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# EXTENSION OF SANTILLI'S ISOTOPIES TO NON-INJECTIVE ISOALGEBRAS

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*Dedicated to the memory of Professor Grigorios F. Tsagas in admiration*

Abstract. Santilli's isotopies constitute a new branch of mathematics characterized by axiom-preserving isotopic lifting of units, products, numbers, fields, topologies, geometries, algebras, groups, etc., with numerous novel applications in physics, chemistry and other quantitative sciences. The continuation of these studies require deeper research on non-injective isotopies. The main goal of this paper is to generalize the usual isotopic construction model to obtain non-injective isoalgebras, by using so many new laws $*$ and isounities in the general set associated with the Santilli's isotopy, as laws has the initial structure. In this way, the study of the properties of this general set are very useful. In fact, this study constitutes the MCIM isotopic construction model, which has been studied by the authors since 2001. In this model, there are a main isounit and a main $*$-law, which determine the mathematical isostructure, and some secondaries ones, which determine the laws in this isostructure. So, the study of all these elements can determine how to build a non-injective isotopy, by taking into consideration the different factors on which the main isounit depends.

## 1. Introduction

The mathematics generally used in quantitative sciences of the 20-th century were based on ordinary fields with characteristic zero and trivial (left, right) unit $I = +1$ and ordinary associative product $a * b$ between generic quantities $a, b$ of a given set, such as matrices, vector fields, etc. Such a mathematics is known to be linear, local differential (beginning from its topology),

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and Hamiltonian, thus solely representing a finite number of isolated point-particles with action-at-a-distance forces derivable from a potential. Such a mathematics was proved to provide an exact and invariant representation of planetary and atomic systems as well as, more generally, of all the so-called *exterior dynamical systems* in which all constituents can be well approximated as being point-like.

By contrast, the great majority of systems in the physical reality are nonlinear, nonlocal (of integral and other type) and not entirely representable with a Hamiltonian in the coordinates of the experimenter. This is the case for all systems historically called *interior dynamical systems*, such as the structure of: planets; strongly interacting particles (such as protons and neutrons); nuclei; molecules; stars; and other systems. The latter systems cannot be consistently reduced to a finite number of isolated point-particles. Therefore, the mathematics so effective for exterior systems is only approximate at best for interior systems. Rather than adapting reality to pre-existing insufficient mathematics, in 1978 the Italian-American physicist R. M. Santilli [13] proposed the construction of a new mathematics, today known as *Santilli's isomathematics,* specifically built for the invariant representation of nonlinear, nonlocal and non-Hamiltonian systems. The proposal was essentially based on the construction of axiom preserving isotopic liftings of all branches of mathematics with trivial (left and right) unit $I = +1$.

Santilli's fundamental isotopy was that of the basic unit that was lifted from the trivial value $I = +1$ to a matrix or operator $\widehat{I} = \widehat{I}(t, x, v, \mu, \tau, \cdots) = 1/T$ that preserves the topological property of $I$ in order to qualify as an isotopy (i.e., nowhere singular, Hermitean and positive-definite), but possesses an otherwise unrestricted, generally nonlinear, nonlocal and non-Hamiltonian functional depend on all needed local variables, such as time $t$, coordinates $x$, velocities $v$, density $\mu$, temperature $\tau$, etc. Jointly, Santilli lifted the conventional associative product $a * b$ with unit $I = +I$ into the new form $a\hat{\times}b = a*T*b$ under which $\widehat{I} = 1/T$ is indeed the correct (left and right) unit of the set considered. It is evident that the new product $a\hat{\times}b$ remains associative and, therefore, the lifting $* \to \hat{\times}$ is an isotopy. Under these conditions $\widehat{I}$ is called *Santilli's isounit,* $T$ is called the *isotopic element,* and $\hat{\times}$ is called the *isoproduct.*

According to these lines, the representation of interior systems via Santilli's isomathematics requires the knowledge of *two* quantities, the conventional; Hamiltonian $H$ for the representation of conventional linear, local and potential forces, and the isounit $\widehat{I}$ for the representation of all nonlinear, nonlocal and non-Hamiltonian effects. Note that Santilli's generalization of the unit is the *unique* choice for the invariant representation of the latter effects because, whether conventional or generalized, the unit is the basic invariant of all mathematics.

The isotopies of the basic unit and product evidently require compatible liftings of *all* branches of mathematics admitting the trivial unit $I = +1$ as the left and right unit, with no possible exclusion to avoid intrinsic inconsistencies. Therefore, Santilli's isomathematics is characterized by mutually compatible isotopic liftings of numbers and fields, vector and metric spaces, topology, differential calculus, functional analysis, manifolds, geometries, algebras, groups, etc.

To outline this significant scientific journey, we here recall the following main advances. As it is well known, Lie's theory has a fundamental role in physics. Nevertheless, Lie's theory also has clear limitations due to its linear, local-differential and Hamiltonian character. Therefore, already in the original proposal of 1978 Santilli [13] proposed the isotopies of all branches of Lie's theory, including the isotopic lifting of universal enveloping associative algebras, Lie algebras, Lie groups,and representation theory, resulting in a generated theory applicable to nonlinear, nonlocal and non-Hamiltonian systems that is today called the *Lie-Santilli isotheory*. Santilli then continued these studies in numerous works (see monographs [14], [15], [16] and papers quoted therein).

To understand the literature, one should note that most of the results on isotopies were presented by Santilli [*loc. cit.*] as a particular case of a broader theory based on the notion of *Lie-admissibility* by A. A. Albert [1], and today known as *Lie-Santilli genotheory* that will not be considered in this paper.

Due to its evident mathematical and physical relevance, the Lie-Santilli isotheory has been studied in numerous papers by other mathematicians and physicists. For brevity, we only quote here the monographs by: H. C. Myung [12] of 1982; A. K. Aringazin. A. Jannussis, D. F./ Lopez, M. Nishioka and B. Veljanovski [2] of 1991; Gr. Tsagas and D. S. Sourlas [21] of 1993;J. Lôhmus, E. Paal and L. Sorgsepp [11] of 1994; J. V. Kadeisvili [8] of 1997; and R. M. Falcón Ganfornina and J. Núñez Valdés [4] of 2001.

Another important advance by Santilli has been the isotopies of numbers that he initiated also in 1978, but presented in mathematical form in memoir [17] of 1993. Santilli's isonumbers were also studied by various authors and a comprehensive presentation of *Santilli's isonumber theory* was provided by C.-X. Jiang [7] in 2002. Yet another significant advance by Santilli has been the isotopic lifting of the Euclidean and Minkowskian spaces and related geometries achieved by Santilli in the mid 1980's (see the review in [15] and papers quoted therein). These studies too were continued by various authors, among whom we quote J. V. Kadeisvili [10], Gr. Tsagas and D. S. Sourlas [22], [23], and R. M. Falcón Ganfornina and J. Núñez Valdés [4], [5].

J. V. Kadeisvili [10] provided the first formulation of *isocontinuity* and *isofunctional analysis* that were continued by various authors, subsequently studied by A. K. Aringazin, D. A. Kirushin and R. M. Santilli [3] and various other authors.

Gr. Tsagas and D. S. Sourlas [22], [23] provided the first formulation of the *isotopology* on isospaces over ordinary fields, a formulation that was extended by R. M. Santilli [20] to isospaces over isofields. Comprehensive studies of the latter advances have been conducted by the authors in the above quoted references, as outlined in this and in the next paper.

In 2001, Falcón and Núñez [4] generalized the isotopic model proposed by Santilli in 1978 although this generalization put stress on the use of several $*$-*laws* and isounits as operations existing in the initial mathematical structure. Such a model, which from now on will be called MCIM (*isoproduct construction model based on the multiplication*), was later generalized in [5] and [6], where Tsagas-Sourlas-Santilli isotopology was improved.

