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**GENERICITY OF THE FIXED POINT
PROPERTY UNDER RENORMING**

Ph. D. Thesis presented by

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Memoria presentada por
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para optar al grado de
Doctora en Matemáticas
por la Universidad de Sevilla

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de la Universidad de Sevilla

Sevilla, Julio de 2010

RESUMEN

Una aplicación T definida de un espacio métrico M en M se dice no expansiva si $d(Tx, Ty) \leq d(x, y)$ para todo $x, y \in M$. Diremos que un espacio de Banach X tiene la *Propiedad Débil del Punto Fijo* (w-FPP) si para toda aplicación no expansiva T definida de un subconjunto débilmente compacto convexo C de X en C tiene un punto fijo. En esta disertación, estudiamos principalmente la w-FPP como una propiedad genérica en el conjunto de todas las normas equivalentes de un espacio de Banach reflexivo dado. Una propiedad P se dice genérica en un conjunto A si todos los elementos de A satisfacen P excepto aquellos pertenecientes a un conjunto de tamaño pequeño. Con el fin de establecer los resultados de este trabajo, consideraremos varias nociones de conjuntos pequeños, como por ejemplo los conjuntos de Baire de primera categoría, conjuntos porosos, conjuntos nulos Gaussianos o conjuntos direccionalmente porosos.

M. Fabian, L. Zajíček y V. Zizler probaron que casi todos los renormamientos de un espacio uniformemente convexo en cada dirección (UCED), en el sentido de la categoría de Baire, son también UCED. Debido al resultado de M.M. Day, R.C. James y S. Swaminathan, todo espacio de Banach separable admite una norma equivalente que es uniformemente convexa en cada dirección. Puesto que esta propiedad geométrica implica la FPP, obtenemos la siguiente conclusión: Si X es un espacio de Banach reflexivo separable, entonces casi todos los renormamientos de X satisfacen la w-FPP. Este método no es válido para el caso de los espacios reflexivos no separables. Sin embargo, recientemente T. Domínguez Benavides ha probado que todo espacio de Banach que pueda ser sumergido en $c_0(\Gamma)$, donde Γ es un conjunto arbitrario (en particular, todo espacio reflexivo) puede ser renormado para tener la w-FPP. Nótese que que el espacio $c_0(\Gamma)$ no es renormable UCED cuando Γ es no numerable, pero satisface la w-FPP porque $R(c_0(\Gamma)) < 2$, donde $R(\cdot)$

es el coeficiente de García-Falset y todo espacio de Banach X con $R(X) < 2$ satisface la w-FPP. Usando la misma inmersión, obtenemos el siguiente resultado: Sea X un espacio de Banach tal que para algún conjunto Γ existe una aplicación continua lineal uno a uno $J : X \rightarrow c_0(\Gamma)$. Entonces, casi todas las normas equivalentes q en X (en el sentido de la categoría de Baire) satisfacen la siguiente propiedad: Toda aplicación q -no-expansiva, definida desde un subconjunto convexo débilmente compacto C de X , en C , tiene un punto fijo. En particular, si X es reflexivo, entonces el espacio (X, q) satisface la FPP. Además, extendemos este resultado a cualquier espacio de Banach que pueda ser sumergido en un espacio de Banach Y , más general que $c_0(\Gamma)$ y que satisfaga $R(Y) < 2$. Probamos que si X es un espacio de Banach satisfaciendo $R(Y) < 2$ y X un espacio de Banach que pueda ser sumergido en Y de manera continua, entonces X puede ser renormado para satisfacer la w-FPP y el conjunto de todas las renormas en X , que no satisfacen la w-FPP, es de primera categoría. En el caso del espacio $C(K)$, donde K es un conjunto disperso tal que $K^{(\omega)} = \emptyset$, obtendremos que existe una norma $|\cdot|$ que es equivalente a la norma del supremo y $R(C(K), |\cdot|) < 2$ (luego tiene la w-FPP). Además, casi todas las normas equivalentes a la norma del supremo (en el sentido de la porosidad) también satisfacen la w-FPP.

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Introduction

Assume that A is a set and \mathcal{P} a property which can be either satisfied or not by the elements of A . The property \mathcal{P} is said to be generic in A if “almost all” elements of A satisfy \mathcal{P} . When speaking about almost all elements we mean all of them except those in a “negligible set”. There are different ways to define the notion of negligible set, according to the setting where we are interested. For instance, in measure theory, a set with null measure can be considered as negligible and many generic results are well known in this theory. Consider, for instance, the following example: Let f and g be Bochner integrable functions. If $\int_E f d\mu = \int_E g d\mu$ for every μ -measurable set E , then $f = g$ μ -almost everywhere, i.e. the set $\{x : f(x) \neq g(x)\}$ is negligible. In topological space, a Baire first category set can be considered as a negligible set. The interest of this notion depends on the size of the whole set, because if the whole set were of Baire first category, then all subsets would be negligible. Thus, this notion is only interesting in second category topological spaces, for instance, in complete metric spaces, according to Baire Theorem.

It must be noted that negligibility in the sense of null measure and in the sense of Baire category can be different in spaces where both notions can be simultaneously considered. For instance, the real line \mathbb{R} is the disjoint union of a set of first category and a set of Lebesgue measure zero. To avoid this problem, we can use the concept of “porosity” as a refined notion of Baire first category. Every σ -porous set is of first category and in a finite dimensional space, it has Lebesgue measure zero.

Many generic results have appeared concerning different subjects. One of the first generic result was obtained by W. Orlicz [62], who proved that the uniqueness of solution of the Cauchy problem for an ordinary differential equations is a generic property in the space of all bounded continuous functions mapping from \mathbb{R}^{n+1} into \mathbb{R}^n . Later, this result was extended to

the generic uniqueness of solutions of different equations in infinite dimensional spaces by A. Lasota and J.A. Yorke [53]. In this dissertation, we study generic property concerning Metric Fixed Point Theory.

Fixed point theory has been usually used to study the existence of solutions of differential equations and also has been applied in many branches of mathematics. The most well-known fixed point theorem is the Contraction Mapping Principle, due to S. Banach [5]. The statement is the following:

Theorem. (Banach Fixed Point Theorem) Let X be a complete metric space and $T : X \rightarrow X$ a contraction, i.e. there exists $k \in [0, 1)$ such that $d(Tx, Ty) \leq kd(x, y)$ for every $x, y \in X$. Then T has a (unique) fixed point x_0 . Furthermore $x_0 = \lim_{n \rightarrow \infty} T^n x$ for every $x \in X$.

However, Banach Contraction Principle fails for non-expansive mapping, i.e. a mapping $T : M \rightarrow M$, where M is a metric space, such that $d(Tx, Ty) \leq d(x, y)$ for every $x, y \in M$. But in 1965, F. E. Browder, D. Göhde and W. A. Kirk proved the existence of fixed points for non-expansive mappings in Banach spaces which satisfy some geometrical properties. Browder [8] proved that every non-expansive mapping defined from a convex closed bounded subset C of a Hilbert space into C , has a fixed point. In the same year, Browder [10] and Göhde [39] simultaneously proved that every non-expansive mapping defined from a convex closed bounded subset C of a uniformly convex Banach space into C , has a fixed point, and Kirk [46] observed that a geometrical property weaker than uniform convexity, called normal structure, guaranteed the same result in a reflexive Banach space.

These results have started the search of more general conditions for a Banach space and for a subset C which assure the existence of fixed points. In order to formulate the problem, it was introduced the following definition in a natural way: A Banach space X is said to have the *fixed point property (FPP)* if for every nonempty closed bounded convex subset C of X , every non-expansive mapping $T : C \rightarrow C$ has a fixed point. Similarly, a Banach space

X is said to have the *weak fixed point property (w-FPP)* if for every nonempty weakly compact convex subset C of X , every non-expansive mapping $T : C \rightarrow C$ has a fixed point.

In the last forty years, many papers have appeared proving the FPP and the w-FPP for different classes of Banach spaces according to the geometrical structure.

Furthermore, many fixed point theorems have been extended to multi-valued mappings, which has useful applications in Applied Sciences, in particular, in Game Theory and Economical Mathematics. For instance, S. B. Nadler [61] extended the Banach Contraction Principle to multi-valued contraction mappings in 1969. He proved that for a complete metric space X , if $T : X \rightarrow 2^X$ is a contraction with closed bounded values, then T has a fixed point. In 1974, Lim [54] proved the existence of a fixed point for a non-expansive multi-valued mapping defined from a closed bounded convex subset C of a uniformly convex Banach space X into the set of all compact subsets of C . Similarly, we say that a Banach space X has the *weak multi-valued fixed point property (w-MFPP)* if every multi-valued non-expansive mapping $T : C \rightarrow K(C)$ has a fixed point, where C is a weakly compact convex subset of X and $K(C)$ is the family of all nonempty compact subsets of C .

Concerning Metric Fixed Point Theory and Genericity, there are some classical results about generic existence of fixed points for non-expansive mappings which are defined on a given Banach space. One of the first generic fixed point results was proved by G. Vidossich [89] saying that if C is a bounded closed convex subset of a Banach space X and \mathcal{F} is the complete metric space of all non-expansive mappings from C into itself endowed with the uniform convergence metric, then almost all mappings in \mathcal{F} (in the sense of Baire category) do have a (unique) fixed point. Afterward, many mathematicians have studied the generic fixed point property within the space of

all non-expansive mappings. For instance, F.S. De Blasi and J. Myjak [15] proved the stronger result that the set of all mappings f such that f has unique fixed point x_0 and $f^n x \rightarrow x_0$ as $n \rightarrow \infty$ for all $x \in C$, is residual in the set of all non-expansive mappings. S. Reich and A.J. Zaslavski (in [79]) have improved De Blasi and Myjak's result by showing that almost all (in the sense of Baire category) non-expansive mappings are, in fact, contractive. We recall that a mapping T is called *contractive* if there exists a decreasing function $\phi : [0, d(C)] \rightarrow [0, 1]$ such that

$$\phi(t) < 1 \quad \text{for all } t \in (0, d(C)]$$

and

$$\rho(Tx, Ty) \leq \phi(\rho(x, y))\rho(x, y) \quad \text{for all } x, y \in C$$

where C is a metric space. Contractive mappings are known to have a unique fixed point and their iterates do converge in all complete metric spaces.

Additionally, Reich and Zaslavski proved later that the complement of the set of all non-contractive mappings is not only of the first category but also σ -porous [81]. In Chapter 3, we extend this result to the case of multi-valued non-expansive mappings: we prove that the complement of the set of all multi-valued contractive mappings, which are known by the result of H. Kaneko [42] to have a fixed point, is σ -porous.

Besides the generic fixed point property on the set of all non-expansive mappings, we are interested in the generic fixed point property concerning Renorming Theory. Assume that $(X, \|\cdot\|)$ is a Banach space. The most common aim of the Renorming Theory is to find an equivalent norm $|\cdot|$ which satisfies (or which does not satisfy) some specific properties. Deep and wide information about this subject can be found in the monographs [18], [29], [33]. It is remarkable that the FPP and the w-FPP are not preserved under isomorphisms. Indeed, P.K. Lin [55] has proved that ℓ_1 can be renormed to have the FPP. We recall that this space does not satisfy the FPP for the

usual norm. On the other hand, it is well known that the space $L_1([0, 1])$ does not satisfy the w-FPP as proved by D.E. Alspach [1]. However this space (and any separable Banach space) can be renormed to have normal structure [88] and so the w-FPP [46].

Thus, very natural questions relating to the three topics: Genericity, Metric Fixed Point Property and Renorming Theory could be the following

(1-a) Let X be a Banach space. Is it possible to renorm X so that the resultant space has the FPP or the w-FPP?

(2-a) Do almost all renormings of X satisfy the FPP or the w-FPP?

The answer, in general, is negative if X is nonreflexive. Indeed it was proved by P. Dowling, C. Lennard and B. Turett [28], that every renorming of $c_0(\Gamma)$ when Γ is uncountable contains an asymptotically isometric copy of c_0 and so it fails to have the FPP. Analogously, any renorming of $\ell_1(\Gamma)$ (Γ uncountable) contains an asymptotically isometric copy of ℓ_1 so it also fails to satisfy the FPP. In the case of the w-FPP, J. Partington [63], [64] has proved that every renorming of $\ell_\infty(\Gamma)$ for Γ uncountable and any renorming of ℓ_∞/c_0 contain an isometric copy of ℓ_∞ and so they fail the w-FPP (again due to Alspach's example). For more examples of nonreflexive Banach spaces which cannot be renormed to satisfy the FPP see for instance [25].

Hence, we restrict the above questions to reflexive Banach spaces. Note that for reflexive spaces, the FPP and the w-FPP coincide. In other words, we consider the following problems:

(1) Let X be a reflexive Banach space. Can X be renormed in such a way that the resultant space has the FPP?

(2) Do almost all renormings of X satisfy the FPP?

The problem (1) was, in fact, long time open (see in [44], Open problem VI and [18], Problem VII.3).

The answers to both problems have been known in the case of separable spaces for a long time. Indeed, M.M. Day, R.C. James and S. Swaminathan [14] have proved that every separable Banach space has a uniform convex in every direction (UCED) renorming. Since, uniform convexity in every direction implies normal structure (proved in [14]) and so the FPP for reflexive spaces (see [46]) we easily obtain: Every separable reflexive space can be renormed to satisfy the FPP. Uniform convexity in every direction is not only used as a condition to obtain the FPP but it can be also used as a property for obtaining the MFPP. According to the result proved by Kirk and Massa [45], by using the asymptotic center method, UCED spaces have the w-MFPP. Hence every separable reflexive space can be renormed to have the MFPP.

For the problem (2), it was proved by M. Fabian, L. Zajíček and V. Zizler [30] the following: Let $(X, \|\cdot\|)$ be a UCED Banach space and (\mathcal{P}, ρ) be the space of all equivalent norms on X , endowed with the metric $\rho(p, q) := \sup\{\|p(x) - q(x)\| : x \in B_X\}$, for each $p, q \in \mathcal{P}$, where B_X is the closed unit ball of X . Then there is a residual subset \mathcal{R} of \mathcal{P} such that every norm $p \in \mathcal{R}$ is UCED. Consequently, if X is a separable reflexive space and \mathcal{P} is defined as above then almost all norms in \mathcal{P} satisfy the FPP and the MFPP.

These arguments do not work for non-separable reflexive spaces. Indeed, D. Kutzarova and S. Troyanski [49] have proved that there are reflexive spaces without equivalent norms which are UCED. It is also known that the space $c_0(\Gamma)$, where Γ is uncountable, has no UCED renormings. However, $c_0(\Gamma)$ enjoys the w-FPP by using a different approach. The geometric coefficient $R(X)$ of a Banach space X is defined in the following way:

$$R(X) := \sup \left(\liminf_{n \rightarrow \infty} \|x_n + x\| \right)$$

where the supremum is taken over all weakly null sequences (x_n) of the unit ball and all points x of the unit ball. J. García-Falset [31] has proved that

a Banach space X with $R(X) < 2$ satisfies the w-FPP. Accordingly, $c_0(\Gamma)$ satisfies the w-FPP since $R(c_0(\Gamma)) = 1$.

Since some Banach spaces which cannot be renormed with a UCED norm satisfy the w-FPP, it would be interesting to know that if X is a Banach space which satisfies the w-FPP, then is the w-FPP generic on the space of all equivalent norms on X ? The answer to this question is true when X satisfies $R(X) < 2$. We prove that if a Banach space X satisfies $R(X) < 2$, then all equivalent norms p also satisfy $R(X, p) < 2$ (so that (X, p) enjoys the w-FPP), except those equivalent norms a σ -porous set.

Apart from the fact that the space $c_0(\Gamma)$, which is not UCED renormable, has the w-FPP, some interesting renorming results have been obtained for non-separable spaces. For instance, D. Amir and J. Lindenstrauss [2] have proved that every WCG Banach space has an equivalent norm which is strictly convex, and S. Troyanski [87] has proved that every WCG Banach space has an equivalent norm which is locally uniformly convex. An important tool in the proofs of these results is the following fact (proved in [2]): For any WCG Banach space X , there exist a set Γ and a bounded one-to-one linear operator $J : X \rightarrow c_0(\Gamma)$. This property is satisfied by a very general class of Banach spaces, for instance subspaces of a space with Markushevich basis, as WCG spaces (and so either separable or reflexive spaces), duals of separable spaces as ℓ_∞ , etc (see [29]). Using this embedding, in [20] the following result is proved: Assume that X is a Banach space such that there exists a bounded one-one linear operator from X into $c_0(\Gamma)$. Then, X has an equivalent norm such that every non-expansive mapping T for the new norm defined from a convex weakly compact set C into C has a fixed point. As a consequence, we obtain: Every reflexive space can be renormed in such a way that the resultant norm has the FPP.

Following this method, we prove the generic fixed point result as follows: Let X be a Banach space such that for some set Γ there exists a one-to-one

linear continuous mapping $J : X \rightarrow c_0(\Gamma)$. Let \mathcal{P} be defined as before. Then, there exists a residual subset \mathcal{R} in \mathcal{P} such that for every $q \in \mathcal{R}$, every q -non-expansive mapping T defined from a weakly compact convex subset C of X into C has a fixed point. Particularly, if X is reflexive, then the space (X, q) satisfies the FPP. This result positively answers the problem (2).

It would be interesting to extend the result in [20] and the above result to any Banach space which can be embedded in more general Banach spaces than $c_0(\Gamma)$, but still satisfying that their García-Falset coefficient is less than 2. In this dissertation, we actually prove this extension in the following sense: Assume that Y is a Banach space such that $R(Y) < 2$ and X is a Banach space which can be continuously embedded in Y . Then, X can be renormed to satisfy the w-FPP. For the generic fixed point result, following the same assumption on the space X , let \mathcal{P} be defined as before. We obtain that the set of all renormings on X which do not satisfy the w-FPP is of the first category.

On the other hand, if we endow Γ with the discrete topology and denote by K the one-point compactification of Γ , then $c_0(\Gamma)$ is isomorphic to $C(K)$ (and isometrically contained in $C(K)$) where K is a topological compact space which satisfies $K^{(2)} = \emptyset$. Thus, any space which can be continuously embedded in $c_0(\Gamma)$ can also be embedded in $C(K)$ where K is a scattered compact topological space such that $K^{(\omega)} = \emptyset$. Since $C(K)$ satisfies the w-FPP [22] when K is a scattered compact topological space K such that $K^{(\omega)} = \emptyset$, another natural question would be the following: Assume that X is a Banach space which can be continuously embedded in $C(K)$ for some K as above. Can X be renormed to satisfy the w-FPP? In this work we prove that there is the case. Nominally we prove the following: Let $C(K)$ be the space of real continuous functions defined on a scattered compact topological space K such that $K^{(\omega)} = \emptyset$. Then, it can be renormed in such a way that $R(C(K), \|\cdot\|) < 2$ where $\|\cdot\|$ is the new norm. By applying this result with

the previous results, we can easily obtain the following two results

- (i) Let X be a Banach space which can be continuously embedded into $C(K)$ where K is a scattered compact topological space such that $K^{(\omega)} = \emptyset$. Then, it can be renormed in such a way that X satisfies the w-FPP.
- (ii) Assume that \mathcal{P} is the set of all norms in $C(K)$ which are equivalent to the supremum norm with the metric $\rho(p, q) = \sup\{|p(x) - q(x)| : x \in B\}$. Then, there exists a σ -porous set $\mathcal{A} \subset \mathcal{P}$ such that if $q \in \mathcal{P} \setminus \mathcal{A}$ the space $(C(K), q)$ satisfies the w-FPP.

The first result is a strict improvement of the result in [20], because, as proved in [34], when K is a Ciesielski-Pol's compact, then $K^{(3)} = \emptyset$, but $C(K)$ cannot be continuously embedded in $c_0(\Gamma)$ for any set Γ . And the second result can be understood as a continuation of the result in [22] which says that $C(K)$ endowed with the supremum norm, K as above, satisfies the w-FPP. Now we can add that $C(K)$ satisfies the w-FPP for almost all norms which are equivalent to the supremum one.

The answers to our problems (1) and (2) seem to be much more difficult in the case of multi-valued fixed point property on non-separable reflexive spaces. Until now, it is still unknown if a reflexive space can be renormed to have the w-MFPP. However, if a given Banach space satisfies some geometrical properties, it can satisfy the w-MFPP. For instance, by the result of T. Domínguez and P. Lorenzo, every nearly uniform convex (NUC) Banach space enjoys the w-MFPP for compact convex valued mappings. Following this assumption, we prove the generic multi-valued fixed point property: if X is a NUC Banach space and \mathcal{P} is defined as before, then all norms in \mathcal{P} , except those in a set of first category, are also NUC (so, they satisfy the w-MFPP for compact convex valued mappings). And by using the Szlenk index, which has been introduced in order to show that there is no universal

space for the class of separable reflexive spaces, we can determine the w-MFPP renormability and genericity in the following way: if X is a reflexive space such that $S_z(X^*) \leq \omega$, where $S_z(\cdot)$ denotes the Szlenk index and X^* is the dual space of X , then X can be renormed to satisfy the w-MFPP. Furthermore, if \mathcal{P} is defined as before, there exists a residual subset \mathcal{R} of \mathcal{P} such that for every $p \in \mathcal{R}$, the space (X, p) satisfies the w-MFPP for compact convex valued mappings.

We must note that for a general reflexive Banach space, both problem (1) and (2) are still open in the case of the multi-valued fixed point property.

The dissertation is divided into six chapters and organized in the following way:

Chapter 1 deals with basic notations, definitions, concepts and background which will be needed in the following chapters.

Chapter 2 is devoted to the concept of negligible sets. Some notions of negligible sets are studied with detail. For instance Baire category, porosity, Gaussian null sets, Aronszajn null sets and directional porosity. There are several ways to define the notions of porosity and directional porosity. In this chapter, we present two different definitions of each notions and give some examples to show the non-equivalence among those definitions.

Chapter 3 contains some classical generic fixed point existence results on the complete metric space of all non-expansive mappings. One of the main result is proved by Reich and Zaslavski: all non-expansive mappings are contractive except those in the σ -porous set. We consider the same setting but equipped with another metric, which introduces a stronger topology, the same results still holds.

Chapter 4 is dedicated to the generic fixed point existence on the set of all equivalent norms of a separable reflexive space. The main important tool used in this chapter is the uniform convexity in every direction. The porosity version of the result of Fabian-Zaiček-Zizler is proved in this chapter under

some equivalent metrics.

Chapter 5 incorporates two main subjects: Fixed point renormability on (non-separable) reflexive spaces and generic fixed point property on renormings of a (non-separable) reflexive Banach space. We begin the chapter by proving that if X is a Banach space with $R(X) < 2$, then almost all renormings of X (in the sense of porosity) have the w-FPP. The answers to our main problems (1) and (2) and also their extensions are proved in this setting. The chapter ends with the fixed point renormability and genericity on the space $C(K)$.

Chapter 6 handles with the multi-valued fixed point property on non-separable reflexive spaces and the Szlenk index.

Chapter 1

Notations and Preliminaries

In this first chapter, we recall some notations, definitions, background and results that we will need in the following chapters.

1.1 Hyperbolic metric spaces

Let (X, ρ) be a metric space. We say that a mapping $c : \mathbb{R} \rightarrow X$ is a *metric embedding* of \mathbb{R} into X if

$$\rho(c(s), c(t)) = |s - t|$$

for all real numbers s and t . The images of \mathbb{R} and a real interval $[a, b]$ under a metric embedding will be called a *metric line* and a *metric segment*, respectively.

Assume that (X, ρ) contains a family M of metric lines such that for each points $x \neq y$ in X there is a unique metric line in M which passes through x and y . This metric line determines a unique metric segment joining x and y and we denote this segment by $[x, y]$. For each $0 \leq t \leq 1$ there is a unique point z in $[x, y]$ such that

$$\rho(x, z) = t\rho(x, y) \quad \text{and} \quad \rho(z, y) = (1 - t)\rho(x, y).$$

We denote the point z by $(1 - t)x \oplus ty$.

Definition. 1.1. X , or more precisely (X, ρ, M) , is said to be *hyperbolic space* if

$$\rho\left(\frac{1}{2}x \oplus \frac{1}{2}y, \frac{1}{2}x \oplus \frac{1}{2}z\right) \leq \frac{1}{2}\rho(y, z)$$

for all x, y and z in X .

From the definition of a hyperbolic space, it is equivalent to say that

$$\rho\left(\frac{1}{2}x \oplus \frac{1}{2}y, \frac{1}{2}w \oplus \frac{1}{2}z\right) \leq \frac{1}{2}(\rho(x, w) + \rho(y, z))$$

for all x, y, z and w in X . The previous inequality immediately implies that

$$\rho((1 - t)x \oplus tz, (1 - t)y \oplus tz) \leq (1 - t)\rho(x, y) + t\rho(z, w)$$

for all x, y, z and w in X and all $0 \leq t \leq 1$.

It is clear that all normed linear spaces are hyperbolic.

1.2 Fixed points for non-expansive mappings

Fixed point theory has many applications in many branches of mathematics. The most simplest fixed point result is a consequence of the Intermediate Valued Theorem. Indeed, let $f : [a, b] \rightarrow [a, b]$ be a continuous function. Since $f(a) \geq a$ and $f(b) \leq b$, we have $a - f(a) \leq 0 \leq b - f(b)$. By applying the Intermediate Valued Theorem to the continuous function $x - f(x)$, there exists $x \in [a, b]$ such that $x - f(x) = 0$. Another well known fixed point result is the Contraction Mapping Principle which first appeared in S. Banach's 1922 thesis [5].

Definition. 1.2. Let (M, d) be a complete metric space. A mapping $T : M \rightarrow M$ is said to be *contraction* if there exists $k \in [0, 1)$ such that

$$d(Tx, Ty) \leq kd(x, y) \quad \text{for all } x, y \in M.$$

T is called *non-expansive* if $k = 1$.

Theorem 1.3 (Banach Contraction Mapping Theorem). *Let (M, d) be a complete metric space and $T : M \rightarrow M$ be a contraction mapping. Then T has a unique fixed point in M . For any $x_0 \in M$ the sequence of iterates $x_0, T(x_0), T^2(x_0), \dots$ converges to the fixed point of T .*

The Banach contraction mapping principle is a basic tool in functional analysis, nonlinear analysis and differential equation. However, there are some trivial examples which show that the Banach Theorem does not hold for non-expansive mappings.

Example 1.4. Fix $a \in \mathbb{R}$. Let $T_a : \mathbb{R} \rightarrow \mathbb{R}$ be a mapping defined by

$$T_a x = x + a, \quad x \in \mathbb{R}.$$

Then for every $x, y \in \mathbb{R}$, we have $|Tx - Ty| = |x - y|$, hence T is a non-expansive mapping. But T has no fixed point in \mathbb{R} .

The following surprising theorem was proved in 1965 independently by F. Browder [8], D. Göhde [39], and by W. Kirk [46] in a more general form.

Theorem 1.5 (Browder-Göhde Theorem). *If C is a closed, bounded, convex subset of a uniformly convex Banach space X , and if T is a non-expansive mapping from C into C , then T has a fixed point.*

We recall that a Banach space X is called *uniformly convex* if for any two sequences $(x_n), (y_n)$ in X such that

$$\|x_n\| = \|y_n\| = 1 \text{ and } \lim_{n \rightarrow \infty} \|x_n + y_n\| = 2,$$

$\lim_{n \rightarrow \infty} \|x_n - y_n\| = 0$ holds.

The above result was the starting point for applying the geometric theory of Banach spaces to fixed point theory. However, the next example demonstrates the failure of the Browder-Göhde Theorem in general Banach spaces.

Example 1.6. Let $C := B_{c_0}^+ = \{(x_n) \in c_0 : 0 \leq x_n \leq 1 \text{ for all } n\}$. Define mappings $T_1, T_2 : C \rightarrow C$ by, for $(x_n) \in C$

$$T_1(x_n) := (1, x_1, x_2, \dots) \quad \text{and} \quad T_2(x_n) = (1 - x_1, x_1, x_2, \dots).$$

We easily see that $\|T_i(x_n) - T_i(y_n)\| = \|(x_n) - (y_n)\|$, $i = 1, 2$, for any $(x_n), (y_n) \in C$. Thus T_1 and T_2 are non-expansive mappings, indeed metric isometries. But the only possible fixed point for T_1 is $(1, 1, \dots)$ while the only possible fixed point for T_2 is $(\frac{1}{2}, \frac{1}{2}, \dots)$ neither of which is in C .

In this case, the non-existence of fixed point is due to the fact that the set $B_{c_0}^+$ is not weakly compact, because B. Maurey [59] proved that any non-expansive mapping T defined from a weakly compact convex subset C of c_0 into C has a fixed point. For many years, it was an open problem to determine if Browder's Theorem can fail for some weakly compact convex subset of a Banach space. This question was solved by Alspach [1] who proved the failure of Browder's Theorem in a weakly compact convex subset of $L^1([0, 1])$. To simplify the notation, we give the following definition.

Definition. 1.7. Let X be a Banach space and C a closed bounded convex subset of X . We say that the space X enjoys the *fixed point property (FPP)* if every non-expansive mapping $T : C \rightarrow C$ has a fixed point.

Analogously, X is said to have the *weak fixed point property (w-FPP)* if for every nonempty weakly compact convex subset C of X , every non-expansive mapping $T : C \rightarrow C$ has a fixed point.

Note that for reflexive Banach spaces, the FPP and the w-FPP are equivalent.

As we mentioned before, the fixed point result for non-expansive mapping by Kirk is more general than the result by Browder and Göhde. This result was proved by using the concept of normal structure which implies uniform convexity, in fact, he proved that every Banach space with the weak normal

structure does satisfy the w-FPP. Normal structure was firstly introduced in 1984 by M.S. Brodskii and D.P. Milman [7] in order to study the existence of common fixed points of certain sets of isometries.

Definition. 1.8. A convex subset C of a Banach space X is said to have *normal structure* (respectively *weak normal structure*) if for each closed bounded (respectively *weakly compact*) convex subset K of C which contains more than one point, there is some point $x \in K$ which is not a diametral point of K .

Recall that x is a diametral point of K if $\sup \{\|x - y\| : y \in K\} = \text{diam}(K)$.

After the publication of Kirk's celebrated result, many papers appeared concerning different geometric properties of a Banach space which imply normal structure. We are interested in uniform convexity in every direction.

Definition. 1.9. Let X be a Banach space. Given $z \in S_X$ and $\epsilon > 0$, the following convexity modulus can be considered:

$$\delta_X(z; \epsilon) = \inf \left\{ 1 - \left\| \frac{x + y}{2} \right\| : x, y \in B_X, \|x - y\| \geq \epsilon, x - y = tz \right\}.$$

The space X is said to be *uniformly convex in every direction (UCED)* if $\delta_X(z; \epsilon) > 0$ for all $\epsilon > 0$ and every $z \in S_X$.

UCED can be also characterized without using the modulus $\delta_X(z; \epsilon)$. Several equivalent conditions to UCED are given as in the next theorem.

Theorem 1.10.

- (a) A Banach space X is UCED if and only if $\lim_{n \rightarrow \infty} \|x_n - y_n\| = 0$ whenever $(x_n), (y_n)$ are sequences in X such that $\lim_{n \rightarrow \infty} \|x_n\| = 1, \lim_{n \rightarrow \infty} \|y_n\| = 1, \lim_{n \rightarrow \infty} \|x_n + y_n\| = 2$ and there is $z \in S_X$ with $x_n - y_n \in \text{span}(\{z\})$ for each n .

- (b) A Banach space X is UCED if and only if whenever $(x_n), (y_n)$ are sequences in B_X such that $\lim_{n \rightarrow \infty} \|x_n + y_n\| = 2$ and there is a real sequence (λ_n) and $z \in S_X$ such that $x_n - y_n = \lambda_n z$ for each n , then $\lim_{n \rightarrow \infty} \lambda_n = 0$.
- (c) A Banach space X is UCED if and only if whenever $x_n, y_n \in X, n = 1, 2, \dots$ are such that $\lim (2\|x_n\|^2 + 2\|y_n\|^2 - \|x_n + y_n\|^2) = 0$, $\{x_n\}$ is bounded and there is a $z \in X \setminus \{0\}$ and real numbers $\lambda_n, n = 1, 2, \dots$ which satisfy $x_n - y_n = \lambda_n z$ for each n , then $\lim \lambda_n = 0$.

