Classifying partial Latin rectangles

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Outline

1. Introduction.
2. Enumeration of $\mathcal{PLR}_{r,s,n}$.
3. Distribution into isomorphism and isotopism classes.

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Introduction

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- Algebraic Geometry.

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$\mathcal{PLR}_{r,s,n}$: Set of $r \times s$ partial Latin rectangles based on $[n] = \{1, ..., n\}$.

1. each cell is empty or contains one symbol of $[n]$.
2. each symbol occurs at most once in each row and each column.

$$P \equiv \begin{array}{|c|c|c|}\hline 1 & & 4 \\ \hline 4 & 3 & 2 \\ \hline \end{array} \in \mathcal{PLR}_{2,3,4} \subset \mathcal{PLR}_{2,3,5} \subset \dots$$

$\mathcal{PLR}_{r,s,n;m}$: Subset of $\mathcal{PLR}_{r,s,n}$ whose elements have m non-empty cells.

$$P \in \mathcal{PLR}_{2,3,4;5} \subset \mathcal{PLR}_{2,3,5;5} \subset \dots$$

Entry set: $\{(\text{row}, \text{column}, \text{symbol})\}$

$$E(P) = \{(1,1,1),(1,3,4),(2,1,4),(2,2,3),(2,3,2)\}.$$

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Symmetries on $\mathcal{PLR}_{r,s,n}$.

S_n: Symmetric group on $[n]$.

$S_r \times S_s \times S_n$: **Isotopism group**.

$$\begin{cases} P = \begin{array}{|c|c|c|}\hline 1 & & 3 \\ \hline & 3 & 2 \\ \hline \end{array} \\ \Theta = ((1\ 2), (2\ 3), \mathrm{Id}) \in S_2 \times S_3 \times S_3 \end{cases} \Rightarrow P^\Theta = \begin{array}{|c|c|c|}\hline & 2 & 3 \\ \hline 1 & 3 & \\ \hline \end{array}$$

$E(P) \to E(P^\Theta)$
$(1,1,1) \to (2,1,1)$
$(1,3,3) \to (2,2,3)$
$(2,2,3) \to (1,3,3)$
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$$P \sim P^\Theta \Rightarrow \begin{cases} \textbf{Isotopism class}: \mathfrak{I}_{n,P}. \\ \mathfrak{I}_n(P,Q) = \{\Theta \in S_r \times S_s \times S_n \mid P^\Theta = Q\} \end{cases}$$

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- $r = s = n$ and $\alpha = \beta = \gamma \Rightarrow \Theta$ is an **isomorphism** ($\simeq$)

$$\Rightarrow \begin{cases} \textbf{Isomorphism class}: \mathcal{I}_{n,P}. \\ \mathcal{I}_n(P,Q) = \{\alpha \in S_n \mid P^{(\alpha,\alpha,\alpha)} = Q\}. \end{cases}$$

Symmetries on $\mathcal{PLR}_{r,s,n}$.

Stabilizer groups:

- Autotopism group: $\mathfrak{A}_n(P) = \mathfrak{I}_n(P,P)$.
- Automorphism group: $\mathcal{A}_n(P) = \mathcal{I}_n(P,P)$.
- $\mathcal{PLR}_{\Theta} = \{P \in \mathcal{PLR}_{r,s,n} \mid \Theta \in \mathfrak{A}_n(P)\}$.
- $\mathcal{PLR}_{\Theta;m} = \{P \in \mathcal{PLR}_{r,s,n;m} \mid \Theta \in \mathfrak{A}_n(P)\}$.

Lemma

A) $|\mathfrak{A}_n(P)| = |\mathfrak{A}_n(Q)|, \forall Q \in \mathfrak{I}_n(P)$.

B) $|\mathfrak{I}_{n,P}| = \frac{r!\cdot s!\cdot n!}{|\mathfrak{A}_n(P)|}$.

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b) $|\mathfrak{I}_{n,P}| = \frac{r! \cdot s! \cdot n!}{|\mathfrak{A}_n(P)|}$.

Enumerating and classiying PLRs.

The enumeration and classification problems of PLRs are open.

- **Case** $r = s = n$ **and** $m = n^2$: Latin squares.
  - $n \leq 11$: [McKay, Wanless; 2005][Hulpke, Kaski, Östergård; 2011].
- **Case** $r \leq s = n$ **and** $m = rn$: Latin rectangles.
  - $n \leq 11$: [Stones; 2010].
- **General case** $r \leq s \leq n$ **and** $m \leq rs$: Partial Latin rectangles.
  - $r, s, n \leq 4$: Enumeration [Falcón; 2012; 2015].
  - $r, s, n \leq 7$: Enumeration and classification [Falcón, Stones; 2015. Under preparation].