It should be finally indicated that all these mathematical advances have permitted a structural generalization of quantum mechanics and chemistry known under the name of *hadronic mechanics and chemistry* [18], [19] that have achieved today numerous experimental verifications and have permitted a number of new industrial applications, as expected from the novelty of the used mathematics.

In this paper we extend Santilli's isotopies to non-injective isoalgebras. It should be noted that some results appearing in this paper will not be proved, due to restrictions on length.

## 2. The MCIM isotopic model

In a schematic way, every isotopy can be described starting from the following diagram:

$$
\begin{array}{ccc}
 & & \text{General level} \\
\text{Conventional level} & \longrightarrow & (V, *, \star, \cdots) \\
 & & \cup \\
(E, +, \times, \cdots) & & (E, *, \star, \cdots) \\
\downarrow & \sharp & \downarrow \mathbf{I} \\
\text{Projection level} & \xleftarrow{\ \pi\ } & \text{Isotopic level} \\
(\overline{\widehat{E}}, \overline{\widehat{+}}, \overline{\widehat{\times}}, \cdots) & & (\widehat{E}, \widehat{+}, \widehat{\times}, \cdots)
\end{array}
$$

where, by construction:

a) The mapping $\mathbf{I}$: $(E, \star, *, \cdots) \to (\widehat{E}, \widehat{+}, \widehat{\times}, \cdots) : X \to \widehat{X}$ is an isomorphism.

b) The *isotopic projection* is onto:
$\pi : (\widehat{E}, \widehat{+}, \widehat{\times}, \cdots) \to (\overline{\widehat{E}}, \overline{\widehat{+}}, \overline{\widehat{\times}}, \cdots) : \widehat{a} \to \pi(\widehat{a}) = \overline{\widehat{a}} = a * \widehat{I}$.

c) $\widehat{a}\widehat{+}\widehat{b} = \widehat{a \star b}$; $\widehat{a}\widehat{\times}\widehat{b} = \widehat{a * b}; \cdots$ .

d) $\overline{\widehat{a}} = a * \widehat{I}$; $\alpha\overline{\widehat{+}}\beta = ((\alpha * T) \star (\beta * T)) * \widehat{I}$; $\alpha\overline{\widehat{\times}}\beta = \alpha * T * \beta; \cdots$

By the other way, an isotopic lifting of the structure $E$ will be *injective* if $a = b$, for all $a, b \in E$ such that $\overline{\widehat{a}} = \overline{\widehat{b}}$. It is equivalent, by construction, to say that the projection $\pi : \widehat{E} \to \overline{\widehat{E}} : \widehat{a} \to \pi(\widehat{a}) = \overline{\widehat{a}}$ is an injective mapping.

An isotopic lifting will be *compatible with respect to* a law $\circ$ on $E$ if $\overline{\widehat{a}\widehat{\circ}\widehat{b}} = \overline{\widehat{a \circ b}}$ for all $a, b \in E$.

**Proposition 2.1.** *They are verified*:

a) *The isotopic projection associated with each injective isotopic lifting is an isomorphism.*

b) *If the isotopic lifting used is compatible with respect to all of initial operations, then the isostructure $\overline{\widehat{E}}$ is isomorphic to the initial $E$.*

c) *The relation of being isotopically equivalents is of equivalence.*

d) *Every isotopic projection $\pi \circ \mathbf{I} : (E, +, \times, \circ, \bullet, \cdots) \to (\overline{\widehat{E}}, \overline{\widehat{+}}, \overline{\widehat{\times}}, \overline{\widehat{\circ}}, \overline{\widehat{\bullet}}, \cdots)$ can be considered as an isotopic lifting which follows the MCIM, that is, every mathematical isostructure is an isostructure with respect to the multiplication.*

e) *The general set can be defined as*:

$$V = E \cup \overline{\widehat{E}} \cup E_T \cup \{T\}.$$

*Where $E_T = \{\overline{\widehat{x}}_T = \overline{\widehat{x}} * T : x \in E\}$ and it must be for all $\overline{\widehat{a}}_T \in E_T$* :

$$\overline{\widehat{a}}_T * \widehat{I} = a * \widehat{I}.$$

*Besides, for all $a, b \in E$* :

$$\overline{\widehat{a}}\overline{\widehat{\times}}\overline{\widehat{b}} = [\overline{\widehat{a}}_T * \overline{\widehat{b}}_T] * \widehat{I}.$$

f) *If the isotopic lifting is injective, then $E_T = E$.*

## 3. Isoalgebras

It has a perfect sense to considerer each one of the isostructures which result of applying the MCIM to conventional structures. Particularly, we can consider the construction of *isoalgebras* (as the isotopic lifting of each algebra, which is endowed with a structure of algebra) by using this isotopic model. Let us see some results about it (see [4]).

**Proposition 3.1.** *Let $U$ be a $K$-algebra and let $\widehat{U}$ be a $\widehat{K}$-isovectorspace. If a $K(a,\star,*)$-algebra $(U,\diamond,\square,\cdot)$ is used in the general level, then the isotopic lifting $\widehat{U}$ corresponding to the isotopy of primary elements $\widehat{I}$ and $\square$ and secondary ones $\widehat{S}$ and $\diamond$, when MCIM is used, has a structure of isoalgebra on $\widehat{K}$, and it preserves the initial type of the algebra.*

A particular type of isoalgebra is the *Lie isoalgebra* (see [13]), which is the one that has a structure of Lie algebra. Particularly, if $\overline{\widehat{U}}$ is the isotopic projection of a Lie isoalgebra, $\widehat{I}=\widehat{I}(x,dx,d^2x,t,T,\mu,\tau,\cdots)$ is an isounit and a basis $\overline{\widehat{U}}$, $\{\overline{\widehat{e_1}},\ldots,\overline{\widehat{e_n}}\}$ is fixed, where $\overline{\widehat{e_i}}\widehat{\cdot}\overline{\widehat{e_j}}=\sum\overline{\widehat{c_{ij}^h}}\widehat{\bullet}\overline{\widehat{e_h}}$, $\forall\ 1\leq i,j\leq n$, then coefficients $\overline{\widehat{c_{ij}^h}}\in\overline{\widehat{K}}$ are the *Maurer-Cartan coefficients* of the isoalgebra, which constitute a generalization of the conventional case, since they are not constants in general, but functions dependent of the factors of $\widehat{I}$.

In 1948, the American mathematician A. A. Albert gave the notion of Lie-admissibility in [1], as a nonassociative algebra whose attached antisymmetric algebra is a Lie algebra. In this way, another interesting isoalgebra is the *isoadmissible Lie algebra*, by Santilli in [13], that is, the isoalgebra $\widehat{U}$ such that endowed with the commutator bracket $[.,.]_{\widehat{U}}:[\widehat{X},\widehat{Y}]_{\widehat{U}}=(\widehat{X}\widehat{\cdot}\widehat{Y})-(\widehat{Y}\widehat{\cdot}\widehat{X})$ is an isotopic Lie isoalgebra. The following result is then satisfied:

**Proposition 3.2.** *Under conditions of Proposition* 3.1*, let us suppose that the law $\widehat{\circ}$ of the isoalgebra $\widehat{U}$ is defined according $\widehat{X}\widehat{\circ}\widehat{Y}=(X\circ Y)\square\widehat{I}$, for all $X,Y\in U$. If $U$ is a Lie* (*admissible*) *algebra, then $\widehat{U}$ is an Lie* (*isoadmissible*) *isoalgebra.*