The class of all UCED spaces is quite large. For instance, all Hilbert spaces are UCED due to the approaching example.

Example 1.11. Let X be a Hilbert space. By using the parallelogram law, it is not difficult to figure out

$$\delta_X(z; \epsilon) = 1 - \sqrt{1 - \left(\frac{\epsilon}{2}\right)^2}$$

for every $z \in X$ and every $\epsilon \in [0, 2]$.

By following the remarkable result proved by M.M. Day, R.C. James and S. Swaminathan in [14] and independently by V. Zizler in [95], every separable Banach space can be renormed to be UCED.

Theorem 1.12. *Let X be a separable Banach space. Then there exists an equivalent norm in X which is UCED.*

The proof of this theorem is based on the fact that every separable Banach space is isometric to a subspace of $C([0, 1])$ and there is a bounded one-to-one linear operator from $C([0, 1])$ into ℓ_2 which is known to be UCED.

It is well known by the result in the same paper of Day-James-Swaminathan [14] that UCED spaces have normal structure, so that UCED spaces enjoy the w-FPP.

Theorem 1.13 (Day-James-Swaminathan). *UCED spaces have normal structure.*

Uniform convexity in every direction is not only used as a property for obtaining the w-FPP but it can also be used as a condition to obtain the w-FPP for multi-valued non-expansive mapping which will be considered in the next section.

1.3 Fixed points for non-expansive multi-valued mappings

The study of the fixed point theory for multi-valued contractions and non-expansive maps was initiated by J.T. Markin [56] by using the Hausdorff metric. We first recall some definitions and notations.

Let (X, d) be a metric space and we denote by $CB(X)$ the set of all nonempty closed bounded subsets of X . For $A, B \in CB(X)$, let

$$H(A, B) = \max \left\{ \sup_{a \in A} \text{dist}(a, B), \sup_{b \in B} \text{dist}(b, A) \right\}$$

where $\text{dist}(a, B) = \inf\{d(a, b) : b \in B\}$. We obtain that H is a metric on $CB(X)$ and it is called the *Hausdorff metric*.

Definition. 1.14. Let T be a mapping of a metric space X into $CB(X)$. Then T is called *contraction* if there exists $k \in [0, 1)$ such that

$$H(Tx, Ty) \leq kd(x, y) \quad \text{for all } x, y \in X.$$

T is said to be *non-expansive* if $k = 1$.

Denote by $K(C)$ and $KC(C)$ the family of all nonempty compact and compact convex subsets of C , respectively.

Definition. 1.15. Let X be a Banach space and C a closed bounded convex (weakly compact) subset of X . Then X is said to have the *(weak) multi-valued fixed point property ((w)-MFPP)* if every multi-valued non-expansive mapping $T : C \rightarrow KC(C)$ has a fixed point, i.e., there exists $x \in C$ such that $x \in Tx$.

Some classical fixed point theorems for single-valued mappings have been extended to multi-valued non-expansive mappings. In 1969, S.B Nadler [61] extended the Banach Contraction Principle to multi-valued contractive mappings in complete metric spaces.

Theorem 1.16 (Nadler). *Let E be a nonempty closed subset of a Banach space X and $T : E \rightarrow CB(E)$ a contraction. Then T has a fixed point.*

T.C. Lim [54], by using Edelstein's method of asymptotic centers, has proved the existence of a fixed point for multi-valued mapping when X is a uniformly convex Banach space in 1974. In order to state Lim's result, we recall some definitions and notations relating to asymptotic centers.

Let C be a nonempty weakly compact convex subset of a Banach space X and $\{x_n\}$ a bounded sequence in X .

Definition. 1.17. Denote by $r(C, \{x_n\})$ and $A(C, \{x_n\})$, the *asymptotic radius* and the *asymptotic center* of $\{x_n\}$ in C , respectively, defining by

$$r(C, \{x_n\}) = \inf \left\{ \limsup_{n \rightarrow \infty} \|x_n - x\| : x \in C \right\}$$

and

$$A(C, \{x_n\}) = \left\{ x \in C : \limsup_{n \rightarrow \infty} \|x_n - x\| = r(C, \{x_n\}) \right\}.$$

Definition. 1.18. The sequence $\{x_n\}$ is called *regular with respect to C* if $r(C, \{x_n\}) = r(C, \{x_{n_i}\})$ for all subsequences $\{x_{n_i}\}$ of $\{x_n\}$, and $\{x_n\}$ is called *asymptotically uniform with respect to C* if $A(C, \{x_n\}) = A(C, \{x_{n_i}\})$ for all subsequences $\{x_{n_i}\}$ of $\{x_n\}$.

It is known that $A(C, \{x_n\})$ is a nonempty weakly compact convex set whenever C is.

Theorem 1.19 (Lim). *Let E be a nonempty closed bounded convex subset of a uniformly convex Banach space X and $T : E \rightarrow K(E)$, where $K(E)$ is the family of nonempty compact subsets of E , a non-expansive mapping. Then T has a fixed point.*

This result was later extended by W.A. Kirk and S. Massa [45].

Theorem 1.20 (Kirk and Massa). *Let C be a nonempty weakly compact convex subset of a Banach space X and $T : C \rightarrow KC(C)$ a non-expansive mapping. Suppose that the asymptotic center in C of each bounded sequence of X is nonempty and compact. Then T has a fixed point.*

In particular, this result holds for UCED spaces, because in this case asymptotic centers are singleton.

Theorem 1.21. *Let X be a UCED space. Then, X has the w -MFPP.*

1.4 Weakly compactly generated spaces

The study of weakly compactly generated spaces was initiated by D. Amir and J. Lindenstrauss in [2] to prove the existence of a bounded linear one-to-one map from such a space into $c_0(\Gamma)$.

Definition. 1.22. A Banach space X is said to be weakly compactly generated (WCG) if there exists a weakly compact subset K of X which generates X , that is, X is a closed linear span of K .

Let us show some examples of WCG spaces and also non-WCG spaces.

Example 1.23.

- (1) All separable spaces are WGC spaces. Indeed, if X is a separable space, define $K = \{x_n/n : n \in \mathbb{N}\} \cup \{0\}$ where (x_n) is a dense sequence in a unit ball of X . It is clear that X is the closed linear span of K .
- (2) Every reflexive space is WCG space because it is a closed linear space generated by its unit ball.
- (3) Spaces $L_1(\mu)$, μ is a finite or σ -finite measure, are WCG spaces. By taking $K = \{f : \int |f|^2 \leq 1\}$, i.e., the unit ball of $L_2(\mu)$ can be considered as a subset of $L_1(\mu)$, then K is a weakly compact set which generates $L_1(\mu)$.
- (4) The non-separable spaces ℓ_∞ , $L_\infty[0, 1]$ and $\ell_1(\Gamma)$, Γ is uncountable, are not WCG.

As mention before, the most remarkable result on WCG spaces is concerning an injection into $c_0(\Gamma)$ as follows:

Theorem 1.24 (Amir-Lindenstrauss). *Let X be a weakly compactly generated Banach space. Then there exist a set Γ and a bounded one-to-one linear operator T from X into $c_0(\Gamma)$.*

The injection of WCG spaces into $c_0(\Gamma)$ leads to some renorming results. For instance, since $c_0(\Gamma)$ has a strictly convex equivalent norm, hence every WCG space can be renormed to be strictly convex (see for instance [60]).

1.5 Cardinal numbers and Ordinal numbers

The term cardinality and ordinal were first used by G. Cantor. Cardinality was used as an instrument to compare finite sets while ordinal is the type of well-ordered sets. In order to define both numbers, we need some definitions and notations.

Definition. 1.25. Let \leq be a partial ordering on a set P . This order is said to be a *well ordering* if every non-empty subset A of P has the smallest element, i.e. there exists $a \in A$ such that $a \leq x$ for every $x \in A$.

The following theorem, which is also known as Zermelo's Theorem, is equivalent to the Axiom of Choice.

Theorem 1.26 (Well-ordering Theorem). *Every set can be well ordered, that is, if S is a set then there exist some well-orderings on S .*

In the original definition, an ordinal number is genuinely an equivalence class of well-ordered sets. However, this definition must be abandoned in some systems because these equivalence classes are too large to form a set. Hence to avoid this problem, we can define directly ordinal and cardinal numbers following von Neumann's definition, where an ordinal is the set of all preceding ordinals.

Definition. 1.27. A set S is said to be an *ordinal* if S is strictly well-ordered with respect to set membership and every element of S is also a subset of S .

In this way, the first ordinal, zero, is the empty set 0 . The second ordinal is the set $1 = \{0\}$ consisting of one element. The third ordinal is the set $2 = \{0, 1\} = \{0, \{0\}\}$, and so on. The first infinite ordinal is the set of all finite ordinals, i.e. $\omega = \{0, 1, 2, 3, \dots\}$. The next is $\omega + 1 = \omega \cup \{\omega\} = \{0, 1, 2, 3, \dots, \omega\}$, and so on.

An ordinal number α is called *compact ordinal* if $\alpha = \beta + 1$ for some ordinal number β . Otherwise α is said to be a *limit ordinal*.

Definition. 1.28. Two sets A and B are said to have the same *cardinality* if there exists a one-to-one mapping from A onto B . A cardinal number is an ordinal number which is the first ordinal between all ordinals with the same cardinality. It is equivalent to say that an ordinal number α is cardinal if for every ordinal $\gamma \neq \alpha$ which has the same cardinality as α we have $\alpha \leq \gamma$ (equivalently: $\alpha \in \gamma$ or $\alpha \subset \gamma$).

1.6 Gâteaux and Fréchet differentiability

The Gâteaux derivative and Fréchet derivative are a generalization of the concept of directional derivative and total derivative in differential calculus. They are commonly used to formalize the functional derivative in mathematical analysis, calculus of variations and nonlinear functional analysis.

Definition. 1.29. Let X be a Banach space. The function $f : X \rightarrow \mathbb{R}$ is said to be *Gâteaux differentiable* at a point x_0 if for each $h \in X$ the limit

$$f'(x_0)(h) = \lim_{t \rightarrow 0} \frac{f(x_0 + th) - f(x_0)}{t} \quad (1.1)$$

exists and $f'(x_0) \in X^*$. Then $f'(x_0)$ is called the *Gâteaux derivative* of f at x_0 . In addition, if the limit in (1.1) is uniform in $h \in S_X$, we say that f is *Fréchet differentiable* at x_0 , and then $f'(x_0)$ is called the *Fréchet derivative* of f at x_0 .

If f is Fréchet differentiable, then it is also Gâteaux differentiable and its Fréchet and Gâteaux derivatives agree. The converse is clearly not true.

Lemma 1.30. *A real function f defined on a Banach space X is Fréchet differentiable at x_0 if and only if for each $\epsilon > 0$, there exists $\delta > 0$ such that if $0 < \|h\| < \delta$ then $x_0 + h \in X$ and*

$$|f(x_0 + h) - f(x_0) - f'(x_0)(h)| < \epsilon \|h\|.$$

Example 1.31. Let X be a Banach space and $f : X \rightarrow \mathbb{R}$ be defined by $f(x) := \|x\|^2$ for every $x \in X$. Then f is Fréchet differentiable at $0 = 0_X$, with $f'(0) = 0_{X^*}$. Indeed, given $\epsilon > 0$, take $\delta = \epsilon$. If $h \in X$ with $0 < \|h\| < \delta$, then

$$|f(0 + h) - f(0) - 0_{X^*}(h)| = \|h\|^2 < \epsilon \|h\|.$$

Thus f is Fréchet (and also Gâteaux) differentiable at 0.

Example 1.32. Let $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ be defined as follow

$$f(x, y) := \begin{cases} \frac{x^3 y}{x^4 + y^2} + x + y, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0). \end{cases}$$

For every $h = (h_1, h_2) \in \mathbb{R}^2$, we have

$$f'(0, 0)(h) = \lim_{t \rightarrow 0} \frac{f((0, 0) + th) - f(0, 0)}{t} = h_1 + h_2.$$

Hence, $f'(0, 0)(h)$ exists and the mapping $h \mapsto f'(0, 0)(h)$ is linear and continuous. Thus f is Gâteaux differentiable at $(0, 0)$. However, f is not Fréchet differentiable at $(0, 0)$.

Notice that a norm $\|\cdot\|$ of a space X is Gâteaux differentiable if and only if

$$\lim_{t \rightarrow 0^-} \frac{\|x + ty\| - \|x\|}{t} = \lim_{t \rightarrow 0^+} \frac{\|x + ty\| - \|x\|}{t}$$

for any vectors $x, y \in X, x \neq 0$. There are some other obvious differentiation rules. For instance, if f is continuous and linear, then f is its own Fréchet derivative. If the composition $f \circ g$ is defined and if g and f are Fréchet differentiable at x_0 and at $g(x_0)$, respectively, then $f \circ g$ is Fréchet differentiable at x_0 and the usual chain rule formula holds

$$(f \circ g)'(x_0) = f'(g(x_0)) \circ g'(x_0).$$

If g is Gâteaux differentiable at x_0 and if f is Fréchet differentiable at $g(x_0)$, then $f \circ g$ is Gâteaux differentiable at x_0 and the same formula holds.

Chapter 2

Negligible sets

In mathematics, negligibility concerns the global behavior of a set, of a function, of a class of sets or of functions or of another entity. In this setting we consider negligibility related to the behavior of a set. *Negligible set* is a set which is “small” enough. But, *how to know the size of a set?* In set theory, negligibility refers usually to cardinality. In some other settings, measure, topology, other meanings are also possible. Consider the following examples: Any monotonous real function $f : [a, b] \rightarrow \mathbb{R}$ can only have countably many discontinuities in $[a, b]$, i.e. f is continuous in each point of $[a, b]$, except a countable set; f is also differentiable in each point of $[a, b]$, except a set of Lebesgue measure zero. Any real function defined on $[a, b]$, which is in $[a, b]$ the limit of a sequence of continuous functions is continuous in each point of $[a, b]$, except a set of first Baire category. In the first example, the exceptional set is negligible with respect to cardinality; in the second example it is negligible with respect to Lebesgue measure; and in the third example it is negligible with respect to Baire category (which is of a topological nature).

It turns out that there are several nonequivalent ways to define appropriate notions of negligible sets. These notions are of interest in themselves. In this chapter we present a rather detailed study of some interesting notions

of negligible sets which are interesting for fixed point problems.

2.1 Porosity

In a metric space, a natural class of negligible sets is that of sets of first category. Let us recall here some definitions about Baire category. Let X be a topological space.

Definition. 2.1. A subset E of X is called *nowhere dense* if the interior of its closure \bar{E} is empty, that is, $\text{Int}(\bar{E}) = \emptyset$.

Definition. 2.2. A subset M of X is said to be of the *first Baire category* if M is a countable union of nowhere dense subsets of X .

In the sense of Baire, the set M can be considered negligible. All other sets are said to be of the second Baire category and are considered “large” (or at least non small). However, the notion of first category is not suitable enough for defining negligible sets in some senses as it is apparent on the real line.

Example 2.3 (Decomposition on the real line). Let $(r_i)_{i \in \mathbb{N}} \subset \mathbb{R}$ be an enumeration of all rational numbers on the real line \mathbb{R} . For each pair of natural numbers i and j , let I_{ij} be the open interval of length $l(I_{ij}) = 2^{-(i+j)}$ centered at r_i . Set

$$G_j := \bigcup_{i=1}^{\infty} I_{ij}, \quad j = 1, 2, \dots$$

and

$$A := \bigcap_{j=1}^{\infty} G_j.$$

Then each G_j is open and dense. Thus its complement G'_j is nowhere dense.

Put

$$B := \bigcup_{j=1}^{\infty} G'_j.$$

It follows that $B = A'$ and B is also of the first Baire category. On the other hand, given $\epsilon > 0$, there is a natural number j such that $2^{-j} < \epsilon$. Then $A \subset \bigcup\{I_{ij} : i = 1, 2, \dots\}$ with

$$m(A) = \sum_{i=1}^{\infty} l(I_{ij}) = \sum_{i=1}^{\infty} 2^{-(i+j)} = 2^{-j} < \epsilon$$

where m is the Lebesgue measure. Hence A is of Lebesgue measure zero. Therefore the real line \mathbb{R} can be decomposed into a disjoint union $\mathbb{R} = A \cup B$ of two small sets: A is a null set and B is of the first category.

There is also another interesting example showing that a first Baire category subset of the real line needs not be Lebesgue null.

Example 2.4 (First category sets of positive measure). Consider the traditional Cantor-like construction. Let $0 < t < 1$ and let $B_0 = [0, 1]$. Suppose we have constructed $B_k, 0 \leq k < n$ and each B_k consists of 2^k disjoint uniformly distributed closed intervals of lengths $2^{-k}(1 - 2^{-1}t - \dots - 2^{-k}t) = 2^{-k}(1 - t(1 - 2^{-k})) > t2^{-2k}$. Each of the closed intervals making up B_{n-1} has length $> t2^{2-2n}$ and therefore we can remove from each the middle open interval of length $t2^{1-2n} < t2^{2-2n}$. This construction yields B_n . Now let

$$A(t) = \bigcap_{n=1}^{\infty} B_n.$$

The longest interval in B_n has length $2^{-n}(1 - 2^{-1}t - \dots - 2^{-n}t) < 2^{-n}$. Hence $A(t)$ contains no nonempty intervals and therefore $\text{Int}A(t) = \emptyset$. Since $B_n \downarrow A(t)$ and B_n has Lebesgue measure $2^n \times 2^{-n}(1 - 2^{-1}t - \dots - 2^{-n}t)$ we see that $A(t)$ has measure $1 - t$. Since $A(t)$ is closed we have $A(t)$ is a closed nowhere dense subset of $[0, 1]$ with Lebesgue measure $1 - t$. Now let

$$A_n = A\left(\frac{1}{n}\right)$$

and let

$$A = \bigcup_{n \geq 1} A_n.$$

Then A is a first Baire category subset of $[0,1]$ and A has Lebesgue measure 1. Consider now the complement $C = [0,1] \setminus A$. Since $[0,1]$ is a complete metric space, $[0,1] = A \cup C$ and A is of the first Baire category, we see by Baire's theorem that C cannot be of the first category. Thus C is a dense second Baire category subset of $[0,1]$ and C has Lebesgue measure 0.

A natural way for a strengthening of the notion of the first category is the concept of “porosity”. The concept of porosity in the real line \mathbb{R} was used, under different terminology, already by A. Denjoy in 1920. But the theory of σ -porous sets was started in 1967 by E. Dolženko who applied σ -porous sets in the theory of boundary behavior of functions [19]. The term “porous set” was used the first time also by Dolženko.

After Dolženko introduced the term porosity, the definition of porous set was modified in several different ways. In this present chapter, we give two different types of porosity. Both concepts give porosity of a set at a point and uniform porosity for a set. All notions of porosity presented in here are in the setting of normed linear spaces but they also work for metric spaces.

We start with the notion of porosity which was first considered by L. Zajíček (see in [92] or [93]).

Definition. 2.5. Let $(X, \|\cdot\|)$ be a normed linear space, $M \subset X$ and $a \in X$. We denote a ball center at $x \in X$ with a radius $s > 0$ by $B(x, s)$. Then we say that

- (i) M is said to be *porous at a* if there exists $c > 0$ such that for each $\epsilon > 0$ there exist $b \in X$ and $s > c\|a - b\|$ such that $\|a - b\| < \epsilon$ and $M \cap B(b, s) = \emptyset$.
- (ii) M is *porous* if M is porous at each of its points.

- (iii) If the number c in (i) is fixed for every $x \in M$, then M is called c -porous.
- (iv) M is σ -porous (σ - c -porous) if it is a countable union of porous (c -porous) sets.

The next concept of porosity is stronger than the one in Definition 2.5. To avoid misunderstanding, we call porosity in the following definition, very porosity.

Definition. 2.6. Let $(X, \|\cdot\|)$ be a normed linear space, $M \subset X$ and $a \in X$. We say that

- (i) M is *very porous at a* if there exist $\alpha > 0$ and $r_0 > 0$ such that for each $r \in (0, r_0]$, there exists $b \in X$ for which $B(b, \alpha r) \subset B(a, r) \setminus E$.
- (ii) M is *very porous* if M is porous at each of its points.
- (iii) If the numbers α and r_0 are fixed for every $x \in M$, then M is called *globally very porous*.
- (iv) M is σ -*very porous* (σ -*globally very porous*) if it is a countable union of very porous (globally very porous) sets.

One easily see that every very porous set is a porous set and every globally very porous set is c -porous for some fixed $c > 0$. The following example indicates that there is a porous set which is not very porous.

Example 2.7. Consider the closed unit interval $[0, 1]$, define a set A as a subset of $[0, 1]$ in the following way:

Step 1. Divide the interval $[0, 1]$ into 4 disjoint subintervals, each subinterval has length $\frac{1}{4}$ (see Figure A). Define a set A_1 to be the set of right end points of the 2nd, the 3rd and the 4th subinterval, that is, $A_1 = \{\frac{1}{2}, \frac{3}{4}, 1\}$.

Step 2. Consider the interval $[0, \frac{1}{4}]$, divide it into 8 disjoint subintervals, each subinterval has length $\frac{1}{2^5}$. Let A_2 be the set of right end points of the 2nd, the 3rd, ... , the 7th and the 8th subinterval. Hence $A_2 = \{\frac{1}{2^4}, \frac{3}{2^5}, \dots, \frac{1}{4}\}$.

Step 3. Construct the set A_3 by consider the interval $[0, \frac{1}{2^5}]$, divide it into 16 disjoint subintervals, each subinterval has length $\frac{1}{2^9}$. Let A_3 be a set of right end points of the 2nd, ... , the 15th and the 16th subinterval. Hence $A_3 = \{\frac{1}{2^8}, \frac{3}{2^9}, \dots, \frac{1}{2^5}\}$.

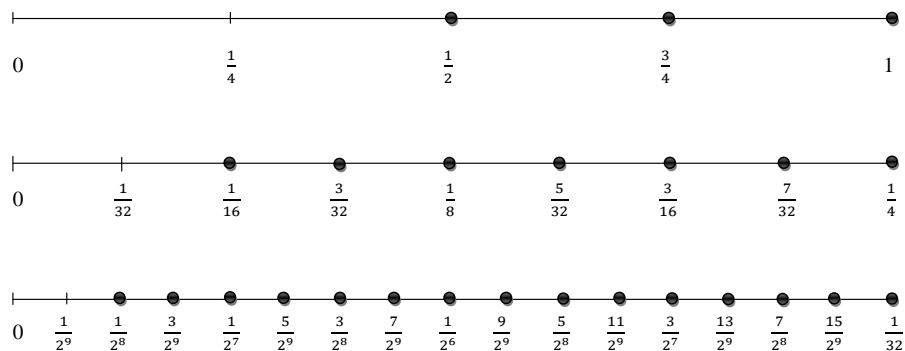


Figure A. Showing Step 1, 2, 3 to construct the sets A_1 , A_2 and A_3

Following the same argument we can construct the set A_n , $n \in \mathbb{N}$ by dividing the interval $[0, \frac{1}{2^{a_n}}]$ where $a_n := \frac{n^2+n-2}{2}$ into 2^{n+1} disjoint subintervals, each subinterval has length $\frac{1}{2^{a(n+1)}}$. Then let $A_n = \{\frac{k}{2^{a(n+1)}} : k = 2, \dots, 2^{n+1}\}$ for each $n \in \mathbb{N}$.

Now define $A := \left(\bigcup_{n=1}^{\infty} A_n \right) \cup \{0\}$.

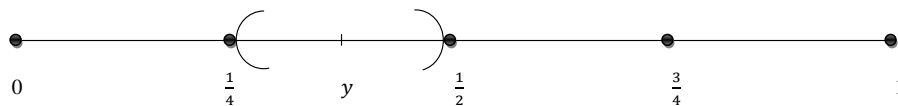


Figure B. An example of how to construct a ball $B(y, s)$ when $\epsilon = 1$

We prove that A is porous at 0. To see this, choose $c = \frac{1}{4}$. For a given $\epsilon > 0$, there exists $n \in \mathbb{N}$ such that $\frac{1}{2^{an}} < \epsilon$. Take b to be a middle point between $\frac{1}{2^{a(n+1)}}$ and $\frac{2}{2^{a(n+1)}}$, that is, $b = \frac{1}{2^{a(n+1)}} + \frac{1}{2 \cdot 2^{a(n+1)}} = \frac{3}{2 \cdot 2^{a(n+1)}}$ and choose $s = \frac{1}{2 \cdot 2^{a(n+1)}}$. Hence

$$|0 - b| = \frac{3}{2 \cdot 2^{a(n+1)}} < \epsilon$$

and

$$s > \frac{1}{4} \cdot \frac{3}{2 \cdot 2^{a(n+1)}} = \frac{1}{4}|0 - b|.$$

It is clear, by the definition of A , that $B(b, s) \cap A = \emptyset$.

However, the set A is not very porous at 0. Indeed, fix $r_0 > 0$ and let $r \in (0, r_0]$. Then we can find $n \in \mathbb{N}$ such that $\frac{1}{2^{an}} < r$. Let I be an interval in $B(0, r) \setminus A$ and denote by $\ell(I)$ the length of I . Then we obtain that $\ell(I) \leq \frac{1}{2^{a(n+1)}}$. Thus the maximum radius of an open ball contained in $B(0, r) \setminus A$ is $\frac{1}{2 \cdot 2^{a(n+1)}}$. Hence we must choose $\alpha \leq \frac{2^{an}}{2 \cdot 2^{a(n+1)}} = \frac{1}{2^{n+2}}$, it follows that the number α depends on n which depends on r .

We go on to the next example which shows that a c -porous set needs not be globally very porous.

Example 2.8. Consider the set $M := \bigcup_{n \in \mathbb{N}} \left\{ n + \frac{k}{n} : k = 0, 1, \dots, n-1 \right\}$.

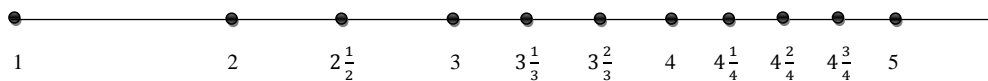


Figure C. The black dots show numbers on the real line which contained in the set M

It is not difficult to see that M is a $\frac{1}{2}$ -porous set. Indeed, fix $\epsilon > 0$ and let $x \in M$. Then there exist $n \in \mathbb{N}$ such that $x = n + \frac{k}{n}$ where $k \in \{0, 1, \dots, n-1\}$. Choose $y = x + \frac{1}{2n}$ and take $s = \frac{1}{2n}$. Then we have $|x - y| = \frac{1}{2n}$ and $s > \frac{1}{2} \cdot \frac{1}{2n} = \frac{1}{2}|x - y|$. Obviously, the ball $B(y, s)$ does not meet any element of the set M .

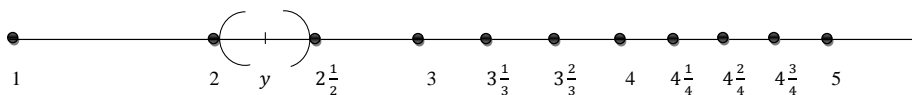


Figure D. Example of how to construct a ball $B(y, s)$ when $x = 2$

To see that the set M is not globally very porous, fix a positive number r . Then for each $x \in M$ can be written as $x = n + \frac{k}{n}$ for some $n \in \mathbb{N}$ and $k \in \{0, 1, \dots, n-1\}$. We also can find a natural number $t \in \mathbb{N}$ such that $\frac{t}{2n} \leq r \leq \frac{t+1}{2n}$. Since the maximum radius of an open ball which can be contained in $B(x, r) \setminus M$ is $\frac{1}{2n}$, we must choose $\alpha \leq \frac{1}{t}$ to obtain $B(y, \alpha r) \subset B(x, r) \setminus M$ for some $y \in (x, x + \frac{1}{n})$. Evidently, the number α is not independent from the element x .

To point out the difference between very porous and nowhere dense sets, note that if a subset E of a metric space (Y, d) is nowhere dense, $y \in Y$

and $r > 0$, then there are a point $z \in Y$ and a number $s > 0$ such that $B_d(z, s) \subset B_d(y, r) \setminus E$. If, however, E is also porous, then for small enough r we can choose $s = \alpha r$, where $\alpha \in (0, 1)$ is a constant which depends only on E .

Some simple examples of globally very porous sets (so porous sets, very porous sets and also c -porous sets) are the following: \mathbb{R} is a globally very porous subset of \mathbb{R}^{p+1} , $p \in \mathbb{N}$ with $r_0 = 1$ and the Cantor-third set is a globally very porous set in \mathbb{R} with $r_0 = \frac{1}{4}$. The next result shows that every σ -porous set is a set of Baire first category and, in a finite-dimensional Euclidean space, σ -porous sets are also of Lebesgue measure zero.

Proposition 2.9.

- i) Every σ -porous (σ -very porous, σ -globally very porous) subset of a metric space is a subset of Baire 1st category.*
- ii) Every porous (very porous, globally very porous) subset of \mathbb{R}^n is a null Lebesgue measurable subset.*

Proof. i) Let E be a σ -porous subset of a metric space (X, d) . Hence E is a countable union of porous sets, say that, $E = \bigcup_{i \in \mathbb{N}} E_i$. According to the definition of porosity, it's clear that every open subset of each E_i contains a nonempty open subset which does not meet E_i . Thus E_i is nowhere dense for all $i \in \mathbb{N}$, which implies E is a set of Baire 1st category.

- ii) We will prove for $n = 1$. A similar proof holds for any dimension. Let m be the Lebesgue measure on \mathbb{R} . Assume that $A \subset \mathbb{R}$ is porous. We recall that the density of A in an ϵ -neighborhood of $x \in \mathbb{R}$ is defined by

$$d_\epsilon(x) = \frac{m(A \cap B_\epsilon(x))}{m(B_\epsilon(x))}$$

where $B_\epsilon(x)$ denotes the ball centered at x with radius ϵ . Let $x \in A$ and $\epsilon > 0$. By the definition of porous set, we obtain a constant $c > 0$, an element $y \in \mathbb{R}$ and a number $s > c\|x - y\|$ such that $y \in B_\epsilon(x)$ and $B_s(y) \cap A = \emptyset$. Denote $\epsilon' = |a - b|$. Then we obtain that

$$\frac{m(A \cap B_{\epsilon'}(x))}{m(B_{\epsilon'}(x))} \leq \frac{m(B_{\epsilon'}(x) \setminus B_{\epsilon'}(y))}{m(B_{\epsilon'}(x))} = \frac{2\epsilon' - c\epsilon'}{2\epsilon'} = 1 - \frac{c}{2}.$$

Hence for each $x \in A$,

$$\liminf_{\epsilon \rightarrow 0} d_\epsilon(x) < 1$$

which implies that the density $d(x) = \lim_{\epsilon \rightarrow 0} d_\epsilon(x) < 1$ when it exists. According to the Lebesgue Density Theorem, for almost every point x of A the density at x exists and is equal to 1. Thus $m(A) = 0$.

□

The existence of a non- σ -porous subset of \mathbb{R}^n which is of the first Baire category and of Lebesgue measure zero was established in [92]. Let \mathcal{E} be the family of all non- σ -porous subsets of \mathbb{R}^n which are of the first Baire category and of Lebesgue measure zero. Suppose that $A \subset \mathbb{R}^n$ is σ -porous, then the set $A \cup P$ belongs to \mathcal{E} for every $P \in \mathcal{E}$. Moreover, if $Q \in \mathcal{E}$ is the countable union of the set $Q_i \subset \mathbb{R}^n, i = 1, 2, \dots$, then there is a natural number j for which the set Q_j is non- σ -porous. Evidently, this set Q_j also belongs to \mathcal{E} . Therefore one can see that the family \mathcal{E} is quite large. Also every complete metric space without isolated points contains a closed nowhere dense set which is not σ -globally very porous [93].

2.2 Gaussian null sets

As we state in Proposition 2.9, every σ -porous subset of a finite-dimensional Banach space is of Lebesgue measure zero. However, in an infinite-dimensional space the situation is different. It turns out that there is no analogous to

Lebesgue measure in this space. In fact, every translation-invariant measure on an infinite-dimensional separable Banach space, which is not identically zero, assigns infinite measure to all open subsets. To see this, suppose that for some $\epsilon > 0$, the open ball of radius ϵ , B_ϵ , has finite measure. Since the space has infinite dimension, we can construct an infinite sequence of disjoint open balls of radius $\frac{\epsilon}{4}$ which are contained in B_ϵ . By the translation invariance, each of these balls has the same measure. Since the sum of their measure is finite, the $\frac{\epsilon}{4}$ -balls must have measure 0. Because the space is separable, so it is second countable and Lindelöf, it can be covered with a countable collection of $\frac{\epsilon}{4}$ -balls. Thus the whole space must have measure 0.