1. Inclusion-exclusion method.
2. Chromatic polynomial method.
3. Sade's method.
4. **Algebraic Geometry method**.

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$|\mathcal{R}_{r,s,n:m}|$

m \ $r.s.n$	1.1.1	1.1.2	1.1.3	1.1.4	1.2.2	1.2.3	1.2.4	1.3.3	1.3.4	1.4.4	2.2.2	2.2.3	2.2.4	2.3.3	2.3.4	2.4.4	3.3.3	3.3.4	3.4.4	4.4.4
0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
1	1	2	3	4	4	6	8	9	12	16	8	12	16	18	24	32	27	36	48	64
2					2	6	12	18	36	72	16	42	80	108	204	384	270	504	936	1,728
3								6	24	96	8	48	144	264	768	2,208	1,278	3,552	9,696	25,920
4										24	2	18	84	270	1,332	6,504	3,078	13,716	58,752	239,760
5														108	1,008	9,792	3,834	29,808	216,864	1,437,696
6														12	264	7,104	2,412	36,216	494,064	5,728,896
7																2,112	756	23,760	691,200	15,326,208
8																216	108	7,776	581,688	27,534,816
9																	12	1,056	283,584	32,971,008
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11																			10,368	13,153,536
12																			576	4,215,744
13																				847,872
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Total	2	3	4	5	7	13	21	34	73	209	35	121	325	781	3,601	28,353	11,776	116,425	2,423,521	127,545,137

Proposition (Falcón, 2015)

A) $|\mathcal{PLR}_{r,s,n:0}| = 1.$

B) $|\mathcal{PLR}_{r,s,n:1}| = rsn.$

C) $|\mathcal{PLR}_{r,s,n:2}| = \frac{1}{2}rsn \cdot (rsn - r - s - n + 2).$

	$\lvert\mathcal{R}_{r,s,n:m}\rvert$																			
	$r.s.n$																			
m	1.1.1	1.1.2	1.1.3	1.1.4	1.2.2	1.2.3	1.2.4	1.3.3	1.3.4	1.4.4	2.2.2	2.2.3	2.2.4	2.3.3	2.3.4	2.4.4	3.3.3	3.3.4	3.4.4	4.4.4
0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
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- Partial Latin rectangles.
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$F[\underline{x}]$: Ring of polynomials in $\underline{x} = \{x_1, \ldots, x_n\}$ over a field F.
I : An ideal in $F[\underline{x}]$.

- **Standard grading**: $F[\underline{x}] = \bigoplus_{0 \leq d} F[\underline{x}]_d$.
- $\mathbf{x}^{\mathbf{a}} = x_1^{a_1} \ldots x_n^{a_n}$ is identified with $\mathbf{a} = (a_1, \ldots, a_n) \in \mathbb{N}^n$.
- A **monomial term order** $<$ is a total order on $\mathbb{N}^n$, where
  A) The zero vector is the unique minimal element.
  B) $\mathbf{a} < \mathbf{b} \rightarrow \mathbf{a} + \mathbf{c} < \mathbf{b} + \mathbf{c}$, for all $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{N}^n$.

  Example: Graded lexicographic term order:

  $$x_1 x_2 x_3^2 <_{\text{lex}} x_1^2 x_2 x_3 <_{\text{lex}} x_1^2 x_2 x_3^2.$$

- The largest monomial of a polynomial is its **initial monomial**.

  $$x_1 x_2 x_3^2 + x_1^2 x_2 x_3 + \mathbf{x_1^2 x_2 x_3^2}.$$

- A monomial is **standard** if it is not contained in the ideal generated by all the initial monomials.

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- **Algebraic set** $\mathcal{V}(I) = \{\underline{a} \in F^n \colon f(\underline{a}) = 0 \text{ for all } f \in I\}$.

- If I is zero-dimensional and radical, then

$$|\mathcal{V}(I)| = \dim_F F[\underline{x}]/I = \sum_{0 \leq d} \mathrm{HF}_{F[\underline{x}]/I}(d),$$

where HF is the **Hilbert function**

$$HF(d) = \dim_F(F[\underline{x}]_d/I_d).$$

It coincides with the number of standard monomials in I of degree d with respect to any term order.

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$\mathbb{F}_2[\mathbf{x}]$: Ring of polynomials in $\underline{x} = \{x_1, \ldots, x_{rsn}\}$ over the finite field $\mathbb{F}_2$.

Theorem (Falcón, 2015)

The set $\mathcal{PLR}_{r,s,n}$ is identified with the set of zeros in $\mathbb{F}_2[\mathbf{x}]$ of the ideal $I_{r,s,n}$ generated by

$$\begin{cases} x_{ijk} \cdot (x_{ijk} - 1), \forall i \in [r], j \in [s], k \in [n], \\ x_{ijk} \cdot x_{ijk'}, \forall i \in [r], j \in [s], k, k' \in [n], k < k', \\ x_{ijk} \cdot x_{ij'k}, \forall i \in [r], j, j' \in [s], k' \in [n], j < j', \\ x_{ijk} \cdot x_{i'jk}, \forall i, i' \in [r], j \in [s], k \in [n], i < i'. \end{cases} \tag{1}$$

Then,

A) $|\mathcal{PLR}_{r,s,n}| = \dim_{\mathbb{F}_2}(\mathbb{F}_2[\mathbf{x}]/I_{r,s,n})$

B) $|\mathcal{R}_{r,s,n:m}| = \mathrm{HF}_{\mathbb{F}_2[\mathbf{x}]/I_{r,s,n}}(m)$, *for all* $m \geq 0$.