In this way, (isoadmissible) Lie isoalgebras inherit the usual properties of conventional (admissible) Lie algebras. In the same way, usual structures related with such algebras have also their analogue ones when isotopies are used. For instance, an *isoideal* of a Lie isoalgebra $\widehat{U}$ is every isotopic lifting of an ideal $\Im$ of $U$, which is by itself an ideal. In particular, the *center* of an Lie isoalgebra $\widehat{U}$, $\{\widehat{X}\in\widehat{U}$ such that $\widehat{X}\widehat{\cdot}\widehat{Y}=\widehat{S}\ \forall\widehat{Y}\in\widehat{U}\}$, is an isoideal of $\widehat{U}$. In fact, it is verified the following result:

**Proposition 3.3.** *Let $\widehat{U}$ be an Lie isoalgebra associated with a Lie algebra $U$ and let $\Im$ be an ideal of $U$. Then, the corresponding isotopic lifting $\widehat{\Im}$ is an isoideal of $\widehat{U}$.*

An isoideal $\widehat{\Im}$ of a Lie isoalgebra $(\widehat{U}, \widehat{\circ}, \widehat{\bullet}, \widehat{\cdot})$, is called *isocommutative* if $\widehat{X}\widehat{\cdot}\widehat{Y} = \widehat{S}$, for all $\widehat{X} \in \widehat{\Im}$ and for all $\widehat{Y} \in \widehat{U}$, being $\widehat{U}$ *isocommutative* if it is so as an isoideal.

**Proposition 3.4.** *$\widehat{U}$ is isocommutative if and only if $U$ is commutative.*

Lie-Santilli algebras [13], studied by Tsagas in [25], constitute a particular example of Lie isoalgebras. Fixed an $\widehat{K}$-isoassociative isoalgebra $(\widehat{U}, \widehat{\circ}, \widehat{\bullet}, \widehat{\cdot})$, the commutator in $\widehat{U}$ associated with $\widehat{\cdot}$ : $[\widehat{X}, \widehat{Y}]_S = (\widehat{X}\widehat{\cdot}\widehat{Y}) - (\widehat{Y}\widehat{\cdot}\widehat{X})$, for all $\widehat{X}, \widehat{Y} \in \widehat{U}$ is called *Lie-Santilli bracket product* $[.,.]_S$ with respect to $\widehat{\cdot}$. The isoalgebra $(\widehat{U}, \widehat{\circ}, \widehat{\bullet}, [.,.]_S)$ is then called *Lie-Santilli algebra.*

**Proposition 3.5.** *Let $\widehat{U}$ be an $\widehat{K}$-isoassociative isoalgebra associated with a $K$-algebra $U$, under conditions of Proposition* 3.2*. Then, the Lie-Santilli algebra associated with $\widehat{U}$ is an Lie isoalgebra if the algebra $U$ is either associative or Lie admissible.*

Apart from that, a Lie isoalgebra $\widehat{U}$ is said to be *isosimple* if, being an isotopy of a simple Lie algebra, it is not isocommutative and the only isoideals which contains are trivial ones. In an analogous way, $\widehat{U}$ is called *isosemisimple* if, being an isotopy of a semisimple Lie algebra, it does not contain non trivial isocommutative isoideals. Note that, this definition involves that every isosemisimple Lie isoalgebra is also isosimple. Moreover, it is verified:

**Proposition 3.6.** *Under conditions of Proposition* 3.2*, the isotopic lifting of a* (*semi*)*simple Lie algebra is an iso*(*semi*)*simple Lie isoalgebra. Particularly, every isosemisimple Lie isoalgebra is a direct sum of isosimple Lie isoalgebras.*

A Lie isoalgebra $(\widehat{U}, \widehat{\circ}, \widehat{\bullet}, \widehat{\cdot})$ is said to be *isosolvable* [9] if, being an isotopy of a solvable Lie algebra, in the *isosolvability series*:

$$\widehat{U}_1 = \widehat{U}, \quad \widehat{U}_2 = \widehat{U}\widehat{\cdot}\widehat{U}, \quad \widehat{U}_3 = \widehat{U}_2\widehat{\cdot}\widehat{U}_2, \quad \cdots, \quad \widehat{U}_i = \widehat{U}_{i-1}\widehat{\cdot}\widehat{U}_{i-1}, \quad \cdots$$

there exists a natural integer $n$ such that $\widehat{U}_n = \{\widehat{S}\}$. The minor of such integers is called *isosolvability index* of the isoalgebra.

**Proposition 3.7.** *Under conditions of Proposition* 3.2*, the isotopic lifting of a solvable Lie algebra is an isosolvable Lie isoalgebra.*

An easy example of isosolvable Lie isolgebras are the isocommutative isotopic Lie isoalgebras, since they verify, by definition, that $\widehat{U}\widehat{\cdot}\widehat{U} = \widehat{U}_2 = \{\widehat{S}\}$. It implies that every nonzero isocommutative Lie isoalgebra has an isosolvability index equals 2, being 1 the corresponding to the trivial isoalgebra $\{\widehat{S}\}$.

**Proposition 3.8.** *Let $\widehat{U}$ be a Lie isoalgebra associated with a Lie algebra $U$. Under conditions of Proposition* 3.1*, they are verified*:

1. *$\widehat{U}_i$ is an isoideal of $\widehat{U}$ and of $\widehat{U}_{i-1}$, for all $i \in \mathbb{N}$.*
2. *If $\widehat{U}$ is isosolvable and $U$ is solvable, then every isosubalgebra of $\widehat{U}$ es isosolvable.*
3. *La intersection and the product of a finite number of isosolvable isoideals of $\widehat{U}$ are isosolvable isoideals. Moreover, under conditions of Proposition* 3.2*, the sum of a finite number of isosolvable isoideals is also an isosolvable isoideal.*

By using this last result (3), it is deduced that the sum of all isosolvable isoideals of $\widehat{U}$ is another isosolvable isoideal, which is called *isoradical* of $\widehat{U}$. Note that it is different from the *radical* of $\widehat{U}$, which would be the sum of all solvable ideals of $\widehat{U}$. The isoradical is denoted by *isorad* $\widehat{U}$, to not be confused with *rad* $\widehat{U}$, and it will always contain $\{\widehat{S}\}$, because this last one is a trivial isosolvable isoideal of every isoalgebra. Note also that as every isosolvable isoideal of $\widehat{U}$ is a solvable ideal of $\widehat{U}$, then *isorad* $\widehat{U} \subset$ *rad* $\widehat{U}$. So, if $\widehat{U}$ is isosolvable, then $\widehat{U} =$ *isorad* $\widehat{U} =$ *rad* $\widehat{U}$, due to $\widehat{U}$ is solvable in particular.

**Proposition 3.9.** *If $\widehat{U}$ is a semisimple Lie isoalgebra over a field of zero characteristic, then isorad $\widehat{U} = \{\widehat{S}\}$.*

A Lie isoalgebra $(\widehat{U}, \widehat{\circ}, \widehat{\bullet}, \widehat{\cdot})$ is called *isonilpotent* [9] if, being an isotopy of a nilpotent Lie algebra, in the series:

$$\widehat{U}^1 = \widehat{U}, \quad \widehat{U}^2 = \widehat{U}\widehat{\cdot}\widehat{U}, \quad \widehat{U}^3 = \widehat{U}^2\widehat{\cdot}\widehat{U}, \quad \cdots, \quad \widehat{U}^i = \widehat{U}^{i-1}\widehat{\cdot}\widehat{U}, \quad \cdots$$

(which is called *isonilpotency series*), there exists a natural integer $n$ such that $\widehat{U}^n = \{\widehat{S}\}$. The minor of such integers is denominated *nilpotency index* of the isoalgebra.