In the absent of a reasonable translation-invariant measure, we cannot simply use the class of null sets of some fixed measures. We can, however, obtain a natural generalization of the class of Lebesgue null sets. In 1978 R.R. Phelps introduced the class of negligible sets which is called “family of Gaussian null sets” [66].

Definition. 2.10. A *non-degenerated Gaussian measure* μ on the real line \mathbb{R} is one having the form

$$\mu(B) = (2\pi b)^{-\frac{1}{2}} \int_B \exp\left(-\frac{(t-a)^2}{2b}\right) dt \quad (2.1)$$

where B is a Borel subset of \mathbb{R} and the constant b is positive. The point a is called the *mean* of μ .

Definition. 2.11. Let X be a separable Banach space. A probability measure λ on the Borel subsets of X is said to be a *non-degenerated Gaussian measure of mean* $x_0 \in X$ if for each $f \in X^*$, $f \neq 0$, the measure $\mu = \lambda \circ f^{-1}$ has the form (2.1), where $a = f(x_0)$.

A Borel subset B of X is called a *Gaussian null set* if $\mu(B) = 0$ for every non-degenerated Gaussian measure μ on E . The family of all Gaussian null sets will be denoted by \mathcal{G} .

Proposition 2.12. *The family \mathcal{G} of Gaussian null sets has the following properties:*

- (i) *The countable union of elements in \mathcal{G} is an element in \mathcal{G} and a Borel subset of an element of \mathcal{G} is in \mathcal{G} .*
- (ii) *For all $B \in \mathcal{G}$ and $x \in X$, the translation $x + B$ is in \mathcal{G} .*
- (iii) *If $S : X \rightarrow X$ is an isomorphism, then $S(B) \in \mathcal{G}$ for every $B \in \mathcal{G}$.*
- (iv) *If $U \subset X$ is open and nonempty, then $U \notin \mathcal{G}$.*
- (v) *If X is finite-dimensional, then a Borel set B is in \mathcal{G} if and only if B has Lebesgue measure zero.*

It is known that a σ -porous subset of a finite-dimensional space is Lebesgue null, so it is Gaussian null. However, a σ -porous subset of an infinite-dimensional space needs not be Gaussian null. Indeed, it was proved in [69] that if X is a Banach space with separable dual then any convex continuous function $f : X \rightarrow \mathbb{R}$ is Fréchet differentiable outside a σ -porous set. On the other hand, it was shown in [57] and [58] that in every infinite dimensional super-reflexive space, in particular in ℓ_2 , there is an equivalent norm which is Fréchet differentiable only on a Gaussian null set. It follows from these two facts that an infinite dimensional separable super-reflexive space X can be decomposed into the union of two Borel sets $A \cup B$ with A Gaussian null and B σ -porous. Such a decomposition was earlier and directly known to hold in every infinite dimensional separable space as the following [68]:

Theorem 2.13 (Preiss and Tišer). *Every infinite dimensional separable Banach space X can be decomposed into two sets A and B such that A is negligible in the Gaussian sense and B is a countable union of closed porous sets.*

2.3 Aronszajn null sets

There is another class of negligible sets which is equivalent to the class of Gaussian null sets, namely the class of Aronszajn null sets. Suggestively by its name, it was introduced by N. Aronszajn in 1976 [3]. We first need some notations and definitions.

Definition. 2.14. Let X be a separable Banach space. Fix $0 \neq v \in X$.

- (i) Define $\mathcal{A}(v)$ as the system of all Borel sets $B \subset X$ such that $B \cap (a + \mathbb{R}v)$ is Lebesgue null on each line $a + \mathbb{R}v$, $a \in X$.
- (ii) If $\{x_n\}$ is a sequence of nonzero elements in X , we denote by $\mathcal{A}(\{x_n\})$ the collection of all Borel sets A which can be decomposed as $A = \bigcup_{n \in \mathbb{N}} A_n$, where $A_n \in \mathcal{A}(x_n)$ for every n .
- (iii) A set A is called *Aronszajn null* if for every given complete sequence $\{x_n\}$ in X , i.e., $\overline{\text{span}\{x_1, x_2, x_3, \dots\}} = X$, the set A belongs to $\mathcal{A}(\{x_n\})$.

Aronszajn has shown that the family of Aronszajn null sets has all the properties listed for the family \mathcal{G} of Gaussian null sets in Proposition 2.12. For their equivalence, it was first observed by Phelps that every Aronszajn null set is Gaussian null [66].

Proposition 2.15 (Phelps). *If a Borel subset E is Aronszajn null, then it is a Gaussian null set.*

The remarkable result that Gaussian null sets are Aronszajn null was later proved by M. Csörnyei [13].

Proposition 2.16 (Csörnyei). *In every separable Banach space, Aronszajn null set and Gaussian null set are coincident.*

Connectively with the next upcoming section, we state one noteworthy theorem of Aronszajn [3] concerning with the existence of Gâteaux derivative and Aronszajn null set. But first we recall the definition of a notion which enters into its statement.

A Banach space Y is said to have the *Radon-Nikodým property (RNP)* if every Lipschitz function $f : \mathbb{R} \rightarrow Y$ is differentiable almost everywhere.

Theorem 2.17 (Aronszajn). *Let X be a Banach space. If Y is a Banach space with Radon-Nykodým property, then every Lipschitz function $f : U \rightarrow Y$, where U is an open subset of X , is Gâteaux differentiable outside an Aronszajn null set.*

2.4 Directional porosity

We describe one more class of negligible sets which is stronger than the class of σ -porous sets, Gaussian null sets and Aronszajn null sets, it is called *the class of directionally porous sets*. This class was introduced by L. Zajíček [91] and was also studied by D. Preiss and Zajíček himself in [70] and [71]. There are also certain different ways to define the notion of directional porosity as well as the notion of porosity. We start with the commonest concept.

Definition. 2.18. Let $(X, \|\cdot\|)$ be a normed linear space, $M \subset X$ and $a \in X$. We denote a ball center at $x \in X$ with a radius $s > 0$ by $B(x, s)$. Then we say that

- (i) M is said to be *directionally porous at a* if there exists $c > 0$ such that for each $\epsilon > 0$ there exist a direction $h \in X$ and a positive real number s such that $s > ct$ and $M \cap B(a + tv, s) = \emptyset$ for some $t \in (0, \epsilon)$.
- (ii) M is *directionally porous* if M is directionally porous at each of its points.

- (iii) If the number c in (i) is fixed for every $a \in M$, then M is called *c-directionally porous*.
- (iv) M is *σ -directionally porous* (*σ -c-directionally porous*) if it is a countable union of directionally porous (*c-directionally porous*) sets.

Clearly every directionally porous set (σ -directionally porous set) is also porous (σ -porous). The additional requirement is that the vector b in 2.5 (i) is restricted to be of the form $b = a + tv$ for a fixed v (depending on a). By following the notion of directional porosity in the above definition, it can be proved that porous sets and directionally porous sets of finite-dimensional Banach spaces are coincident.

Proposition 2.19. *Let X be a finite-dimensional Banach space and $M \subset X$. Then M is porous if and only if M is directionally porous.*

Proof. Assume that M is porous. Hence for every $x \in M$, there exist $c = c(x)$ such that for each $\epsilon > 0$, there are $y \in M$ and a number $s > c\|x - y\|$ such that $\|x - y\| \leq \epsilon$ and $B(y, s) \cap M = \emptyset$.

Let $t = \|x - y\|$, i.e., $t \in (0, \epsilon)$. Since the space X is finite-dimensional, the unit ball B is compact. Thus there exists $\{w_1, w_2, \dots, w_n\} \subset B$ such that $B \subset \bigcup_{i=1}^n B\left(w_i, \frac{ct}{2}\right)$. Then we can find $w_k \in \{w_1, w_2, \dots, w_n\}$ such that $\left\|w_k - \frac{y-x}{\|y-x\|}\right\| \leq \frac{c}{2}$. We obtain that

$$\|(x + tw_k) - y\| = t \left\|w_k - \frac{y-x}{\|y-x\|}\right\| \leq \frac{ct}{2}. \quad (2.2)$$

Take $r^* = \frac{ct}{2}$. Then $r^* > \frac{c}{3}t = \frac{c}{3}\|x - (x + tw_k)\|$. Since the distance between y and $x + tw_k$, by (2.2), is less than $\frac{ct}{2}$ and $\frac{ct}{2} < \frac{s}{2}$, the ball $B(x + tw_k, r^*)$ is contained in the ball $B(y, s)$. Hence $B(x + tw_k, r^*) \cap M = \emptyset$ which implies that M is porous at x in the direction of w_k with the porous constant $\acute{c} = \frac{c}{3}$. It completes the proof. □

However, we give another notion of directional porosity which is stronger than the notion in Definition 2.18 and does not coincide with the notion of porosity even in a finite-dimensional space.

Definition. 2.20. Let $(X, \|\cdot\|)$ be a normed linear space, $M \subset X$ and $a \in X$. Then we say that

- (i) M is said to be *directionally very porous at a* if there exist $\lambda \in (0, 1)$, a number $r_0 > 0$ and a direction $h \in X$ such that for each $r \in (0, r_0]$ there exists $t \in (0, r)$ for which $B(a + th, \lambda r) \subset B(a, r) \setminus M$.
- (ii) M is *directionally very porous* if M is directionally porous at each of its points.
- (iii) If the numbers λ and r_0 , and the direction h in (i) are fixed for every $a \in M$, then M is called *globally directionally very porous*.
- (iv) M is *σ -directionally very porous* (*σ -globally directionally very porous*) if it is a countable union of directionally very porous (globally directionally very porous) sets.

It is clear that directionally very porosity implies very porosity. Even so in finite dimensional spaces, the converse is not true. The following example shows that there is a subset of the real line \mathbb{R} which is very porous at 0 but not directionally very porous at this point.

Example 2.21. Let (A_n) be sequences of subsets of the interval $[0, 1]$ defined as in the Example 2.7. For each $n \in \mathbb{N}$, define a set B_n in the following way: $B_n = A_n$ if n is an odd number and $B_n = -A_n$ if n is an even number. Hence, for instance, $B_1 = \{\frac{1}{2}, \frac{3}{4}, 1\}$, $B_2 = \{-\frac{1}{4}, -\frac{7}{25}, \dots, -\frac{1}{24}\}$ and $A_3 = \{\frac{1}{2^8}, \frac{3}{2^9}, \dots, \frac{1}{2^5}\}$ as shown in Figure E. Let $B = \left(\bigcup_{n \in \mathbb{N}} B_n\right) \cup \{0\}$.

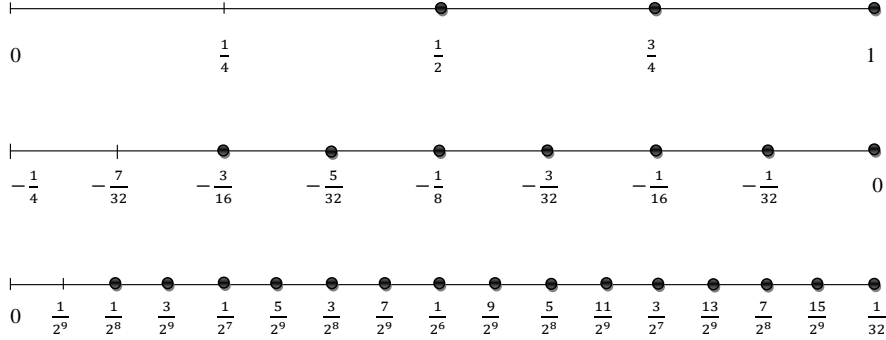


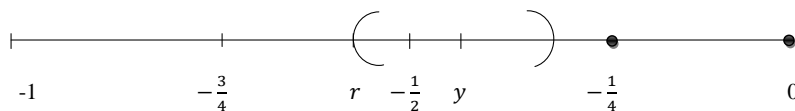
Figure E. Showing Step 1, 2, 3 to construct the sets B_1 , B_2 and B_3

First we show that B is a very porous subset of the interval $[-1, 1]$ at 0. Fix $r_0 = 1$ and $\alpha = \frac{1}{4}$. Then for each $r \in (0, r_0]$ there exists $n \in \mathbb{N}$ such that $\frac{1}{2^{a(n+1)}} < r < \frac{1}{2^{a_n}}$, recall that $a_n = \frac{n^2+n-2}{2}$. Assume that n is an odd number. We consider 2 possible situations.

- (1) If $\frac{i}{2^{a(n+1)}} < r < \frac{i+1}{2^{a_n}}$, $i = 2, 3, \dots, 2^{n+1}$, then by the construction of B , we know that the interval $\left[-\frac{1}{2^{a_n}}, -\frac{1}{2^{a(n+1)}}\right)$ does not contain any element of B . Choose $y = -\frac{3}{4}r$. Since $\frac{r}{2} > \frac{i}{2 \cdot 2^{a(n+1)}} \geq \frac{1}{2^{a(n+1)}}$, $B(y, \alpha r) = B\left(-\frac{3}{4}r, \frac{1}{4}r\right) \subset \left[-\frac{1}{2^{a_n}}, -\frac{1}{2^{a(n+1)}}\right)$, i.e., it does not meet any element of B . Hence $B(y, \frac{1}{4}r) \subset B(0, r) \setminus B$.
- (2) If $\frac{1}{2^{a(n+1)}} < r < \frac{2}{2^{a_n}}$, then consider the positive side of the interval $[-1, 1]$. We obtain that the intersection between the set B and the interval $\left(\frac{1}{2^{a(n+2)}}, \frac{1}{2^{a(n+1)}}\right]$ is empty. Let $y = \frac{3}{4}r$. Then $B(y, \alpha r) = B\left(\frac{3}{4}r, \frac{1}{4}r\right) \subset \left(\frac{1}{2^{a(n+2)}}, \frac{1}{2^{a(n+1)}}\right]$ because $\frac{r}{2} > \frac{1}{2 \cdot 2^{a(n+1)}} \geq \frac{1}{2^{a(n+2)}}$. Thus $B(y, \frac{1}{4}r) \subset B(0, r)$ does not meet the set B .

The following figure shows how to construct a ball $B(y, \alpha r)$ when $n = 1$, that is, $\frac{1}{4} < r < 1$.

1) Assume that $\frac{1}{2} < r < \frac{3}{4}$, then we can choose y and construct an ball $B(y, \frac{1}{4}r)$ as the following:



2) Assume that $\frac{1}{4} < r < \frac{1}{2}$, then consider the positive side of the interval $[-1, 1]$ and choose $B(y, \frac{1}{4}r)$ as follow:

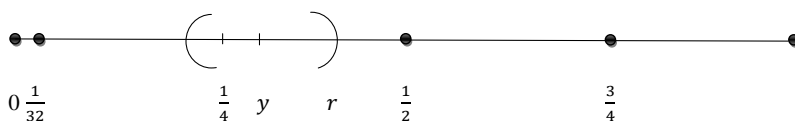


Figure F. Showing how to construct a ball $B(y, \frac{1}{4}r)$ which contained in the ball $B(0, r)$ and does not meet any elements of B when $\frac{1}{4} < r < 1$

From both cases, we can see that there exists $y \in [-1, 1]$ such that $B(y, \alpha r) \subset B(0, r) \setminus B$. If n is an even number, we can apply the same argument. Thus B is very porous at 0.

On the other hand the set B is not directionally very porous at 0. To see this, note that there are only 2 possible directions on the interval $[-1, 1]$: the unit vectors 1 and -1. Consider the direction of the unit vector 1, as it was explained in Example 2.7, the valued of α is depending on r , i.e., we cannot choose an appropriate α which holds for every r . The same problem happens as well in the case of the direction of the unit vector -1.

Let us list some facts about σ -directionally very porous sets.

Proposition 2.22.

- (i) *Every σ -directionally very porous subset of a separable Banach space is Gaussian null.*

(ii) Every Banach space contains a σ -very porous set which is not σ -directionally very porous.

Proof. (i) Let X be a separable Banach space and E a σ -directionally very porous subset of X . Let $d_E(x)$ be defined as the distance of the point $x \in X$ to the set E . Since d attains a minimum at $a \in E$, there are two possibilities: the directional derivative of d_E at $a \in E$ in the direction $h \in S_X$ fails to exist or d_E is Gâteaux differentiable at $a \in E$ if and only if

$$\lim_{t \rightarrow 0^+} \frac{d_E(a + th) - d_E(a)}{t} = 0$$

for any $h \in X$.

If $x \in E$, $h \in X$, $\lambda = \lambda(x) \in (0, 1)$ and $r_0 = r_0(x) > 0$ such that $E \cap B(x + th, \lambda r) = \emptyset$ for every $r \in (0, r_0]$ and $t \in (0, r)$, then we have $d_E(x + th) - d_E(x) = d_E(x + th) \geq \lambda r > \lambda t$. Hence

$$\limsup_{t \rightarrow 0^+} \frac{d_E(x + th) - d_E(x)}{t} \geq \lambda > 0$$

which implies that d_E is not Gâteaux differentiable at $x \in E$. Due to Aronszajn's theorem (Theorem 2.17) E is an Aronszajn null set. So it is Gaussian null.

(ii) It is a consequence of the first fact. □

All properties in the above proposition show that the notion of σ -directional porosity is really a very strong notion of smallness of sets.

Remark 2.23. For σ -directionally porous sets, property (i) (Proposition 2.22) is also true but property (ii) holds only in the case of infinite dimensional Banach spaces.

Chapter 3

Generic fixed point results in a classic sense

Assume that A is a set and \mathcal{P} a property which can be either satisfied or not by the elements of A . The property \mathcal{P} is said to be generic in A if “almost all” elements of A satisfy \mathcal{P} . When speaking about almost all elements we mean all of them except those in a negligible set. As we present in Chapter 2, there are different ways to define the notion of negligible sets. In this dissertation, negligibility refers to Baire category or porosity. When we mention about porosity, from now on we have in mind that porous set means globally very porous set and σ -porous set means σ -globally very porous set (see definition 2.6) which are the strongest notions of porosity.

3.1 Generic fixed point results on the set of non-expansive mappings

As far as we know, the first generic result concerning metric fixed point theory was obtained by G. Vidossich in 1974 [89]. This result shows that

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certain extension of the classical Browder-Göhde-Kirk fixed point theorem, obtained within the framework of spaces with normal structure, still remain valid for almost all mappings in general Banach spaces. From the starting of Vidossich's result, the generic fixed point property on the set of all non-expansive mappings has been widely studied by many mathematicians. We will call the generic fixed point results in this sense, "generic fixed point results in a classic sense". Vidossich's original result was proved on a normed linear space. In this setting we will state it on a hyperbolic complete metric space which is more general than a linear space.

Let (X, ρ) be a hyperbolic complete metric space, C a bounded, closed, and ρ -convex subset of X . Denote by \mathcal{A} the set of all non-expansive self-mappings of C equipped with the metric h defined by, for each $A, B \in \mathcal{A}$,

$$h(A, B) := \sup\{\rho(Ax, Bx) : x \in C\}.$$

It's clear that (\mathcal{A}, h) is complete.

Theorem 3.1 (Vidossich). *Let \mathcal{F}_0 be the subset of all $F \in \mathcal{A}$ which have a unique fixed point. Then \mathcal{F}_0 is a residual subset in \mathcal{A} .*

By the celebrated theorem of Banach, it is well-known that a contraction mapping which maps a complete metric space into itself has a unique fixed point and the successive approximations of any point in the space converge to its fixed point. A generic result of constructive type stating that, for almost all (in the sense of Baire category) non-expansive self-mappings on C the sequence of successive approximations actually do converge to their fixed points, was first proved by F. S. De Blasi and J. Myjak [15]. Later the same authors obtained a stronger result [16] in the sense of porosity as the following:

Theorem 3.2 (De Blasi and Myjak). *There exists a subset $\mathcal{F}_1 \subset \mathcal{A}$ such that the complement $\mathcal{A} \setminus \mathcal{F}_1$ is σ -porous in (\mathcal{A}, h) and for each $A \in \mathcal{F}_1$ the following property holds:*

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There exists a unique $x_A \in C$ for which $Ax_A = x_A$ and $A^n x \rightarrow x_A$ as $n \rightarrow \infty$ uniformly on C .

However, the iterates of non-expansive mappings do not converge in general. In contrast with this fact, The first significant generalization of Banach theorem was obtained by E. Rakotch [74] who replaced Banach's strict contraction by a mapping which satisfies a weaker condition. To avoid misunderstanding, from now on we will call a contraction due to Banach contraction principle, a strict contraction, and a weaker contraction introduced by Rakotch, a contractive mapping. In contrast with the fact that the iterates of non-expansive mappings do not converge in general, the iterates of contractive mappings do converge in all complete metric spaces.

Definition. 3.3. A mapping $A \in \mathcal{A}$ is called *contractive* if there exists a decreasing function $\phi : [0, d(C)] \rightarrow [0, 1]$ such that

$$\phi(t) < 1 \quad \text{for all } t \in (0, d(C))$$

and

$$\rho(Ax, Ay) \leq \phi(\rho(x, y))\rho(x, y) \quad \text{for all } x, y \in C.$$

It is clear that every strict contraction is a contractive mapping.

Theorem 3.4 (Rakotch). *Assume that $A \in \mathcal{A}$ is contractive. Then A has a unique fixed point $x_A \in C$ and the iterates $A^n x \rightarrow x_A$ as $n \rightarrow \infty$ uniformly on C .*

S. Reich and A.J. Zaslavski (in [79]) have improved the result of De Blasi and Myjak by showing that almost all (in the sense of Baire category) non-expansive mappings are, in fact, contractive. Afterward, they shown that the complement of the set of all non-contractive mappings is not only of the first category but also σ -porous [81].

Theorem 3.5 (Reich and Zaslavski). *There exists a set $\mathcal{F}_2 \subset \mathcal{A}$ such that $\mathcal{A} \setminus \mathcal{F}_2$ is σ -porous in (\mathcal{A}, h) and each $A \in \mathcal{F}_2$ is contractive (so that A has a fixed point).*

3.2 Generic fixed point results on the set of non-expansive mappings equipped with another metric

Let $(X, |\cdot|)$ be a Banach space and C a closed convex bounded subset of X . Assume that \mathcal{L} is the set of all Lipschitz mappings $T : C \rightarrow C$, and endowed this set with a metric $\|\cdot\|$ defined by

$$\|T\| = \|T\|_\infty + \|T\|_{Lip}$$

where $\|T\|_{Lip}$ is the Lipschitz constant of T . It is not difficult to verify that $(\mathcal{L}, \|\cdot\|)$ is a Banach space.

Let \mathcal{F} be a subset of \mathcal{L} formed by all non-expansive self-mappings of C . Then $(\mathcal{F}, \|\cdot\|)$ is a closed subspace of $(\mathcal{L}, \|\cdot\|)$, hence a complete metric space. To see this, let T be a limit point of \mathcal{F} , then there exists a sequence $(T_n) \subset \mathcal{F}$ such that $T_n \rightarrow T$. Given $\epsilon > 0$. Then there exists $n_0 \in \mathbb{N}$ for which $\|T - T_n\| < \epsilon$ for all $n \geq n_0$. Hence for every $n \geq n_0$, we obtain that

$$\begin{aligned} \epsilon &> \|T - T_n\| \\ &= \|T - T_n\|_\infty + \|T - T_n\|_{Lip} \\ &\geq \|T - T_n\|_{Lip} \\ &\geq \left| \|T\|_{Lip} - \|T_n\|_{Lip} \right| \end{aligned}$$

which implies $\|T\|_{Lip} = 1$. Thus T belongs to \mathcal{F} , so that \mathcal{F} is closed in \mathcal{L} .

Furthermore, the topology induced by $\|\cdot\|$ is stronger than the one we used in the previous section.

Let \mathcal{C} and $\hat{\mathcal{C}}$ be subsets of \mathcal{L} formed by all contractive mappings and strict contractions, respectively. Recall that both strict contraction and contractive mapping are well-known to have a fixed point.

Theorem 3.6. *$\hat{\mathcal{C}}$ is an open dense subset of \mathcal{L} .*

Proof. Let $T \in \mathcal{L}$ and $\epsilon > 0$. Fix $\theta \in C$. Consider a mapping

$$Sx := \left(1 - \frac{\epsilon}{1 + d(c)}\right)Tx + \frac{\epsilon}{1 + d(c)}\theta, \quad x \in C$$

where $d(C)$ denote the diameter of C . It immediately implies that S is a contraction mapping with a Lipschitz constant $k = 1 - \frac{\epsilon}{1 + d(c)} < 1$, thus $S \in \hat{\mathcal{C}}$.

Then for each $x \in C$, we have

$$|Tx - Sx| = \left| \frac{\epsilon}{1 + d(c)}(Tx - \theta) \right| = \frac{\epsilon}{1 + d(c)}|Tx - \theta| \leq \frac{\epsilon d(C)}{1 + d(c)}$$

which implies

$$\|T - S\|_{\infty} = \sup \{|Tx - Sx| : x \in C\} \leq \frac{\epsilon d(C)}{1 + d(c)}.$$

Moreover, for every $x, y \in C$

$$|(T - S)x - (T - S)y| = \left(\frac{\epsilon}{1 + d(c)} \right) |Tx - Ty| \leq k_1 \left(\frac{\epsilon}{1 + d(c)} \right) |x - y|$$

where k_1 is a Lipschitz constant of T . Thus

$$\begin{aligned} \|T - S\| &= \|T - S\|_{\infty} + \|T - S\|_{Lip} \\ &\leq \frac{\epsilon d(C)}{1 + d(c)} + k_1 \left(\frac{\epsilon}{1 + d(c)} \right) \\ &\leq \frac{\epsilon d(C)}{1 + d(c)} + \frac{\epsilon}{1 + d(c)} \\ &= \epsilon. \end{aligned}$$

Therefore the subset $\hat{\mathcal{C}}$ is dense in \mathcal{L} .

Additionally, we obtain that $\hat{\mathcal{C}}$ is open. Indeed, for every $T \in \hat{\mathcal{C}}$ with a contraction constant $k < 1$, the open ball with radius $r < 1 - k$ around T is a subset of $\hat{\mathcal{C}}$. To see this, assume that $S \in B(T, r)$, then since $\|S - T\|_{Lip} \leq \|S - T\| \leq r$ we have

$$\|S\|_{Lip} \leq \|T\|_{Lip} + r < k + (1 - k) < 1.$$

□

As a consequence of the previous result, $\mathcal{L} \setminus \hat{\mathcal{C}}$ is nowhere dense. Since $\hat{\mathcal{C}} \subset \mathcal{C}$ and every subset of nowhere dense set is also nowhere dense, we obtain the following result on \mathcal{C} .

Corollary 3.7. *$\mathcal{L} \setminus \mathcal{C}$ is a nowhere dense subset of \mathcal{L} .*

Usually a generic result is obtained when it is shown that the set of “good” points in a complete metric space contains a dense G_δ subset. Note that our result is, in fact, stronger because we construct an open everywhere dense subset of “good” points. Moreover we can show that the complement of non-contractions is not only nowhere dense in \mathcal{L} but also σ -porous in \mathcal{L} as follow:

Theorem 3.8. *$\mathcal{L} \setminus \mathcal{C}$ is σ -porous in \mathcal{L} .*

Proof. Let \mathcal{C}_n , $n \in \mathbb{N}$ be a set of all $T \in \mathcal{L}$ satisfying the following property:

P(1) There exists $k \in (0, 1)$ such that $|Tx - Ty| \leq k|x - y|$ for all $x, y \in C$ satisfying $\|x - y\| \geq \frac{d(C)}{2n}$.

For each $n \in \mathbb{N}$, we claim that $\mathcal{L} \setminus \mathcal{C}_n$ is porous in \mathcal{L} with $\alpha = \frac{\min\{d(C), 1\}}{24n(d(C)+1)}$ and $r_0 = 1$. Fix $\theta \in C$. For $T \in \mathcal{L}$ and $r \in (0, 1]$ define a mapping $T_\lambda : C \rightarrow C$ by

$$T_\lambda x = (1 - \lambda)Tx + \lambda\theta$$

3.2 Generic non-expansive mappings with another metric

when $\lambda = \frac{r}{2(d(C)+1)}$. It is easy to check that T_λ is a contraction mapping for $k = 1 - \lambda$. For each $x, y \in C$, we obtain that

$$|(T - T_\lambda)x| = \lambda|Tx - \theta| \leq \lambda d(C)$$

and

$$|(T - T_\lambda)(x - y)| = \lambda|T(x - y)| \leq \lambda|x - y|.$$

Thus

$$\|T - T_\lambda\| = \|T - T_\lambda\|_\infty + \|T - T_\lambda\|_{Lip} \leq \lambda(d(C) + 1).$$

Let $S \in B_{\|\cdot\|}(T_\lambda, \alpha r)$. Then $S \in \mathcal{C}_n$. Indeed, let $x, y \in C$ for which $|x - y| \geq \frac{d(C)}{2n}$. Since

$$\begin{aligned} |Sx - Sy| &\leq |Sx - T_\lambda x| + |T_\lambda x - T_\lambda y| + |T_\lambda y - Sy| \\ &\leq 2\|T_\lambda - S\| + |T_\lambda x - T_\lambda y| \\ &\leq 2\alpha r + (1 - \lambda)|x - y| \end{aligned}$$

we have

$$\begin{aligned} |x - y| - |Sx - Sy| &\geq \lambda|x - y| - 2\alpha r \\ &\geq \lambda \frac{d(C)}{2n} - 2\alpha r \\ &\geq \frac{r}{2} \left(\frac{d(C)}{2n(d(C) + 1)} - \frac{4d(C)}{16n(d(C) + 1)} \right) \\ &= \frac{rd(C)}{8n(d(C) + 1)} \\ &\geq \frac{r}{8n(d(C) + 1)}|x - y|. \end{aligned}$$

This implies $|Sx - Sy| \leq \left(1 - \frac{r}{8n(d(C)+1)}\right)|x - y|$. Thus $B_{\|\cdot\|}(T_\lambda, \alpha r) \subset \mathcal{C}_n$.

Furthermore, for each $S \in B_{\|\cdot\|}(T_\lambda, \alpha r)$, we obtain that

$$\begin{aligned} \|T - S\| &\leq \|T - T_\lambda\| + \|T_\lambda - S\| \\ &\leq \lambda(d(C) + 1) + \alpha r \\ &\leq \frac{r}{2} + \frac{r}{16} \\ &< r \end{aligned}$$

which gives us

$$B_{\|\cdot\|}(T_\lambda, \alpha r) \subset B_{\|\cdot\|}(T, r) \cap \mathcal{C}_n.$$

Therefore the porosity of $\mathcal{L} \setminus \mathcal{C}_n$ is proved.

Put $\mathcal{C}_0 = \bigcap_{n \in \mathbb{N}} \mathcal{C}_n$. Then $\mathcal{L} \setminus \mathcal{C}_0 = \bigcup_{n \in \mathbb{N}} \mathcal{L} \setminus \mathcal{C}_n$ is a σ -porous set in \mathcal{L} .

Furthermore, we claim that any mapping $T \in \mathcal{C}_0$ is contractive. To see this, define $\varphi : [0, d(K)] \rightarrow [0, 1]$ by

$$\varphi(0) := 1$$

and, for $0 < t \leq d(K)$

$$\varphi(t) := \sup \left\{ \frac{|Tx - Ty|}{|x - y|} : |x - y| \geq t \right\}.$$

Clearly, φ is decreasing and $|Tx - Ty| \leq \varphi(|x - y|)|x - y|$ for all $x, y \in K$. Given $0 < t \leq d(K)$, let n be an integer satisfying $\frac{d(K)}{2^n} \leq t$. If $|x - y| \geq t$, then $|x - y| \geq \frac{d(K)}{2^n}$. By the property P(1), there exists $k \in (0, 1)$ such that $|Tx - Ty| \leq k|x - y|$. Thus

$$\varphi(t) = \sup \left\{ \frac{|Tx - Ty|}{|x - y|} : |x - y| \geq t \right\} \leq k < 1$$

which implies that T is contractive.

