SOME REMARKS ON OPTIMALITY CONDITIONS FOR FUZZY OPTIMIZATION PROBLEMS

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ABSTRACT
In this article we present a new concept of stationary point for gH-differentiable fuzzy functions which generalize previous concepts that exist in the literature. Also, we give a concept of generalized convexity for gH-differentiable fuzzy functions more useful than level-wise generalized convexity (generalized convexity of the endpoint functions). Then we give optimatily conditions for fuzzy optimization problems.

KEYWORDS: Fuzzy optimization, stationary point, gH-differentiable function.

MSC: 90C70.

RESUMEN
En este artículo presentamos un nuevo concepto de punto estacionario para funciones difusas gH-diferenciables que generalizan los conceptos previos que existen en la literatura. También damos un concepto de convexidad generalizada para funciones difusas gH-diferenciables más útil que los basados en las funciones extremos. A partir de esos conceptos, damos condiciones de optimalidad para problemas de optimización difusos.

1. PRELIMINARIES

Let $\mathcal{K}_C$ denote the family of all bounded closed intervals in $\mathbb{R}$ and let $\mathcal{F}_C$ denote the family of all fuzzy intervals. So, for any $u \in \mathcal{F}_C$ we denote its α-levels by $[u]^\alpha = [\underline{u}_\alpha, \overline{u}_\alpha] \in \mathcal{K}_C$, for all $\alpha \in [0,1]$. For fuzzy intervals $u, v \in \mathcal{F}_C$, with α-levels represented by $[\underline{u}_\alpha, \overline{u}_\alpha]$ and $[\underline{v}_\alpha, \overline{v}_\alpha]$, respectively, and for any real number λ, the addition $u+v$ and the scalar multiplication λu are defined as follows

$$(u+v)(x) = \sup_{y+z=x} \min\{u(y), v(z)\} \;\; , \;\; (\lambda u)(x) = \begin{cases} u(\lambda^{-1}x), & \text{if } \lambda \neq 0, \\ 0, & \text{if } \lambda = 0. \end{cases}$$

Equivalently,

$$[u+v]^\alpha = [(\underline{u+v})_\alpha, (\overline{u+v})_\alpha] = [\underline{u}_\alpha + \underline{v}_\alpha \;,\; \overline{u}_\alpha + \overline{v}_\alpha],$$
$$[\lambda u]^\alpha = \left[(\underline{\lambda u})_\alpha, (\overline{\lambda u})_\alpha\right] = [\min\{\lambda \underline{u}_\alpha, \lambda \overline{u}_\alpha\}, \max\{\lambda \underline{u}_\alpha, \lambda \overline{u}_\alpha\}],$$

for every $\alpha \in [0,1]$.

Definition 1 *([17]) Given two fuzzy intervals u, v, the generalized Hukuhara difference (gH-difference, for short) of u and v is the fuzzy interval w, if it exists, such that*

$$u \ominus_{gH} v = w \Leftrightarrow \begin{cases} & (i)\; u = v + w, \\ or & (ii)\; v = u + (-1)w. \end{cases}$$

It is easy to show that (i) and (ii) are both valid if and only if w is a crisp number.
If $u \ominus_{gH} v$ exists then, in terms of α-levels, we have

$$[u \ominus_{gH} v]^\alpha = [u]^\alpha \ominus_{gH} [v]^\alpha = [\min\{\underline{u}_\alpha - \underline{v}_\alpha, \overline{u}_\alpha - \overline{v}_\alpha\}, \max\{\underline{u}_\alpha - \underline{v}_\alpha, \overline{u}_\alpha - \overline{v}_\alpha\}],$$

for all $\alpha \in [0,1]$, where $[u]^\alpha \ominus_{gH} [v]^\alpha$ denotes the gH-difference between two intervals (see [16, 17]). We use the Pompeiu-Hausdorff metric to define the distance between two fuzzy numbers. Given $u, v \in \mathcal{F}_C$ we define the distance between u and v by

$$D(u,v) \quad = \quad \sup_{\alpha\in[0,1]} H\left([u]^\alpha,[v]^\alpha\right) = \sup_{\alpha\in[0,1]} \max\left\{\left|\underline{u}_\alpha - \underline{v}_\alpha\right|, \left|\overline{u}_\alpha - \overline{v}_\alpha\right|\right\}.$$

We recall the usual order for fuzzy intervals [9, 11, 12, 14, 15].