$P =$

1		3
	3	2

$\equiv \langle x_{111} - 1, x_{113} - 1, x_{223} - 1, x_{232} - 1, x_{112}, x_{113}, \ldots \rangle$

Algebraic Geometry

		$\lvert\mathcal{PLR}_{r,s,n}\rvert$							
		n							
r	s	1	2	3	4	5	6	7	8
1	1	2	3	4	5	6	7	8	9
	2		7	13	21	31	43	57	73
	3			34	73	136	229	358	529
	4				209	501	1,045	1,961	3,393
	5					1,546	4,051	9,276	19,081
	6						13,327	37,633	93,289
	7							130,922	394,353
	8								1,441,729
2	2		35	121	325	731	1,447	2,605	4,361
	3			781	3,601	12,781	37,273	93,661	209,761
	4				28,353	162,661	720,181	2,599,185	7,985,761
	5					1,502,171	10,291,951	54,730,201	236,605,001
	6						108,694,843	864,744,637	5,376,213,193
	7							10,256,288,925	92,842,518,721
	8								1,219,832,671,361
3	3			11,776	116,425	805,366	4,193,269	17,464,756	60,983,761
	4				2,423,521	33,199,561	317,651,473	2,263,521,961	12,703,477,825
	5					890,442,316	15,916,515,301	199,463,431,546	1,854,072,020,881
	6						526,905,708,889	11,785,736,969,413	*
4	4				127,545,137	4,146,833,121	87,136,329,169	1,258,840,124,753	*
	5					313,185,347,701	*	*	*

*Excessive cost of computation for a computer system i7-2600, 3.4 GHz.

Max. time of computation: 4,180 seconds ($\mathcal{PLR}_{2,9,13}$).

		$\lvert\mathcal{PLR}_{r,s,n}\rvert$				
		n				
r	s	9	10	11	12	13
1	1	10	11	12	13	14
	2	91	111	133	157	183
	3	748	1,021	1,354	1,753	2,224
	4	5,509	8,501	12,585	18,001	25,013
	5	36,046	63,591	106,096	169,021	259,026
	6	207,775	424,051	805,597	1,442,173	2,456,299
	7	1047,376	2,501,801	5,470,158	11,109,337	21,204,548
	8	4,596,553	12,975,561	32,989,969	76,751,233	165,625,929
	9	17,572,114	58,941,091	175,721,140	472,630,861	1,163,391,958
	10		234,662,231	824,073,141	258,128,454	7,307,593,151
	11			3,405,357,682	12,470,162,233	40,864,292,184
	12				53,334,454,417	202,976,401,213
	13					896,324,308,634
2	2	6,985	10,411	15,137	21,325	29,251
	3	28,941	815,161	1,458,733	2,482,801	4,050,541
	4	21,582,613	52,585,221	117,667,441	245,278,945	481,597,221
	5	864,742,231	2,756,029,891	7,846,852,421	20,336,594,221	48,689,098,771
	6	27,175,825,171	115,690,051,951	426,999,864,193	1,398,636,508,477	4,141,988,637,463
	7	661,377,377,305	3,836,955,565,101	18,712,512,041,917	78,819,926,380,945	293,220,109,353,081
	8	12,372,136,371,721	99,423,049,782,601	652,303,240,153,313	3,595,671,023,722,081	17,076,864,830,330,761
	9	178,156,152,706,483	2,000,246,352,476,311	17,908,872,286,407,301	131,297,226,011,020,765	808,986,548,443,056,751
	10		31,296,831,902,738,931	385,203,526,838,449,441	*	*
	11			*	*	*
3	3	184,952,170	500,317,981	1,231,810,504	2,803,520,281	5,970,344,446
	4	58,737,345,481	231,769,858,321	802,139,572,873	2,487,656,927,521	7,030,865,002,825
	5	13,451,823,665,776	*	*	*	*

*Excessive cost of computation for a computer system i7-2600, 3.4 GHz.

Max. time of computation: 4,180 seconds ($\mathcal{PLR}_{2,9,13}$).

Outline

1. Introduction.
2. Enumeration of $\mathcal{PLR}_{r,s,n}$.
3. Distribution into isomorphism and isotopism classes.

The new approach is based on the symmetry of those polynomials in (1) related to each row of a partial Latin rectangle.

$$\begin{cases} x_{ijk}\cdot(x_{ijk}-1), \forall i\in[r], j\in[s], k\in[n],\\ x_{ijk}\cdot x_{ijk'}, \forall i\in[r], j\in[s], k,k'\in[n], k<k',\\ x_{ijk}\cdot x_{ij'k}, \forall i\in[r], j,j'\in[s], k'\in[n], j<j',\\ x_{ijk}\cdot x_{i'jk}, \forall i,i'\in[r], j\in[s], k\in[n], i<i'. \end{cases}$$

1. For each $i \in [r]$, let

$$I^{(i)}_{r,s,n} = \langle\, x_{ijk}x_{ij'k},\ x_{ijk}x_{ijk'} :\ j, j' \in [s],\ k, k' \in [n], j < j', k < k' \,\rangle$$

2. Let $\{J_{1,1}, \ldots, J_{1,t}\}$ be a finite set of t subideals of $I^{(1)}_{r,s,n}$
   - generated by triangular systems of polynomial equations
   - and whose affine varieties constitute a partition of $V(I^{(1)}_{r,s,n})$.

   [Moeller, 1993] [Hillebrand, 1999]

3. For $i > 1$, let $J_{i,l}$ be the subideal of $I^{(i)}_{r,s,n}$ whose generators coincide with those of $J_{1,l}$ after replacing each variable x_{1jk} by x_{ijk}.

4. For each tuple $(t_1, \ldots, t_r) \in [t]^r$, let

$$K_{t_1,\ldots,t_r} = J_{1,t_1} + \ldots + J_{r,t_r} + \langle\, x_{ijk}x_{i'jk} :\ i, i' \in [r], j \in [s], k \in [n], i < i' \,\rangle$$

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The triangularity of the systems involve each subideal J_{i,t_j} to have at least one generator of the form $x_{ij'k}$ or $x_{ij'k}-1$.

The number $m_{t_1,\ldots,t_r}$ of generators $x_{ij'k}-1$ in $K_{t_1,\ldots,t_r}$ constitutes the minimum number of cells that are necessarily non-empty in any partial Latin rectangle identified with a point of the affine variety $V(K_{t_1,\ldots,t_r})$.