Since $\mathcal{C}_0 \subset \mathcal{C}$, we have $\mathcal{L} \setminus \mathcal{C} \subset \mathcal{L} \setminus \mathcal{C}_0$ and thus $\mathcal{L} \setminus \mathcal{C}$ is σ -porous in \mathcal{L} .

□

3.3 Generic fixed point results in the set of multi-valued non-expansive mappings

Assume that $(X, \|\cdot\|)$ is a Banach space. Denote by $S(X)$ the set of all nonempty closed convex subsets of X . For the set $S(X)$ we consider the uniformity determined by the following base:

$$\mathcal{G}(n) = \left\{ (A, B) \in S(X) \times S(X) : H(A, B) \leq \frac{1}{n} \right\}$$

where H is the Hausdorff metric on $S(X)$ and $n = 1, 2, \dots$. The set $S(X)$ with this uniformity is metrizable and complete. We endow the set $S(X)$ with the topology induced by this uniformity.

Assume now that K is a nonempty closed convex subset of X and denote by $S(K)$ the set of all $A \in S(X)$ such that $A \subset K$. It is clear that $S(K)$ is a closed subset of $S(X)$. We equip the topological subspace $S(K) \subset S(X)$ with its relative topology.

Let \mathcal{N} be the set of all non-expansive set-valued self-mappings of K which have nonempty bounded closed convex point images. Fix $\theta \in K$. For the set \mathcal{N} we consider the uniformity determined by the following base:

$$\mathcal{E}(n) = \left\{ (T_1, T_2) \in \mathcal{N} \times \mathcal{N} : H(T_1(x), T_2(x)) \leq \frac{1}{n}, \text{ for all } x \in K; \|x - \theta\| \leq n \right\}$$

$n = 1, 2, \dots$. The space \mathcal{N} with this uniformity is also metrizable and complete.

The following result, proved by S. Reich and A.J. Zaslavski [82], shows that a generic non-expansive mapping does have a fixed point.

Theorem 3.9 (Reich and Zaslavski). *Assume that $\text{Int}(K) \neq \emptyset$. Then there exists an open everywhere dense set $\mathcal{M} \subset \mathcal{N}$ with the following property:*

For each $\hat{T} \in \mathcal{M}$ there exists $\bar{x} \in K$ and a neighborhood \mathcal{U} of \hat{T} in \mathcal{N} such that $\bar{x} \in T(\bar{x})$ for each $T \in \mathcal{U}$.

Next we turn back to the concept of contractive mappings. In the following we will assume that K is a bounded closed convex set. Thus we can use the Hausdorff metric instead of a uniformity. The definition of multi-valued contractive mapping is the following:

Definition. 3.10. A mapping $T \in \mathcal{N}$ is called *contractive* if there exists a decreasing function $\phi : [0, d(K)] \rightarrow [0, 1]$ such that

$$\phi(t) < 1 \quad \text{for all } t \in (0, d(K)]$$

and

$$H(Tx, Ty) \leq \phi(\|x - y\|)\|x - y\| \quad \text{for all } x, y \in K.$$

As we know from the case of single-valued mappings, a single-valued contractive mapping does have a fixed point and its successive approximations converge to the fixed point. It turns out with the same result in the case of multi-valued mapping due to the result by H. Kaneko ([42]).

Theorem 3.11 (Kaneko). *Let (X, d) be a complete metric space and $T : X \rightarrow P(X)$. If ϕ is a monotone increasing function such that $0 \leq \phi(t) < 1$ for each $t \in (0, \infty)$ and if $H(Tx, Ty) \leq \phi(d(x, y))d(x, y)$ for each $x, y \in X$, then T has a fixed point.*

In 2002, Reich and Zaslavski [80] have shown that the set of all non-expansive non-contractive mappings from K into $S(K)$ is a Baire set of first category in the complete metric space of all non-expansive mappings from K into $S(K)$. In other words, generically all non-expansive mappings are contractive. In this section, we will prove that the set of all non-expansive non-contractive mappings from K into $S(K)$ is smaller than the set of Baire first category, in fact it is σ -porous.

Theorem 3.12. *There exists a subset \mathcal{M} of \mathcal{N} such that the complement $\mathcal{N} \setminus \mathcal{M}$ is σ -porous and each $T \in \mathcal{M}$ is contractive (so that it has a fixed point).*

Proof. For each $n \in \mathbb{N}$, let \mathcal{N}_n be the set of all $T \in \mathcal{N}$ which have the following property:

P(2) There exists $k \in (0, 1)$ such that $H(Tx, Ty) \leq k\|x - y\|$ for all $x, y \in K$ satisfying $\|x - y\| \geq \frac{d(K)}{2n}$.

Let $n \geq 1$ be an integer. We will show that $\mathcal{N} \setminus \mathcal{N}_n$ is porous in (\mathcal{N}, h) . Take

$$\alpha = \frac{\min\{d(K), 1\}}{(16n(d(K) + 1))}.$$

Fix $\theta \in K$, let $T \in \mathcal{N}$ and $r \in (0, 1]$ and put

$$\gamma = \frac{r}{2(d(K) + 1)}.$$

Consider the mapping $T_\gamma : K \rightarrow S(K)$ defined by

$$T_\gamma x := \{(1 - \gamma)a + \gamma\theta : a \in Tx\}, \quad x \in K.$$

It is clear that $T_\gamma \in \mathcal{N}$. We also obtain that

$$h(T_\gamma, T) \leq \gamma d(K)$$

and for all $x, y \in K$

$$H(T_\gamma x, T_\gamma y) \leq (1 - \gamma)H(Tx, Ty) \leq (1 - \gamma)\|x - y\|. \quad (3.1)$$

Assume that $S \in \mathcal{N}$ satisfies $h(S, T_\gamma) \leq \alpha r$. Then $S \in \mathcal{N}_n$. Indeed, let $x, y \in K$ for which $\|x - y\| \geq \frac{d(K)}{2n}$. It follows from (3.1) that

$$\|x - y\| - H(T_\gamma x, T_\gamma y) \geq \gamma\|x - y\| \geq \frac{\gamma d(K)}{2n}.$$

We also have

$$\begin{aligned} H(Sx, Sy) &\leq H(Sx, T_\gamma x) + H(T_\gamma x, T_\gamma y) + H(T_\gamma y, Sy) \\ &\leq 2\alpha r + H(T_\gamma x, T_\gamma y). \end{aligned}$$

It now implies that

$$\begin{aligned}
 \|x - y\| - H(Sx, Sy) &\geq \|x - y\| - H(T_\gamma x, T_\gamma y) - 2\alpha r \\
 &\geq \frac{\gamma d(K)}{2n} - 2\alpha r \\
 &= \frac{rd(K)}{2n(2(d(K) + 1))} - 4\alpha \frac{r}{2} \\
 &= \frac{r}{2} \left(\frac{d(K)}{2n(d(K) + 1)} - 4\alpha \right) \\
 &= \frac{r}{2} \frac{d(K)}{4n(d(K) + 1)}.
 \end{aligned}$$

Thus $H(Sx, Sy) \leq \left(1 - \frac{r}{8n(d(K)+1)}\right) \|x - y\|$.

Since this inequality holds for all $x, y \in K$ satisfying $\|x - y\| \geq \frac{d(K)}{2n}$, we conclude that $S \in \mathcal{N}_n$. Hence

$$B(T_\gamma, \alpha r) = \{S \in \mathcal{N} : h(S, T_\gamma) \leq \alpha r\} \subset \mathcal{N}_n. \quad (3.2)$$

Moreover, if $S \in B(T_\gamma, \alpha r)$ we have

$$\begin{aligned}
 h(S, T) &\leq h(S, T_\gamma) + h(T_\gamma, T) \\
 &\leq \alpha r + \gamma d(K) \\
 &\leq \frac{r}{16} + \frac{r}{2} \\
 &< r.
 \end{aligned}$$

Thus

$$\{S \in \mathcal{N} : h(S, T_\gamma) \leq \alpha r\} \subset \{B \in \mathcal{N} : h(T, S) \leq r\}. \quad (3.3)$$

From (3.2) and (3.3), we have

$$\{S \in \mathcal{N} : h(S, T_\gamma) \leq \alpha r\} \subset \{S \in \mathcal{N} : h(T, S) \leq r\} \setminus (\mathcal{N} \setminus \mathcal{N}_n)$$

that is, $\mathcal{N} \setminus \mathcal{N}_n$ is porous in (\mathcal{N}, h) .

Let $\mathcal{M} = \bigcap \{\mathcal{N}_n : n \in \mathbb{N}\}$. Then $\mathcal{N} \setminus \mathcal{M} = \bigcup_{n=1}^{\infty} \mathcal{N} \setminus \mathcal{N}_n$ is σ -porous.

3.3 Generic multi-valued non-expansive mappings

By follow the same argument used in Theorem 3.8, we shall have each mapping $T \in \mathcal{M}$ is contractive. This completes the proof. \square

Chapter 4

Generic fixed point property in separable reflexive spaces

In this chapter, our framework of the generic fixed point existence is varied from the set of all non-expansive mappings to the set of all renormings of a Banach space. Note that the FPP neither the w-FPP is not preserved under isomorphisms. Indeed, P.K. Lin [55] has proved that ℓ_1 can be renormed to have the FPP (recall that this space does not satisfy the FPP for the usual norm). On the other hand, the space $L_1([0, 1])$ does not satisfy the w-FPP as proved by D.E. Alspach [1]. However this space (and any separable Banach space) can be renormed to have normal structure [88] and so the w-FPP [46].

It is known that there are some nonreflexive Banach spaces which cannot be renormed to have the FPP or the w-FPP. For instance, as proved by P. Dowling, C. Lennard and B. Turett [28], every renorming of $c_0(\Gamma)$ when Γ is uncountable contains an asymptotically isometric copy of c_0 and so it fails to have the FPP. Analogously, any renorming of $\ell_1(\Gamma)$ (Γ uncountable) contains an asymptotically isometric copy of ℓ_1 so it also fails to satisfy the FPP. In the case of the w-FPP, J. Partington [63], [64] has proved that every renorming of $\ell_\infty(\Gamma)$ for Γ uncountable and any renorming of ℓ_∞/c_0 contain

an isometric copy of ℓ_∞ and so they fail the w-FPP (again due to Alspach's example). Hence we restrict our framework to study the generic existence of fixed points on renormings of a reflexive Banach space. Recall that in reflexive spaces, the FPP and the w-FPP are equivalent.

We divide our work into two parts: generic fixed point property in separable reflexive spaces and generic fixed point property in nonseparable reflexive spaces. Methods using for the separable case and nonseparable case are quite different. In this chapter, we present the generic fixed points existence on renormings of a separable reflexive Banach space.

4.1 Generic fixed point results on the set of all equivalent norms on a Banach space

If a reflexive Banach space X is separable, it is well known that X can be renormed to have the w-FPP. Indeed, it was independently studied in the paper by V. Zizler [95] and another paper by M.M Day, R.C. James and S. Swaminathan [14] that every separable Banach space admits an equivalent uniformly convex in every direction (UCED) norm (See Chapter 1, Definition 1.2.8 and Theorem 1.2.11). Recall that UCED is a geometrical property which implies normal structure (Chapter 1, Theorem 1.2.12) and so the FPP for reflexive spaces (or the FPP for Banach spaces). Concerning fixed point existence renorming and genericity, it comes out that almost all renormings (in the sense of Baire category) of a separable reflexive Banach space satisfy the FPP. Indeed, let \mathcal{P} denote the space of all equivalent norms on a given Banach space $(X, \|\cdot\|)$. Define a metric ρ on \mathcal{P} by: for each $p, q \in \mathcal{P}$

$$\rho(p, q) = \sup \{|p(x) - q(x)| : x \in B_X\}$$

where $B_X = \{x \in X : \|x\| \leq 1\}$. Then (\mathcal{P}, ρ) is an open subset of the space (\mathcal{Q}, ρ) of all continuous semi-norms on $(X, \|\cdot\|)$ endowed with the

metric ρ defined as above. Since (\mathcal{Q}, ρ) is a complete metric space, by Baire Theorem, it is a Baire space. Hence (\mathcal{P}, ρ) is also a Baire space due to the fact that every open subset of a Baire space is also a Baire space. Following this approach, M. Fabian, L. Zajíček and V. Zizler [30] proved the following result:

Theorem 4.1 (Fabian-Zajíček-Zizler). *Let (X, r) be a uniformly convex in every direction Banach space and \mathcal{P} be as above. Then, there exists a residual subset \mathcal{R} (in fact a dense- G_δ) of \mathcal{P} , such that for all $p \in \mathcal{R}$, the space (X, p) is uniformly convex in every direction.*

Proof. For $p \in \mathcal{P}$ and $j \in \mathbb{N}$, let

$$\mathcal{R}(p, j) = \left\{ q \in (\mathcal{P}, \rho) : \sup \left\{ \left| p^2(x) + \frac{1}{j} r^2(x) - q^2(x) \right| : x \in B_X(r) \right\} < \frac{1}{j^2} \right\}$$

For $k \in \mathbb{N}$, define

$$\mathcal{R}_k = \bigcup \{ \mathcal{R}(p, j) : p \in \mathcal{P}, j \geq k \}$$

and set

$$\mathcal{R} = \bigcap_{k=1}^{\infty} \mathcal{R}_k.$$

It is clear that \mathcal{R}_k is open in (\mathcal{P}, ρ) . We show that \mathcal{R} is a dense G_δ in (\mathcal{P}, ρ) . Let $k \in \mathbb{N}$ and $p \in \mathcal{P}$ be given. Observe that for each $j \geq k$,

$$\left(p^2 + \frac{1}{j} r^2 \right)^{\frac{1}{2}} \in \mathcal{R}(p, j) \subset \mathcal{R}_k.$$

Since $\rho\left(\left(p^2 + \frac{1}{j} r^2\right)^{\frac{1}{2}}, p\right) \rightarrow 0$ as $j \rightarrow \infty$, we have $p \in \overline{\mathcal{R}_k}$. It follows that \mathcal{R}_k is dense in (\mathcal{P}, ρ) and also \mathcal{R} is a G_δ set in (\mathcal{P}, ρ) . The density of \mathcal{R} in \mathcal{P} follows from the fact that (\mathcal{P}, ρ) is a Baire space.

It remains to show that for each $p \in \mathcal{R}$, (X, p) is UCED. Let $p_0 \in \mathcal{R}$. Let $(x_n), (y_n) \in X$ be sequences such that

$$\lim(2p_0(x_n)^2 + 2p_0(y_n)^2 - p_0(x_n + y_n)^2) = 0, \quad (4.1)$$

(x_n) is bounded and there is a $z \in X \setminus \{0\}$ and real numbers $\lambda_n, n = 1, 2, \dots$ which satisfy $x_n - y_n = \lambda_n z$ for each n . We need to show that $\lim \lambda_n = 0$.

Since (4.1) implies the boundedness of (y_n) , and from the assumption (x_n) is bounded, we can assume without loss of generality that $\lim_n r(x_n) \leq \frac{1}{2}$ and $\lim_n r(y_n) \leq \frac{1}{2}$ for each n .

Since $p_0 \in \mathcal{R} = \bigcap_{k=1}^{\infty} \mathcal{R}_k$, for every k , there exist $j_k \geq k$ and $p_k \in \mathcal{P}$ such that $p_0 \in \mathcal{R}(p_k, j_k)$, i.e.,

$$\sup \left\{ \left| p_k^2(x) + \frac{1}{j_k} r^2(x) - p_0^2(x) \right| : x \in B_X(r) \right\} < \frac{1}{j_k^2}. \quad (4.2)$$

According to the convexity of p_k^2 and (4.2), for each $k, n \in \mathbb{N}$, we obtain

$$\begin{aligned} & \frac{1}{j_k} (2r^2(x_n) + 2r^2(y_n) - r^2(x_n + y_n)) \\ & \leq 2p_k^2(x_n) + 2p_k^2(y_n) - p_k^2(x_n + y_n) + \frac{1}{j_k} (2r^2(x_n) + 2r^2(y_n) - r^2(x_n + y_n)) \\ & = 2(p_k^2 + \frac{1}{j_k} r^2)(x_n) + 2(p_k^2 + \frac{1}{j_k} r^2)(y_n) - (p_k^2 + \frac{1}{j_k} r^2)(x_n + y_n) \\ & \leq \frac{5}{j_k^2} + 2p_0^2(x_n) + 2p_0^2(y_n) - p_0^2(x_n + y_n). \end{aligned}$$

Then for each $k \in \mathbb{N}$,

$$\begin{aligned} & \frac{1}{j_k} \limsup_n (2r^2(x_n) + 2r^2(y_n) - r^2(x_n + y_n)) \\ & \leq \frac{5}{j_k^2} + \limsup_n (2p_0^2(x_n) + 2p_0^2(y_n) - p_0^2(x_n + y_n)). \end{aligned}$$

By using (4.1), we have for each $k \in \mathbb{N}$

$$\limsup_n (2r^2(x_n) + 2r^2(y_n) - r^2(x_n + y_n)) \leq \frac{5}{j_k}.$$

Thus

$$\lim (2r^2(x_n) + 2r^2(y_n) - r^2(x_n + y_n)) = 0.$$

From the uniform convexity in every direction of r , it follows that $\lim \lambda_n = 0$. \square

By applying the fact that every separable reflexive Banach space is UCED renormable, we obtain the following result:

Corollary 4.2. *Let X be a separable reflexive Banach space and \mathcal{P} be as above. Then, there exists a residual subset \mathcal{R} (in fact a dense- G_δ) of \mathcal{P} , such that for all $p \in \mathcal{R}$, the space (X, p) has the w-FPP.*

In particular, this result can be applied for some classical Banach spaces for instance $L^1([0, 1])$, $C([0, 1])$ which do not satisfy the w-FPP. In fact, we can obtain a stronger result as a consequence of Fabian-Zajícěk-Zizler's result. We recall that a Banach space $(X, \|\cdot\|)$ is *strictly convex* if $\|\frac{x+y}{2}\| < 1$ whenever $x, y \in S_X, x \neq y$. It is equivalent to say that X is strictly convex if and only if the unit sphere S_X does not contain a nontrivial segment. It is not difficult to see that a UCED space is strictly convex. In 1965, F.E. Browder [9] published his well known fixed point result as follow:

Theorem 4.3 (Browder). *Let C be a nonempty, closed, convex subset of a strictly convex Banach space X and let $T : C \rightarrow C$ be a non-expansive map. Then the set $F(T)$ of fixed points of T is closed and convex.*

Consequently, we obtain the following generic fixed point result on the set \mathcal{P} .

Corollary 4.4. *Let X be a separable reflexive Banach space, \mathcal{P} as in Theorem 4.1. Then, there exists a residual subset \mathcal{F} of \mathcal{P} , such that for all $p \in \mathcal{F}$, and every non-expansive mapping T defined from a convex weakly compact subset C of (X, p) into itself, the set of fixed point of T is non-empty and convex.*

The problem of determining “how large” the first category subset of $\mathcal{P} \setminus \mathcal{R}$ formed by the norms which do not have the FPP or the w-FPP is, seems to be very difficult, because, for instance, it is unknown if ℓ_2 can be renormed

in such a way that the new norm does not have the FPP, or, more generally, it is unknown if there exists a reflexive space which does not satisfy the FPP. It can be interesting to know if the Fabian-Zajícěk-Zizler result can be reformulated in the sense of porosity.

Theorem 4.5. *Let X be a separable reflexive Banach space. Assume that \mathcal{P} is defined as above with the metric ρ . Then, there exists a σ -porous subset \mathcal{R} of \mathcal{P} such that for every norm $p \in \mathcal{P} \setminus \mathcal{R}$, (X, p) is UCED (and so, it has the w -FPP).*

Proof. Since X is separable, there exists a norm r on X such that (X, r) is UCED. For any $p \in \mathcal{P}$, denote $m(p) = \inf_{r(x)=1} p(x)$ and let $p_j = \sqrt{p^2 + (r^2/j)}$. It is easy to check that $d(p, p_j) \leq \frac{1}{jm(p)}$. Indeed, for every $x \in S_X$ we have

$$|p_j(x) - p(x)| = \frac{|p_j^2(x) - p^2(x)|}{p_j(x) + p(x)} \leq \frac{r^2(x)}{jp(x)} \leq \frac{1}{jm(p)}.$$

Denote

$$A_n = \left\{ p \in \mathcal{P} : \frac{1}{n} \leq m(p) \right\}$$

and

$$G_k = \bigcup_{p \in \mathcal{P}, j \geq k} B\left(p_j, \frac{1}{kj}\right).$$

We claim that $A_n \setminus G_k$ is porous for $r_0 = \frac{1}{k}$ and $\alpha = \frac{1}{4kn}$. Indeed, let $s < \frac{1}{k}$, then $\frac{2n}{s} \geq \frac{1}{s} > k \geq 1$. This implies that there exists an integer $j \geq k$ such that $j \in \left(\frac{2n}{s}, \frac{4n}{s}\right)$. Thus $\frac{s}{4n} \leq \frac{1}{j} < \frac{s}{2n}$. Assume $p \in A_n \setminus G_k$. If $q \in B(p_j, \frac{s}{4kn})$, we have

$$\begin{aligned} \rho(p, q) &\leq \rho(p, p_j) + \rho(p_j, q) \\ &\leq \frac{1}{jm(p)} + \frac{s}{4kn} \\ &\leq \frac{n}{j} + \frac{s}{4kn} \\ &\leq \frac{s}{2} + \frac{s}{2} \\ &= s. \end{aligned}$$

Thus $B(p_j, \frac{s}{4kn}) \subset B(p, s)$. Furthermore, we have that $B(p_j, \frac{1}{kj})$ lies in G_k . Since $\frac{s}{4kn} < \frac{1}{kj}$, the ball $B(p_j, \frac{s}{4kn})$ does not meet $A_n \setminus G_k$.

Hence,

$$\mathcal{R} = \bigcup_{n,k=1}^{\infty} A_n \setminus G_k$$

is a σ -porous set. We claim that p is UCED if $p \in \mathcal{P} \setminus \mathcal{R}$. Indeed, note that

$$\mathcal{R} = \bigcup_{n,k=1}^{\infty} (A_n \setminus G_k) = \bigcup_{k=1}^{\infty} (\mathcal{P} \setminus G_k) = \mathcal{P} \setminus \bigcap_{k=1}^{\infty} G_k$$

which implies that $\mathcal{P} \setminus \mathcal{R} = \bigcap_{k=1}^{\infty} G_k$. Assume that $q \in \bigcap_{k=1}^{\infty} G_k$. Since, for every k , q belongs to G_k , there exist $p = p(k) \in \mathcal{P}$ and $j \geq k$ such that q belongs to $B(p_j, \frac{1}{kj})$. Note that for every $x \in X$ such that $r(x) \leq 1$ we have

$$|q^2(x) - p_j^2(x)| = |q(x) - p_j(x)||q(x) + p_j(x)| \leq \frac{2}{kj}(M(q) + 1).$$

Let $(x_n), (y_n)$ be sequences in X such that $r(x_n) \leq \frac{1}{2}$, $r(y_n) \leq \frac{1}{2}$ and $x_n - y_n = \lambda_n z$ for some $z \in X$ and

$$\liminf_n 2q^2(x_n) + 2q^2(y_n) - q^2(x_n + y_n) = 0.$$

Thus

$$\liminf_n 2p_j^2(x_n) + 2p_j^2(y_n) - p_j^2(x_n + y_n) \leq \frac{10}{kj}(M(q) + 1).$$

Hence

$$\liminf_n 2r^2(x_n) + 2r^2(y_n) - r^2(x_n + y_n) \leq \frac{10}{k}(M(q) + 1).$$

Since k is arbitrary and r is UCED we obtain that $\lambda_n \rightarrow 0$.

□

4.2 Equivalent metrics on the set of all equivalent norms of a Banach space

We can define in the natural way another metric on \mathcal{P} . Denote by $CB(X)$ the set of all nonempty closed bounded subsets of X . Define a metric h on \mathcal{P} in the following way

$$h(p, q) := H(B_p, B_q)$$

where H is the Hausdorff metric, B_p and B_q are the unit balls in (X, p) and (X, q) , respectively. It is not difficult to see that (\mathcal{P}, h) is a complete metric space. We obtain also the equivalence between ρ and h .

Proposition 4.6. *The metric spaces (\mathcal{P}, ρ) and (\mathcal{P}, h) are equivalent.*

Proof. Let $p, q \in \mathcal{P}$ and ϵ a given positive real number. If $h(p, q) \leq \epsilon$, then $B_p \subset B(B_q, \epsilon) = \{y \in X : \text{there exist } x \in B_q \text{ such that } |x - y| \leq \epsilon\}$ and $B_q \subset B(B_p, \epsilon)$. Since p and q are equivalent to $|\cdot|$, there exist $a_i, b_i \in \mathbb{R}$ ($i = 1, 2$) such that for every $x \in X$

$$a_1|x| \leq p(x) \leq b_1|x| \quad \text{and} \quad a_2|x| \leq q(x) \leq b_2|x|.$$

Let $x \in X$. If $x \in B_p$, then $x \in B(B_q, \epsilon)$ and there exists $y \in B_q$ such that $|x - y| \leq \epsilon$. Thus $q(x) - q(y) \leq q(x - y) \leq \epsilon b_2$. Since $\frac{x}{p(x)} \in B_p$, we have

$$q(x) \leq (1 + \epsilon b_2)p(x). \tag{4.3}$$

In the same way, we obtain that

$$p(x) \leq (1 + \epsilon b_1)q(x). \tag{4.4}$$

From (4.3) and (4.4) we have, for $x \in S_X$

$$p(x) - q(x) \leq \epsilon b_1 q(x) \leq \epsilon b_1 b_2$$

and

$$p(x) - q(x) \geq -\epsilon b_2 p(x) \geq -\epsilon b_1 b_2.$$

Hence $|p(x) - q(x)| \leq \epsilon b_1 b_2$ which implies that $\rho(p, q) \leq \epsilon b_1 b_2$.

In the other hand, assume that $\rho(p, q) \leq \epsilon$. Thus for every $x \in B_X$ we have $|p(x) - q(x)| \leq \epsilon$. Let $x \in B_p$. Then we have

$$q(x) \leq p(x) + \epsilon|x| \leq 1 + \epsilon|x|$$

which implies $\frac{x}{1+\epsilon|x|} \in B_q$ and

$$\left| x - \frac{x}{1+\epsilon|x|} \right| = |x| \left| 1 - \frac{1}{1+\epsilon|x|} \right| \leq \frac{1}{a_1} \cdot \frac{\epsilon|x|}{1+\epsilon|x|} \leq \frac{\epsilon}{a_1^2}.$$

Using the same argument, for each $x \in B_q$, we obtain that

$$\left| x - \frac{x}{1+\epsilon|x|} \right| \leq \frac{\epsilon}{a_2^2}$$

where $\frac{x}{1+\epsilon|x|} \in B_p$. Denote $d = \min \{a_1^2, a_2^2\}$. It follows that $h(p, q) \leq \frac{\epsilon}{d}$.

□

Remark 4.7. Even though the metrics ρ and h are equivalent, they are not uniformly equivalent. Consider the following easy example: Assume that X is \mathbb{R}^2 with the maximum norm. Define

$$p_n((x_1, x_2)) = \max \left\{ |x_1|, \frac{1}{n}|x_2| \right\}.$$

Then $\{p_n\}$ is a Cauchy sequence in (\mathcal{E}, ρ) , but $h(p_n, p_m) = |n - m|$ which implies that $\{p_n\}$ is not h -Cauchy.

In the general theory of Banach space, it is usual to identify isometric spaces. However, when we consider the metric introduced in [30], we have $\rho(p, q) = 2$ if p is the original norm and $q = 2p$. To avoid this, we can restrict to “normalized” norms, i.e. norms which satisfy $\sup_{x \in B_X} p(x) = 1$. Denote by \mathcal{E}

the set formed by all equivalent normalized norms endowed with the metric ρ . Again \mathcal{E} is an open subset of the complete metric space formed by all continuous normalized semi-norms defined on $(X, \|\cdot\|)$, and so \mathcal{E} is a Baire space.

On the other hand, following the idea of the Banach-Mazur distance, we can also define a metric in \mathcal{E} by

$$d(p, q) = \log \frac{b_{p,q}}{a_{p,q}} = \log \|i\| \|i^{-1}\|$$

where

$$a_{p,q} = \inf \left\{ \frac{p(x)}{q(x)} : x \in S_X \right\}; \quad b_{p,q} = \sup \left\{ \frac{p(x)}{q(x)} : x \in S_X \right\}$$

and i is the identity mapping from (X, p) into (X, q) .

Assume that $\epsilon < 1$. Then $e^\epsilon < 1 + 3\epsilon$. Hence, if $d(p, q) < \epsilon$, we have $\frac{b_{p,q}}{a_{p,q}} < 1 + 3\epsilon$. Furthermore, the normalization condition implies that $b_{p,q} \geq 1 \geq a_{p,q}$, so that we obtain $b_{p,q} < 1 + 3\epsilon$ and $a_{p,q} > 1 - 3\epsilon$. Thus for every $x \in B_X$ we have

$$(1 - 3\epsilon)q(x) \leq p(x) \leq (1 + 3\epsilon)q(x) \tag{4.5}$$

which implies

$$|p(x) - q(x)| \leq 3\epsilon q(x) \leq 3\epsilon.$$

Proposition 4.8. (\mathcal{E}, d) is a complete metric space.

Proof. It is easy to see that d is a metric on \mathcal{E} . Furthermore (\mathcal{E}, d) is complete. Indeed, let $\{p_n\}$ be a Cauchy sequence in \mathcal{E} and let $\epsilon > 0$. Then there exists $n_0 \in \mathbb{N}$ such that $d(p_m, p_n) < \epsilon$ for all $m, n \geq n_0$. According to (4.5), for every $x \in S_X$ we have

$$|p_m(x) - p_n(x)| \leq 3\epsilon$$

for every $m, n \geq n_0$. Hence $\{p_n\}$ is a uniform Cauchy sequence on S_X . If $x_0 \in X$, we have $p_n(x_0) = \|x_0\| p_n \left(\frac{x_0}{\|x_0\|} \right) \rightarrow \|x_0\| \lim_n p_n \left(\frac{x_0}{\|x_0\|} \right)$. Thus

$\{p_n(x)\}$ is convergent for all $x \in X$. Define $p(x) = \lim_n p_n(x)$. We will show that p belongs to \mathcal{E} . For any $x, y \in X$ and a scalar α , we obtain that

$$p(x + y) = \lim_n p_n(x + y) \leq \lim_n p_n(x) + \lim_n p_n(y) = p(x) + p(y)$$

and

$$p(\alpha x) = \lim_n p_n(\alpha x) = |\alpha| \lim_n p_n(x) = |\alpha| p(x)$$

which implies p is a semi-norm.

It remains to show that p is equivalent to $\|\cdot\|$. According to the normalized condition of p_n , for each $n \in \mathbb{N}$, $p_n(x) \leq \|x\|$ for all $x \in X$. This implies $p(x) \leq \|x\|$ for every $x \in X$. Furthermore, $\sup_{x \in S_X} p(x) = 1$ because $p_n(x) \rightarrow p(x)$ uniformly on S_X and $\sup_{x \in S_X} p_n(x) = 1$ for every n .

On the other hand, denote

$$a_n = a_{p_n, \|\cdot\|} = \inf_{x \in S_X} \frac{p_n(x)}{\|x\|}.$$

Then

$$d(p_n, \|\cdot\|) = \log \frac{b_{p_n, \|\cdot\|}}{a_{p_n, \|\cdot\|}} = \log \frac{1}{a_n}.$$

Since (p_n) is a Cauchy sequence, it is bounded. So that there exists a positive number M such that

$$d(p_n, \|\cdot\|) = \log \frac{1}{a_n} \leq M, \quad \text{for every } n \in \mathbb{N}.$$

Due to the property of the logarithm function, $\frac{1}{a_n} \leq K$ for every $n \in \mathbb{N}$, where K is a positive real number. Thus $a_n \geq \frac{1}{K} > 0$ for every $n \in \mathbb{N}$, and we obtain that $p(x) \geq \frac{1}{K} \|x\|$ for every $x \in X$. Hence p is an equivalent normalized norm and so $p \in \mathcal{E}$. Moreover, the uniform convergence of $p_n(x)$ to $p(x)$ on S_X easily implies that $p_n \rightarrow p$ on \mathcal{E} . \square

To point out the relation between the metrics ρ , h and d , we will check that these metrics are equivalent in the next proposition.