As a consequence immediate of this definition it is deduced that every isonilpotent Lie isoalgebra is isosolvable and that every nonzero isocommutative Lie isoalgebra has an isonilpotency index equals 2, being 1 the corresponding of the isoalgebra $\{\widehat{S}\}$. Moreover, they are verified:

**Proposition 3.10.** *Under conditions of Proposition* 3.2*, the isotopic lifting of a nilpotent Lie algebra is an isonilpotent isotopic Lie isoalgebra.*

**Proposition 3.11.** *Let $\widehat{U}$ be a Lie isoalgebra associated with a Lie algebra $U$. They are verified*:

1. *Under conditions of Proposition* 3.2*, the sum of a finite number of isonilpotent isoideals of $\widehat{U}$ is another isonilpotent isoideal.*

2. *If* $\widehat{U}$ *is also isonilpotent and* $U$ *is nilpotent, then*
   (a) *Every isosubalgebra of* $\widehat{U}$ *is isonilpotent.*
   (b) *Under conditions of Proposition* 3.2*, if* $\widehat{U}$ *is nonzero isonilpotent, then its center is non null.*

In a similar way as the case isosolvable, the result (1) involves that the sum of all isonilpotent isoideals of $\widehat{U}$ is another isonilpotent isoideal, which is denoted by *isonihil-radical* of $\widehat{U}$, to be distinguished from the nihil-radical of $\widehat{U}$, which is the sum of the radicals ideals. It will be represented by *isonil-rad* $\widehat{U}$, which allows to distinguish it from the *nil- rad* $\widehat{U}$. It is immediate that *isonil-rad* $\widehat{U} \subset$ *nil-rad* $\widehat{U} \cap$ *isorad* $\widehat{U} \subset$ *nil-rad* $\widehat{U} \subset$ *rad* $\widehat{U}$.

Apart from that, it is possible to relate an isosolvable isotopic Lie isoalgebra with its derived Lie isoalgebra, by using the following:

**Proposition 3.12.** *Under conditions of Proposition* 3.2*, an Lie isotopic isoalgebra is isosolvable if and only if its derived Lie isoalgebra is isonilpotent.*

Finally, an isonilpotent Lie isoalgebra $(\widehat{U}, \widehat{\circ}, \widehat{\bullet}, \widehat{\cdot})$ is called *isofiliform* if, being an isotopy of a filiform Lie algebra, it is verified that dim $\widehat{U}^2 = n-2, \cdots$ , dim $\widehat{U}^i = n-i, \cdots$ , dim $\widehat{U}^n = 0$, where dim $\widehat{U} = n$.

Note that the theory related with a filiform Lie algebra $U$ is based on the use of a basis of such an algebra. So, starting from a basis $\{e_1, \cdots, e_n\}$ de $U$, which is preferably an *adapted basis*, we can deal with lots of concepts of it, such that dimensions of $U$ and of elements of the nilpotency series, invariants $i$ and $j$ of $U$ and, in general, the resting properties, starting from its structure coefficients, which are, in fact, responsible for the complete study of filiform Lie algebras.

## 4. Non-injective isotopies

Two clases of non-injective isotopies can be considered:

I) The first type is that in which the isotopic projection can be inverted, by using the corresponding isotopic element.

II) The second type is that in which the isotopic projection cannot be inverted.

The first of them is dealt in [6], although it is only dealt the construction of the corresponding isotopic set, not being considered the construction os the isooperations. It appears when the isounits depends on more than one factor, that is, it depends on a factor different from the factor coordinate. Such a dependence allows earlier the use of the isotopic element to invert the isotopic projection, by giving concrete values to factors from which the isotopic lifting given depends to obtain it.

So, if under usual notations, we have the isounit $\widehat{I} = \widehat{I}(x, t, \delta, \cdots)$ and we consider an isotopic lifting of a mathematical structure $E$, such that there exist $x, y \in E$ verifying that:

$$\alpha = \overline{\overline{\widehat{x}}} = x * \widehat{I}(x, t_x, \delta_x, \cdots) = y * \widehat{I}(y, t_y, \delta_y, \cdots) = \overline{\overline{\widehat{y}}}$$

where, at least, one of the values in $\{t_x, \delta_x, \cdots\}$ does not coincide with the corresponding one in $\{t_y, \delta_y, \cdots\}$, then we can assure the existence of the isotopic element $T = T(x, t, \delta, \cdots)$, such that:

$$\alpha * T(\alpha, t_x, \delta_x, \cdots) = x; \quad \alpha * T(\alpha, t_y, \delta_y, \cdots) = y$$

The second type of non-injective isotopic lifting is the one in which we can assure the non existence of an isotopic element which allows to undo the isotopic projection. It will occur when there exist two different elements $x, y \in E$, verifying that:

$$\alpha = \overline{\overline{\widehat{x}}} = x * \widehat{I}(x, t_x, \delta_x, \cdots) = y * \widehat{I}(y, t_y, \delta_y, \cdots) = \overline{\overline{\widehat{y}}}$$

in the way that all of the values assigned to factors coincide, up to the factor coordinate. That is, $t_x = t_y$, $\delta_x = \delta_y$, etc.

Note that by proceeding in a similar way as the previous case with respect to the possible isotopic element $T = T(x, t, \delta, \cdots)$, we would obtain that:

$$\alpha * T(\alpha, t_x, \delta_x, \cdots) = \{x, y\}$$

and thus, we cannot give an unique value, which it would be desirable, to invert the isotopic projection.

We will now give some notations to be used later. $F$ will denote the setof factors from which the isounit used in an isotopic, like factors coordinates, speed or acceleration. That is, if we have the isounit $\widehat{I} = \widehat{I}(x, x', x'', t, \delta, T, D, \cdots)$, then $F = \{t, \delta, T, D, \cdots\}$. So, to simplify the notation, we will write $\widehat{I} = \widehat{I}(x, x', x'', F)$. Similarly, the corresponding isotopic element will be denoted by $T = T(x, x', x'', F)$. Finally, a subindex will be adjoint to $F$, $F_a$, when we wish to indicate that to the factors of $F$ have been particularized with a concrete values. So, when we denote by $\overline{\overline{\widehat{a}}}$ an element in the corresponding projection level $\overline{\overline{\widehat{E}}}$ (where $a \in E$), we will suppose that these concrete values are known. That is,

$$\overline{\overline{\widehat{a}}} = a * \widehat{I}(a, x'_a, x''_a, F_a).$$

However, when we denote an element without the symbol $\overline{\widehat{\ }}$ overlined, we will suppose that the values assigned to $F$, needed for obtaining that element are not known. They will be only known when they are the unique possible ones, that is, when there not exist $F_a$ and $F_b$ such that:

$$\alpha = a * \widehat{I}(a, x'_a, x''_a, F_a) = b * \widehat{I}(b, x'_b, x''_b, F_b)$$

Otherwise, such values will be assigned as we previously fix, according to the conditions with which we wish work with the isotopic lifting used.

It is interesting to note the following result:

**Proposition 4.1.** *Under usual notations, it is verified that*:

a) *Let $F$ be an empty set. If the isotopic lifting is non-injective, then such a lifting is of the type II. In particular, those of the type I must verify that $F$ is non empty.*

b) *If $F$ is a non empty set, then a non-injective isotopic lifting is of the type II if and only if there exist some concrete values assigned to factors constituting $F$, $F_0$, for which the restriction of the isotopic projection $\pi \circ \mathbf{I}_{|F=F_0} : x \to \overline{\widehat{x}} = x * \widehat{I}(x, x', x'', F_0)$ is non-injective.*

*Proof.* **a)** This result is immediate, because if we fix $a, b \in E$ such that $\overline{\widehat{a}} = \overline{\widehat{b}} = \alpha$, then:

$$\alpha * T = \alpha * T(\alpha, x'_\alpha, x''_\alpha) \supseteq \{a, b\}$$

And we cannot give explicitly an unique value. So, we have a non-injective isotopic lifting of the type II. So, the assert is immediate.