PROPOSITION

Let m be a non-negative integer. Then

$$|\mathcal{PLR}_{r,s,n;m}| = \mathrm{HF}_{\mathbb{F}_2[\mathbf{x}]/I_{r,s,n}}(m) = \sum_{\substack{(t_1,\ldots,t_r)\in[t]^r \\ m_{t_1,\ldots,t_r}\leq m}} \mathrm{HF}_{\mathbb{F}_2[\mathbf{x}]/K_{t_1,\ldots,t_r}}(m-m_{t_1,\ldots,t_r}).$$

Advantages: Less storage memory. Parallel computation.

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Advantages: Less storage memory. Parallel computation.

Example: $|\mathcal{PLR}_{3,3,3}|$

The ideal $I^{(1)}_{3,3,3}$ can be decomposed into the next six disjoint subideals

- $J_{1,1} = I^{(1)}_{3,3,3} + \langle\, x_{111}, x_{121}, x_{131} \,\rangle$.
- $J_{1,2} = I^{(1)}_{3,3,3} + \langle\, x_{111}, x_{121}, x_{131} - 1, x_{132}, x_{133} \,\rangle$.
- $J_{1,3} = I^{(1)}_{3,3,3} + \langle\, x_{111}, x_{121} - 1, x_{122}, x_{123}, x_{131} \,\rangle$.
- $J_{1,4} = I^{(1)}_{3,3,3} + \langle\, x_{111} - 1, x_{112}, x_{113}, x_{121}, x_{122}, x_{131}, x_{132} \,\rangle$.
- $J_{1,5} = I^{(1)}_{3,3,3} + \langle\, x_{111} - 1, x_{112}, x_{113}, x_{121}, x_{122}, x_{131}, x_{132} - 1, x_{133} \,\rangle$.
- $J_{1,6} = I^{(1)}_{3,3,3} + \langle\, x_{111} - 1, x_{112}, x_{113}, x_{121}, x_{122} - 1, x_{123}, x_{131}, x_{132} \,\rangle$.

For each triple $(t_1, t_2, t_3) \in [6]^3$, let

$$K_{t_1,t_2,t_3} = J_{1,t_1} + J_{2,t_2} + J_{3,t_3} + \langle\, x_{ijk} x_{i'jk} :\ i, i', j, k \in [3], i < i' \,\rangle$$

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$$K_{t_1,t_2,t_3} = J_{1,t_1} + J_{2,t_2} + J_{3,t_3} + \langle\, x_{ijk}x_{i'jk} : \ i, i', j, k \in [3], i < i' \,\rangle$$

Example: $|\mathcal{PLR}_{3,3,3}|$

	$\mathrm{HF}_{\mathbb{F}_2[\mathbf{x}]/K_{t_1,t_2,t_3}}(m)$															
	$t_1.t_2.t_3$															
m	1.1.1	1.1.2	1.1.3	1.1.4	1.1.5	1.1.6	1.2.3	1.2.4	1.2.5	1.2.6	1.3.4	1.3.5	1.3.6	2.3.4	2.3.5	2.3.6
0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
1	18	16	16	14	11	11	14	12	10	9	12	9	10	10	8	8
2	108	84	84	62	36	36	64	45	29	24	45	24	29	32	19	19
3	264	176	176	104	42	42	116	63	29	23	63	23	29	38	16	16
4	270	150	150	66	18	18	84	32	11	8	32	8	11	16	5	5
5	108	48	48	12	2	2	24	5	1	1	5	1	1	2	1	1
6	12	4	4	0	0	0	2	0	0	0	0	0	0	0	0	0

Example:

$$|\mathcal{PLR}_{3,3,3:2}| = \sum_{\substack{(t_1,t_2,t_3)\in[6]^3\\ m_{t_1,t_2,t_3}\leq 2}} \mathrm{HF}_{\mathbb{F}_2[\mathbf{x}]/K_{t_1,t_2,t_3}}(2-m_{t_1,t_2,t_3}) =$$

$$\mathrm{HF}_{\mathbb{F}_2[\mathbf{x}]/K_{1,1,1}}(2) + 3\ \mathrm{HF}_{\mathbb{F}_2[\mathbf{x}]/K_{1,1,2}}(1) + 3\ \mathrm{HF}_{\mathbb{F}_2[\mathbf{x}]/K_{1,1,3}}(1) +$$
$$3\ \mathrm{HF}_{\mathbb{F}_2[\mathbf{x}]/K_{1,1,4}}(1) + 3\ \mathrm{HF}_{\mathbb{F}_2[\mathbf{x}]/K_{1,1,5}}(0) + 3\ \mathrm{HF}_{\mathbb{F}_2[\mathbf{x}]/K_{1,1,6}}(0) +$$
$$6\ \mathrm{HF}_{\mathbb{F}_2[\mathbf{x}]/K_{1,2,3}}(0) + 6\ \mathrm{HF}_{\mathbb{F}_2[\mathbf{x}]/K_{1,2,4}}(0) + 6\ \mathrm{HF}_{\mathbb{F}_2[\mathbf{x}]/K_{1,3,4}}(0) =$$
$$108 + 3\cdot 16 + 3\cdot 16 + 3\cdot 14 + 3\cdot 1 + 3\cdot 1 + 6\cdot 1 + 6\cdot 1 + 6\cdot 1 = 270.$$