Proposition 4.9. *The metric spaces (\mathcal{E}, ρ) , (\mathcal{E}, h) and (\mathcal{E}, d) are equivalent.*

Proof. The equivalence of ρ and h was proved in proposition 4.6. It remains to show that either ρ or h is equivalent to the metric d . We will prove here that ρ and d are equivalent.

Let $p, q \in \mathcal{E}$. Assume that $d(p, q) < d < 1$, then we obtain that

$$\begin{aligned}
 \varrho(p, q) &= \sup_{x \in B_X} |p(x) - q(x)| \\
 &\leq \sup_{x \in B_X} q(x) \left| \frac{p(x)}{q(x)} - 1 \right| \\
 &\leq \sup_{x \in B_X} \left| \frac{p(x)}{q(x)} - 1 \right| \\
 &\leq \max\{b_{p,q} - 1, 1 - a_{p,q}\} \\
 &\leq b_{p,q} - a_{p,q} \\
 &= a_{p,q} \left(\frac{b_{p,q}}{a_{p,q}} - 1 \right) \\
 &\leq \frac{b_{p,q}}{a_{p,q}} - 1 \\
 &\leq e^d - 1 \\
 &\leq 2d.
 \end{aligned}$$

Conversely, let $m(p) = \inf_{x \in S_X} p(x)$. Assume that $\rho(p, q) < \varrho < \frac{m(p)}{2}$. For every $x \in S_X$ we have

$$1 - \frac{\varrho}{m(p)} \leq 1 - \frac{\varrho}{p(x)} \leq \frac{q(x)}{p(x)} = \frac{q(x) - p(x)}{p(x)} + 1 \leq 1 + \frac{\varrho}{m(p)}.$$

Thus

$$\frac{b_{p,q}}{a_{p,q}} \leq \frac{1 + \frac{\varrho}{m(p)}}{1 - \frac{\varrho}{m(p)}}$$

which implies

$$d(p, q) \leq \log \frac{1 + \frac{\varrho}{m(p)}}{1 - \frac{\varrho}{m(p)}}.$$

□

Remark 4.10. The metric d is not uniformly equivalent to the metric ρ nor to the metric h . Indeed, assume that X is \mathbb{R}^2 with the maximum norm. Consider the following sequences:

- (i) Define the Cauchy sequence $\{p_n\}$ as in remark 4.7,

$$p_n((x_1, x_2)) = \max\{|x_1|, n|x_2|\}.$$

It was shown that $\{p_n\}$ is a Cauchy sequence in (\mathcal{E}, ρ) . But we obtain that for every $m, n \in \mathbb{N}$, $d(p_n, p_m) = \left|\log\left(\frac{m}{n}\right)\right|$ which implies that $\{p_n\}$ is not d -Cauchy.

- (ii) For the nonuniform equivalence of d and h . Consider as well the Cauchy sequence $\{p_n\}$ as in remark 4.7. It is not difficult to see that for every $n \in \mathbb{N}$, $H(B_{p_n}, B_{p_{n+1}}) = 1$ where $B_{p_n}, B_{p_{n+1}}$ are the unit balls corresponding to the norm p_n and p_{n+1} , respectively. But $d(p_n, p_{n+1}) = \log\frac{n+1}{n}$ which converges to 0. Thus for $\epsilon = 1$, we cannot obtain $\delta > 0$ such that if $p, q \in \mathcal{P}$ and $d(p, q) < \delta$ then $h(p, q) < \epsilon$.

4.3 Porosity version of Fabian-Zajíček-Zizler's result on the set \mathcal{E} with equivalent metrics

In this setting we prove the result of Fabian-Zajíček-Zizler in the sense of porosity on the set \mathcal{E} of all normalization norms equipped with the equivalent metrics ρ and d . Since these metrics are not uniformly equivalent, we must prove our result independently on (\mathcal{E}, ρ) and (\mathcal{E}, d) .

Theorem 4.11. *Let X be a separable reflexive Banach space. Assume that \mathcal{E} is defined as above with the metric ρ (resp.: d). Then, there exists a σ -porous*

subset \mathcal{R}_ρ (Resp.: \mathcal{R}_d) of \mathcal{E} such that for every norm $p \in \mathcal{E} \setminus \mathcal{R}$, (X, p) is UCED (and so, it has the FPP).

Proof. Due to the separability of X , assume that r is a norm on X which is UCED. We prove first the result for the metric ρ . For any $p \in \mathcal{E}$ denote

$$m(p) = \inf_{r(x)=1} p(x).$$

Let $h_p = \sqrt{1 + (m(p)^3/j)}$ and $p_j = \frac{1}{h_p} \sqrt{p^2 + (r^2 m(p)^3/j)}$. It is easy to check that $p_j \in \mathcal{E}$. Furthermore $\rho(p, p_j) \leq \frac{1}{j}$. Indeed, for every $x \in S_X$ we have

$$\begin{aligned} |p_j(x) - p(x)| &= \frac{|p_j^2(x) - p^2(x)|}{p_j(x) + p(x)} \\ &\leq \frac{\left| p^2(x) + \frac{m(p)^3}{j} - h_p^2 p^2(x) \right|}{h_p^2 m(p)} \\ &= \frac{m(p)^3 |1 - p^2(x)|}{j h_p^2 m(p)} \\ &\leq \frac{1}{j}. \end{aligned}$$

Denote

$$G_k = \bigcup_{p \in \mathcal{E}, j \geq k} B\left(p_j, \frac{1}{kj}\right).$$

We claim that $\mathcal{E} \setminus G_k$ is porous for $r_0 = \frac{1}{k}$ and $\alpha = \frac{1}{4k}$. Indeed, for $s < \frac{1}{k}$ choose $j > k$ such that $\frac{s}{4} \leq \frac{1}{j} < \frac{s}{2}$. Assume $p \in \mathcal{E} \setminus G_k$. Then, $p_j \in B(p, \frac{s}{2})$. Furthermore, we have that $B(p_j, \frac{1}{kj})$ lies in G_k and the ball $B(p_j, \frac{s}{4k})$ does not meet $\mathcal{E} \setminus G_k$.

Hence,

$$\mathcal{R} = \bigcup_{k=1}^{\infty} (\mathcal{E} \setminus G_k)$$

is a σ -porous set, and we claim that p is UCED if $p \in \mathcal{E} \setminus \mathcal{R}$. Indeed, assume that $q \in \bigcap_{k=1}^{\infty} G_k$. For every k , q belongs to G_k , there exist $p = p(k) \in \mathcal{E}$ and $j \geq k$ such that q belongs to $B(p_j, \frac{1}{kj})$. Note that for every $x \in X$ such that $r(x) \leq 1$ we have $|q^2(x) - p_j^2(x)| = |q(x) - p_j(x)||q(x) + p_j(x)| \leq \frac{2}{kj}$. Let $(x_n), (y_n)$ be sequences in X such that $r(x_n) \leq \frac{1}{2}$, $r(y_n) \leq \frac{1}{2}$ and $x_n - y_n = \lambda_n z$ for some $z \in X$ and

$$\liminf_n 2q^2(x_n) + 2q^2(y_n) - q^2(x_n + y_n) = 0.$$

Thus

$$\liminf_n 2p_j^2(x_n) + 2p_j^2(y_n) - p_j^2(x_n + y_n) \leq \frac{10}{kj}.$$

Hence

$$\liminf_n 2r^2(x_n) + 2r^2(y_n) - r^2(x_n + y_n) \leq \frac{10h_p^2}{km(p)^3} \leq \frac{20}{km(p)^3}.$$

Since $q \in B(p_j, \frac{1}{kj})$ and $\rho(p, p_j) \leq \frac{1}{j}$, we obtain that $|p(x) - q(x)| \leq \frac{2}{j}$ for every $x \in B_X$. This implies $\frac{1}{m(p)} \leq \frac{1}{|m(q) - \frac{2}{j}|}$.

Thus

$$\liminf_n 2r^2(x_n) + 2r^2(y_n) - r^2(x_n + y_n) \leq \frac{20}{k \left| m(q) - \frac{2}{j} \right|^3}.$$

Since k is arbitrary and r is UCED we obtain that $\lambda_n \rightarrow 0$.

Let us turn to the metric d . Let $t_p = \sqrt{1 + \frac{(m(p))^2}{j}}$ and $p_j = \frac{1}{t_p} \sqrt{p^2 + \frac{(m(p))^2}{j} r^2}$. It is easy to check that $p_j \in \mathcal{E}$. Furthermore $d(p, p_j) \leq \frac{1}{j}$. Indeed,

$$b_{p_j, p} = \sup_{x \in S_1} \frac{\sqrt{p^2(x) + \frac{(m(p))^2}{j}}}{t_p p(x)} = \sup_{x \in S_1} \frac{\sqrt{1 + \frac{(m(p))^2}{j p^2(x)}}}{t_p} \leq \frac{\sqrt{1 + \frac{1}{j}}}{t_p}.$$

On the other hand

$$a_{p_j, p} = \inf_{x \in S_1} \frac{\sqrt{p^2(x) + \frac{(m(p))^2}{j}}}{t_p p(x)} \geq \inf_{x \in S_1} \frac{\sqrt{p^2(x)}}{t_p p(x)} = \frac{1}{t_p}.$$

Thus

$$d(p, p_j) = \log \frac{b_{p,p_j}}{a_{p,p_j}} \leq \log \sqrt{1 + \frac{1}{j}} < \frac{1}{2j}.$$

For $k \geq 2$ define

$$G_k = \bigcup_{p \in \mathcal{E}; j \geq k} B\left(p_j, \frac{1}{kj}\right).$$

If $q \in G_k$ for some k , there exist $p \in \mathcal{E}$ depending on k and $j \geq k$ such that $d(q, p_j) < \frac{1}{kj}$. For each $x \in B_1$, we obtain that

$$\begin{aligned} |q^2(x) - p_j^2(x)| &= |q(x) - p_j(x)||q(x) + p_j(x)| \leq \frac{3}{kj} |q(x) + p_j(x)| \\ &\leq \frac{3}{kj} |q(x) + q(x) + \frac{3}{kj}| \\ &\leq \frac{9}{kj}. \end{aligned}$$

Assume that $(x_n), (y_n) \in B_{(\frac{1}{2})}$, $x_n - y_n = \lambda_n z$ where $z \in X$ and

$$\lim 2q^2(x_n) + 2q^2(y_n) - q^2(x_n + y_n) = 0.$$

An argument as above lets obtain

$$\liminf 2r^2(x_n) + 2r^2(y_n) - r^2(x_n + y_n) \leq \frac{45t_p^2}{km(p)} \leq \frac{90}{km(p)}.$$

Furthermore, since $j \geq k \geq 2$

$$\frac{1}{m(p)} = b_{r,p} \leq b_{r,q} b_{q,p_j} b_{p_j,p} \leq b_{r,q} \left(1 + \frac{3}{kj}\right) \sqrt{1 + \frac{1}{j}} \leq \frac{3}{m(q)}.$$

Thus

$$\liminf 2r^2(x_n) + 2r^2(y_n) - r^2(x_n + y_n) \leq \frac{810}{k(m(q))^2}.$$

Let $G = \bigcap_{k \geq 2} G_k$. If $q \in G$ and $(x_n), (y_n)$ are sequences as above, we obtain that

$$\liminf 2r^2(x_n) + 2r^2(y_n) - r^2(x_n + y_n) = 0.$$

By the uniform convexity in every direction of r , $\lambda_n \rightarrow 0$. Therefore q is UCED.

Next we will show that for each k , $\mathcal{E} \setminus G_k$ is porous for $\rho_0 = \frac{1}{k}$ and $\alpha = \frac{1}{2k}$. Assume $p \in \mathcal{E}$ and $\rho < \frac{1}{k}$. Choose $j \geq k$ such that $\frac{\rho}{2} \leq \frac{1}{j} < \rho$. Then we have

$$B\left(p_j, \frac{\rho}{2k}\right) \subset B\left(p_j, \frac{1}{kj}\right) \subset B(p, \rho) \cap G_k = B(p, \rho) \setminus G_k^c.$$

Take $\mathcal{R} = \bigcup_k (\mathcal{E} \setminus G_k)$. Then \mathcal{R} is σ -porous and this completes the proof. □

Chapter 5

Generic fixed point results on nonseparable reflexive Banach spaces

It seems to be more difficult to prove the generic fixed point results on non-separable reflexive spaces. The arguments used in Chapter 4 do not work for non-separable reflexive spaces. Indeed, D. Kutzarova and S.L. Troyanski [49] have proved that there are reflexive spaces without equivalent norms which are UCED. However, some interesting renorming results have been obtained for non-separable spaces. In this chapter we present some interesting fixed point renorming results on non-separable Banach spaces and later prove that such fixed point properties are generic on the set of all renormings.

5.1 Generic fixed point results on spaces with the coefficient $R(X)$ less than 2

It was proved by M. Day, R. James and S. Swaminathan [14] that there is no UCED renorming of $c_0(\Gamma)$ when Γ is uncountable. However, $c_0(\Gamma)$ enjoys the FPP. In fact, this is a consequence of a more general result. In 1997, J. García-Falset introduced a Banach space coefficient and proved that a Banach space with this coefficient less than 2 satisfies the w-FPP [31].

Definition 5.1. Let X be a Banach space. The coefficient $R(X)$ is defined by

$$R(X) = \sup\{\liminf \|x_n + x\| : x_n \text{ is weakly null with } \|x_n\| \leq 1; \\ \lim \|x_n\| = 1, \|x\| = 1\}.$$

Theorem 5.2 (J. García-Falset). *Let X be a Banach space, such that the coefficient $R(X) < 2$. Then, X satisfies the w-FPP.*

By using Garcia-Falset coefficient we can obtain that the space $c_0(\Gamma)$, when Γ is uncountable, has the w-FPP.

Lemma 5.3. $R(c_0(\Gamma)) = 1$.

Proof. Let $X = c_0(\Gamma)$. Assume that (x_n) is a weakly null sequence in the unit ball of X and x is an element in X with $\|x\| = 1$. Choose an arbitrary $\epsilon > 0$. There exists a finite subset F of Γ such that $|x(t)| < \epsilon$ if $t \notin F$. Since the evaluation functionals are linear and continuous on X we have $x_n(t) \rightarrow 0$ for each $t \in \Gamma$. Thus, $x_n(t) < \epsilon$ for every $t \in F$ and n large enough and $\liminf \|x_n + x\| \leq 1 + \epsilon$ which implies $R(X) = 1$ because ϵ is arbitrary. \square

Since, we can prove that some Banach spaces which cannot be renormed with a UCED norm satisfy the w-FPP, it would be interesting to obtain that for a general Banach space which can be a renormed to have the FPP, almost

all renormings of it are able to be renormed to satisfy the w-FPP as well.

Conjecture. Let X be a Banach space which satisfies the w-FPP. Is the w-FPP a generic property for all renormings of X ?

First we give a partial answer for spaces which satisfy the Garcia-Falset coefficient less than 2. In this chapter, the set \mathcal{P} will refer to the set of all equivalent norms on a Banach space equipped with the metric ρ as it was defined in the previous chapter. We need some technical lemmas.

Lemma 5.4. *Let X be a Banach space. Then*

$$R(X) = \sup\{\liminf \|x_n + x\| : x_n \text{ is weakly null with } \|x_n\| \leq 1; \|x\| \leq 1\}.$$

Proof. Denote a the number defined in the statement of this lemma. It is clear that $a \geq R(X)$. On the other hand, assume that $a > c > R(X)$. There exists a weakly null sequence (x_n) such that $\|x_n\| \leq 1$ and x such that $\|x\| \leq 1, \liminf \|x_n + x\| > c$. Taking a subsequence we can assume that $\lim \|x_n\| = d \leq 1$. Denote $b = \|x\|$. We claim that

$$\liminf_n \left\| \frac{x_n}{\|x_n\|} + \frac{x}{b} \right\| > c.$$

Indeed, otherwise we have, if $d > b$

$$\begin{aligned} \liminf_n \|x_n + x\| &\leq \lim_n \left(\left\| (\|x_n\| - b) \frac{x_n}{\|x_n\|} \right\| \right) + \liminf_n \left(\left\| b \left(\frac{x_n}{\|x_n\|} + \frac{x}{b} \right) \right\| \right) \\ &\leq (d - b) + bc \\ &\leq dc \\ &\leq c. \end{aligned}$$

Analogously, if $d < b$ we have

$$\begin{aligned} \liminf_n \|x_n + x\| &\leq \lim_n \left(\left\| (b - \|x_n\|) \frac{x}{b} \right\| \right) + \liminf_n \left(\left\| \|x_n\| \left(\frac{x_n}{\|x_n\|} + \frac{x}{b} \right) \right\| \right) \\ &\leq (b - d) + dc \\ &\leq bc \\ &\leq c. \end{aligned}$$

Thus $\liminf_n \left\| \frac{x_n}{\|x_n\|} + \frac{x}{b} \right\| > c > R(X)$ when $\left(\frac{x_n}{\|x_n\|} \right)$ is a weakly null sequence with $\left\| \frac{x_n}{\|x_n\|} \right\| \leq 1$, $\lim_n \left\| \frac{x_n}{\|x_n\|} \right\| = 1$ and $\left\| \frac{x}{b} \right\| = 1$. This contradicts to the definition of $R(X)$. Therefore $R(X) = a$. □

Lemma 5.5. *Let $(X, \|\cdot\|)$ be a Banach space and $p \in \mathcal{P}$ with $m(p) = \inf\{p(x) : x \in B_X\}$, $M(p) = \sup\{p(x) : x \in B_X\}$, where B_X is the unit ball corresponding with the norm $\|\cdot\|$. Assume that $\lambda \in (0, 1)$ and $p_\lambda(x) = p(x) + \lambda\|x\|$. Then*

$$R(X, p_\lambda) \leq 2 - \frac{\lambda}{M(p) + \lambda} (2 - R(X)).$$

Proof. Assume that (x_n) and x are a sequence and an element in X such that $p_\lambda(x_n) \leq 1$, $p_\lambda(x) = 1$, $p_\lambda(x_n) \rightarrow 1$. We can assume that $\|x_n\| \rightarrow a$, $\|x\| = b$. Then $p(x_n) \rightarrow 1 - \lambda a$, $p(x) = 1 - \lambda b$. Denote $c = \max\{a, b\}$, $d = \min\{a, b\}$. Note that, if $a < b$

$$\begin{aligned} \liminf_n \|x_n + x\| &\leq \liminf_n \left(\left\| (b - a) \frac{x}{b} \right\| + \left\| a \left(\frac{x_n}{a} + \frac{x}{b} \right) \right\| \right) \\ &\leq (b - a) + aR(X) \\ &= (c - d) + dR(X) \end{aligned}$$

and if $a > b$,

$$\begin{aligned} \liminf_n \|x_n + x\| &\leq \liminf_n \left(\left\| (a-b) \frac{x_n}{a} \right\| + \left\| b \left(\frac{x_n}{a} + \frac{x}{b} \right) \right\| \right) \\ &\leq (a-b) + bR(X) \\ &= (c-d) + dR(X). \end{aligned}$$

Thus we have

$$\begin{aligned} \liminf (p(x_n + x) + \lambda \|x_n + x\|) &\leq \liminf (p(x_n) + p(x) + \lambda \|x_n + x\|) \\ &= 1 - \lambda c + 1 - \lambda d + \liminf \lambda \|x_n + x\| \\ &\leq 1 - \lambda c + 1 - \lambda d + \lambda(c-d) + \lambda dR(X) \\ &= 2 - \lambda d(2 - R(X)). \end{aligned}$$

Since $p(y) \leq M(p)\|y\|$ for every $y \in X$, we have $p(y) + \lambda\|y\| \leq (M(p) + \lambda)\|y\|$ which implies $\|y\| \geq \frac{p_\lambda(y)}{(M(p) + \lambda)}$, for every $y \in X$. Thus we obtain that

$$\frac{1}{M(p) + \lambda} = \lim_n \frac{p_\lambda(x_n)}{M(p) + \lambda} \leq \lim_n \|x_n\| = a$$

and

$$\frac{1}{M(p) + \lambda} = \frac{p_\lambda(x)}{M(p) + \lambda} \leq \|x\| = b.$$

Hence $d \geq \frac{1}{M(p) + \lambda}$ and we have the result. \square

Lemma 5.6. *Let $p, q \in \mathcal{P}$ such that $\rho(p, q) < \epsilon$. Denote $m(p)$ as above. Then*

$$R(X, q) \leq \frac{m(p)R(X, p) + 2\epsilon}{m(p) - \epsilon}.$$

Proof. Assume $q(x) = 1$, $q(x_n) \leq 1$ and $q(x_n) \rightarrow 1$. Since $p(x) \leq q(x) + \epsilon\|x\|$ for every $x \in X$, so that if $x \in X$ with $\|x\| = 1$ we have $p(x) - \epsilon \leq q(x)$.

Thus $\frac{p(x)}{q(x)} \leq \frac{p(x)}{p(x)-\epsilon}$. Since the function $t \mapsto \frac{t}{t-\epsilon}$ is a decreasing function and $p(x) \geq m(p)$ for all $x \in S_X$,

$$p(x) \leq \frac{p(x)}{p(x)-\epsilon} q(x) \leq \frac{m(p)}{m(p)-\epsilon} q(x)$$

which implies that $p\left(\frac{m(p)-\epsilon}{m(p)}y\right) \leq 1$, if $q(y) \leq 1$. Thus we have $p\left(\frac{m(p)-\epsilon}{m(p)}x_n\right) \leq 1$ and $p\left(\frac{m(p)-\epsilon}{m(p)}x\right) \leq 1$ because $q(x_n) \leq 1$ and $q(x) \leq 1$. Moreover, since $m(p) \leq p\left(\frac{x}{\|x\|}\right)$ for all $x \in X$, $\|x\| \leq \frac{p(x)}{m(p)}$. Therefore

$$\begin{aligned} \liminf q(x_n + x) &\leq \liminf (p(x_n + x) + \epsilon\|x_n + x\|) \\ &= \liminf \left(\frac{m(p)}{m(p)-\epsilon} p\left(\frac{m(p)-\epsilon}{m(p)}(x_n + x)\right) + \epsilon\|x_n + x\| \right) \\ &\leq \frac{m(p)}{m(p)-\epsilon} R(X, p) + \epsilon \cdot \frac{p(x_n + x)}{m(p)} \\ &\leq \frac{m(p)}{m(p)-\epsilon} R(X, p) + \epsilon \cdot \frac{q(x_n + x)}{m(p)-\epsilon} \\ &\leq \frac{m(p)R(X, p) + 2\epsilon}{m(p)-\epsilon}. \end{aligned}$$

□

Theorem 5.7. *Let X be a Banach space such that $R(X) < 2$. Then, there exists a σ -porous subset \mathcal{R} of (\mathcal{P}, ρ) such that for every norm $p \in \mathcal{P} \setminus \mathcal{R}$, we have $R(X, p) < 2$ (and so (X, p) has the w-FPP).*

Proof. Denote $B_n = \{p \in \mathcal{P} : \frac{1}{n} < m(p) < M(p) < n\}$ where $m(p)$ and $M(p)$ are defined as in Lemma 5.5 and

$$A_n = B_n \setminus \bigcup_{\lambda \in (0,1), p \in \mathcal{P}} B\left(p_\lambda, \frac{2-R(X)}{8n(n+2)}\lambda\right).$$

We claim that A_n is a porous set for $r_0 = 1$ and $\beta = \frac{2-R(X)}{16n(n+2)}$. Indeed, let r be any positive number less than 1. Take $\lambda = \frac{r}{2}$. Note that for each $x \in B_X$,

$|p(x) - p_{\frac{r}{2}}(x)| = \frac{r}{2}\|x\| \leq \frac{r}{2}$. Hence $\rho(p, p_{\frac{r}{2}}) \leq \frac{r}{2}$. Then if $q \in B(p_{\frac{r}{2}}, \beta r) = B\left(p_{\frac{r}{2}}, \frac{2-R(X)}{16n(n+2)}r\right)$, we have

$$\rho(p, q) \leq \rho(p, p_{\frac{r}{2}}) + \rho(p_{\frac{r}{2}}, q) \leq \frac{r}{2} + \frac{2-R(X)}{16n(n+2)}r \leq \frac{r}{2} + \frac{r}{2} = r.$$

Thus the ball $B(p_{\frac{r}{2}}, \beta r)$ is contained in $B(p, r)$. The conclusion is clear because for $\beta = \frac{2-R(X)}{16n(n+2)}$ we have $B(p_{\frac{r}{2}}, \beta r) = B\left(p_{\lambda}, \frac{2-R(X)}{8n(n+2)}\lambda\right)$. Thus $\mathcal{R} =$

$\bigcup_{n=1}^{\infty} A_n$ is a σ -porous set.

It remains to show that $R(X, q) < 2$ if $q \in \mathcal{P} \setminus \mathcal{R}$. Assume that $q \in B_n$. Hence $\frac{1}{n} < m(q) \leq M(q) < n$. Since $q \in \mathcal{P} \setminus A_n$, we have that q belongs to $B\left(p_{\lambda}, \frac{2-R(X)}{8n(n+2)}\lambda\right)$ for some $p \in \mathcal{P}$ and $\lambda \in (0, 1)$. Hence

$$\begin{aligned} \rho(q, p_{\lambda}) &\leq \frac{2-R(X)}{8n(n+2)}\lambda \\ &\leq \frac{2-R(X)}{8n(M(q)+2)}\lambda \\ &\leq \frac{2-R(X)}{8n(M(p)+1)}\lambda \\ &\leq \frac{2-R(X)}{8n(M(p)+\lambda)}\lambda. \end{aligned}$$

By lemma 5.5, we obtain

$$\rho(q, p_{\lambda}) \leq \frac{2-R(X, p_{\lambda})}{8n} \leq \frac{(2-R(X, p_{\lambda}))(m(p)+\lambda)}{8}.$$

By Lemma 5.6, we obtain that

$$\begin{aligned}
 R(X, q) &\leq \frac{m(p_\lambda)R(X, p_\lambda) + 2 \left(\frac{(2-R(X, p_\lambda))(m(p)+\lambda)}{8} \right)}{m(p_\lambda) - \left(\frac{(2-R(X, p_\lambda))(m(p)+\lambda)}{8} \right)} \\
 &= \frac{2m(p_\lambda) (4R(X, p_\lambda) + (2 - R(X, p_\lambda)))}{m(p_\lambda) (8 - (2 - R(X, p_\lambda)))} \\
 &= \frac{6R(X, p_\lambda) + 4}{R(X, p_\lambda) + 6}.
 \end{aligned}$$

Since the function $t \mapsto \frac{6t+4}{t+6}$ is less than 2 if $t < 2$ and $R(X, p_\lambda) < 2$ from Lemma 5.5, hence $R(X, q) < 2$. \square

A similar result can be proved for the space (\mathcal{P}, h) , (\mathcal{E}, ρ) , (\mathcal{E}, d) and (\mathcal{E}, h) .

Remark 5.8. In fact, the above result holds in the sense of directional porosity. According to the definition of directionally porous set (in this case we refer to Definition 2.20), our space \mathcal{P} does not have any direction because it is just a metric space not a normed space. However, the sum of two equivalent norms is still an equivalent norm. Hence by following the proof in Theorem 5.7, we obtain that for each $n \in \mathbb{N}$, A_n is directionally porous for $r_0 = 1$, $\beta = \frac{2-R(X)}{16n(n+2)}$ and $h = \|\cdot\|$ in the sense that for each $p \in \mathcal{P}$ and $r \in (0, r_0]$, there exists $t = \frac{r}{2}$ for which $B(p + th, \beta r) = B(p + \frac{r}{2}\|\cdot\|, \beta r) \subset B(p, r) \setminus M$.

Thus $\mathcal{R} = \bigcup_{n=1}^{\infty} A_n$ is a σ -directionally porous set.

5.2 Generic fixed point results on a Banach space which can be embedded into $c_0(\Gamma)$

There are some other interesting fixed point renorming results for non-separable spaces. For instance, D. Amir and J. Lindenstrauss [2] have proved that ev-

ery WCG Banach space has an equivalent norm which is strictly convex, and S.L. Troyanski [87] has proved that every WCG Banach space has an equivalent norm which is locally uniformly convex. An important tool in the proofs of these results is the following fact (proved in [2]): For any WCG Banach space X , there exist a set Γ and a bounded one-to-one linear operator $J : X \rightarrow c_0(\Gamma)$. This property is satisfied by a very general class of Banach spaces, for instance subspaces of a space with Markushevich basis, as WCG spaces (and so either separable or reflexive spaces), duals of separable spaces as ℓ_∞ , etc (see [29]). By using this embedding, T. Domínguez-Benavides [20] has proved the following result:

Theorem 5.9 (Domínguez-Benavides). *Assume that X is a Banach space such that there exists a bounded one-one linear operator from X into $c_0(\Gamma)$. Then, X has an equivalent norm such that every non-expansive mapping T for the new norm defined from a convex weakly compact set C into C has a fixed point.*

Consequently, we obtain that every reflexive Banach space can be renormed in such a way that the resultant norm has the FPP. We extend the above result to obtain a generic fixed point result on a reflexive Banach space.

Let X be a Banach space. Assume that C is a weakly compact convex subset of X and $T : C \rightarrow C$ is a non-expansive mapping. By using Zorn's lemma it is easy to prove that there exists a convex closed subset K of C which is T -invariant and minimal for these conditions. This set must be separable (see [37] pages 35-36, for details). If K is not a singleton (i.e. a fixed point), then by multiplication we can assume that $\text{diam}(K) = 1$. Furthermore, we can easily construct a sequence $\{x_n\}$ in K formed by approximated fixed points, i.e. $\lim_n (Tx_n - x_n) = 0$, and, by using the weak compactness and a translation, we can assume that the sequence is weakly null. The following lemmas are basic tools for proving our result.

Lemma 5.10 (Goebel-Karlovit's Lemma [36], [43]). *Let K be a weakly compact convex subset of a Banach space X , and $T : K \rightarrow K$ a non-expansive mapping. Assume that K is minimal under these conditions and $\{x_n\}$ is an approximated fixed point sequence for T . Then, $\lim_{n \rightarrow \infty} \|x_n - x\| = \text{diam}(K)$ for every $x \in K$.*

Lemma 5.11. *Let K be a weakly compact convex subset of a Banach space X , and $T : K \rightarrow K$ a non-expansive mapping. Assume that K is minimal under these conditions, $\text{diam}(K) = 1$ and $\{x_n\}$ is a approximated fixed point sequence for T which is weakly null. Then, for every $\varepsilon > 0$ and $t \in [0, 1]$, there exist a subsequence of $\{x_n\}$, denoted again $\{x_n\}$, and a sequence $\{z_n\}$ in K such that:*

- (i) $\{z_n\}$ is weakly convergent to a point $z \in K$.
- (ii) $\|z_n\| > 1 - \varepsilon$ for every $n \in \mathbb{N}$.
- (iii) $\|z_n - z_m\| \leq t$ for every $n, m \in \mathbb{N}$.
- (iv) $\limsup_n \|z_n - x_n\| \leq 1 - t$.

The proof of Lemma 5.11 is implicitly contained in the proof of Theorem 1 in [41], and explicitly proved in [20].

Lemma 5.12. *Let $\{x_n\}$ be a weakly null sequence and x a vector in $c_0(\Gamma)$, where Γ is an arbitrary set. Assume that $\lim_n \|x_n\|$ exists. Then,*

$$\lim_n \|x_n + x\| = \max \left\{ \lim_n \|x_n\|, \|x\| \right\}.$$

Proof. For an arbitrary positive number ϵ , there exists a finite subset F of Γ such that $|x(t)| < \epsilon$ if $t \in \Gamma \setminus F$. Since $x_n(t) \rightarrow 0$ at any $t \in \Gamma$, we can choose n_0 large enough such that $|x_n(t)| < \epsilon$ for every $n \geq n_0$ and $t \in F$. Thus $|x_n(t) + x(t)| < \max\{\|x_n\|, \|x\|\} + \epsilon$ for every $n \geq n_0$ and

$t \in \Gamma$, which implies $\limsup_n \|x_n + x\| \leq \max \left\{ \lim_n \|x_n\|, \|x\| \right\}$. Analogously, $\lim_n \|x_n + x\| \geq \max \left\{ \lim_n \|x_n\|, \|x\| \right\}$ \square

We state our result in a setting more general than reflexive Banach spaces:

Theorem 5.13. *Let X be a Banach space such that for some set Γ there exists a one-to-one linear continuous mapping $J : X \rightarrow c_0(\Gamma)$. Then, there exists a residual subset \mathcal{R} in \mathcal{P} such that for every $q \in \mathcal{R}$, every q -non-expansive mapping T defined from a weakly compact convex subset C of X into C has a fixed point.*

In particular, if X is reflexive, then the space (X, q) satisfies the FPP.