**b)** To check the sufficient condition, we can use the same reasoning for $F = F_0$. So, there will exist $a, b \in E$ such that $a * \widehat{I}(a, x'_a, x''_a, F_0) = b * \widehat{I}(b, x'_b, x''_b, F_0) = \alpha$. Then:

$$\alpha * T = \alpha * T(\alpha, x'_\alpha, x''_\alpha, F_0) \supseteq \{a, b\}$$

We have again a non-injective isotopic lifting of the type II:

Now, to check the necessary condition, we suppose the existence of an element $\alpha$ in the projection level, such that there exist concrete values $F_0$ to be assigned to factors from which the isotopic element depends, such that $\alpha * T(\alpha, x'_\alpha, x''_\alpha, F_0)$ is not an unique value. Therefore, there exist al least two elements in the initial structure $E$, $a$ and $b$, such that $a * \widehat{I}(a, x'_a, x''_a, F_0) = b * \widehat{I}(b, x'_b, x''_b, F_0) = \alpha$. In particular, it is obtained that the map $\pi \circ \mathbf{I}_{|F=F_0}$ is non-injective, which completes the proof. $\square$

When defining the isooperations associated with the projection level obtained from a non-injective isotopic lifting with $F$ non empty, it is needed, for each of such isooperations, a mapping of the type:

$$\Phi : F \times F \to F : (F_\alpha, F_\beta) \to \Phi(F_\alpha, F_\beta)$$

So, fixed and given a mathematical structure $(E, \circ)$, which is isotopically lifted by using the main isotopic elements $*$ and $\widehat{I}$, the isooperation $\overline{\widehat{\circ}}$ will be defined, for all $a * \widehat{I}(a, F_a), b * \widehat{I}(b, F_b) \in \overline{\widehat{E}}$ according to:

$$(a * \widehat{I}(a, F_a))\overline{\widehat{\circ}}(b * \widehat{I}(b, F_b)) = (a * b) * \widehat{I}(a * b, \Phi_\circ(F_a, F_b)).$$

If we take into consideration that $x * \widehat{I}(x, F_x) = y * \widehat{I}(y, F_y) \in \overline{\overline{\widehat{E}}}$, then, fixed $a * \widehat{I}(a, F_a) \in \overline{\overline{\widehat{E}}}$, it must be verified that:

$$(x * a) * \widehat{I}(x * a, \Phi_\circ(F_x, F_a)) = (y * a) * \widehat{I}(y * a, \Phi_\circ(F_y, F_a))$$

## 5. Non-injective isoalgebras

We will begin this section by giving an example of a non-injective isotopic lifting of the type II with $F$ empty:

**Example 5.1.** *Let us consider the* $(\mathbb{R}, +, \times)$*-Lie algebra* $(L, +, \times, [.,.])$*, with basis* $\beta = \{X, Y, Z\}$ *and Cartan coefficients* $[X, Y] = X = [X, Z]$*, and the isofield* $(\widehat{\mathbb{R}}, \widehat{+}, \widehat{\times})$ *obtained from the identity isotopy.*

*Let us now consider the* $\widehat{L}$*-isoalgebra* $(\widehat{L}, \widehat{+}, \widehat{\times}, \widehat{[.,.]})$*, constructed from the isotopic lifting associated with the main elements* $\diamondsuit$ *and* $\widehat{I}' = \widehat{I}'(x)$ *and secondary ones* $\square \equiv \times$*,* $\diamondsuit \equiv +y$ $\widehat{S}' = \mathbf{0}$ *(the null vector in* $L$*), where, for all* $x = aX + bY + cZ \in L$*, the element* $x \diamondsuit \widehat{I}'(x)$ *will be defined as:*

$$x \diamondsuit \widehat{I}'(x) = a \times \overline{\overline{\widehat{X}}} + b \times \overline{\overline{\widehat{Y}}} + c \times \overline{\overline{\widehat{Z}}}$$

*where:*

$$\overline{\overline{\widehat{X}}} = X, \quad \overline{\overline{\widehat{Y}}} = Y = \overline{\overline{\widehat{Z}}}$$

*In this way we will get that a basis of* $\overline{\overline{\widehat{L}}}$ *is* $\overline{\overline{\widehat{\beta}}} = \{X, Y\}$*, since, fixed* $aX + bY$*, we have, for instance, that* $(aX + bY) \diamondsuit \widehat{I}'(aX + bY) = aX + bY$*.*

*It is easy to see that such a lifting is non-injective, due to, for instance,* $\overline{\overline{\widehat{Y}}} = \overline{\overline{\widehat{Z}}} = Y$*, and it is also of the type II, for being* $F$ *empty.*

*We will now define the isooperations associated with* $\overline{\overline{\widehat{L}}}$*. To do this, fixed* $a, b, c, d, e \in \mathbb{R}$*, it is verified that:*

$$\begin{aligned}(aX + bY)\overline{\overline{\widehat{+}}}(cX + dY) &= ((a + c)X + be_Y + de_Z) \diamondsuit \widehat{I}'((a + c)X + be_Y + de_Z)\\ &= (a + c)X + (b + d)Y, \text{ where } e_Y, e_Z \in \{Y, Z\}\end{aligned}$$

$$\begin{aligned}e\overline{\overline{\widehat{\times}}}(aX + bY) &= \overline{\overline{\widehat{e}}}\overline{\overline{\widehat{\times}}}(aX + be_Y) \diamondsuit \widehat{I}'(aX + be_Y)\\ &= (eaX + ebe_Y) \diamondsuit \widehat{I}'(eaX + ebe_Y) = eaX + ebY, \text{ where } e_Y \in \{Y, Z\}\end{aligned}$$

$$\begin{aligned}&\overline{\overline{\widehat{[}}} aX + bY, cX + dY \overline{\overline{\widehat{]}}}\\ &= \overline{\overline{\widehat{[}}}(aX + be_Y) \diamondsuit \widehat{I}'(aX + be_Y), (cX + de'_Y) \diamondsuit \widehat{I}'(cX + de'_Y)\overline{\overline{\widehat{]}}}\\ &= [aX + be_Y, cX + de'_Y] * \widehat{I}([aX + be_Y, cX + de'_Y])\\ &= (ad - bc)X \diamondsuit \widehat{I}'((ad - bc)X) = (ad - bc)X, \text{ where } e_Y \in \{Y, Z\}\end{aligned}$$

*As a consequence, we have obtained that* $(\overline{\widehat{L}}, \overline{\widehat{+}}, \overline{\widehat{\times}}, \bar{\widehat{[.,.]}})$ *is the subalgebra of dimension* 2 *of* $L$ *having a basis* $\{X, Y\}$ *and Cartan coefficients* $[X, Y] = X$, *which implies that it has a structure of isoalgebra.*

*It is important to note that in this way the possibility of isotopically relate between themselves Lie algebras of different dimension has been proved.*