$K_{1,2,3} \hookrightarrow$

		1
	1	

$\Rightarrow m_{1,2,3} = 2.$

Example: $|\mathcal{PLR}_{3,3,3}|$

	$\mathrm{HF}_{\mathbb{F}_2[\mathbf{x}]/K_{t_1,t_2,t_3}}(m)$															
	$t_1.t_2.t_3$															
m	1.1.1	1.1.2	1.1.3	1.1.4	1.1.5	1.1.6	1.2.3	1.2.4	1.2.5	1.2.6	1.3.4	1.3.5	1.3.6	2.3.4	2.3.5	2.3.6
0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
1	18	16	16	14	11	11	14	12	10	9	12	9	10	10	8	8
2	108	84	84	62	36	36	64	45	29	24	45	24	29	32	19	19
3	264	176	176	104	42	42	116	63	29	23	63	23	29	38	16	16
4	270	150	150	66	18	18	84	32	11	8	32	8	11	16	5	5
5	108	48	48	12	2	2	24	5	1	1	5	1	1	2	1	1
6	12	4	4	0	0	0	2	0	0	0	0	0	0	0	0	0

Example:

$$|\mathcal{PLR}_{3,3,3:2}| = \sum_{\substack{(t_1,t_2,t_3)\in[6]^3 \\ m_{t_1,t_2,t_3}\leq 2}} \mathrm{HF}_{\mathbb{F}_2[\mathbf{x}]/K_{t_1,t_2,t_3}}(2-m_{t_1,t_2,t_3}) =$$
$$\mathrm{HF}_{\mathbb{F}_2[\mathbf{x}]/K_{1,1,1}}(2) + 3\ \mathrm{HF}_{\mathbb{F}_2[\mathbf{x}]/K_{1,1,2}}(1) + 3\ \mathrm{HF}_{\mathbb{F}_2[\mathbf{x}]/K_{1,1,3}}(1) +$$
$$3\ \mathrm{HF}_{\mathbb{F}_2[\mathbf{x}]/K_{1,1,4}}(1) + 3\ \mathrm{HF}_{\mathbb{F}_2[\mathbf{x}]/K_{1,1,5}}(0) + 3\ \mathrm{HF}_{\mathbb{F}_2[\mathbf{x}]/K_{1,1,6}}(0) +$$
$$6\ \mathrm{HF}_{\mathbb{F}_2[\mathbf{x}]/K_{1,2,3}}(0) + 6\ \mathrm{HF}_{\mathbb{F}_2[\mathbf{x}]/K_{1,2,4}}(0) + 6\ \mathrm{HF}_{\mathbb{F}_2[\mathbf{x}]/K_{1,3,4}}(0) =$$
$$108 + 3\cdot 16 + 3\cdot 16 + 3\cdot 14 + 3\cdot 1 + 3\cdot 1 + 6\cdot 1 + 6\cdot 1 + 6\cdot 1 = 270.$$

$$K_{1,2,3} \hookrightarrow \begin{array}{|c|c|c|}\hline & & \\ \hline & & 1 \\ \hline & 1 & \\ \hline\end{array} \Rightarrow m_{1,2,3} = 2.$$

Enumeration of $\mathcal{PLR}_{r,s,n;m}$

	$\lvert\mathcal{R}_{r,s,n:m}\rvert$																			
	r.s.n																			
m	1.1.1	1.1.2	1.1.3	1.1.4	1.2.2	1.2.3	1.2.4	1.3.3	1.3.4	1.4.4	2.2.2	2.2.3	2.2.4	2.3.3	2.3.4	2.4.4	3.3.3	3.3.4	3.4.4	4.4.4
0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
1	1	2	3	4	4	6	8	9	12	16	8	12	16	18	24	32	27	36	48	64
2					2	6	12	18	36	72	16	42	80	108	204	384	270	504	936	1,728
3								6	24	96	8	48	144	264	768	2,208	1,278	3,552	9,696	25,920
4										24	2	18	84	270	1,332	6,504	3,078	13,716	58,752	239,760
5														108	1,008	9,792	3,834	29,808	216,864	1,437,696
6														12	264	7,104	2,412	36,216	494,064	5,728,896
7																2,112	756	23,760	691,200	15,326,208
8																216	108	7,776	581,688	27,534,816
9																	12	1,056	283,584	32,971,008
10																			75,744	25,941,504
11																			10,368	13,153,536
12																			576	4,215,744
13																				847,872
14																				110,592
15																				9,216
16																				576
Total	2	3	4	5	7	13	21	34	73	209	35	121	325	781	3,601	28,353	11,776	116,425	2,423,521	127,545,137