Proof. For any $p \in \mathcal{P}$ and $k \in \mathbb{N}$, we denote by p_k the norm defined by $p_k^2(x) = p^2(x) + \frac{1}{k^2} \|Jx\|^2$ and choose a positive number $\delta = \delta(k) < 1/(400k^7)$. It is straightforward to prove that

$$16k^2\delta + \frac{1 + \delta k}{2} \sqrt{1 + 12\delta k - \frac{1}{k^4}} < \frac{1}{2}. \quad (5.1)$$

Define

$$\mathcal{R} = \bigcap_{j=1}^{\infty} \bigcup_{p \in \mathcal{P}; k \geq j} B(p_k, \delta(k)).$$

It is clear that \mathcal{R} is a dense G_δ -set and so a residual set. We shall prove that for every $q \in \mathcal{R}$, the space (X, q) satisfies the properties in the statement of this theorem. By contradiction, assume that there exists a weakly compact convex and separable set $K \subset X$, which is not a singleton, and a q -non-expansive mapping $T : K \rightarrow K$ such that K is minimal under these conditions. We can assume that $q\text{-diam}(K) = 1$ and there exists a weakly null approximated fixed point sequence $\{x_n\}$ for T .

Denote $a = \sup \left\{ \left\| \frac{Jx}{4} \right\| : x \in K \right\}$. Choose a positive integer k such that $\sup\{q(x) : x \in B\} < k$, $\inf\{q(x) : x \in S\} > \frac{1}{k}$ and $\frac{1}{a} < k$. Since $q \in \mathcal{R}$ there exists $p \in \mathcal{P}$ such that q belongs to $B(p_k, \delta(k))$. In order to simplify the

proof and using the separability of K and $J(K)$, we assume that $\lim_n p(x_n - x)$, $\lim_n q(x_n - x)$ and $\lim_n \|Jx_n - Jx\|$ do exist for every $x \in K$ (see Lemma 1.1 [76] for the existence of a subsequence satisfying this property).

For every $x \in X$, we have

$$|q(x) - p_k(x)| \leq \delta \|x\| \leq \delta k q(x) \quad (5.2)$$

and since $\delta k < 1$, we have from (5.2)

$$|q^2(x) - p_k^2(x)| \leq \delta k q(x)(q(x) + p_k(x)) \leq 3k\delta q^2(x).$$

Since $q(x - x_n) \leq 2$ for $x, x_n \in K$, we have

$$|q^2(x - x_n) - p_k^2(x - x_n)| \leq 12k\delta. \quad (5.3)$$

Claim. For any weakly null approximated fixed point sequence $\{x_n\}$ for T in K , we have $\lim_n \|Jx_n\| \geq 2a$.

Assume, by contradiction, that $\lim_n \|Jx_n\| < 2a$. We can choose $x \in K$ such that $\left\| \frac{Jx}{2} \right\| > \lim_n \|Jx_n\|$. Since $\{Jx_n\}$ is weakly null in $c_0(\Gamma)$, from Lemma 5.12 we have

$$\lim_n \left\| Jx_n - \frac{Jx}{2} \right\| = \max \left\{ \lim_n \|Jx_n\|, \left\| \frac{Jx}{2} \right\| \right\} = \left\| \frac{Jx}{2} \right\| \quad (5.4)$$

and, in the same way,

$$\lim_n \|Jx_n - Jx\| = \|Jx\|. \quad (5.5)$$

From Goebel-Karlovitz' lemma, (5.3) and (5.5), we have

$$\begin{aligned} 1 &= \lim_n q^2(x - x_n) \\ &\geq \lim_n p_k^2(x - x_n) - 12\delta k \\ &= \lim_n p^2(x - x_n) + \frac{1}{k^2} \lim_n \|J(x - x_n)\|^2 - 12\delta k \\ &= \lim_n p^2(x - x_n) + \frac{1}{k^2} \|J(x)\|^2 - 12\delta k \end{aligned}$$

which implies

$$\lim_n p^2(x - x_n) \leq 1 + 12k\delta - \frac{1}{k^2} \|Jx\|^2. \quad (5.6)$$

Since $1 = \lim_n q(x_n)$ by Goebel-Karlovitz' lemma, we have from (5.2)

$$\lim_n p(x_n) \leq \lim_n p_k(x_n) \leq (\delta k + 1) \lim_n q(x_n) = \delta k + 1. \quad (5.7)$$

Thus (5.1), (5.3), (5.4), (5.6) and (5.7) imply

$$\begin{aligned} & \lim_n q^2\left(x_n - \frac{x}{2}\right) \\ & \leq 12\delta k + \lim_n p_k^2\left(x_n - \frac{x}{2}\right) \\ & \leq 12\delta k + \lim_n p^2\left(x_n - \frac{x}{2}\right) + \frac{1}{k^2} \left\| \frac{Jx}{2} \right\|^2 \\ & \leq 12\delta k + \lim_n \left(\frac{p(x_n - x) + p(x_n)}{2} \right)^2 + \frac{1}{k^2} \left\| \frac{Jx}{2} \right\|^2 \\ & = 12\delta k + \frac{1}{4} \lim_n (p^2(x_n - x) + 2p(x_n - x)p(x_n) + p^2(x_n)) + \frac{1}{k^2} \left\| \frac{Jx}{2} \right\|^2 \\ & \leq 12\delta k + \frac{1}{k^2} \left\| \frac{Jx}{2} \right\|^2 \\ & \quad + \frac{1}{4} \left(1 + 12\delta k - \frac{1}{k^2} \|Jx\|^2 + 2(\delta k + 1) \sqrt{1 + 12\delta k - \frac{\|Jx\|^2}{k^2}} + (\delta k + 1)^2 \right) \\ & = 12\delta k + \frac{1}{2} + \frac{1}{4} \left(14\delta k + \delta^2 k^2 + 2(\delta k + 1) \sqrt{1 + 12\delta k - \frac{\|Jx\|^2}{k^2}} \right) \\ & \leq 16\delta k^2 + \frac{1}{2} + \frac{1 + \delta k}{2} \sqrt{1 + 12\delta k - \frac{1}{k^4}} \\ & < 1 \end{aligned}$$

which is a contradiction according to Goebel-Karlovitz' lemma, because $\frac{x}{2}$ belongs to K .

Denote $b = \frac{a}{k}$. Note that $b \in (0, 1/2)$. Indeed, for every $x \in K$ we have $q(x) \leq 1$ which implies

$$\frac{\|Jx\|}{4k} \leq \frac{p_k(x)}{4} \leq \frac{\sqrt{1 + \delta k}}{4} < \frac{1}{2}.$$

Choose a positive number

$$\epsilon < \min \left\{ \frac{b^2}{32}, b \left(1 - b - \frac{b^2}{18} - \sqrt{1 + \frac{7}{8}b^2 - 2b} \right), \frac{1}{2} \left(\frac{97}{50} - 2\sqrt{\frac{9}{10}} \right) b(1 - b) \right\}.$$

We can apply Lemma 5.11 to the sequence $\{x_n\}$, $t = 1 - b$, and ϵ as above, to obtain a sequence $\{z_n\}$ satisfying (i) to (iv). Denote $z = w - \lim_n z_n$. Taking a subsequence of $\{z_n\}$ (and also of $\{x_n\}$) and using the separability of K we can assume as above that $\lim_n p(z_n - y)$, $\lim_n q(z_n - y)$ and $\lim_n \|Jz_n - Jy\|$ do exist for every $y \in K$. Since $\lim_n \|Jx_n\| \geq 2a$, $\limsup_n q(x_n - z_n) \leq b$ by Lemma 5.11 (iv) and using (5.2), we have

$$\begin{aligned} \frac{1}{k} \lim_n \|Jz_n\| &\geq \frac{1}{k} \left(\lim_n \|Jx_n\| - \limsup_n \|Jx_n - Jz_n\| \right) \\ &\geq \frac{2a}{k} - \limsup_n p_k(x_n - z_n) \\ &\geq \frac{2a}{k} - (\delta k + 1) \limsup_n q(x_n - z_n) \\ &\geq \frac{2a}{k} - (\delta k + 1)b \\ &\geq (1 - \delta k)b. \end{aligned}$$

Since $\delta \leq \frac{1}{10k}$, we have

$$\lim_n \|Jz_n\| \geq a(1 - \delta k) \geq \frac{9}{10}a. \quad (5.8)$$

Moreover, from Lemma 5.11 (iii) and the weak lower semi-continuity of the norm q we have

$$q(z_n - z) \leq \lim_m q(z_n - z_m) \leq 1 - b$$

and so

$$\lim_n q(z_n - z) \leq 1 - b. \quad (5.9)$$

Analogously from Lemma 5.11 (iv), we obtain

$$q(z) \leq \lim_n q(z_n - x_n) \leq b. \quad (5.10)$$

Hence, by (5.3), Lemma 5.11 (ii) and Lemma 5.12 we have

$$\begin{aligned} (1 - \epsilon)^2 &\leq \lim_n q^2(z_n) \\ &\leq 12\delta k + \lim_n p_k^2(z_n) \\ &\leq 12\delta k + \lim_n (p(z_n - z) + p(z))^2 + \frac{1}{k^2} \lim_n \|J(z_n - z) + Jz\|^2 \\ &= 12\delta k + \lim_n (p(z_n - z) + p(z))^2 + \frac{1}{k^2} \left(\max \left\{ \lim_n \|J(z_n - z)\|, \|Jz\| \right\} \right)^2. \end{aligned} \quad (5.11)$$

We split the proof into two cases:

Case A. Assume that $\lim_n \|Jz_n - Jz\| \geq \|Jz\|$. In this case, (5.11) becomes

$$(1 - \epsilon)^2 \leq 12\delta k + \lim_n p_k^2(z_n - z) + p(z) \left(p(z) + 2 \lim_n p(z_n - z) \right). \quad (5.12)$$

We still have two possibilities:

Case A1. Assume $\|Jz\| \geq \frac{a}{2}$. In this setting we have from (5.12) and (5.3)

$$(1 - \epsilon)^2 \leq 24\delta k + \lim_n q^2(z_n - z) + p(z) \left(p(z) + 2 \lim_n p(z_n - z) \right). \quad (5.13)$$

Since $\frac{1}{a} < k$ by the choice of k , we have

$$12\delta k < \frac{12}{400k^6} < \frac{1}{8k^4} < \frac{a^2}{8k^2}.$$

Thus, from (5.3) and (5.10) we have

$$b^2 \geq q^2(z) \geq p^2(z) + \frac{a^2}{4k^2} - 12k\delta \geq p^2(z) + \frac{a^2}{8k^2}$$

which implies

$$p^2(z) \leq b^2 - \frac{a^2}{8k^2} = \frac{7b^2}{8}. \quad (5.14)$$

According to the fact that $b \in (0, \frac{1}{2})$, note that $b(1-b) < 1$ and $26\delta k < \frac{26b^3}{400} < \frac{26b^2}{800} < \frac{b^2}{16}$. Hence by (5.13), (5.14), (5.2), (5.9) we obtain the contradiction

$$\begin{aligned} (1 - \epsilon)^2 &\leq 24\delta k + (1 - b)^2 + \sqrt{\frac{7}{8}}b \left(\sqrt{\frac{7}{8}}b + 2(1 + \delta k) \lim_n q(z_n - z) \right) \\ &\leq 24\delta k + (1 - b)^2 + \sqrt{\frac{7}{8}}b \left(\sqrt{\frac{7}{8}}b + 2(1 + \delta k)(1 - b) \right) \\ &< 26\delta k + (1 - b)^2 + \frac{7}{8}b^2 + 2\sqrt{\frac{7}{8}}b(1 - b) \\ &< 26\delta k + (1 - b)^2 + b^2 + 2b(1 - b) - \frac{b^2}{8} \\ &= 26\delta k + (1 - b + b)^2 - \frac{b^2}{8} \\ &= 1 + 26\delta k - \frac{b^2}{8} \\ &< 1 + \frac{b^2}{16} - \frac{b^2}{8} \\ &= 1 - \frac{b^2}{16} < 1 - 2\epsilon \end{aligned}$$

Case A2. Assume $\|Jz\| \leq \frac{a}{2}$. In this case (5.8) implies $\lim_n \|Jz_n - Jz\| \geq \lim_n \|Jz_n\| - \|Jz\| \geq \frac{9a}{10} - \frac{a}{2} = \frac{2a}{5}$. Furthermore, from (5.3) and (5.9) we have

$$\begin{aligned} (1 - b)^2 &\geq \lim_n q^2(z_n - z) \\ &\geq \lim_n p_k^2(z_n - z) - 12\delta k \\ &\geq \lim_n p^2(z_n - z) - 12\delta k + \frac{4b^2}{25}. \end{aligned}$$

Since $12\delta k < \frac{3}{100k^6} < \frac{3b^3}{100} < \frac{3b^2}{200} < \frac{7b^2}{200}$, we have

$$\lim_n p^2(z - z_n) \leq (1 - b)^2 - \frac{4}{25}b^2 + 12\delta k < 1 - 2b + \frac{7}{8}b^2. \quad (5.15)$$

Note that $\delta k < \frac{1}{100k^6} < \frac{a^3}{100k^3} < \frac{b^3}{100}$ which implies

$$p(z) \leq q(z) + \delta k \leq b + \delta k \leq b + \frac{b^3}{100}. \quad (5.16)$$

As above $24\delta k < \frac{24}{400k^6} < \frac{a^3}{16k^3} = \frac{b^3}{16}$. Furthermore, $1 - 2b + \frac{7b^2}{8} < 1$. Thus (5.12), (5.9), (5.15) and (5.16) imply the contradiction

$$\begin{aligned} (1 - \epsilon)^2 &\leq 24k\delta + (1 - b)^2 + \left(b + \frac{b^3}{100}\right) \left(b + \frac{b^3}{100} + 2\sqrt{1 - 2b + \frac{7b^2}{8}}\right) \\ &\leq \frac{b^3}{16} + (1 - b)^2 + \left(b + \frac{b^3}{100}\right) \left(b + \frac{b^3}{100} + 2\sqrt{1 - 2b + \frac{7b^2}{8}}\right) \\ &\leq 1 - 2b + b^2 + \frac{b^3}{16} + b^2 + \frac{4b^3}{100} + 2b\sqrt{1 - 2b + \frac{7b^2}{8}} \\ &\leq 1 - 2b + b^2 + \frac{b^3}{9} + b^2 + 2b\sqrt{1 - 2b + \frac{7b^2}{8}} \\ &< 1 - 2\epsilon. \end{aligned}$$

Case B. Assume $\lim_n \|Jz_n - Jz\| \leq \|Jz\|$. Since $\lim_n \|Jz_n\| \geq \frac{9a}{10}$ by (5.8), we have

$$\|Jz\| \geq \lim_n \|Jz_n\| - \lim_n \|Jz_n - Jz\| \geq \frac{9a}{10} - \|Jz\|.$$

Thus $\|Jz\| \geq \frac{9a}{20}$. Since $q(z) \leq b$ by (5.10), we have by (5.3) that

$$p_k(z) \leq \sqrt{b^2 + 12\delta k} < \sqrt{b^2 + \frac{7b^2}{200}} \leq \frac{21b}{20}.$$

Hence

$$\frac{441b^2}{400} \geq p^2(z) + \frac{\|Jz\|}{k^2} \geq p^2(z) + \frac{81b^2}{400}$$

which implies

$$p^2(z) \leq \frac{9b^2}{10}$$

and so

$$p(z) \leq \sqrt{\frac{9}{10}}b. \quad (5.17)$$

Note that the choice of $k > \frac{1}{a}$ implies $k \geq 2$, so that $24\delta k < \frac{6b}{100k^4} \leq \frac{6b}{(100)(2^4)} < \frac{b}{(100)(2)} < \frac{b(1-b)}{100}$. From Lemma 5.11 (ii), Lemma 5.12, (5.9), (5.10) and (5.17) we have the contradiction

$$\begin{aligned} (1 - \epsilon)^2 &\leq \lim_n q^2(z_n) \\ &\leq 12\delta k + \lim_n p_k^2(z_n) \\ &= 12\delta k + \lim_n p^2(z_n) + \frac{1}{k^2} \lim_n \|Jz_n\|^2 \\ &\leq 12\delta k + \lim_n (p(z_n - z) + p(z))^2 + \frac{1}{k^2} \lim_n \|J(z_n - z) + Jz\|^2 \\ &\leq 12\delta k + \lim_n (p(z_n - z) + p(z))^2 + \frac{1}{k^2} \left(\max \left\{ \lim_n \|J(z_n - z)\|, \|Jz\| \right\} \right)^2 \\ &= 12\delta k + \lim_n p(z_n - z) (p(z_n - z) + 2p(z)) + p^2(z) + \frac{1}{k^2} \|Jz\|^2 \\ &= 12\delta k + \lim_n p(z_n - z) (p(z_n - z) + 2p(z)) + p_k^2(z) \\ &\leq 24\delta k + q^2(z) + \lim_n p(z_n - z) (p(z_n - z) + 2p(z)) \\ &\leq 24\delta k + b^2 + (1 - b)(1 + \delta k) \left((1 - b)(1 + \delta k) + 2\sqrt{\frac{9}{10}}b \right) \\ &\leq 24\delta k + b^2 + (1 - b + \delta k) \left(1 - b + \delta k + 2\sqrt{\frac{9}{10}}b \right) \\ &\leq \frac{1}{100}b(1 - b) + b^2 + \left(1 - b + \frac{1}{100}b(1 - b) \right) \left(1 - b + \frac{1}{100}b(1 - b) + 2\sqrt{\frac{9}{10}}b \right) \end{aligned}$$

$$\begin{aligned}
 &= \frac{1}{100}b(1-b) + b^2 + (1-b)^2 + \frac{1}{100}b(1-b)^2 + 2\sqrt{\frac{9}{10}}b(1-b) + \frac{1}{100}b(1-b)^2 \\
 &\quad + \frac{1}{100}b(1-b)^2 + \frac{1}{10^4}b^2(1-b)^2 + \frac{2}{100}\sqrt{\frac{9}{10}}b^2(1-b) \\
 &\leq \frac{6}{100}b(1-b) + b^2 + (1-b)^2 + 2\sqrt{\frac{9}{10}}b(1-b) \\
 &= 1 - \left(\frac{97}{50} - 2\sqrt{\frac{9}{10}}\right)b(1-b) \\
 &< 1 - 2\epsilon.
 \end{aligned}$$

□

We obtain also the following result.

Corollary 5.14. *Let X be a reflexive Banach space. Then, for almost all $q \in \mathcal{P}$, the space (X, q) has the FPP and for every q -non-expansive mapping T defined from a convex closed bounded set C into C , the set of fixed points of T is convex.*

Proof. According to the results in [2], every reflexive Banach space has a strictly convex renorming. By Theorem 4.1.1, if there is a strictly convex renorming, then almost all norms in \mathcal{P} are strictly convex. Thus, from Theorem 5.13 almost all norms in \mathcal{P} are strictly convex and satisfy the FPP. The convexity of the set of fixed points is a consequence of the strict convexity of the space [37]. □

5.3 Generic fixed point results on a Banach space which can be embedded into a Banach space Y satisfying $R(Y) < 2$

According to the result of T. Domínguez-Benavides (Theorem 5.9), every Banach space which can be embedded into the space $c_0(\Gamma)$ can be renormed to have the w-FPP. The proof of this result is strongly based upon some specific properties of the space $c_0(\Gamma)$, specially the equality $R(c_0(\Gamma)) = 1$. Thus, a natural conjecture could be to extend the above result to any Banach space which can be embedded in more general Banach spaces than $c_0(\Gamma)$, but still satisfying $R(Y) < 2$. In this section, we actually prove this extension. In order to stating our result, we give a necessary lemma.

Lemma 5.15. *Let $(Y, \|\cdot\|_Y)$ be a Banach space with $R(Y) < 2$, $\{y_n\}$ a sequence in Y weakly convergent to 0. Assume that $0 < \alpha = \lim_n \|y_n\|_Y$ and $0 < \beta = \|y\|_Y$. Then*

$$\limsup_n \|y_n + y\|_Y \leq c \left(\lim_n \|y_n\|_Y + \|y\|_Y \right)$$

$$\text{where } c = \frac{R(Y)-1+\max\left\{\frac{\alpha}{\beta}, \frac{\beta}{\alpha}\right\}}{1+\max\left\{\frac{\alpha}{\beta}, \frac{\beta}{\alpha}\right\}} < 1.$$

Proof. It is clear that $c < 1$. Assume that $\alpha > \beta$. Then we obtain that

$$\begin{aligned} \limsup_n \|y_n + y\|_Y &\leq \beta \limsup_n \left\| \frac{y_n}{\alpha} + \frac{y}{\beta} \right\|_Y + \left(1 - \frac{\beta}{\alpha}\right) \lim_n \|y_n\|_Y \\ &\leq \beta R(Y) + \alpha - \beta \\ &= \beta(R(Y) - 1) + \alpha \\ &= \frac{\beta(R(Y) - 1) + \alpha}{\alpha + \beta} (\alpha + \beta) \\ &= \frac{R(Y) - 1 + \frac{\alpha}{\beta}}{1 + \frac{\alpha}{\beta}} (\alpha + \beta). \end{aligned} \tag{5.18}$$

Slight modification of the above argument shows that if $\beta \geq \alpha$, then

$$\limsup_n \|y_n + y\|_Y \leq \frac{R(Y) - 1 + \frac{\beta}{\alpha}}{1 + \frac{\beta}{\alpha}} (\alpha + \beta).$$

It follows that

$$\limsup_n \|y_n + y\|_Y \leq c \left(\lim_n \|y_n\|_Y + \|y\|_Y \right)$$

because the inequality $R(Y) < 2$ implies that the function $t \mapsto \frac{R(Y)-1+t}{1+t}$ is increasing on the interval $[0, +\infty)$. □

Theorem 5.16. *Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be Banach spaces. Assume that $R(Y) < 2$ and there exists a one-to-one linear continuous mapping $J : X \rightarrow Y$. Then there exists an equivalent norm in X such that X endowed with the new norm satisfies the w-FPP.*

Proof. Define, for each $x \in X$,

$$|x|^2 = \|x\|_X^2 + \|Jx\|_Y^2.$$

It is not difficult to check that $|\cdot|$ is an equivalent norm on X . We will show that the space $(X, |\cdot|)$ enjoys the w-FPP.

Assume that C is a weakly compact convex subset of X and $T : C \rightarrow C$ is a $|\cdot|$ -non-expansive mapping. By Zorn's lemma, there exists a convex closed subset K of X which is T -invariant and minimal under these conditions. This set must be separable (see [37], page 35-36) and each point of K is diametral. We will assume by contradiction that K is not a singleton. Then by multiplication, we can assume that $|\cdot|$ -diam $K = 1$ and we can also assume that there exists a weakly convergent approximated fixed point sequence $\{x_n\}$ for T in K . By translation, we can assume that $\{x_n\}$ is weakly null and so, $0 \in K$. By Goebel-Karlovitz' lemma, $\lim_n |x_n| = \lim_n |x_n - 0| = \text{diam}K = 1$.

Due to the separability of K , we can assume that $\lim_n |x_n - x|$, $\lim_n \|x_n - x\|_X$ and $\lim_n \|J(x_n - x)\|_Y$ do exist for every $x \in K$.

We claim that for the weakly null approximated fixed point sequence $\{x_n\}$, $\lim_n \|Jx_n\|_Y > 0$. To see this, assume by contradiction that $\lim_n \|Jx_n\|_Y = 0$. Since $\text{diam}K = 1$ and J is a one-to-one linear mapping, we can choose x in K such that $\|Jx\|_Y = a > 0$. According to Goebel-Karlovitz' lemma,

$$\begin{aligned} 1 &= \lim_n |x_n - x|^2 \\ &= \lim_n (\|x_n - x\|_X^2 + \|J(x_n - x)\|_Y^2) \\ &= \lim_n \|x_n - x\|_X^2 + a^2. \end{aligned} \tag{5.19}$$

Thus,

$$\lim_n \|x_n - x\|_X^2 = 1 - a^2. \tag{5.20}$$

By (5.20), we obtain that

$$\begin{aligned} \lim_n \left| x_n - \frac{x}{2} \right|^2 &= \lim_n \left(\left\| x_n - \frac{x}{2} \right\|_X^2 + \left\| J\left(x_n - \frac{x}{2}\right) \right\|_Y^2 \right) \\ &\leq \frac{1}{4} \lim_n ((\|x_n - x\|_X + \|x_n\|_X)^2 + (\|J(x_n - x)\|_Y + \|Jx_n\|_Y)^2) \\ &= \frac{1}{4} \lim_n (|x_n - x|^2 + |x_n|^2 + 2(\|x_n - x\|_X \|x_n\|_X + \|J(x_n - x)\|_Y \|Jx_n\|_Y)) \\ &\leq \frac{1}{4} (1 + 1 + 2\sqrt{1 - a^2}) \\ &< 1 \end{aligned}$$

which contradicts to Goebel-Karlovitz' lemma since $\frac{x}{2}$ belongs to K . Thus, $\lim_n \|Jx_n\|_Y > 0$.

Assume that $\lim_n \|Jx_n\|_Y = 3b$ for some positive real number b . Note that $b \in (0, \frac{1}{3}]$. Indeed, since J is one-to-one linear continuous and $|\cdot| - \text{diam}K = 1$, then $3b = \lim_n \|Jx_n\|_Y \leq 1$. Choose $0 < \gamma < \frac{b(1-b-\sqrt{1-2b})}{2}$ and let $\hat{c} = \max \left\{ 1, \frac{1-b}{\gamma} \right\}$. Denote $c = \frac{R(Y)-1+\hat{c}}{1+\hat{c}} < 1$.

We apply lemma 5.11 for $t = 1 - b$ and

$$\epsilon < \min \left\{ \frac{\gamma^2}{2}(1 - c^2), \frac{b(1 - b - \sqrt{1 - 2b})}{2} \right\}$$

to obtain a subsequence of $\{x_n\}$, denoted again by $\{x_n\}$, and a sequence $\{z_n\}$ in K which satisfy (i)-(iv).

Assume that $z = w - \lim z_n$. By (iv) and the weak lower semi-continuity of the norm, we have

$$|z| \leq \liminf_n |z_n - x_n| \leq 1 - t = b \quad (5.21)$$

and by (iii),

$$\lim_n |z_n - z| \leq \lim_n \lim_m |z_n - z_m| \leq t = 1 - b. \quad (5.22)$$

Moreover, again by using (iv), we obtain

$$b^2 \geq \limsup_n |x_n - z_n|^2 \geq \limsup_n \|J(x_n - z_n)\|_Y^2.$$

Thus

$$b \geq \limsup_n \|J(x_n - z_n)\|_Y \geq \limsup_n (\|Jx_n\|_Y - \|Jz_n\|_Y)$$

which implies

$$\lim_n \|Jz_n\|_Y \geq \lim_n \|Jx_n\|_Y - b = 2b. \quad (5.23)$$

We will split the proof into two cases:

Case I. Assume that $\|Jz\|_Y \leq \gamma$.

In this case, we have, by (5.23)

$$\lim_n \|J(z_n - z)\|_Y \geq \lim_n \|Jz_n\|_Y - \|Jz\|_Y \geq 2b - \gamma.$$

By (5.22), we have

$$\begin{aligned} (1 - b)^2 &\geq \lim_n |z_n - z|^2 \\ &= \lim_n (\|z_n - z\|_X^2 + \|J(z_n - z)\|_Y^2) \\ &\geq \lim_n \|z_n - z\|_X^2 + (2b - \gamma)^2. \end{aligned}$$

Since $\gamma < b$ we have

$$\lim_n \|z_n - z\|_X^2 \leq (1 - b)^2 - (2b - \gamma)^2 < (1 - b)^2 - b^2 = 1 - 2b. \quad (5.24)$$

Thus, by using (ii), (5.21), (5.22) and (5.24), we obtain the following contradiction:

$$\begin{aligned} (1 - \epsilon)^2 &\leq \lim_n |z_n|^2 \\ &= \lim_n (\|z_n\|_X^2 + \|Jz_n\|_Y^2) \\ &\leq \lim_n ((\|z_n - z\|_X + \|z\|_X)^2 + (\|J(z_n - z)\|_Y + \|Jz\|_Y)^2) \\ &= \lim_n (|z_n - z|^2 + |z|^2 + 2(\|z_n - z\|_X \|z\|_X + \|J(z_n - z)\|_Y \|Jz\|_Y)) \\ &\leq (1 - b)^2 + b^2 + 2b\sqrt{1 - 2b} + 2\gamma \\ &\leq (1 - b)^2 + b^2 + 2b\sqrt{1 - 2b} + b(1 - b - \sqrt{1 - 2b}) \\ &\leq (1 - b)^2 + b^2 + 2b\sqrt{1 - 2b} + 2b(1 - b - \sqrt{1 - 2b}) - b(1 - b - \sqrt{1 - 2b}) \\ &< 1 - 2\epsilon. \end{aligned}$$

Case II. Assume that $\|Jz\|_Y \geq \gamma$.

By the assumption, we have

$$0 < \gamma \leq \|Jz\|_Y \leq |z| \leq b.$$

Furthermore,

$$\lim_n \|J(z_n - z)\|_Y \geq \lim_n \|Jz_n\|_Y - \|Jz\|_Y \geq 2b - b = b.$$

Hence

$$0 < b \leq \lim_n \|J(z_n - z)\|_Y \leq \lim_n |z_n - z| \leq 1 - b.$$

Applying lemma 5.15 with $\{y_n\} = \{J(z_n - z)\}$ and $y = Jz$, we obtain that

$$\begin{aligned} \lim_n \|J(z_n - z) + Jz\|_Y &\leq \frac{R(Y) - 1 + \hat{c}}{1 + \hat{c}} \left(\lim_n \|J(z_n - z)\|_Y + \|Jz\|_Y \right) \\ &= c \left(\lim_n \|J(z_n - z)\|_Y + \|Jz\|_Y \right). \end{aligned} \quad (5.25)$$

Then by (ii) and (5.25), we have

$$\begin{aligned}
 (1 - \epsilon)^2 &\leq \lim_n |z_n|^2 \\
 &= \lim_n (\|z_n\|_X^2 + \|Jz_n\|_Y^2) \\
 &\leq \lim_n ((\|z_n - z\|_X + \|z\|_X)^2 + \|J(z_n - z) + Jz\|_Y^2) \\
 &\leq \lim_n (\|z_n - z\|_X + \|z\|_X)^2 + c^2 \left(\lim_n \|J(z_n - z)\|_Y + \|Jz\|_Y \right)^2 \\
 &= \lim_n ((\|z_n - z\|_X + \|z\|_X)^2 + (\|J(z_n - z)\|_Y + \|Jz\|_Y)^2) \\
 &\quad + (c^2 - 1) \left(\lim_n \|J(z_n - z)\|_Y + \|Jz\|_Y \right)^2 \\
 &\leq \lim_n (|z_n - z|^2 + |z|^2 + 2(\|z_n - z\|_X \|z\|_X + \|J(z_n - z)\|_Y \|Jz\|_Y)) \\
 &\quad - (1 - c^2) \|Jz\|_Y^2.
 \end{aligned} \tag{5.26}$$

Assume that $\lim_n |z_n - z| = u$ and $|z| = v$. Denote $t = \lim_n \|z_n - z\|_X$ and $s = \|z\|_X$. Then we have

$$\lim_n \|J(z_n - z)\|_Y = \left(\lim_n |z_n - z|^2 - \lim_n \|z_n - z\|_X^2 \right)^{\frac{1}{2}} = \sqrt{u^2 - t^2}$$

and

$$\|Jz\|_Y = (|z|^2 - \|z\|_X^2)^{\frac{1}{2}} = \sqrt{v^2 - s^2}.$$

Consider the function of two variables

$$f(t, s) = 2ts + 2\sqrt{u^2 - t^2}\sqrt{v^2 - s^2}.$$

By elementary calculus, we obtain that

$$\max_{[0, u] \times [0, v]} f(t, s) = 2uv$$

hence

$$2 \left(\|z\|_X \lim_n \|z_n - z\|_X + \|Jz\|_Y \lim_n \|J(z_n - z)\|_Y \right) \leq 2|z| \lim_n |z_n - z|. \tag{5.27}$$

By applying (5.21), (5.22) and (5.27), then (5.26) becomes

$$\begin{aligned}
 (1 - \epsilon)^2 &\leq \lim_n (|z_n - z|^2 + |z|^2 + 2|z_n - z||z|) - (1 - c^2)\|Jz\|_Y^2 \\
 &\leq (1 - b)^2 + b^2 + 2b(1 - b) - (1 - c^2)\gamma^2 \\
 &= 1 - (1 - c^2)\gamma^2 \\
 &< 1 - 2\epsilon
 \end{aligned}$$

and we again reach the contradiction.