Definition 2 *For $u, v \in \mathcal{F}_C$, it is said that $u \preceqq v$, if for every $\alpha \in [0,1]$, $\underline{u}_\alpha \leq \underline{v}_\alpha$ and $\overline{u}_\alpha \leq \overline{v}_\alpha$. It is said that $u \preceq v$ if $u \preceqq v$ and $\exists \alpha_0 \in [0,1]$, such that $\underline{u}_{\alpha_0} < \underline{v}_{\alpha_0}$ or $\overline{u}_{\alpha_0} < \overline{v}_{\alpha_0}$. It is said that $u \prec v$ if $u \preceqq v$ and $\exists \alpha_0 \in [0,1]$, such that $\underline{u}_{\alpha_0} < \underline{v}_{\alpha_0}$ and $\overline{u}_{\alpha_0} < \overline{v}_{\alpha_0}$.*

Note that $\preceqq$ is a partial order relation on $\mathcal{F}_C$. So $v \succeqq u$ instead of $u \preceqq v$ can be written.

2. DIFFERENTIABLE FUZZY FUNCTIONS

Henceforth, let K be an open subset of $\mathbb{R}$, and $F : K \to \mathcal{F}_C$ be a fuzzy function. For each $\alpha \in [0,1]$, we define the family of interval-valued functions $F_\alpha : K \to \mathcal{K}_C$, associated with F, and given by $F_\alpha(x) = [F(x)]^\alpha$. For any $\alpha \in [0,1]$, we denote

$$F_\alpha(x) = \left[\underline{f}_\alpha(x), \overline{f}_\alpha(x)\right].$$

Here, for each $\alpha \in [0,1]$, the endpoint functions $\underline{f}_\alpha, \overline{f}_\alpha : K \to \mathbb{R}$ are called upper and lower functions of F, respectively.
Next, we present the concept of gH-differentiable fuzzy functions based on the gH-difference of fuzzy intervals.

Definition 3 *([1]) The gH-derivative of a fuzzy function $F : K \to \mathcal{F}_C$ at $x_0 \in K$ is defined as*

$$F'(x_0) = \lim_{h\to 0} \frac{1}{h}\left[F(x_0+h) \ominus_{gH} F(x_0)\right]. \tag{1}$$

If $F'(x_0) \in \mathcal{F}_C$ satisfying (1) exists, we say that F is generalized Hukuhara differentiable (gH-differentiable, for short) at x_0.

Next we characterize the gH-differentiability of a fuzzy function in terms of the differentiability of its endpoint functions.

Theorem 1 *([2]) Let $F : K \to \mathcal{F}_C$ be a fuzzy function and $x \in K$. Then F is gH-differentiable at x if and only if one of the following four cases holds:*

(a) $\underline{f}_\alpha$ and $\overline{f}_\alpha$ are differentiable at x, uniformly in $\alpha \in [0,1]$, $(\underline{f}_\alpha)'(x)$ is monotonic increasing and $(\overline{f}_\alpha)'(x)$ is monotonic decreasing as functions of α and $(\underline{f}_1)'(x) \leq (\overline{f}_1)'(x)$. In this case,

$$F'_\alpha(x) = \left[(\underline{f}_\alpha)'(x), (\overline{f}_\alpha)'(x)\right],$$

for all $\alpha \in [0,1]$.
(b) $\underline{f}_\alpha$ and $\overline{f}_\alpha$ are differentiable at x, uniformly in $\alpha \in [0,1]$, $(\underline{f}_\alpha)'(x)$ is monotonic decreasing and $(\overline{f}_\alpha)'(x)$ is monotonic increasing as functions of α and $(\overline{f}_1)'(x) \leq (\underline{f}_1)'(x)$. In this case,

$$F'_\alpha(x) = \left[(\overline{f}_\alpha)'(x), (\underline{f}_\alpha)'(x)\right],$$

for all $\alpha \in [0,1]$.