ENUMERATION OF $\mathcal{PLR}_{r,s,n;m}$

	$\lvert\mathcal{R}_{r,s,5:m}\rvert$														
	$r.s.5$														
m	1.1.5	1.2.5	1.3.5	1.4.5	1.5.5	2.2.5	2.3.5	2.4.5	2.5.5	3.3.5	3.4.5	3.5.5	4.4.5	4.5.5	5.5.5
0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
1	5	10	15	20	25	20	30	40	50	45	60	75	80	100	125
2		20	60	120	200	130	330	620	1,000	810	1,500	2,400	2,760	4,400	7,000
3			60	240	600	320	1,680	4,800	10,400	7,590	20,520	43,200	54,240	112,800	233,000
4				120	600	260	4,140	20,040	61,400	40,500	169,920	486,000	676,200	1,881,600	5,159,000
5					120		4,680	45,600	211,440	126,900	891,360	3,594,960	5,641,920	21,612,480	80,602,200
6							1,920	54,480	421,200	232,680	3,018,000	17,930,400	32,423,520	176,546,400	920,160,000
7								30,720	465,600	240,840	6,605,280	60,912,000	130,248,960	1,045,147,200	7,845,192,000
8								6,360	262,200	128,520	9,224,280	140,826,600	367,731,360	4,530,640,800	50,648,616,000
9									63,600	27,480	7,983,840	219,307,800	728,440,320	14,444,083,200	249,687,408,000
10									5,280		4,063,680	225,419,040	1,004,380,800	33,852,910,080	944,069,668,800
11											1,100,160	148,010,400	950,238,720	58,065,734,400	2,741,210,616,000
12											120,960	59,047,200	603,722,880	72,278,294,400	6,104,066,712,000
13												13,284,000	249,580,800	64,484,985,600	10,385,299,320,000
14												1,512,000	63,884,160	40,544,726,400	13,420,351,008,000
15												66,240	9,216,000	17,571,260,160	13,065,814,483,200
16													590,400	5,099,169,600	9,486,099,648,000
17														953,107,200	5,073,056,640,000
18														108,288,000	1,970,474,400,000
19														6,681,600	547,608,096,000
20														161,280	107,330,054,400
21															14,667,552,000
22															1,388,160,000
23															91,008,000
24															4,032,000
25															161,280
Total	6	31	136	501	1,546	731	12,781	162,661	1,502,171	805,366	33,199,561	890,442,316	4,146,833,121	313,185,347,701	64,170,718,937,006

ENUMERATION OF $\mathcal{PLR}_{r,s,n;m}$

	$\lvert\mathcal{R}_{r,s,6:m}\rvert$														
	$r.s.6$														
m	1.1.6	1.2.6	1.3.6	1.4.6	1.5.6	1.6.6	2.2.6	2.3.6	2.4.6	2.5.6	2.6.6	3.3.6	3.4.6	3.5.6	3.6.6
0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
1	6	12	18	24	30	36	24	36	48	60	72	54	72	90	108
2		30	90	180	300	450	192	486	912	1,470	2,160	1,188	2,196	3,510	5,130
3			120	480	1,200	2,400	600	3,120	8,880	19,200	35,400	13,896	37,344	78,360	141,840
4				360	1,800	5,400	630	9,990	48,060	146,700	349,650	94,770	392,580	1,115,100	2,547,450
5					720	4,320		15,120	146,880	678,240	2,168,640	389,340	2,676,240	10,667,160	31,419,360
6						720		8,520	245,760	1,899,600	8,546,880	961,380	12,082,680	70,540,800	274,470,480
7									204,480	3,139,200	21,211,200	1,375,920	36,270,720	326,808,000	1,727,352,000
8									65,160	2,881,800	32,189,400	1,038,960	71,633,160	1,064,140,200	7,893,282,600
9										1,303,200	28,267,200	317,760	90,585,600	2,422,568,400	26,212,965,600
10										222,480	13,063,680		69,603,840	3,803,369,040	62,938,898,640
11											2,669,760		29,255,040	4,021,099,200	108,045,861,120
12											190,800		5,112,000	2,756,361,600	130,246,779,600
13														1,152,144,000	107,367,120,000
14														262,828,800	58,252,478,400
15														24,791,040	19,683,613,440
16															3,828,798,720
17															384,652,800
18															15,321,600
Total	7	43	229	1,045	4,051	13,327	1,447	37,273	720,181	10,291,951	108,694,843	4,193,269	317,651,473	15,916,515,301	526,905,708,889