Therefore the space $(X, |\cdot|)$ enjoys the w-FPP.

□

We also prove a stronger result than Theorem 5.16, the w-FPP property is generic on all renormings of a Banach space which can be embedded into a Banach space Y with $R(Y) < 2$.

Theorem 5.17. *Let X be a Banach space and \mathcal{P} as above. Assume that there is a one-to-one linear continuous mapping $J : X \rightarrow (Y, \|\cdot\|_Y)$ where $(Y, \|\cdot\|_Y)$ is a Banach space with $R(Y, \|\cdot\|_Y) < 2$. Then there exists a residual subset \mathcal{R} in \mathcal{P} such that every $q \in \mathcal{R}$, the space (X, q) has the FPP.*

Proof. For each $p \in \mathcal{P}$ and $k \in \mathbb{N}$, define a norm p_k by

$$p_k^2(x) = p^2(x) + \frac{1}{k^2}\|Jx\|_Y^2, \quad x \in X.$$

For each $k \in \mathbb{N}$, let $\gamma = \gamma(k)$ and $\hat{c}_1 = \hat{c}_1(k)$ be positive real numbers such that

$$0 < \gamma < \frac{1}{6k} \left(1 - \frac{1}{3k^2} - \sqrt{\left(1 - \frac{1}{3k^2}\right)^2 - \frac{1}{90k^4}} \right)$$

and

$$\hat{c}_1 = \frac{31k}{30\gamma}.$$

Note that $\gamma < \frac{1}{6} < \frac{1}{2}$. Take $c_1 = c_1(k) = \frac{R(Y)-1+c_1}{1+c_1} < 1$ and choose a positive real number

$$\delta_k < \min \left\{ \frac{1}{30k^5}, \frac{1}{30k^3} \left(1 - \frac{1}{3k^2} - \sqrt{\left(1 - \frac{1}{3k^2}\right)^2 - \frac{1}{90k^4}} \right), \frac{1 - c_1^2}{6k^3} \gamma^2 \right\}.$$

Define

$$\mathcal{R} = \bigcap_{j=1}^{\infty} \bigcup_{p \in \mathcal{P}; k \geq j} B(p_k, \delta_k).$$

It is clear that \mathcal{R} is a dense G_δ -set, hence \mathcal{R} is residual. We will prove that for each $q \in \mathcal{R}$, the space (X, q) enjoys the FPP.

We will assume by contradiction that (X, q) fails to have the w-FPP. Assume that C is a weakly compact convex subset of X and $T : C \rightarrow C$ is a q -non-expansive mapping. Then by Zorn's lemma, there exists a convex closed subset K of X which is not a singleton, T -invariant and minimal for these conditions. This set must be separable and each point of K is diametral. Then by multiplication, we can assume that $q - \text{diam}K = 1$ and we can also assume that there exists a weakly null approximated fixed point sequence $\{x_n\}$ for T in K . Since $0 \in K$, by Goebel-Karlovitz' lemma, $\lim_n q(x_n) = \lim_n q(x_n - 0) = q - \text{diam}K = 1$.

We claim that $\liminf_n \|Jx_n\|_Y > 0$. Assume that $\liminf_n \|Jx_n\|_Y = 0$. Since $q - \text{diam}K = 1$ and J is a one-to-one linear mapping, we can choose $0 \neq x \in K$ such that $\|Jx\|_Y = a > 0$. Let $k_1 \in \mathbb{N}$ such that $k_1 > \sup\{q(x) : x \in B_X\}$, $\frac{1}{k_1} < \inf\{q(x) : x \in S_X\}$ and $\left(\frac{80k_1^4+1}{10k_1^2(10k_1^4+1)}\right)^{\frac{1}{2}} < a$. It is straightforward to verify that the last condition implies

$$\left(1 - \frac{a^2}{k_1^2} + \frac{1}{10k_1^4}\right) \left(1 + \frac{1}{10k_1^4}\right) < 1 - \frac{3}{5k_1^4}. \quad (5.28)$$

Since

$$1 - \frac{3}{5k_1^4} < \left(1 - \frac{3}{10k_1^4}\right)^2$$

we have

$$\sqrt{\left(1 - \frac{a^2}{k_1^2} + \frac{1}{10k_1^4}\right) \left(1 + \frac{1}{10k_1^4}\right)} < 1 - \frac{3}{10k_1^4}. \quad (5.29)$$

Since $q \in \mathcal{R}$, we can assume that there exists $p \in \mathcal{P}$ such that $q \in B(p_{k_1}, \delta_{k_1})$. Then for each $x \in X$, we obtain that

$$|q(x) - p_{k_1}(x)| \leq k_1 \delta_{k_1} q(x) \quad (5.30)$$

and

$$|q^2(x) - p_{k_1}^2(x)| \leq 3k_1 \delta_{k_1} q^2(x). \quad (5.31)$$

By using the separability of K , we can assume that $\lim_n q(x_n - x)$, $\lim_n \|J(x_n - x)\|_Y$ and $\lim_n p(x_n - x)$ do exist for every $x \in K$.

According to Goebel-Karlovitz' lemma and (5.31), we have

$$\begin{aligned} 1 &= \lim_n q^2(x_n - x) \\ &\geq \lim_n p_{k_1}^2(x_n - x) - 3k_1 \delta_{k_1} \\ &> \lim_n \left(p^2(x_n - x) + \frac{1}{k_1^2} \|J(x_n - x)\|_Y^2 \right) - \frac{1}{10k_1^4} \\ &\geq \lim_n \left(p^2(x_n - x) + \frac{1}{k_1^2} (\|Jx\|_Y - \|Jx_n\|)^2 \right) - \frac{1}{10k_1^4} \\ &= \lim_n p^2(x_n - x) + \frac{a^2}{k_1^2} - \frac{1}{10k_1^4}. \end{aligned}$$

Thus

$$\lim_n p^2(x_n - x) < 1 - \frac{a^2}{k_1^2} + \frac{1}{10k_1^4}. \quad (5.32)$$

Now consider

$$\begin{aligned}
 & \lim_n q^2 \left(x_n - \frac{x}{2} \right) \\
 & \leq \lim_n p_{k_1}^2 \left(x_n - \frac{x}{2} \right) + 3k_1 \delta_{k_1} \\
 & \leq \lim_n \left(p^2 \left(x_n - \frac{x}{2} \right) + \frac{1}{k_1^2} \left\| J \left(x_n - \frac{x}{2} \right) \right\|_Y^2 \right) + \frac{1}{10k_1^4} \\
 & \leq \frac{1}{4} \lim_n \left((p(x_n - x) + p(x_n))^2 + \frac{1}{k_1^2} (\|J(x_n - x)\|_Y + \|Jx_n\|_Y)^2 \right) + \frac{1}{10k_1^4} \\
 & = \frac{1}{4} \lim_n (p_{k_1}^2(x_n - x) + p^2(x_n) + 2p(x_n - x)p(x_n)) + \frac{1}{10k_1^4} \\
 & \leq \frac{1}{4} \lim_n \left(q^2(x_n - x) + q^2(x_n) + 6k_1 \delta_{k_1} + 2\sqrt{p^2(x_n - x)(q^2(x_n) + 3k_1 \delta_{k_1})} \right) \\
 & \quad + \frac{1}{10k_1^4} \\
 & < \frac{1}{2} \left(1 + \frac{1}{10k_1^4} + \sqrt{\left(1 - \frac{a^2}{k_1^2} + \frac{1}{10k_1^4} \right) \left(1 + \frac{1}{10k_1^4} \right)} \right) + \frac{1}{10k_1^4} \\
 & < \frac{1}{2} \left(1 + \frac{1}{10k_1^4} + 1 - \frac{3}{10k_1^4} \right) + \frac{1}{10k_1^4} \\
 & = 1
 \end{aligned}$$

which contradicts to Goebel-Karlovitz' lemma since $\frac{x}{2} \in K$. Hence $\liminf_n \|Jx_n\|_Y > 0$.

Thus, we can denote $\liminf_n \|Jx_n\|_Y = 4b$ for some positive real number b . Let k be a natural number which satisfies $k > \sup\{q(x) : x \in B_X\}$, $\frac{1}{k} < \inf\{q(x) : x \in S_X\}$ and $\frac{1}{3k} < b$. We can assume that there exists $p \in \mathcal{P}$ such that $q \in B(p_k, \delta_k)$. For each $x \in X$, we also obtain that

$$|q(x) - p_k(x)| \leq k\delta_k q(x) \tag{5.33}$$

and

$$|q^2(x) - p_k^2(x)| \leq 3k\delta_k. \tag{5.34}$$

Since $\frac{1}{3k} < b$, then

$$\begin{aligned}
 1 - \frac{b}{k} - \sqrt{1 + \frac{1}{10k^4} - \frac{2b}{k}} &= 1 - \frac{b}{k} - \sqrt{\left(1 - \frac{b}{k}\right)^2 - \frac{b^2}{k^2} + \frac{1}{10k^4}} \\
 &> 1 - \frac{b}{k} - \sqrt{\left(1 - \frac{b}{k}\right)^2 - \frac{1}{9k^4} + \frac{1}{10k^4}} \quad (5.35) \\
 &= 1 - \frac{b}{k} - \sqrt{\left(1 - \frac{b}{k}\right)^2 - \frac{1}{90k^4}}.
 \end{aligned}$$

Consider the real valued function $f(t) = t - \sqrt{t^2 - r}$, where r is a positive real number. By using the elementary calculus we can see that this function is decreasing. Since $1 - \frac{b}{k} < 1 - \frac{1}{3k^2}$, from (5.35) we have

$$1 - \frac{1}{3k^2} - \sqrt{\left(1 - \frac{1}{3k^2}\right)^2 - \frac{1}{90k^4}} < 1 - \frac{b}{k} - \sqrt{1 + \frac{1}{10k^4} - \frac{2b}{k}}. \quad (5.36)$$

It follows that

$$\begin{aligned}
 10k\delta_k &< \frac{1}{3k^2} \left(1 - \frac{1}{3k^2} - \sqrt{\left(1 - \frac{1}{3k^2}\right)^2 - \frac{1}{90k^4}}\right) \\
 &< \frac{b}{k} \left(1 - \frac{b}{k} - \sqrt{1 + \frac{1}{10k^4} - \frac{2b}{k}}\right). \quad (5.37)
 \end{aligned}$$

Moreover, note that for every $k \in \mathbb{N}$

$$\frac{b}{k} \left(1 - \frac{b}{k}\right) < \frac{1}{2}. \quad (5.38)$$

By (5.37) and (5.38),

$$0 < \frac{b}{k} \left(1 - \frac{b}{k} - \sqrt{1 + \frac{1}{10k^4} - \frac{2b}{k}}\right) - 10k\delta_k < \frac{1}{2}. \quad (5.39)$$

Let $\hat{c} = \max\left\{\frac{k(1+k\delta_k)(1-\frac{b}{k})}{\gamma}, \frac{1+k\delta_k}{1-k\delta_k}\right\}$ and fix $c = \frac{R(Y)-1+\hat{c}}{1+\hat{c}}$. It can be checked that $c < c_1$. From the choice of choosing δ_k , we have

$$6k\delta_k < \frac{1-c_1^2}{k^2}\gamma^2 < \frac{1-c^2}{k^2}\gamma^2. \quad (5.40)$$

Since $\gamma < \frac{1}{2}$, we also obtain that

$$0 < \frac{1 - c_1^2}{k^2} \gamma^2 - 6k\delta_k < \frac{1}{2}. \quad (5.41)$$

Apply lemma 5.11 for $t = 1 - \frac{b}{k}$ and

$$\epsilon < \min \left\{ \frac{1}{2} \left(\frac{b}{k} \left(1 - \frac{b}{k} - \sqrt{1 + \frac{1}{10k^4} - \frac{2b}{k}} \right) - 10k\delta_k \right), \frac{1}{2} \left(\frac{(1 - c^2)\gamma^2}{k^2} - 6k\delta_k \right) \right\}.$$

We can see that, by (5.39) and (5.41), ϵ is a positive number strictly less than $\frac{1}{2}$ and there exist a subsequence of $\{x_n\}$, denote again by $\{x_n\}$, and a sequence $\{z_n\}$ in K satisfy (i)-(iv).

Taking a subsequence of $\{z_n\}$ and using the separability of K , we can assume that $\lim_n q(z_n - y)$, $\lim_n \|J(z_n - y)\|_Y$ and $\lim_n p(z_n - y)$ do exist for every $y \in K$. Assume that $z = w - \lim z_n$. Since $\{z_n - x_n\}$ converges weakly to z , then by (iv) we obtain that

$$q(z) \leq \liminf_n q(z_n - x_n) \leq \frac{b}{k}. \quad (5.42)$$

Furthermore by using (iii), we have

$$\lim_n q(z_n - z) \leq \lim_n \lim_m q(z_n - z_m) \leq 1 - \frac{b}{k}. \quad (5.43)$$

It follows from (5.33) that

$$\begin{aligned} \frac{1}{k} \lim_n \|Jz_n\|_Y &\geq \frac{1}{k} \lim_n (\|Jx_n\|_Y - \|J(x_n - z_n)\|_Y) \\ &\geq \frac{4b}{k} - \lim_n p_k(x_n - z_n) \\ &\geq \frac{4b}{k} - (1 + k\delta_k) \lim_n q(x_n - z_n) \\ &\geq \frac{4b}{k} - (1 + k\delta_k) \frac{b}{k} \\ &\geq \frac{2b}{k}. \end{aligned}$$

Thus

$$\lim_n \|Jz_n\|_Y \geq 2b. \quad (5.44)$$

Moreover, we obtain

$$\begin{aligned} \lim_n \|J(z_n - z)\|_Y &\leq k \lim_n p_k(z_n - z) \\ &\leq k(1 + k\delta_k) \lim_n q(z_n - z) \\ &\leq k(1 + k\delta_k) \left(1 - \frac{b}{k}\right). \end{aligned} \quad (5.45)$$

Next we will split the proof into two cases:

Case I. Assume that $\|Jz\|_Y \leq \gamma$.

In this case, we have, by (5.44)

$$\lim_n \|J(z_n - z)\|_Y \geq \lim_n \|Jz_n\|_Y - \|Jz\|_Y \geq 2b - \gamma$$

which implies that

$$\begin{aligned} \left(1 - \frac{b}{k}\right)^2 &\geq \lim_n q^2(z_n - z) \\ &\geq \lim_n p_k^2(z_n - z) - 3k\delta_k \\ &\geq \lim_n \left(p^2(z_n - z) + \frac{1}{k^2} \|J(z_n - z)\|_Y^2\right) - 3k\delta_k \\ &\geq \lim_n p^2(z_n - z) + \frac{(2b - \gamma)^2}{k^2} - 3k\delta_k. \end{aligned}$$

Hence

$$\begin{aligned} \lim_n p^2(z_n - z) &\leq \left(1 - \frac{b}{k}\right)^2 - \frac{(2b - \gamma)^2}{k^2} + 3k\delta_k \\ &\leq \left(1 - \frac{b}{k}\right)^2 - \frac{b^2}{k^2} + 3k\delta_k \\ &\leq 1 - \frac{2b}{k} + 3k\delta_k \\ &\leq 1 - \frac{2b}{k} + \frac{1}{10k^4}. \end{aligned} \quad (5.46)$$

We also obtain, from (5.33) and (5.42)

$$p(z) \leq p_k(z) \leq (1 + k\delta_k)q(z) \leq (1 + k\delta_k)\frac{b}{k}. \quad (5.47)$$

Then by (ii), (5.34), (5.42), (5.43), (5.45), (5.46) and (5.47) we obtain that

$$\begin{aligned} (1 - \epsilon)^2 &\leq \lim_n q^2(z_n) \\ &\leq \lim_n p_k^2(z_n) + 3k\delta_k \\ &= \lim_n \left(p^2(z_n) + \frac{1}{k^2} \|Jz_n\|_Y^2 \right) + 3k\delta_k \\ &\leq \lim_n \left((p(z_n - z) + p(z))^2 + \frac{1}{k^2} (\|J(z_n - z)\|_Y + \|Jz\|_Y)^2 \right) + 3k\delta_k \\ &= \lim_n \left(p_k^2(z_n - z) + p_k^2(z) + 2 \left(p(z_n - z)p(z) + \frac{1}{k^2} \|J(z_n - z)\|_Y \|Jz\|_Y \right) \right) \\ &\quad + 3k\delta_k \\ &\leq \lim_n \left(q^2(z_n - z) + q^2(z) + 2 \left(p(z_n - z)p(z) + \frac{1}{k^2} \|J(z_n - z)\|_Y \|Jz\|_Y \right) \right) \\ &\quad + 9k\delta_k \\ &\leq \left(1 - \frac{b}{k} \right)^2 + \left(\frac{b}{k} \right)^2 + 2 \left(\frac{b}{k} (1 + k\delta_k) \sqrt{1 + \frac{1}{10k^4} - \frac{2b}{k}} + \frac{\gamma}{k} (1 + k\delta_k) \left(1 - \frac{b}{k} \right) \right) \\ &\quad + 9k\delta_k. \\ &\leq \left(1 - \frac{b}{k} \right)^2 + \left(\frac{b}{k} \right)^2 + 2 \left(\frac{b}{k} (1 + k\delta_k) \sqrt{1 + \frac{1}{10k^4} - \frac{2b}{k}} + \frac{\gamma}{k} (1 + k\delta_k) \right) + 9k\delta_k. \end{aligned} \quad (5.48)$$

According to (5.36), we obtain that

$$\begin{aligned} \gamma &< \frac{1}{6k} \left(1 - \frac{1}{3k^2} - \sqrt{\left(1 - \frac{1}{3k^2} \right)^2 - \frac{1}{90k^4}} \right) \\ &< \frac{b}{2} \left(1 - \frac{b}{k} - \sqrt{1 + \frac{1}{10k^4} - \frac{2b}{k}} \right). \end{aligned} \quad (5.49)$$

Thus, by (5.49) and (5.38), the inequality (5.48) gives a contradiction as follow:

$$\begin{aligned}
 (1 - \epsilon)^2 &< \left(1 - \frac{b}{k}\right)^2 + \left(\frac{b}{k}\right)^2 + \frac{2b}{k}(1 + k\delta_k)\sqrt{1 + \frac{1}{10k^4} - \frac{2b}{k}} + 9k\delta_k \\
 &\quad + \frac{b}{k}(1 + k\delta_k)\left(1 - \frac{b}{k} - \sqrt{1 + \frac{1}{10k^4} - \frac{2b}{k}}\right) \\
 &< \left(1 - \frac{b}{k}\right)^2 + \left(\frac{b}{k}\right)^2 + \frac{2b}{k}(1 + k\delta_k)\sqrt{1 + \frac{1}{10k^4} - \frac{2b}{k}} + 9k\delta_k \\
 &\quad + \frac{2b}{k}(1 + k\delta_k)\left(1 - \frac{b}{k} - \sqrt{1 + \frac{1}{10k^4} - \frac{2b}{k}}\right) - \frac{b}{k}\left(1 - \frac{b}{k} - \sqrt{1 + \frac{1}{10k^4} - \frac{2b}{k}}\right) \\
 &= \left(1 - \frac{b}{k}\right)^2 + \left(\frac{b}{k}\right)^2 + 2(1 + k\delta_k)\left(1 - \frac{b}{k}\right)\left(\frac{b}{k}\right) + 9k\delta_k \\
 &\quad - \frac{b}{k}\left(1 - \frac{b}{k} - \sqrt{1 + \frac{1}{10k^4} - \frac{2b}{k}}\right) \\
 &= 1 + 2k\delta_k\left(1 - \frac{b}{k}\right)\left(\frac{b}{k}\right) - \frac{b}{k}\left(1 - \frac{b}{k} - \sqrt{1 + \frac{1}{10k^4} - \frac{2b}{k}}\right) + 9k\delta_k \\
 &< 1 - \frac{b}{k}\left(1 - \frac{b}{k} - \sqrt{1 + \frac{1}{10k^4} - \frac{2b}{k}}\right) + 10k\delta_k \\
 &< 1 - 2\epsilon.
 \end{aligned}$$

Case II. Assume that $\|Jz\|_Y > \gamma$.

Due to equation (5.47),

$$\|Jz\|_Y \leq kp_k(z) \leq (1 + k\delta_k)b$$

which implies

$$\|J(z_n - z)\|_Y \geq \|Jz_n\|_Y - \|Jz\|_Y \geq 2b - (1 + k\delta_k)b = (1 - k\delta_k)b.$$

Since we have

$$0 < \gamma < \|Jz\|_Y \leq (1 + k\delta_k)b$$

and, by (5.45),

$$0 < (1 - k\delta_k)b \leq \|J(z_n - z)\|_Y \leq k(1 + k\delta_k) \left(1 - \frac{b}{k}\right),$$

applying lemma 5.15 with the sequence $\{J(z_n - z)\}$ and the element Jz of Y , noting that $\gamma \leq \alpha \leq (1 + k\delta_k)b$, $(1 - k\delta_k)b \leq \beta \leq k(1 + k\delta_k) \left(1 - \frac{b}{k}\right)$ and using that the map $t \mapsto \frac{R(Y)-1+t}{1+t}$ is increasing, then we obtain that

$$\lim_n \|J(z_n - z) + Jz\|_Y \leq c \left(\lim_n \|J(z_n - z)\|_Y + \|Jz\|_Y \right). \quad (5.50)$$

Then by (ii), (5.34) and (5.50), we have

$$\begin{aligned} (1 - \epsilon)^2 &\leq \lim_n q^2(z_n) \\ &\leq \lim_n p_k^2(z_n) + 3k\delta_k \\ &= \lim_n \left(p^2(z_n) + \frac{1}{k^2} \|Jz_n\|_Y^2 \right) + 3k\delta_k \\ &\leq \lim_n \left((p(z_n - z) + p(z))^2 + \frac{c^2}{k^2} (\|J(z_n - z)\|_Y + \|Jz\|_Y)^2 \right) + 3k\delta_k \\ &\leq \lim_n \left((p(z_n - z) + p(z))^2 + \frac{1}{k^2} (\|J(z_n - z)\|_Y + \|Jz\|_Y)^2 \right) + 3k\delta_k \\ &\quad - \frac{1 - c^2}{k^2} (\|J(z_n - z)\|_Y + \|Jz\|_Y)^2 \\ &= \lim_n \left(p_k^2(z_n - z) + p_k^2(z) + 2 \left(p(z_n - z)p(z) + \frac{1}{k^2} \|J(z_n - z)\|_Y \|Jz\|_Y \right) \right) \\ &\quad + 3k\delta_k - \frac{1 - c^2}{k^2} (\|J(z_n - z)\|_Y + \|Jz\|_Y)^2. \end{aligned} \quad (5.51)$$

For fixed $n \in \mathbb{N}$, denote $u = p_k(z_n - z)$, $v = p_k(z)$, $t = p(z_n - z)$ and $s = p(z)$. Then we have

$$\frac{1}{k} \|J(z_n - z)\|_Y = \sqrt{p_k^2(z_n - z) - p^2(z_n - z)} = \sqrt{u^2 - t^2}$$

and

$$\frac{1}{k} \|Jz\|_Y = \sqrt{p_k^2(z) - p^2(z)} = \sqrt{v^2 - s^2}.$$

Consider the function of two variables

$$f(t, s) = 2ts + 2\sqrt{u^2 - t^2}\sqrt{v^2 - s^2}.$$

By the elementary calculus, we obtain that

$$\max_{[0,u] \times [0,v]} f(t, s) = 2uv.$$

Hence

$$\begin{aligned} \lim_n \left(p_k^2(z_n - z) + p_k^2(z) + 2 \left(p(z_n - z)p(z) + \frac{1}{k^2} \|J(z_n - z)\|_Y \|Jz\|_Y \right) \right) \\ \leq \lim_n (p_k^2(z_n - z) + p_k^2(z) + 2p_k(z_n - z)p_k(z)) \\ = \lim_n (p_k(z_n - z) + p_k(z))^2 \\ \leq (1 + k\delta_k)^2 \lim_n (q(z_n - z) + q(z))^2 \\ \leq (1 + k\delta_k)^2. \end{aligned} \tag{5.52}$$

Then according to (5.51) and (5.52), it follows that

$$\begin{aligned} (1 - \epsilon)^2 &\leq (1 + k\delta_k)^2 + 3k\delta_k - \frac{1 - c^2}{k^2} (\|J(z_n - z)\|_Y + \|Jz\|_Y)^2 \\ &< (1 + k\delta_k)^2 + 3k\delta_k - \frac{1 - c^2}{k^2} \|Jz\|_Y^2 \\ &< 1 + 6k\delta_k - \frac{(1 - c^2)\gamma^2}{k^2} \\ &< 1 - 2\epsilon \end{aligned}$$

we reach a contradiction again.

Therefore the space (X, q) satisfies the w-FPP, this completes the proof. \square

Finally, we give some remarks for all generic fixed point results presented in this chapter.

Remark 5.18.

- (1) A natural question would be to study if the word “almost” can be removed from our main result. The answer is unknown even for a Hilbert space because it is unknown if any Banach space isomorphic to a Hilbert space satisfies the FPP. In fact, it is unknown if there exists a reflexive Banach space which do not have the FPP (see [75], [77]).
- (2) It would be also interesting to determine the size of the set of all equivalent norms which do not satisfy the FPP or the w-FPP (if non-empty). We can say that, in general, this set is not dense. This is due to the following fact: There are some results proving that several properties of a Banach space X implying the FPP are stable, in the sense, that if Y is isomorphic to X and the Banach-Mazur distance between X and Y is small, then Y shares this property. For instance, it is known that if H is a Hilbert space and X is a renorming of H such that $\rho(X, H) < .37\dots$, then X satisfies the FPP.
- (3) Recall that, in general, a non-reflexive Banach space cannot be renormed to satisfy the FPP. It would be interesting to determine those non-reflexive Banach spaces, such that our result holds for them. In particular, it would be interesting to know if this result holds for ℓ_1 because this is the unique non-reflexive Banach space which is known to have a renorming which satisfies the FPP [55].
- (4) Generic results can be useful to obtain standard results. For instance Corollary 5.14 assures that every reflexive space can be renormed in such a way that for every non-expansive mapping (for the new norm) defined from a convex bounded closed subset C of X into C , the set of fixed points is convex and nonempty. It seems to be very difficult to give a direct proof of this result.

5.4 Generic fixed point results on space of continuous functions $C(K)$

Let K be a compact metrizable space and let $C(K)$ be the Banach space of all real continuous functions $x = x(t)$ defined on K with the maximum norm $\|x\|_\infty = \max_{t \in K} |x(t)|$. It is well known (see [65], [84]) that many topological properties of K are strongly related to geometrical properties of $C(K)$. It is proved (in [65] Main Theorem) that $C(K)$ isometrically contains $C[0, 1]$, which fails to have the w-FPP by Alpaachs example, if and only if K is a compact set which is not scattered. Thus $C(K)$ fails to have the w-FPP if K is not scattered compact. We recall the definition of scattered compact set.

Definition. 5.19. Let K be a topological space and A a subset of K . The set A is said to be *perfect* if it is closed and has no isolated points, i.e., A is equal to the set of its own accumulation points. The space K is said to be *scattered* if it contains no perfect nonempty subset.

If A is a subset of a topological space M , the derived set of A is the set $A^{(1)}$ of all accumulation points of A . If α is an ordinal number, we define the α^{th} -derived set by transfinite induction:

$$A^{(0)} = A \quad A^{(\alpha+1)} = (A^{(\alpha)})^{(1)} \quad A^{(\lambda)} = \bigcap_{\alpha < \lambda} A^{(\alpha)}$$

when λ is a limit ordinal.

We first show that in some cases the space $C(K)$ can be renormed to have an equivalent norm $\|\cdot\|$ such that the Garcia-Falset coefficient $R(C(K), \|\cdot\|) < 2$, hence it satisfies the w-FPP. We need the following technical lemma.

Lemma 5.20. *Let A be the subset of \mathbb{R}^n formed by all vectors $x = (\alpha_1, \dots, \alpha_n)$ such that $0 \leq \alpha_n \leq \dots \leq \alpha_1$, $\alpha_1^2 + \dots + \alpha_n^2 \leq 1$, and B the subset of \mathbb{R}^n formed by all vectors $y = (\beta_1, \dots, \beta_{n-1}, 0)$ such that $0 \leq \beta_{n-1} \leq \dots \leq \beta_1$,*

$\beta_1^2 + \dots + \beta_{n-1}^2 \leq 1$. Define $\phi : A \times B \rightarrow \mathbb{R}$ by

$$\phi(x, y) = \max\{\alpha_1^2, (\beta_1 + \alpha_2)^2\} + \dots + \max\{\alpha_{n-1}^2, (\beta_{n-1} + \alpha_n)^2\} + \alpha_n^2.$$

Then, $\max\{\phi(x, y) : x \in A, y \in B\} \leq 4 - n^{-1}$.