In the case of $F$ is non empty it will be necessary to suppose the existence of three maps of the type $\Phi$ to define the corresponding isooperations. So, to isotopically lift a $(K, +, \times)$-algebra $(A, \circ, \bullet, \cdot)$, if we suppose the use of some external factors $F^K$ and $F^A$ to obtain the isofield $\overline{\widehat{K}}$ and the isoalgebra $\overline{\widehat{A}}$ respectively, starting from an isotopic lifting of elements $\widehat{I}, \widehat{S}, \widehat{I}', \widehat{S}', *, \star, \Diamond, \Box$ and $\Diamond$, we will impose the existence of the following three maps:

$$\begin{aligned} \Phi_\circ : F^A \times F^A &\to F^A \\ (F_\alpha, F_\beta) &\to \Phi_\circ(F_\alpha, F_\beta) \\ \Phi_\bullet : F^K \times F^A &\to F^A \\ (F_a, F_\alpha) &\to \Phi_\bullet(F_a, F_\alpha) \\ \Phi_. : F^A \times F^A &\to F^A \\ (F_\alpha, F_\beta) &\to \Phi_.(F_\alpha, F_\beta) \end{aligned}$$

in the way that, we define, for all $\alpha, \beta \in A$ y $a \in K$ :

$$\alpha \bar{\bar{\circ}} \beta = (\alpha_{T'_{F_\alpha}} \Diamond \beta_{T'_{F_\beta}}) \Diamond \widehat{I}' \left( \alpha_{T'_{F_\alpha}} \Diamond \beta_{T'_{F_\beta}}, x'_{\alpha_{T'_{F_\alpha}} \Diamond \beta_{T'_{F_\beta}}}, x''_{\alpha_{T'_{F_\alpha}} \Diamond \beta_{T'_{F_\beta}}}, \Phi_\circ(F_\alpha, F_\beta) \right)$$

$$a \bar{\bar{\bullet}} \alpha = (a_{T'_{F_a}} \Box \alpha_{T'_{F_\alpha}}) \Diamond \widehat{I}' \left( a_{T'_{F_a}} \Box \alpha_{T'_{F_\alpha}}, x'_{a_{T'_{F_a}} \Box \alpha_{T'_{F_\alpha}}}, x''_{a_{T'_{F_a}} \Box \alpha_{T'_{F_\alpha}}}, \Phi_\bullet(F_a, F_\alpha)) \right)$$

$$\alpha \bar{\bar{\cdot}} \beta = (\alpha_{T'_{F_\alpha}} \Diamond \beta_{T'_{F_\beta}}) \Diamond \widehat{I}' \left( \alpha_{T'_{F_\alpha}} \Diamond \beta_{T'_{F_\beta}}, x'_{\alpha_{T'_{F_\alpha}} \Diamond \beta_{T'_{F_\beta}}}, x''_{\alpha_{T'_{F_\alpha}} \Diamond \beta_{T'_{F_\beta}}}, \Phi_.(F_\alpha, F_\beta)) \right)$$

As an example, we will now see a non-injective isotopic lifting of the type II, with $F_A$ non empty:

**Example 5.2.** *Let us consider the* $(\mathbb{R}, +, \times)$*-Lie algebra* $(L, +, \times, [.,.])$, *with basis* $\beta = \{X, Y, Z\}$ *and Cartan coefficients* $[X, Y] = Z$ *and the isofield* $(\widehat{\mathbb{R}}, \widehat{+}, \widehat{\times})$ *obtained from the identity isotopy.*

*Let us now consider the* $\widehat{L}$*-isoalgebra* $(\widehat{L}, \widehat{+}, \widehat{\times}, \widehat{[.,.]})$, *constructed from the isotopic lifting associated with the main elements* $\Diamond$ *e* $\widehat{I}' = \widehat{I}'(x, t)$, *where* $t \in \{0, 1\}$ *is the factor time, and the secondary ones* $\Box \equiv \times$, $\Diamond \equiv +$ *and* $\widehat{S}' = \mathbf{0}$ *(the null vector in* $L$*), where for all* $x \in L$ *and* $t \in \{0, 1\}$, *the element*

*$x\diamondsuit\widehat{I}'(x,t)$ is obtained, by using a linear extension of the following database*:

| x \ t | 0 | 1 |
|---|---|---|
| X | X | **0** |
| Y | Y | **0** |
| Z | **0** | X |

*The linear extension above mentioned will be given by:*

$$(a\times X+b\times Y)\diamondsuit\widehat{I}'(a\times X+b\times Y,t)=a\times(X\diamondsuit\widehat{I}'(X,t))+b\times(Y\diamondsuit\widehat{I}'(Y,t))$$

*where $a,b\in\mathbb{R}$. However, for this extension to be coherent, we must define in a suitable way the isooperations associated with $\widehat{\overline{L}}$, which would have the basis $\widehat{\overline{\beta}}=\{X,Y\}$.*

*Moreover, in the above table can be checked that this lifting is non-injective. Indeed, $X\diamondsuit\widehat{I}'(X,0)=Z\diamondsuit\widehat{I}'(Z,1)=X$. Besides, it is of the type II, because if $T'=T'(x,t)$ was the associated isotopic element, we will have, for instance, that $\mathbf{0}\diamondsuit T'(\mathbf{0},1)=\{X,Y\}$.*

*We are now going to define the isooperations associated with $\widehat{\overline{L}}$. To do this, we will impose that in this isotopic lifting we cannot operate with elements obtained in different instants of time. That is, if we wish to operate with $x\diamondsuit\widehat{I}'(x,t_x)$ and $y\diamondsuit\widehat{I}'(y,t_y)$, then it must be verified that $t_x=t_y$.*

*By taking it into consideration, if $F$ denotes the external factors associated with the construction of $\widehat{\overline{L}}$ (that is, the factor time) and $F'=\{(t,t):t\in\{0,1\}\}$, then we define the following maps*:

$$\begin{aligned}\Phi_+ &: F'\to F\\ &(t,t)\to\Phi_+(t,t)=t\\ \Phi_\times &: F\to F\\ &t\to\Phi_\times(t)=t\\ \Phi_{[.,.]} &: F'\to F\\ &(t,t)\to\Phi_{[.,.]}(t,t)=s\neq t\end{aligned}$$

*Now, by taking into consideration these maps we will look for the definition of the isooperations associated with $\widehat{\overline{L}}$. To do this, fix $a,b,c,d,e\in\mathbb{R}$. Then, when operating with vector $aX+bY$ y $cX+dY$, if $b$ o $d$ is non null, then, by construction, we can only operate with the value $t=0$ for the factor time. In* *this case*:

$$
\begin{aligned}
&(aX+bY)\overline{\overline{\hat{+}}}(cX+dY)\\
&=(aX+bY)\Diamond\widehat{I}'(aX+bY,0)\overline{\overline{\hat{+}}}(cX+dY)\Diamond\widehat{I}'(cX+dY,0)\\
&=(aX+bY+cX+dY)\Diamond\widehat{I}'(aX+bY+cX+dY,\Phi_{+}(0,0))\\
&=((a+c)X+(b+d)Y)\Diamond\widehat{I}'((a+c)X+(b+d)Y,0)\\
&=(a+c)\times X\Diamond\widehat{I}'(X,0)+(c+d)\times Y\Diamond\widehat{I}'(Y,0)=(a+c)X+(b+d)Y
\end{aligned}
$$

*If* $b=d=0$, *then it is also possible that* $t=1$. *However, in this case*:

$$
\begin{aligned}
aX\overline{\overline{\hat{+}}}cX &= aZ\Diamond\widehat{I}'(aZ,1)\overline{\overline{\hat{+}}}cZ\Diamond\widehat{I}'(cZ,1)\\
&=(aZ+cZ)\Diamond\widehat{I}'(aZ+cZ,\Phi_{+}(1,1))\\
&=(a+c)Z\Diamond\widehat{I}'((a+c)Z,1)\\
&=(a+c)\times Z\Diamond\widehat{I}'(Z,1)=(a+c)X
\end{aligned}
$$

*So, it is proved that* $\overline{\overline{\hat{+}}}\equiv +$.

*Apart from that, if* $b\neq 0$ :

$$
\begin{aligned}
e\overline{\overline{\hat{\times}}}(aX+bY) &= \overline{\overline{\hat{e}}}\overline{\overline{\hat{\times}}}(aX+bY)\Diamond\widehat{I}'(aX+bY,0)\\
&=(e\times(aX+bY))\Diamond\widehat{I}'(e\times(aX+bY),\Phi_{\times}(0))\\
&=(eaX+ebY)\Diamond\widehat{I}'(eaX+ebY,0)\\
&=ea\times X\Diamond\widehat{I}'(X,0)+eb\times Y\Diamond\widehat{I}'(Y,0)=eaX+ebY
\end{aligned}
$$

*If* $b=0$, *then it could be* $t=1$ :

$$
\begin{aligned}
e\overline{\overline{\hat{\times}}}aX &= \overline{\overline{\hat{e}}}\overline{\overline{\hat{\times}}}aZ\Diamond\widehat{I}'(aZ,1)(e\times aZ)\Diamond\widehat{I}'(e\times aZ,\Phi_{\times}(1))\\
&=eaZ\Diamond\widehat{I}'(eaZ,1)=ea\times Z\Diamond\widehat{I}'(Z,1)=eaZ
\end{aligned}
$$

*So, we obtain that* $\overline{\overline{\hat{\times}}}\equiv\times$.