Enumeration of $\mathcal{PLR}_{r,s,n;m}$

	$\lvert\mathcal{R}_{r,s,6:m}\rvert$					
	$r.s.6$					
m	4.4.6	4.5.6	4.6.6	5.5.6	5.6.6	6.6.6
0	1	1	1	1	1	1
1	96	120	144	150	180	216
2	4,032	6,420	9,360	10,200	14,850	21,600
3	98,016	203,040	364,560	417,600	746,400	1,330,200
4	15,384,24	4,245,120	9,527,220	11,532,600	25,631,100	56,614,950
5	16,476,480	62,189,280	177,310,080	228,154,320	639,260,640	1,771,796,160
6	124,148,160	660,375,600	2,434,907,520	3,352,566,000	12,019,602,000	423,57,620,160
7	669,176,640	5,189,068,800	25,231,996,800	37,450,656,000	174,585,456,000	793,416,600,000
8	2,599,625,880	30,548,079,000	200,165,742,000	322,946,451,000	1,991,858,418,000	11,852,197,317,000
9	7,281,623,040	135,625,603,200	1,226,542,944,000	2,171,483,394,000	18,056,836,776,000	142,993,809,528,000
10	14,618,868,480	455,097,055,680	5,834,154,055,680	11,456,637,616,800	131,095,655,863,200	1,406,144,941,776,000
11	20,771,527,680	1,152,338,169,600	21,579,415,960,320	47,586,889,008,000	766,225,199,808,000	11,344,829,123,448,000
12	20,451,767,040	2,190,542,918,400	62,007,749,812,800	155,763,852,264,000	3,616,441,279,056,000	75,444,662,621,250,000
13	13,491,532,800	3,099,028,723,200	137,935,650,124,800	401,342,211,504,000	13,801,803,749,280,000	414,809,990,051,328,000
14	5,635,215,360	3,221,159,616,000	236,112,048,230,400	811,559,781,792,000	42,582,496,312,944,000	1,888,965,825,155,136,000
15	1,337,610,240	2,415,807,221,760	308,313,104,578,560	1,281,622,863,052,800	106,042,151,250,892,000	7,129,083,890,074,291,200
16	137,116,800	1,274,532,969,600	303,524,671,011,840	1,569,898,647,504,000	212,529,994,957,440,000	22,290,972,757,613,899,200
17		455,792,486,400	221,831,824,435,200	1,478,352,018,528,000	341,378,166,715,776,000	57,672,207,579,205,440,000
18		104,134,464,000	117,967,540,608,000	1,058,153,580,288,000	437,045,603,416,704,000	123,205,370,805,154,944,000
19		13,604,889,600	44,468,899,430,400	567,490,862,592,000	442,874,461,303,296,000	216,689,524,093,737,792,000
20		767,854,080	11,483,903,278,080	223,899,017,011,200	352,217,521,389,081,000	312,570,613,181,156,803,200
21			1,942,917,304,320	63,429,754,752,000	217,606,324,462,848,000	368,084,100,503,749,939,200
22			202,499,481,600	12,467,229,696,000	103,166,400,104,064,000	351,915,364,298,700,288,000
23			11,670,220,800	1,610,606,592,000	36,987,139,952,640,000	271,409,503,369,430,016,000
24			283,046,400	123,628,032,000	9,853,601,458,752,000	167,607,699,757,168,896,000
25				4,356,218,880	1,909,729,461,012,480	82,187,524,303,374,458,880
26					262,267,391,462,400	31,703,766,748,202,926,080
27					24,634,533,888,000	9,523,824,649,261,056,000
28					1,496,724,480,000	2,204,514,949,427,712,000
29					52,752,384,000	389,140,940,150,784,000
30					812,851,200	51,905,194,846,617,600
31						5,196,712,196,505,600
32						389,383,137,792,000
33						21,862,379,520,000
34						925,655,040,000
35						29,262,643,200
36						812,851,200
Total	87,136,329,169	14,554,896,138,901	1,474,670,894,380,885	7,687,297,409,633,551	2,322,817,844,850,427,451	202,7032,853,070,203,981,647

Outline

1. Introduction.
2. Enumeration of $\mathcal{PLR}_{r,s,n}$.
3. Distribution into isomorphism and isotopism classes.

Isotopism and isomorphism clases

We focus on the distribution of $\mathcal{PLR}_{r,s,n}$ into isomorphism and isotopism classes, that is, on the sets $\mathcal{I}_{r,s,n}$ and $\mathfrak{I}_{r,s,n}$.

Lemma

Let r, s and n be three positive integers.

- *If $rs \leq n$, then $|\mathfrak{I}_{r,s,n}| = |\mathfrak{I}_{r,s,rs}|$.*
- *If $s \leq n$, then $|\mathfrak{I}_{1,s,n}| = s + 1$.*

Let $\mathcal{PLR}_\Theta$ denote the set of $r \times s$ partial Latin rectangles based on $[n]$ that have an isotopism $\Theta \in S_r \times S_s \times S_n$ in its autotopism group.

Lemma

Let Θ_1 and Θ_2 be two conjugate isotopisms in $S_r \times S_s \times S_n$. Then,

- $|\mathcal{PLR}_{\Theta_1}| = |\mathcal{PLR}_{\Theta_2}|$.
- *The set of isotopism classes of $\mathcal{PLR}_{\Theta_1}$ coincides with that of $\mathcal{PLR}_{\Theta_2}$.*

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- $|\mathcal{PLR}_{\Theta_1}| = |\mathcal{PLR}_{\Theta_2}|$.
- *The set of isotopism classes of $\mathcal{PLR}_{\Theta_1}$ coincides with that of $\mathcal{PLR}_{\Theta_2}$.*

To be conjugate is an equivalence relation among isotopisms. Each conjugacy class is characterized by the common cycle structure of its elements.

The **cycle structure** of a permutation π in the symmetric group S_m is defined as the expression $z_\pi = m^{d_m^\pi} \ldots 1^{d_1^\pi}$, where d_i^π is the number of cycles of length i in the unique cycle decomposition of the permutation π.

The **cycle structure** of an isotopism $(\alpha, \beta, \gamma) \in S_r \times S_s \times S_n$ is the triple $z_\Theta = (z_\alpha, z_\beta, z_\gamma)$.

$$\Theta = ((1234), (12)(3)(45), (12)(345)(6)) \in S_4 \times S_5 \times S_6 \Rightarrow z_\Theta = (4, 2^2 1, 321).$$

$\mathcal{CS}_m$: The set of cycle structures of the symmetric group S_m.

Let $z \in \mathcal{CS}_m$. Let d_i^z denote the value of d_i^π for all permutation $\pi \in S_m$ of cycle structure z.

Lemma

A triple $z = (z_1, z_2, z_3) \in \mathcal{CS}_r \times \mathcal{CS}_s \times \mathcal{CS}_n$ is the cycle structure of an isotopism of a non-empty $r \times s$ partial Latin rectangle based on $[n]$ if and only if there exists a triple $(i, j, k) \in [r] \times [s] \times [n]$ such that

$$\mathrm{lcm}(i,j) = \mathrm{lcm}(i,k) = \mathrm{lcm}(j,k) = \mathrm{lcm}(i,j,k) \text{ and } d_i^{z_1} \cdot d_j^{z_2} \cdot d_k^{z_3} > 0.$$

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A triple $z = (z_1, z_2, z_3) \in \mathcal{CS}_r \times \mathcal{CS}_s \times \mathcal{CS}_n$ is the cycle structure of an isotopism of a non-empty $r \times s$ partial Latin rectangle based on $[n]$ if and only if there exists a triple $(i, j, k) \in [r] \times [s] \times [n]$ such that

$$\mathrm{lcm}(i, j) = \mathrm{lcm}(i, k) = \mathrm{lcm}(j, k) = \mathrm{lcm}(i, j, k) \text{ and } d_i^{z_1} \cdot d_j^{z_2} \cdot d_k^{z_3} > 0.$$

$\Delta(z_1, z_2, z_3) = |\mathcal{PLR}_\Theta|$ for any isotopism Θ of cycle structure $(z_1, z_2, z_3) \in \mathcal{CS}_r \times \mathcal{CS}_s \times \mathcal{CS}_n$.