Proof. Since $A \times B$ is a compact subset of \mathbb{R}^{2n} and ϕ is a continuous function, we know that ϕ attains a maximum M at a point in $A \times B$. We will check that $M \leq 4 - n^{-1}$. First, we will prove that $M \leq 4 - n^{-1}$ if for some $k \in \{1, \dots, n-1\}$ we have $\max\{\alpha_k, (\beta_k + \alpha_{k+1})\} = \alpha_k$. Indeed, assume that for some $k \in \{1, \dots, n-1\}$ we have that $\max\{\alpha_j, \beta_j + \alpha_{j+1}\} = \beta_j + \alpha_{j+1}$ for $j = 1, \dots, k-1$ and $\max\{\alpha_k, \beta_k + \alpha_{k+1}\} = \alpha_k$. Thus,

$$\begin{aligned} \phi(x, y) &\leq (\beta_1 + \alpha_2)^2 + \dots + (\beta_{k-1} + \alpha_k)^2 \\ &\quad + \alpha_k^2 + (\beta_{k+1} + \alpha_{k+1})^2 + \dots + (\beta_{n-1} + \alpha_n)^2 + \alpha_n^2 \\ &= \|u + v\|^2 \end{aligned}$$

where $u = (\beta_1, \dots, \beta_{k-1}, 0, \beta_{k+1}, \dots, \beta_{n-1}, 0)$, $v = (\alpha_2, \dots, \alpha_k, \alpha_k, \alpha_{k+1}, \dots, \alpha_{n-1}, \alpha_n)$ and $\|\cdot\|$ denotes the Euclidean norm. Since $\|v\|^2 = \|x\|^2 + \alpha_k^2 - \alpha_1^2$ and $\|u - v\| \geq \alpha_k$, the parallelogram identity implies

$$\begin{aligned} \|u + v\|^2 &\leq 2 + 2\|x\|^2 + 2\alpha_k^2 - 2\alpha_1^2 - \alpha_k^2 \\ &\leq 2 + 2\|x\|^2 - \alpha_1^2 \\ &\leq 2 + \left(2 - \frac{1}{n}\right) \|x\|^2 \\ &\leq 4 - \frac{1}{n}. \end{aligned}$$

Now, consider the case $\max\{\alpha_k, \beta_k + \alpha_{k+1}\} = \beta_k + \alpha_{k+1}$ for every $k = 1, \dots, n-1$. Then $\phi(x, y) = (\beta_1 + \alpha_2)^2 + \dots + (\beta_{n-1} + \alpha_n)^2 + \alpha_n^2 = \|u + y\|^2$

where $u = (\alpha_2, \dots, \alpha_n, \alpha_n)$. Since $\|u - y\| \geq \alpha_n$, the parallelogram identity gives us

$$\|u+y\|^2 \leq 2+2\|u\|^2 - \alpha_n^2 \leq 2+2\|x\|^2 + 2\alpha_n^2 - 2\alpha_1^2 - \alpha_n^2 \leq 2+2\|x\|^2 - \alpha_1^2 \leq 4 - \frac{1}{n}.$$

□

Theorem 5.21. *Assume that $K^{(m)} = \emptyset$. Then, there exists a norm $\|\cdot\|$ equivalent to the supremum norm $\|\cdot\|_\infty$ such that $R(C(K), \|\cdot\|) \leq \sqrt{4 - m^{-1}}$.*

Proof. For any $x \in C(K)$ denote by $\alpha_k = \max\{|x(t)| : t \in K^{(k-1)}\}$ and define $\|x\|^2 = \alpha_1^2 + \dots + \alpha_m^2$. It is clear that this norm is equivalent to the supremum norm. Assume that $x \in C(K)$, $\|x\| \leq 1$ and $\{x_n\}$ is a weakly null sequence in $C(K)$ such that $\|x_n\| \leq 1$ for all $n \in \mathbb{N}$. We will prove that $\limsup_n \|x + x_n\| \leq \sqrt{4 - m^{-1}}$. Let ϵ be an arbitrary positive number. By induction, we will define some subsets of K depending on ϵ . Denote $L_0 = K$. Since $K^{(m-1)}$ is a finite set, there exists an open subset U_1 of K , containing $K^{(m-1)}$, such that $|x(t)| \leq \alpha_m + \epsilon$ for $t \in \overline{U_1}$. Denote $L_1 = K \setminus U_1$. We have $L_1^{(m-1)} = \emptyset$ which implies that $L_1^{(m-2)}$ is a finite set contained in $K^{(m-2)}$. Thus, there exists an open set U_2 , containing $L_1^{(m-2)}$ such that $|x(t)| \leq \alpha_{m-1} + \epsilon$ for every $t \in \overline{U_2}$. We define $L_2 = L_1 \setminus U_2$. Then, $L_2^{(m-3)}$ is a finite subset of $K^{(m-3)}$. By induction, we can assume that we have defined open sets U_1, \dots, U_k and compact sets L_1, \dots, L_k such that for $j = 1, \dots, k$ we have $L_j = L_{j-1} \setminus U_j$, $L_j^{(m-j-1)}$ is a finite subset of $K^{(m-j-1)}$, $L_{j-1}^{(m-j)} \subset U_j$ and $|x(t)| \leq \alpha_{m-j+1} + \epsilon$ for every $t \in \overline{U_j}$. Since $L_k^{(m-k-1)}$ is a finite subset of $K^{(m-k-1)}$, there exists an open set U_{k+1} , containing $L_k^{(m-k-1)}$ such that $|x(t)| \leq \alpha_{m-k} + \epsilon$ for every $t \in \overline{U_{k+1}}$. We have that $L_{k+1}^{(m-k-1)} \subset L_k^{(m-k-1)} \setminus U_{k+1} = \emptyset$, which implies that $L_{k+1}^{(m-k-2)}$ is a finite set. Thus, we can construct open sets U_1, \dots, U_{m-1} and compact sets L_1, \dots, L_{m-1} such that for $j = 1, \dots, m-1$ we have $L_j = L_{j-1} \setminus U_j$, $L_j^{(m-j-1)}$ is a finite subset of $K^{(m-j-1)}$, $L_{j-1}^{(m-j)} \subset U_j$ and $|x(t)| \leq \alpha_{m-j+1} + \epsilon$ for every $t \in \overline{U_j}$.

Moreover, $L_{m-1}^{(m-m)} = L_{m-1}$ is a finite set. Since the sequence $\{x_n\}$ is weakly null, we can assume $|x_n(t)| < \epsilon$ for every $t \in L_j^{(m-j-1)}$ for $j = 0, \dots, m-1$ and n large enough. Denote $\beta_j = \beta_j(n) = \max \{|x_n(t)| : t \in K^{(j-1)}\}$. We claim that $\|x + x_n\|^2 \leq \phi(x, y) + O(\epsilon)$, where ϕ is the function in lemma 5.20 and $O(\epsilon) \rightarrow 0$ as $\epsilon \rightarrow 0$. Since $(A \cup B)' = A' \cup B'$ and $K \subset L_j \cup \overline{U_1} \cup \dots \cup \overline{U_j}$ for $j = 1, \dots, m-1$, we have

$$K^{(m-j-1)} \subset L_j^{(m-j-1)} \bigcup_{i=1}^j \overline{U_i}.$$

We should compute $\max \{|(x + x_n)(t)| : t \in K^{(m-j-1)}\}$ for $j = 1, \dots, m-1$. We have two possibilities:

(1) Assume that $t \in \bigcup_{i=1}^j \overline{U_i}$. In this case we have that $|x(t)| \leq \alpha_{m-j+1} + \epsilon$ and so, $|x(t) + x_n(t)| \leq \alpha_{m-j+1} + \beta_{m-j} + \epsilon$.

(2) Assume that $t \in K^{(m-j-1)} \setminus \bigcup_{i=1}^j \overline{U_i}$. In this case $t \in L_j^{(m-j-1)}$ which implies that $|x_n(t)| \leq \epsilon$ and $|x(t) + x_n(t)| \leq \alpha_{m-j} + \epsilon$.

Thus,

$$\max \{|x(t) + x_n(t)| : t \in K^{(m-j-1)}\} \leq \max \{\alpha_{m-j+1} + \beta_{m-j}, \alpha_{m-j}\} + \epsilon$$

for $j = 1, \dots, m-1$ and

$$\max \{|x(t) + x_n(t)| : t \in K^{(m-1)}\} \leq \alpha_m + \epsilon.$$

Hence,

$$\begin{aligned} \|x + x_n\|^2 &\leq \max\{(\alpha_1 + \epsilon)^2, (\beta_1 + \alpha_2 + \epsilon)^2\} \\ &+ \dots + \max\{(\alpha_{m-1} + \epsilon)^2, (\beta_{m-1} + \alpha_m + \epsilon)^2\} + (\alpha_m + \epsilon)^2 \\ &\leq \phi(x, y) + O(\epsilon) \end{aligned}$$

where $O(\epsilon)$ tends to 0 as $\epsilon \rightarrow 0$. Thus, by Lemma 5.20 we have that $\limsup_n \|x + x_n\| \leq \sqrt{4 - m^{-1}} + O(\epsilon)$. Since ϵ is arbitrary we easily obtain the conclusion. \square

Remark 5.22. By Cantor-Bendixson Theorem ([84], page 148), it is known that $K^{(\alpha)} = \emptyset$ for some ordinal number α if and only if K is scattered. Recall $C(K)$ fails to have the w-FPP if K is not scattered.

Assume that Γ is an uncountable set. We can assume that Γ is endowed with the discrete topology. Let K be the one-point compactification of Γ . Then, $(c_0(\Gamma), \|\cdot\|_\infty)$ is isomorphic to $(C(K), \|\cdot\|_\infty)$ by defining $S : C(K) \rightarrow c_0(\Gamma)$ by $S(x(\gamma)) = (x(\gamma) - x(\infty))$. Thus, any space which can be continuously embedded in $(c_0(\Gamma), \|\cdot\|_\infty)$, can be also embedded in $(C(K), \|\cdot\|_\infty)$, where $K^{(2)} = \emptyset$. From, Theorem 5.3.2 we obtain the following result which strictly improves the result of T. Domínguez-Benavides (Theorem 5.2.1) saying that any Banach space that can be continuously embedded into $c_0(\Gamma)$ has an equivalent norm with the w-FPP.

Corollary 5.23. *Let X be a Banach space which can be continuously embedded in $(C(K), \|\cdot\|_\infty)$ for some compact set K such that $K^{(\omega)} = \emptyset$. Then, X can be renormed to satisfy the w-FPP.*

K. Ciesielski and R. Pol [12] have constructed a (non-metrizable) compact set K which satisfies $K^{(3)} = \emptyset$, however, there is no weak-to-weak continuous injective map, in particular no bounded linear injective map from $C(K)$ to any $c_0(\Gamma)$. Hence Corollary 5.23 is a strict improvement of T. Domínguez-Benavides result.

Due to Theorem 5.7 and Theorem 5.21, we also obtain the generic fixed point result on the space $C(K)$ which can be regarded as an improvement of the result in [22].

Corollary 5.24. *Assume that $K^{(\omega)} = \emptyset$ and \mathcal{P} is the set of all norms in $C(K)$ which are equivalent to the supremum norm with the metric $\rho(p, q) = \sup\{|p(x) - q(x)| : x \in B\}$. Then, there exists a σ -porous (in fact, σ -directionally porous) set $\mathcal{A} \subset \mathcal{P}$ such that if $q \in \mathcal{P} \setminus \mathcal{A}$ the space $(C(K), q)$ satisfies the w -FPP.*

Chapter 6

Generic multi-valued fixed point property on renormings of a Banach space

From Chapter 3, in the classic sense, we know that almost all non-expansive multi-valued mappings (in the sense of porosity) do have a fixed point (Theorem 3.12). In this chapter we consider the multi-valued fixed point property on the set of all renormings of a Banach space. Furthermore, we determine the multi-valued fixed point property renormability and genericity on a Banach space by using the value of its Szlenk index.

6.1 Generic multi-valued fixed point result on renormings of a reflexive Banach space

Regarding to some results in Chapter 4 and Chapter 5, we know that every reflexive Banach space can be renormed to have the FPP and also almost all its renormings satisfy the FPP. What happen in the case of the MFPP?

Concerning with renorming theory and genericity, two natural questions show up:

- (i) Does every reflexive Banach space admit an equivalent norm which satisfy the MFPP?
- (ii) If a reflexive Banach space admits an equivalent norm which satisfy the MFPP, then do almost all its renormings also satisfy the MFPP?

In the case of separable reflexive spaces, the answers to both questions are known by using the concept of uniform convexity in every direction. It is known, according to Lim's result [54], every UCED Banach space does satisfy the weak multi-valued fixed point property (w-MFPP). And since every separable Banach space is UCED renormable, hence by Theorem 4.1.6, we obtain the following generic multi-valued fixed point result:

Corollary 6.1. *Let X be a separable reflexive Banach space and \mathcal{P} the set of all equivalent norms on X equipped with the metric ρ . Then, there exists a σ -porous subset \mathcal{R} of \mathcal{P} , such that for all $p \in \mathcal{R}$, the space (X, p) satisfies the MFPP.*

The situation is rather different in the case of non-separable reflexive spaces. We only know that Banach spaces with some geometrical properties, for instance nearly uniform convexity, uniform smoothness and uniform Opial property, satisfy the w-MFPP. We attend to the nearly uniform convexity. By a consequence of the result by T. Domínguez-Benavides and P. Lorenzo [23], a nearly uniformly convex reflexive space does have the MFPP. Let us recall the definition of nearly uniform convexity which was introduced by R. Huff in [40].

Definition. 6.2. A Banach space X is said to be *nearly uniformly convex* (NUC) if for every $\epsilon > 0$ there exists $\delta > 0$ such that if (x_n) is a sequence in B_X such that $\|x_n - x_m\| > \epsilon$ for $n \neq m$, then $co\{x_n\} \cap B(0, 1 - \delta) \neq \emptyset$.

On the other hand, the modulus of convexity of a Banach space X is defined as follows:

$$\Delta_X(\epsilon) = \inf \left\{ 1 - \|x\| : (x_n) \subset B_X, x_n \rightharpoonup x, \liminf_n \|x_n - x\| \geq \epsilon \right\}$$

and the characteristic of convexity as

$$\Delta_0(X) = \sup \{ \epsilon > 0 : \Delta_X(\epsilon) = 0 \}.$$

Remark 6.3. The modulus of convexity of X can also be written as

$$\Delta_X(\epsilon) = \inf \left\{ 1 - \|x\| : (x_n) \subset B_X, x_n \rightharpoonup x, \liminf_n \|x_n - x\| \geq \epsilon, \lim_n \|x_n\| = 1 \right\}$$

because if $\lim_n \|x_n\| < 1$, there exists a subsequence of (x_n) , denote again by (x_n) , satisfying $x_n \rightharpoonup x$, $\liminf_n \|x_n - x\| \geq \epsilon$ and $\lim_n \|x_n\| = r < 1$. Thus the sequence $(y_n) = \left(\frac{x_n}{\|x_n\|} \right)$ satisfies $(y_n) \subset B(0, 1)$, $\lim_n \|y_n\| = 1$, $y_n \rightharpoonup \frac{x}{r}$, $\liminf_n \left\| y_n - \frac{x}{r} \right\| \geq \frac{\epsilon}{r} > \epsilon$ and $\|x\| \leq \frac{\|x\|}{r}$.

It is known that a Banach space X is NUC if and only if X is reflexive and $\Delta_X(\epsilon) > 0$ for each $\epsilon > 0$, or equivalently $\Delta_0(X) = 0$.

Theorem 6.4 (Domínguez-Benavides and Lorenzo). *Let X be a Banach space with $\Delta_0(X) < 1$. Then X satisfies the w-MFPP.*

In particular, every NUC Banach space has the MFPP.

Remark 6.5. It would be nice if every reflexive space could be renormed to have an equivalent NUC norm. The first question would be positively solved. Unfortunately, it was proved in [40] that there exist a reflexive space which does not admit an equivalent NUC norm.

By following the concept of nearly uniform convexity, we could give a partial answer to the second question as well. In order to state our result, we prove some needed lemmas. Let $(X, \|\cdot\|)$ be a Banach space and \mathcal{P} the set of all equivalent norms on X equipped with the metric ρ .

Lemma 6.6. For $p \in \mathcal{P}$, denote by $m_p = \inf \left\{ \frac{1}{p(x)} : x \in S_X \right\}$ and $M_p = \sup \left\{ \frac{1}{p(x)} : x \in S_X \right\}$ and define $p_\lambda(x) = p(x) + \lambda\|x\|$, $\lambda \in (0, 1)$. Then

$$\Delta_{p_\lambda}(\epsilon) \geq \frac{\lambda m_p}{1 + \lambda m_p} \Delta_{\|\cdot\|} \left(\frac{\epsilon m_p}{M_p(1 + \lambda m_p)} \right)$$

for every $\epsilon > 0$.

In particular, the space (X, p_λ) is NUC for every $\lambda \in (0, 1)$ whenever the space $(X, \|\cdot\|)$ is.

Proof. Given $\epsilon > 0$ and let $\{x_n\}$ be a sequence in $B_{(X, p_\lambda)}$ such that $\lim_n p_\lambda(x_n) = 1$, $x_n \rightharpoonup x$ and $\liminf_n p_\lambda(x_n - x) \geq \epsilon$. Note that for each $x \in X$,

$$m_p p(x) \leq \|x\| \leq M_p p(x). \quad (6.1)$$

By taking a subsequence, we can assume that $\lim_n \|x_n\|$ and $\lim_n p(x_n)$ do exist. Denote by $a = \lim_n \|x_n\|$ and $b = \lim_n p(x_n)$. Then, the sequence $\left(\frac{x_n}{\|x_n\|} \right) \subset B_{(X, \|\cdot\|)}$ and $\frac{x_n}{\|x_n\|} \rightharpoonup \frac{x}{a}$. Moreover, since

$$\begin{aligned} \epsilon &\leq \liminf_n p_\lambda(x_n - x) \\ &= \liminf_n (p(x_n - x) + \lambda\|x_n - x\|) \\ &\leq \frac{1 + \lambda m_p}{m_p} \liminf_n \|x_n - x\| \end{aligned}$$

we also have

$$\liminf_n \left\| \frac{x_n}{\|x_n\|} - \frac{x}{a} \right\| = \liminf_n \left\| \frac{x_n - x}{a} \right\| \geq \frac{\epsilon m_p}{a(1 + \lambda m_p)}.$$

Hence

$$\|x\| \leq a \left(1 - \Delta_{\|\cdot\|} \left(\frac{\epsilon m_p}{a(1 + \lambda m_p)} \right) \right).$$

It follows from the weak semi-continuity of the norm p that

$$p(x) \leq \liminf p(x_n) = b = 1 - \lambda a.$$

From (6.1), we have

$$m_p(1 - \lambda a) \leq a \leq M_p(1 - \lambda a)$$

which implies

$$a \geq \frac{m_p}{1 + \lambda m_p} \quad \text{and} \quad a \leq \frac{M_p}{1 + \lambda M_p} \leq M_p.$$

Thus we obtain that

$$\begin{aligned} p_\lambda(x) &= p(x) + \lambda \|x\| \\ &\leq 1 - \lambda a + \lambda a \left(1 - \Delta_{\|\cdot\|} \left(\frac{\epsilon m_p}{a(1 + \lambda m_p)} \right) \right) \\ &= 1 - \lambda a \Delta_{\|\cdot\|} \left(\frac{\epsilon m_p}{a(1 + \lambda m_p)} \right) \\ &\leq 1 - \frac{\lambda m_p}{1 + \lambda m_p} \Delta_{\|\cdot\|} \left(\frac{\epsilon m_p}{M_p(1 + \lambda m_p)} \right). \end{aligned}$$

Thus

$$\Delta_{p_\lambda}(\epsilon) \geq \frac{\lambda m_p}{1 + \lambda m_p} \Delta_{\|\cdot\|} \left(\frac{\epsilon m_p}{M_p(1 + \lambda m_p)} \right).$$

□

In the following lemma we prove a continuity result for $\Delta_p(\epsilon)$ as a function of $p \in \mathcal{P}$.

Lemma 6.7. *Assume that $p, q \in \mathcal{P}$ such that $\rho(p, q) < \delta$ for some $\delta > 0$, then*

$$\Delta_q(\epsilon) \geq \Delta_p \left(\epsilon \frac{1 - \delta M_p}{1 + \delta M_p} \right) - \frac{2\delta M_p}{1 - \delta M_p}.$$

Proof. Denote m_p and M_p as in Lemma 6.6. Since $\rho(p, q) < \delta$, for each $x \in X$ we have

$$|p(x) - q(x)| < \delta \|x\| \leq \delta M_p p(x). \tag{6.2}$$

Fix $\epsilon > 0$ and let $\{x_n\}$ be a sequence in $B_{(X,q)}$ such that $x_n \rightharpoonup x$ and $\liminf_n q(x_n - x) \geq \epsilon$. Since by (6.2) we have

$$\lim_n p(x_n) \leq \frac{1}{1 - \delta M_p} \lim_n q(x_n),$$

it follows that $\{(1 - \delta M_p)x_n\} \subset B_{(X,p)}$. It is clear that $(1 - \delta M_p)x_n \rightharpoonup (1 - \delta M_p)x$. Again by using (6.2) we obtain that

$$\liminf_n p(x_n - x) \geq \frac{1}{1 + \delta M_p} \liminf_n q(x_n - x) \geq \frac{\epsilon}{1 + \delta M_p}.$$

Hence

$$(1 - \delta M_p) \liminf_n p(x_n - x) \geq \epsilon \frac{1 - \delta M_p}{1 + \delta M_p}.$$

It follows from the definition of the modulus of convexity that

$$p(x) \leq \frac{1}{1 - \delta M_p} \left(1 - \Delta_p \left(\epsilon \frac{1 - \delta M_p}{1 + \delta M_p} \right) \right). \quad (6.3)$$

By (6.2) and (6.3), we have

$$\begin{aligned} q(x) &\leq \frac{1 + \delta M_p}{1 - \delta M_p} \left(1 - \Delta_p \left(\epsilon \frac{1 - \delta M_p}{1 + \delta M_p} \right) \right) \\ &= 1 - \left(\frac{1 + \delta M_p}{1 - \delta M_p} \Delta_p \left(\epsilon \frac{1 - \delta M_p}{1 + \delta M_p} \right) - \frac{2\delta M_p}{1 - \delta M_p} \right). \end{aligned}$$

Thus

$$\begin{aligned} \Delta_q(\epsilon) &\geq \frac{1 + \delta M_p}{1 - \delta M_p} \Delta_p \left(\epsilon \frac{1 - \delta M_p}{1 + \delta M_p} \right) - \frac{2\delta M_p}{1 - \delta M_p} \\ &\geq \Delta_p \left(\epsilon \frac{1 - \delta M_p}{1 + \delta M_p} \right) - \frac{2\delta M_p}{1 - \delta M_p} \end{aligned}$$

□

Theorem 6.8. *Assume that $(X, \|\cdot\|)$ is a NUC Banach space. Then there exists a residual subset \mathcal{R} of \mathcal{P} such that for every $q \in \mathcal{R}$, the space (X, q) is NUC.*

Proof. For each $p \in \mathcal{P}$ denote $m_p = \inf \left\{ \frac{1}{p(x)} : x \in S_X \right\}$ and $M_p = \sup \left\{ \frac{1}{p(x)} : x \in S_X \right\}$ and for each $\lambda \in (0, 1)$ define

$$p_\lambda(x) = p(x) + \lambda\|x\|, \quad x \in X.$$

Choose

$$\delta = \delta(p, \lambda, n) < \min \left\{ \frac{1}{3M_p}, \frac{\lambda m_p}{8M_p(1 + \lambda m_p)} \Delta_{\|\cdot\|} \left(\frac{m_p}{2nM_p(1 + \lambda m_p)} \right) \right\}.$$

Define

$$\mathcal{R}_n = \bigcup_{p \in \mathcal{P}, \lambda \in (0, 1)} B(p_\lambda, \delta(p, \lambda, n))$$

and take

$$\mathcal{R} = \bigcap_{n=1}^{\infty} \mathcal{R}_n.$$

It is clear that \mathcal{R} is residual, indeed, it is a dense G_δ subset of \mathcal{P} . Let $q \in \mathcal{R}$. We shall prove that the space (X, q) is NUC. Since $q \in \mathcal{R}$, for each $n \in \mathbb{N}$, there exist $p = p(n) \in \mathcal{P}$ and $\lambda = \lambda(n) \in (0, 1)$ such that $q \in B(p_\lambda, \delta(p, \lambda, n)) = B(p_\lambda, \delta)$. Since $\rho(p_\lambda, q) \leq \delta$, by Lemma 6.7, for every $n \in \mathbb{N}$ we have

$$\Delta_q \left(\frac{1}{n} \right) \geq \Delta_{p_\lambda} \left(\frac{1 - \delta M_{p_\lambda}}{n(1 + \delta M_{p_\lambda})} \right) - \frac{2\delta M_{p_\lambda}}{1 - \delta M_{p_\lambda}}. \quad (6.4)$$

For each $x \in S_X$, we have

$$\frac{1}{p_\lambda(x)} = \frac{1}{p(x) + \lambda} < \frac{1}{p(x)}$$

which implies

$$m_{p_\lambda} \leq m_p \quad \text{and} \quad M_{p_\lambda} \leq M_p.$$

Hence

$$\Delta_q \left(\frac{1}{n} \right) \geq \Delta_{p_\lambda} \left(\frac{1 - \delta M_p}{n(1 + \delta M_p)} \right) - \frac{2\delta M_p}{1 - \delta M_p}. \quad (6.5)$$

Since $p_\lambda = p + \lambda \|\cdot\|$, it follows from Lemma 6.6 that

$$\Delta_{p_\lambda} \left(\frac{1 - \delta M_p}{n(1 + \delta M_p)} \right) \geq \frac{\lambda m_p}{1 + \lambda m_p} \Delta_{\|\cdot\|} \left(\frac{m_p}{M_p(1 + \lambda m_p)} \frac{1 - \delta M_p}{n(1 + \delta M_p)} \right). \quad (6.6)$$

By the choice of $\delta < \frac{1}{3M_p}$, it is straightforward to prove that

$$\frac{1 - \delta M_p}{1 + \delta M_p} > \frac{1}{2}. \quad (6.7)$$

Equation (6.7) implies that

$$1 - \delta M_p > \frac{1}{2}. \quad (6.8)$$

According to (6.5), (6.6), (6.7), (6.8) and the choice of δ , we obtain that

$$\begin{aligned} \Delta_q \left(\frac{1}{n} \right) &\geq \frac{\lambda m_p}{1 + \lambda m_p} \Delta_{\|\cdot\|} \left(\frac{m_p}{M_p(1 + \lambda m_p)} \frac{1 - \delta M_p}{n(1 + \delta M_p)} \right) - \frac{2\delta M_p}{1 - \delta M_p} \\ &\geq \frac{\lambda m_p}{1 + \lambda m_p} \Delta_{\|\cdot\|} \left(\frac{m_p}{2nM_p(1 + \lambda m_p)} \right) - 4\delta M_p \\ &> \frac{\lambda m_p}{1 + \lambda m_p} \Delta_{\|\cdot\|} \left(\frac{m_p}{2nM_p(1 + \lambda m_p)} \right) \\ &\quad - \frac{\lambda m_p}{2(1 + \lambda m_p)} \Delta_{\|\cdot\|} \left(\frac{m_p}{2nM_p(1 + \lambda m_p)} \right) \\ &= \frac{\lambda m_p}{2(1 + \lambda m_p)} \Delta_{\|\cdot\|} \left(\frac{m_p}{2nM_p(1 + \lambda m_p)} \right) \\ &> 0. \end{aligned}$$

It is equivalent to say that $\Delta_q(\epsilon) > 0$ for every $\epsilon > 0$, thus the space (X, q) is NUC.

□

6.2 Generic multi-valued fixed point property concerning with the Szlenk index

In this section, we use the value of the Szlenk index of a Banach space to determine its multi-valued fixed point renormability and genericity. The Szlenk index $S_z(X)$ is an ordinal number which was introduced by W. Szlenk ([86]) to prove that there is no separable reflexive Banach space universal for the class of all separable reflexive Banach spaces. Later this index has been used in various areas of the geometry of Banach spaces (see [52]). By following the survey [52] we consider the definition of Szlenk index which is more general than the Szlenk original index. However, both definitions are equivalent for separable spaces which do not contain ℓ_1 ([50] Proposition 3.3).

Definition. 6.9. Let X be a Banach space and X^* its dual. For any closed bounded subset $A \subset X^*$ and $\epsilon > 0$, we define a *Szlenk derivation* by

$$\langle A \rangle'_\epsilon = \{x^* \in A : \forall U \text{ } \omega^*\text{-neighborhood of } x^*, \text{ diam}(A \cap U) > \epsilon\}.$$

By iteration, the set $\langle A \rangle_\epsilon^\alpha$ are defined for any ordinal α , taking intersection in the case of limit ordinals, i.e., $\langle A \rangle_\epsilon^\alpha = \bigcap_{\beta < \alpha} \langle A \rangle_\epsilon^\beta$. The indices $S_z(X)_\epsilon$ are ordinal numbers defined as

$$S_z(X)_\epsilon = \inf \{ \alpha : \langle B_{X^*} \rangle_\epsilon^\alpha = \emptyset \}$$

if such an ordinal exists. Otherwise we write $S_z(X)_\epsilon = \infty$. The *Szlenk index* is defined by $S_z(X) = \sup_{\epsilon > 0} S_z(X)_\epsilon$.

In connection with the Szlenk index, it is not difficult to see that if the dual space X^* is NUC, then $S_z(X)_\epsilon$ is finite for every $\epsilon > 0$. We can see this through the concept of uniform Kadec Klee which was first introduced also by R. Huff in [40].

Definition. 6.10. A Banach space X is said to be *uniformly Kadec Klee (UKK)* if for every $\epsilon > 0$ there exists $\delta > 0$ such that if (x_n) is a sequence in the unit ball B_X such that $\|x_n - x_m\| > \epsilon$ for $n \neq m$ and (x_n) converges weakly to x , then $x \in B(0, 1 - \delta)$.

Later G. Lancien formulated the different concept of UKK in [51]. However Lanciens UKK definition coincides with Huffs definition in reflexive spaces in the following way.

Lemma 6.11. *Let X be a reflexive Banach space. Then X is UKK if and only if for every $\epsilon > 0$ there is $\theta(\epsilon) \in (0, 1)$ such that every $x \in B_X$ with $\|x\| > 1 - \theta(\epsilon)$ has a weak open neighborhood U with $\text{diam}(B_X \cap U) < \epsilon$.*

Proof. Assume that X is UKK. Let $\epsilon > 0$ and $x \in B_X$ such that for every weak neighborhood U of x , $\text{diam}(U \cap B_X) > \epsilon$. We shall prove that $\|x\| < 1 - \delta$ where δ is a positive real number depending on ϵ .

First, note that $x \in \overline{B_X \setminus B(x, \frac{\epsilon}{2})}^\omega$. Otherwise, if $V = X \setminus \left(\overline{B_X \setminus B(x, \frac{\epsilon}{2})}^\omega \right)$, then V is a weakly open set which contains x and $\text{diam}(V \cap B_X) < \epsilon$. By the reflexivity of the space X , the unit ball B_X is weakly compact. Then it follows that B_X is weakly sequentially compact (by Eberlein-Šmulian Theorem). Thus there exists a sequence $(x_n) \subset B_X \setminus B(x, \frac{\epsilon}{2})$ converging weakly to x . The weak lower semi-continuity of the norm implies that, for each $n \in \mathbb{N}$

$$\liminf_m \|x_n - x_m\| \geq \|x_n - x\| > \frac{\epsilon}{2}.$$

Choose $y_1 = x_1$. Then there exists $n_0 \in \mathbb{N}$ such that $\|y_1 - x_m\| > \frac{\epsilon}{2}$ for every $m \geq n_0$. Then choose $y_2 = x_{n_0}$, we have $\|y_1 - y_2\| > \frac{\epsilon}{2}$. Following the same argument, since $\liminf_m \|y_2 - x_m\| > \frac{\epsilon}{2}$, there exist $n_1 \in \mathbb{N}, n_1 > n_0$ such that $\|y_2 - x_m\| > \frac{\epsilon}{2}$ for every $m \geq n_1$. Denote $y_3 = x_{n_1}$. Then we obtain that $\|y_i - y_j\| > \frac{\epsilon}{2}$, for $i, j = 1, 2, 3$ and $i \neq j$. By induction, we can construct a sequence $(y_n) \subset B_X$ which is $\frac{\epsilon}{2}$ -separated and $y_n \rightharpoonup x$. Since X is UKK, $\|x\| < 1 - \delta$ where $\delta \in (0, 1)$ and depends on ϵ .

In the other hand, let $\epsilon > 0$ and assume that $\theta(\epsilon) \in (0, 1)$ satisfies the condition stated in the lemma. Let $(x_n) \subset B_X$ with $\text{sep}(x_n) > \epsilon$ and $x_n \rightarrow x$, $x \in B_X$. Then for every weak neighborhood U of x , for $n, m \in \mathbb{N}$ large enough,

$$\text{diam}(B_X \cap U) \geq \|x_n - x_m\| \geq \text{sep}(x_n) > \epsilon.$$

It follows that $x \in B(0, 1 - \theta(\epsilon))$, i.e., X is UKK. □

Note that any Banach space X satisfying the condition in the previous lemma is UKK. We have shown that both conditions are equivalent for reflexive Banach spaces. It is known by Huff that nearly uniform convexity is equivalent to reflexivity and uniform Kadec Klee.

Theorem 6.12 (Huff). *A norm on a Banach space X is NUC if and only if it is UKK and X is reflexive.*

Assume that X^* is NUC. Hence it is reflexive and UKK (in case of dual spaces, some authors prefer to call it UKK*), then for each $\epsilon > 0$, there exists $\theta(\epsilon) \in (0, 1)$ such that

$$\langle B_{X^*} \rangle'_\epsilon \subset (1 - \theta(\epsilon))B_{X^*}.$$

By iteration of this inclusion, we can deduce that $S_z(X)_\epsilon$ is finite for every $\epsilon > 0$.

The converse was later proved in separable spaces in [48], the authors also shown that the equivalent NUC norm satisfies $\theta(\epsilon) = c\epsilon^p$ for some $c, p > 0$. Analogously to G. Pisier's result about super-reflexive spaces [67]. Recently, M. Raja [73] completely solved this problem.

Theorem 6.13 (Raja). *A Banach space X with $S_z(X) \leq \omega$ can be renormed in such a way that the dual norm on X^* is UKK (hence, it is NUC) with modulus of power type $\theta(\epsilon) = c\epsilon^p$.*

Particularly, a reflexive Banach space with $S_z(X^*) \leq \omega$ is NUC renormable. Thus Domínguez-Benavides and Lorenzo' result together with Theorem 6.8 can be restated in the following way:

Corollary 6.14. *Let X be a reflexive Banach space and X^* the dual space of X . Assume that $S_z(X^*) \leq \omega$. Then there exists a residual subset \mathcal{R} of \mathcal{P} such that for every $q \in \mathcal{R}$, the space (X, q) satisfies the w -MFPP.*

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