(c) $(\underline{f}_\alpha)'_{+/-}(x)$ *and* $(\overline{f}_\alpha)'_{+/-}(x)$ *exist uniformly in* $\alpha \in [0,1]$, $(\underline{f}_\alpha)'_+(x) = (\overline{f}_\alpha)'_-(x)$ *is monotonic increasing and* $(\overline{f}_\alpha)'_+(x) = (\underline{f}_\alpha)'_-(x)$ *is monotonic decreasing as functions of* α *and* $(\underline{f}_1)'_+(x) \leq (\overline{f}_1)'_+(x)$*. In this case,*

$$F'_\alpha(x) = \left[(\underline{f}_\alpha)'_+(x), (\overline{f}_\alpha)'_+(x)\right] = \left[(\overline{f}_\alpha)'_-(x), (\underline{f}_\alpha)'_-(x)\right],$$

for all $\alpha \in [0,1]$.

(d) $(\underline{f}_\alpha)'_{+/-}(x)$ *and* $(\overline{f}_\alpha)'_{+/-}(x)$ *exist uniformly in* $\alpha \in [0,1]$, $(\underline{f}_\alpha)'_+(x) = (\overline{f}_\alpha)'_-(x)$ *is monotonic decreasing and* $(\overline{f}_\alpha)'_+(x) = (\underline{f}_\alpha)'_-(x)$ *is monotonic increasing as functions of* α *and* $(\overline{f}_1)'_+(x) \leq (\underline{f}_1)'_+(x)$*. In this case,*

$$F'_\alpha(x) = \left[(\overline{f}_\alpha)'_+(x), (\underline{f}_\alpha)'_+(x)\right] = \left[(\underline{f}_\alpha)'_-(x), (\overline{f}_\alpha)'_-(x)\right],$$

for all $\alpha \in [0,1]$.

There is another one definition of differentiability in terms of the differentiability of its endpoint functions.

Definition 4 *Given* $F : K \to \mathcal{F}_C$ *a fuzzy function, it is said to be a level-wise differentiable at* $x_0 \in K$ *if both endpoint functions* $\underline{f}_\alpha$ *and* $\overline{f}_\alpha$ *are differentiable at* x_0 *for each* $\alpha \in [0,1]$.

We can see that the concept of level-wise differentiable fuzzy functions is very restrictive due to fuzzy arithmetic. The gH-differentiability is more general. For more details see [1, 2, 4].

3. NECESSARY OPTIMALITY CONDITIONS FOR FUZZY OPTIMIZATION PROBLEMS

Let us consider $F : K \subseteq \mathbb{R} \to \mathcal{F}_C$ a fuzzy function and let us suppose that F is gH-differentiable on K, an open subset of $\mathbb{R}$. We introduce the following minimum definition for F.

Definition 5 $x^* \in K$ *is said to be a minimum for* F *if there does not exist* $x \in K$ *such that* $F(x) \preceq F(x^*)$.

Note that in other articles minimum is also called non-dominated solution or efficient solution [3, 5, 9, 14, 15].
In the literature there are different definitions of stationary point for fuzzy functions which generalize the stationary point for classical mathematical programming. Some of these definitions use the defuzzification techniques and others that are very restrictive.
We propose a new fuzzy stationary point definition that generalizes the existing ones.

Definition 6 $x^* \in K$ *is said to be a fuzzy stationary point for* F *if* $0 \in [F'(x^*)]^\alpha$ *for some* $\alpha \in [0,1]$.

Using different concepts of differentiability, definitions of stationary point have been introduced. For instance, using level-wise differentiability [5, 9, 14, 15] and GH-differentibility [3, 8]. Since gH-differentiability is more general than level-wise and GH-differentiability, then Definition 6 is more general.

Theorem 2 *Let* F *be a* gH*-differentiable fuzzy function. If* x^* *is a minimum of* F *then* x^* *is a fuzzy stationary point for* F.

Proof Since F is gH-differentiable then taking into account Theorem 1 we have the cases (a), (b), (c) and (d). So, by the same arguments in the proof of Theorem 3 in [8] we obtain the result.□

Corollary 1 $x^* \in K$ *is a fuzzy stationary point if and only if* $0 \in [F'(x^*)]^0$

Of course, a stationary point is not always a minimum.