*Finally, if* $b$ *or* $d$ *is non null, then*:

$$
\begin{aligned}
\overline{\overline{\hat{[}}}aX+bY,cX+dY\overline{\overline{\hat{]}}} &= \overline{\overline{\hat{[}}}(aX+bY)\Diamond\widehat{I}'(aX+bY,0),(cX+dY)\Diamond\widehat{I}'(cX+dY,0)\overline{\overline{\hat{]}}}\\
&=[aX+bY,cX+dY]\Diamond\widehat{I}'([aX+bY,cX+dY],\Phi_{[.,.]}(0,0))\\
&=((ad-bc)Z)\Diamond\widehat{I}'((ad-bc)Z,1)\\
&=(ad-bc)\times Z\Diamond\widehat{I}'(Z,1)=(ad-bc)X
\end{aligned}
$$

*If* $b = d = 0$, *it could be* $t = 1$ :

$$\begin{aligned}\hat{\bar{\hat{[}}}aX, cX\hat{\bar{\hat{]}}} &= \hat{\bar{\hat{[}}}aX \diamondsuit \widehat{I}'(aX, 1), cX \diamondsuit \widehat{I}'(cX, 1)\hat{\bar{\hat{]}}} \\ &= [aX, cX] \diamondsuit \widehat{I}'([aX, cX], \Phi_{[.,.]}(1, 1)) = \mathbf{0} \diamondsuit \widehat{I}'(\mathbf{0}, 0) = \mathbf{0}\end{aligned}$$

*So, we obtain that* $\hat{\bar{\hat{[}}}.,.\hat{\bar{\hat{]}}} \equiv [.,.]$.

*As a consequence, it has been obtained that* $(\hat{\bar{\hat{L}}}, \hat{\bar{\hat{+}}}, \hat{\bar{\hat{\times}}}, \hat{\bar{\hat{[}}}.,.\hat{\bar{\hat{]}}})$ *is the subalgebra of dimension* 2 *of* $L$ *having a basis* $\{X, Y\}$ *and Cartan coefficients* $[X, Y] = X$, *which implies that it has a structure of isoalgebra.*

*We note again that Lie algebras of different dimension have been isotopically related between themselves.*

To finish the paper, we are going to show an example of a non-injective isotopic lifting of the type I:

**Example 5.3.** *Let us consider the* $(\mathbb{R}, +, \times)$*-Lie algebra* $(L, +, \times, [.,.])$, *with basis* $\beta = \{e_1, e_2, e_3\}$ *and Cartan coefficients* $[e_1, e_3] = e_2$ *and the isofield* $(\widehat{\mathbb{R}}, \widehat{+}, \widehat{\times})$ *obtained from the identity isotopy.*

*Let us now consider the* $\widehat{L}$*-isoalgebra* $(\widehat{L}, \widehat{+}, \widehat{\times}, \widehat{[}.,.\widehat{]})$, *constructed from the isotopy associated with the main elements* $\diamondsuit$ *and* $\widehat{I}' = \widehat{I}'(x, t, s)$, *being* $t \in \{0, 1, 2\}$ *and* $s \in \mathbb{R}$, *and secondary ones* $\square \equiv \times$, $\diamondsuit \equiv +$ *and* $\widehat{S}' = \mathbf{0}$ *(the null vector in L), where for all* $x = ae_1 + be_2 + ce_3 \in L$, $t \in \{0, 1, 2\}$ *and* $s \in \mathbb{R}$, *we will define the element* $x \diamondsuit \widehat{I}'(x, t, s)$ *as*:

$$x \diamondsuit \widehat{I}'(x, t, s) = a \times \overline{\widehat{e_1}}_t + b \times \overline{\widehat{e_2}}_t + c \times \overline{\widehat{e_3}}_t + s e_{3(t+1)}$$

*where the elements* $\overline{\widehat{e_i}}_t$ *are determined in the way as it is shown in the following table*:

| $e_i \setminus t$ | 0 | 1 | 2 |
|---|---|---|---|
| $e_1$ | $e_1$ | $e_1$ | $e_1$ |
| $e_2$ | $e_2$ | $e_2$ | $e_3$ |
| $e_3$ | $e_3$ | $e_4$ | $e_2$ |

*where* $e_4$ *has to be a linearly independent vector with respect to* $\{e_1, e_2, e_3\}$.

*In this way, with this definition we will get that* $\overline{\widehat{L}}$ *has the basis* $\overline{\widehat{\beta}} = \{e_1, e_2, e_3, e_4\}$, *because, fixed* $ae_1 + be_2 + ce_3 + de_4$, *we can take, for instance, that* $(ae_1 + be_2 + ce_3) \diamondsuit \widehat{I}'(ae_1 + be_2 + ce_3, 0, d) = ae_1 + be_2 + ce_3 + de_4$.

*Moreover, in the above table can be checked that this lifting is non-injective. Indeed,* $e_2 \diamondsuit \widehat{I}'(e_2, 0, 0) = e_3 \diamondsuit \widehat{I}'(e_2, 2, 0) = e_2$.

*Besides, it is of the type I. To see it we will have to define an isotopic element* $T' = T'(x,t,s)$ *such that, given* $x = ae_1 + be_2 + ce_3 + de_4 \in \overline{\widehat{L}}$, *it is verified*:

$$x \diamondsuit T(x,t,s) = \left\{ \begin{array}{l} ae_1 + be_2 + ce_3, \ \textit{if } t = 0 \ (\textit{it must be } s = d) \\ ae_1 + (b-s)e_2 + de_3, \ \textit{if } t = 1 \ (\textit{it must be } c = 0) \\ ae_1 + (c-s)e_2 + be_3, \ \textit{if } t = 2 \ (\textit{it must be } d = 0) \end{array} \right\}$$

*We will now define the isooperations associated with* $\overline{\widehat{L}}$. *To do this, as we already did in the previous example, we will impose some restrictions when doing such operations, which are referred to the domain of the maps of the type* $\Phi$ *which are needed to define such isooperations.*

*Let us consider then the following sets*:

$$\begin{aligned} F &= \{(t,s) : t \in \{\{0,1,2\}, s \in \mathbb{R}\} \\ F_+ &= \{((t,s_1),(t,s_2)) : t \in \{\{0,1,2\}, s_1, s_2 \in \mathbb{R}\} \\ F_{[.,.]} &= \{((t,0),(t,0)) : t \in \{\{0,1,2\}\} \end{aligned}$$

*In this way, we define the following three maps*:

$$\begin{aligned} \Phi_+ &: F_+ \to F \\ ((t,s_1),(t,s_2)) &\to \Phi_+((t,s_1),(t,s_2)) = (t, s_1 + s_2) \\ \Phi_\times &: \mathbb{R} \times F \to F \\ (a,(t,s)) &\to \Phi_\times(a,(t,s)) = (t, as) \\ \Phi_{[.,.]} &: F_{[.,.]} \to F \\ ((t,0),(t,0)) &\to \Phi_{[.,.]}((t,0),(t,0)) = (t+1, 0) \end{aligned}$$

*Now, by taking into consideration these maps we will look for the definition of the isooperations associated with* $\overline{\widehat{L}}$. *To do this, fix* $a, b, c, d, e, f, g, h, i, \in \mathbb{R}$. *To check the coherence of the definition searched we have to distinguish some cases, according to the factor time*:

*By beginning with* $\overline{\widehat{+}}$ *we have that*:

a) $t = 0$

$$\begin{aligned}&(ae_1 + be_2 + ce_3 + de_4)\overline{\widehat{+}}(ee_1 + fe_2 + ge_3 + he_4)\\&= (ae_1 + be_2 + ce_3)\diamondsuit\widehat{I}'(ae_1 + be_2 + ce_3, 0, d)\\&\quad\overline{\widehat{+}}(ee_1 + fe_2 + ge_3)\diamondsuit\widehat{I}'(ee_1 + fe_2 + ge_3, 0, h)\\&= ((a+e)e_1 + (b+f)e_2 + (c+g)e_3)\diamondsuit\widehat{I}'((a+e)e_1\\&\quad + (b+f)e_2 + (c+g)e_3, 0, d+h)\\&= (a+e)e_1 + (b+f)e_2 + (c+g)e_3 + (d+h)e_4\end{aligned}$$

b) $t = 1$

$$\begin{aligned}&(ae_1 + be_2 + de_4)\overline{\widehat{+}}(ee_1 + fe_2 + he_4)\\&= (ae_1 + (b-s_1)e_2 + de_3)\diamondsuit\widehat{I}'(ae_1 + (b-s_1)e_2 + de_3, 1, s_1)\\&\quad\overline{\widehat{+}}(ee_1 + (f-s_2)e_2 + he_3)\diamondsuit\widehat{I}'(ee_1 + (f-s_2)e_2 + he_3, 1, s_2)\\&= ((a+e)e_1 + (b+f-s_1-s_2)e_2 + (d+h)e_3)\\&\quad\diamondsuit\widehat{I}'((a+e)e_1 + (b+f-s_1-s_2)e_2 + (d+h)e_3, 0, s_1+s_2)\\&= (a+e)e_1 + (b+f)e_2 + (d+h)e_4\end{aligned}$$

c) $t = 2$

$$\begin{aligned}&(ae_1 + be_2 + ce_3)\overline{\widehat{+}}(ee_1 + fe_2 + ge_3)\\&= (ae_1 + (c-s_1)e_2 + be_3)\diamondsuit\widehat{I}'(ae_1 + (c-s_1)e_2 + be_3, 2, s_1)\\&\quad\overline{\widehat{+}}(ee_1 + (g-s_2)e_2 + fe_3)\diamondsuit\widehat{I}'(ee_1 + (g-s_2)e_2 + fe_3, 2, s_2)\\&= ((a+e)e_1 + (c+g-s_1-s_2)e_2 + (b+f)e_3)\\&\quad\diamondsuit\widehat{I}'((a+e)e_1 + (c+g-s_1-s_2)e_2 + (b+f)e_3, 0, s_1+s_2)\\&= (a+e)e_1 + (b+f)e_2 + (c+g)e_3\end{aligned}$$

*So,* $\overline{\widehat{+}} \equiv +$.

*Apart from that we have, with respect to* $\overline{\widehat{\times}}$ *that:*

a) $t = 0$

$$\begin{aligned}i\overline{\widehat{\times}}(ae_1 + be_2 + ce_3 + de_4) &= \overline{\widehat{i}} \times (ae_1 + be_2 + ce_3)\diamondsuit\widehat{I}'(ae_1 + be_2 + ce_3, 0, d)\\&= (iae_1 + ibe_2 + ice_3)\diamondsuit\widehat{I}'(iae_1 + ibe_2 + ice_3, 0, id)\\&= iae_1 + ibe_2 + ice_3 + ide_4\end{aligned}$$

b) $t = 1$

$$\begin{aligned} i\hat{\overline{\times}}(ae_1+be_2+de_4) &= \hat{\overline{i}}\times(ae_1+(b-s)e_2+de_3)\Diamond\widehat{I}'(ae_1+(b-s)e_2+de_3,1,s) \\ &= (iae_1+(ib-is)e_2+ide_3)\Diamond\widehat{I}'(iae_1+(ib-is)e_2+ice_3,1,is) \\ &= iae_1+ibe_2+ide_4 \end{aligned}$$

c) $t = 2$

$$\begin{aligned} i\hat{\overline{\times}}(ae_1+be_2+ce_3) &= \hat{\overline{i}}\times(ae_1+(c-s)e_2+be_3)\Diamond\widehat{I}'(ae_1+(c-s)e_2+be_3,2,s) \\ &= (iae_1+(ic-is)e_2+ice_3)\Diamond\widehat{I}'(iae_1+(ib-is)e_2+ice_3,2,is) \\ &= iae_1+ibe_2+ice_3 \end{aligned}$$

*So,* $\hat{\overline{\times}} \equiv \times$.

*Finally, to define* $\hat{\overline{[.,.]}}$ *we will impose to such an isooperation to be bilinear so that it can originate a Lie algebra in the projection level. In this way and as we have already proved that* $\hat{\overline{+}} \equiv +$ *y* $\hat{\overline{\times}} \equiv \times$, *we finally have that:*

$$\begin{aligned} &\hat{\overline{[}}ae_1+be_2+ce_3+de_4, ee_1+fe_2+ge_3+he_4\hat{\overline{]}} \\ &= \hat{\overline{[}}\hat{\overline{a}}\hat{\overline{\times}}e_1\hat{\overline{+}}\hat{\overline{b}}\hat{\overline{\times}}e_2\hat{\overline{+}}\hat{\overline{c}}\hat{\overline{\times}}e_3\hat{\overline{+}}\hat{\overline{d}}\hat{\overline{\times}}e_4, \hat{\overline{e}}\hat{\overline{\times}}e_1\hat{\overline{+}}\hat{\overline{f}}\hat{\overline{\times}}e_2\hat{\overline{+}}\hat{\overline{g}}\hat{\overline{\times}}e_3\hat{\overline{+}}\hat{\overline{h}}\hat{\overline{\times}}e_4\hat{\overline{]}} \\ &= \hat{\overline{a}}\hat{\overline{\times}}\hat{\overline{e}}\hat{\overline{\times}}\hat{\overline{[}}e_1,e_1\hat{\overline{]}}\hat{\overline{+}}\cdots\hat{\overline{+}}\hat{\overline{d}}\hat{\overline{\times}}\hat{\overline{h}}\hat{\overline{\times}}\hat{\overline{[}}e_4,e_4\hat{\overline{]}} = ae\hat{\overline{[}}e_1,e_1\hat{\overline{]}}\hat{\overline{+}}\cdots\hat{\overline{+}}dh\hat{\overline{[}}e_4,e_4\hat{\overline{]}} \end{aligned}$$

*By taking into consideration the definition of* $\Phi_{[.,.]}$ *and the table of the elements* $e_i\Diamond\widehat{I}'(x,t,s)$, *it is easy checked that the Cartan coefficients of the resulting algebra in the projection level are* $\hat{\overline{[}}e_1,e_3\hat{\overline{]}} = e_2$ *and* $\hat{\overline{[}}e_1,e_4\hat{\overline{]}} = e_3$. *So, we finally deduced that*

$$\hat{\overline{[}}ae_1+be_2+ce_3+de_4, ee_1+fe_2+ge_3+he_4\hat{\overline{]}} = (ag-ce)e_2+(ah-de)e_3$$

*As a consequence, we have obtained that* $(\hat{\overline{L}}, \hat{\overline{+}}, \hat{\overline{\times}}, \hat{\overline{[.,.]}})$ *is the superalgebra of dimension* 4 *of* $L$ *with basis and Cartan coefficients given, which has a structure of isoalgebra.*

*As in the previous examples, in this one have been also isotopically related between themselves two filiform Lie algebras of different dimensions.*

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