a subsequence (relabeled the same) and a weak solution $u \in \Phi(\tau, u_\tau)$ such that $u^n(s) \to u(s)$ strongly in $L^2(\Omega)$ for any $s \geq \tau$. Moreover, from Theorem 4 we have that $\{u^n\}$ is (uniformly) bounded in $L^\infty(\tau+\varepsilon, t; W^{1,p}_0(\Omega))$. This, combined with (24), allows to apply again [26, Lemma 11.2] or [19, Lemma 4.9] and to deduce that $\{u^n(t)\}$ is bounded in $W^{1,p}_0(\Omega)$.

As far as $W^{1,p}_0(\Omega)$ is compactly embedded into $L^p(\Omega)$, we conclude that $u^n(t) \to u(t)$ in $L^p(\Omega)$, which is a contradiction. The fact that U has closed values in $\mathcal{P}(L^p(\Omega))$ follows immediately with the same arguments. □

Now, as a consequence of the regularizing effect of the equation (cf. Theorem 4) and the existence of an absorbing family in $\mathcal{D}_\mu^{L^2}$ (cf. Corollary 11), it is immediate to deduce the existence of a pullback $\mathcal{D}_\mu^{L^2,L^p}$-absorbing family.

Corollary 17. *Assume that (2)-(3) hold,* f *fulfils (5),* $h \in L^2_{loc}(\mathbb{R}; L^2(\Omega))$ *and there exists* $\mu \in (0, 2m)$ *such that* h *satisfies (28). Then, the family* $\widehat{D}_{0,L^p} = \{D_{0,L^p}(t) : t \in \mathbb{R}\}$ *defined by* $D_{0,L^p}(t) = D_0(t) \cap L^p(\Omega)$ *is pullback* $\mathcal{D}_\mu^{L^2,L^p}$*-absorbing for the multi-valued process* U.

The existence of attractor in $L^p(\Omega)$ is now established.

Theorem 18. *Assume (2)-(3) and (5) hold and* $h \in L^2_{loc}(\mathbb{R}; L^2(\Omega))$ *satisfies the condition (29) for some* $\mu \in (0, 2m)$. *Then, there exist the minimal pullback* $\mathcal{D}_F^{L^p}$*-attractor* $\mathcal{A}_{\mathcal{D}_F^{L^p}} = \{\mathcal{A}_{\mathcal{D}_F^{L^p}}(t) : t \in \mathbb{R}\}$ *and the minimal pullback* $\mathcal{D}_\mu^{L^2,L^p}$*-attractor* $\mathcal{A}_{\mathcal{D}_\mu^{L^2,L^p}} = \{\mathcal{A}_{\mathcal{D}_\mu^{L^2,L^p}}(t) : t \in \mathbb{R}\}$ *for the process* $U : \mathbb{R}^2_d \times L^p(\Omega) \to \mathcal{P}(L^p(\Omega))$. *The minimal pullback* $\mathcal{D}_\mu^{L^2,L^p}$*-attractor* $\mathcal{A}_{\mathcal{D}_\mu^{L^2,L^p}}$ *belongs to* $\mathcal{D}_\mu^{L^2,L^p}$ *and the following relationships hold*

$$\mathcal{A}_{\mathcal{D}_F^{L^p}}(t) \subset \mathcal{A}_{\mathcal{D}_F^{L^2}}(t) \subset \mathcal{A}_{\mathcal{D}_\mu^{L^2}}(t) = \mathcal{A}_{\mathcal{D}_\mu^{L^2,L^p}}(t) \quad \forall t \in \mathbb{R}. \tag{33}$$

In addition, if the function h *fulfils*

$$\sup_{s \leq 0} \left(e^{-\mu s} \int_{-\infty}^{s} e^{\mu\theta} \|h(\theta)\|_2^2 d\theta \right) < \infty, \tag{34}$$

then all attractors in (33) coincide.

Proof. Observe that all the necessary ingredients to apply [4, Theorem 2] in $L^p(\Omega)$ have already been established. Namely, in Proposition 16 we ensured that $U : \mathbb{R}^2_d \times L^p(\Omega) \to \mathcal{P}(L^p(\Omega))$ is upper-semicontinuous with closed values. Corollary 17 showed the existence of an absorbing family in $\mathcal{D}_\mu^{L^2,L^p}$. The final key is the pullback $\widehat{D}_{0,L^p}$-asymptotic compactness of U. Indeed, this is again a straightforward consequence of Lemma 13. Particularly, the third inequality in (30) provides the absorbing family in

$W_0^{1,p}(\Omega)$, constituted by balls of center zero and radii $\rho_3(t)^{1/p}$. Therefore, the existence of $\mathcal{A}_{\mathcal{D}_F^{L^p}}$ and $\mathcal{A}_{\mathcal{D}_\mu^{L^2,L^p}}$ follows.

The relations in (33) are consequence of a slight variation of [19, Theorem 3.15], adapted to the multivalued framework, and Corollary 11.

Finally, the equality of these families of attractors can be deduced using again [4, Corollary 2] since all the radii $\rho_i(\cdot)$ $(i = 1, 2, 3)$ in (12) are bounded for $t \le T$, thanks to the assumption (34). □

The arguments of this section can be generalised replacing the space $L^p(\Omega)$ by a general Banach space X such that the chain of embeddings (compact and continuous, respectively) hold

$$W_0^{1,p}(\Omega) \subset\subset X \subset L^2(\Omega). \tag{35}$$

Corollary 19. *Under the assumptions of Theorem 18, all its theses remain valid for a phase space X fulfilling (35).*

Remark 20.

(i) For instance, one can take $X = L^{p^*-\epsilon}(\Omega)$ for all $\epsilon \in (0, p^* - 2]$ if $p < N$, improving the norm of attraction of the attractor.

(ii) All results are valid in the autonomous case (when h is time independent). In particular, Theorem 18 and Corollary 19 improve the main results given in [5].

(iii) Observe that the values of γ provided for the assumption (5) can be improved if the interpolation result is applied not only to $L^\infty(\tau, T; L^2(\Omega)) \cap L^p(\tau, T; W_0^{1,p}(\Omega))$, but also to $L^p(\tau, T; W_0^{1,p}(\Omega)) \cap L^q(\tau, T; L^q(\Omega))$. Namely, the largest values of γ would be

$$\gamma : \begin{cases} = \max\left\{\dfrac{p}{2}, \dfrac{q-2}{2}\right\} & \text{if } p > N, \\ \begin{cases} = \dfrac{q-2}{2} & \text{if } N - q + 2 \le 0 \\ < \dfrac{N}{2} & \text{if } N - q + 2 > 0 \end{cases} & \text{if } p = N, \\ = \max\left\{\dfrac{2p + pN - 2N}{2N}, \dfrac{q-2}{2}\right\} & \text{if } p < N. \end{cases}$$

However, for the sake of clarity in the presentation (e.g. see (6) which depends on the interpolation spaces) and taking into account that the proofs of the results are completely analogous but changing the spaces, we have chosen to use just $L^\infty(\tau, T; L^2(\Omega)) \cap L^p(\tau, T; W_0^{1,p}(\Omega))$ in order to provide a clearer idea about how to tackle the problem.

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