4. SUFFICIENT OPTIMALITY CONDITIONS FOR FUZZY OPTIMIZATION PROBLEMS

In order to ensure that all stationary points are minima, we need to introduce convexity or generalized convexity notions for fuzzy functions, similar to classical optimization problems.
The following convexity definition for fuzzy function is well-known in the literature, for instance see [3, 6, 7, 9, 11, 13, 14, 15].

Definition 7 *A fuzzy function $F : K \to \mathcal{F}_C$ is said to be convex on a convex set $K \subset \mathbb{R}$ if for all $x, y \in K$, $\lambda \in [0,1]$,*

$$F(\lambda x + (1-\lambda)y) \preceq \lambda F(x) + (1-\lambda)F(y).$$

A convex fuzzy function F is closely relate to its endpoint functions $\underline{f}_\alpha$ and $\overline{f}_\alpha$ convexity. More precisely, we have the following result.

Proposition 1 *Let K be a nonempty convex subset of $\mathbb{R}$ and let $F : K \to \mathcal{F}_C$ be a fuzzy function. Then, F is convex on K if and only if, for each $\alpha \in [0,1]$, $\underline{f}_\alpha$, $\overline{f}_\alpha$ are convex functions on K.*

We remark that the concept of convexity is very restrictive by two reasons: due to fuzzy arithmetic, since the space of fuzzy interval is not a linear space, we have that usual fuzzy functions such as $F(x) = u \cdot x$, which is the extension of a linear function, is not convex. On the other hand, convex and gH-differentiable fuzzy function cannot be characterized by inequality (2)

$$F(y) \succeq F'(x) \cdot (y - x) + F(x), \tag{2}$$

for all $x, y \in K$ [8]. This is the main difference with the characterization of convex and differentiable functions in classical optimization. Consequently, we can not extend the well-known results of classic convex optimization problems in a natural form to fuzzy optimization problems.
Recently, concept of invex fuzzy functions as a generalization of convex fuzzy functions have been introduced [5, 10]. By same reasons to convexity these concepts are very restrictive.
Also, In the literature we can find articles where convex (invex, pseudoinvex, preinvex) level-wise fuzzy functions are introduced. This concepts are based on the convexity (invex, pseudoinvex, preinvex) of the endpoint functions [5, 6, 7, 9, 11, 12, 13, 14, 15]. These concepts are also very restrictive due to fuzzy arithmetic such as we have said above. In addition, as it was also said above, level-wise differentiability is not good tool to differentibility fuzzy.
In the article [8] was considered a concept more general than previous concepts of generalized convexity and, such as we can see next, this tool is more useful.

Definition 8 *Let F be a gH-differentiable fuzzy mapping. F is said to be pseudoinvex on K if for all $x, y \in K$, there exists $\eta : K \times K \to \mathbb{R}$ such that*

$$F(y) \preceq F(x) \Rightarrow F'(x)\eta(x,y) \prec 0.$$

Note that in the previous Definition 8 we consider the gH-differentiability instead of GH-differentiability as was used in [8].

Theorem 3 *If F is pseudoinvex on K then every fuzzy stationary point is a minimum.*

Proof The demonstration follows by the same arguments of Theorem 4 in [8].□
In the next example we present a fuzzy function that is not convex and is not level-wise differentiable, but it is gH-differentiable and pseudoinvex.

Example 1 *Let us consider $F : \mathbb{R} \to \mathcal{F}_C$ defined by its level sets*

$$[F(x)]^\alpha = [-(1-\alpha), (1-\alpha)]x$$

Its endpoint functions are:

$$\underline{f}_\alpha(x) \begin{cases} (1-\alpha)x & if \quad x \leq 0 \\ -(1-\alpha)x & if \quad x > 0 \end{cases} \qquad and\ \overline{f}_\alpha(x) \begin{cases} -(1-\alpha)x & if \quad x \leq 0 \\ (1-\alpha)x & if \quad x > 0 \end{cases}$$

In this case, F is not level-wise differentiable, but it is gH-differentiable. Also F is not convex, since $\underline{f}_\alpha$ is not convex, but F is pseudoinvex and all the points are stationary points and by Theorem 3 these are also minima.

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