Since the isotopism and the isomorphism groups are finite groups that acts on $\mathcal{R}_{r,s,n}$ and $\mathcal{R}_{n,n,n}$, respectively, **Burnside's lemma** and the number of permutations with a given cycle structure involve:

$$|\mathfrak{I}_{r,s,n}| = \sum_{\substack{\alpha\in S_r\\ \beta\in S_s\\ \gamma\in S_n}} \frac{\Delta(z_\alpha, z_\beta, z_\gamma)}{r!s!n!} = \sum_{\substack{z_1\in\mathcal{CS}_r\\ z_2\in\mathcal{CS}_s\\ z_3\in\mathcal{CS}_n}} \frac{\Delta(z_1, z_2, z_3)}{\prod_{\substack{i\in[r]\\ j\in[s]\\ k\in[n]}} d_i^{z_1}!d_j^{z_2}!d_k^{z_3}!\, i^{d_i^{z_1}} j^{d_j^{z_2}} k^{d_k^{z_3}}}.$$

$$|\mathcal{I}_n| = \sum_{\pi\in S_n} \frac{\Delta(z_\pi, z_\pi, z_\pi)}{n!} = \sum_{z\in\mathcal{CS}_n} \frac{\Delta(z, z, z)}{\prod_{i\in[n]} d_i^z!\, i^{d_i^z}}.$$

Isotopism and isomorphism clases

$\Delta(z_1, z_2, z_3) = |\mathcal{PLR}_\Theta|$ for any isotopism Θ of cycle structure $(z_1, z_2, z_3) \in \mathcal{CS}_r \times \mathcal{CS}_s \times \mathcal{CS}_n$.

Since the isotopism and the isomorphism groups are finite groups that acts on $\mathcal{R}_{r,s,n}$ and $\mathcal{R}_{n,n,n}$, respectively, **Burnside's lemma** and the number of permutations with a given cycle structure involve:

$$|\mathfrak{I}_{r,s,n}| = \sum_{\substack{\alpha\in S_r\\ \beta\in S_s\\ \gamma\in S_n}} \frac{\Delta(z_\alpha, z_\beta, z_\gamma)}{r!s!n!} = \sum_{\substack{z_1\in\mathcal{CS}_r\\ z_2\in\mathcal{CS}_s\\ z_3\in\mathcal{CS}_n}} \frac{\Delta(z_1, z_2, z_3)}{\prod_{\substack{i\in[r]\\ j\in[s]\\ k\in[n]}} d_i^{z_1}!d_j^{z_2}!d_k^{z_3}!\, i^{d_i^{z_1}} j^{d_j^{z_2}} k^{d_k^{z_3}}}.$$

$$|\mathcal{I}_n| = \sum_{\pi\in S_n} \frac{\Delta(z_\pi, z_\pi, z_\pi)}{n!} = \sum_{z\in\mathcal{CS}_n} \frac{\Delta(z, z, z)}{\prod_{i\in[n]} d_i^z!\, i^{d_i^z}}.$$

Theorem

Let $\Theta = (\alpha, \beta, \gamma)$ be an isotopism of $\mathcal{PLR}_{r,s,n}$. The set $\mathcal{PLR}_\Theta$ is identified with the affine variety defined by the ideal in $\mathbb{F}_2[x_{111}, \ldots, x_{rsn}]$

$$I_\Theta = I_{r,s,n} \cup \langle\, x_{ijk} - x_{\alpha(i)\beta(j)\gamma(k)} \colon\ i \in [r],\, j \in [s],\, k \in [n] \,\rangle.$$

Further,

$$\Delta(z_\alpha, z_\beta, z_\gamma) = |\mathcal{PLR}_\Theta| = \dim_{\mathbb{F}_2}(\mathbb{F}_2[x_{111}, \ldots, x_{rsn}]/I_\Theta).$$

Isotopism and isomorphism clases

n	$\lvert\mathcal{I}_n\rvert$
1	2
2	20
3	2029
4	5319934
5	534759300182
6	281532435872410905

r	s	n	$\lvert\mathfrak{I}_{r,s,n}\rvert$
2	2	2	8
		3	10
		4	11
		5	11
		6	11
	3	3	20
		4	27
		5	29
		6	30
	4	4	54
		5	70
		6	78
	5	5	125
		6	166
	6	6	292
3	3	3	81
		4	184
		5	279

r	s	n	$\lvert\mathfrak{I}_{r,s,n}\rvert$
3	3	6	325
	4	4	839
		5	2227
		6	3825
	5	5	11194
		6	33299
	6	6	177892
4	4	4	9878
		5	61955
		6	218558
	5	5	914969
		6	7074338
	6	6	118883849
5	5	5	37202840
		6	742190170
	6	6	37349106398
6	6	6	5431010366322

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THANK YOU!!!

Classifying partial Latin rectangles

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