Mutation graphs of asexual diploid organisms

Óscar J. Falcón[a], Raúl M. Falcón[b], Juan Núñez[a]

(a) *Departamento de Geometría y Topología, Universidad de Sevilla.*
E-mail: oscfalgan@yahoo.es, jnvaldes@us.es

(b) *Departamento de Matemática Aplicada I, Universidad de Sevilla*
E-mail: rafalgan@us.es

Abstract. The mutation graph of an asexual diploid organism is introduced as an edge-coloured graph derived from the genetic pattern by isotopisms of an evolution algebra over a finite field. We describe the step-by-step construction of this graph and establish some of its basic properties. In order to illustrate this construction, we focus on the spectrum of genetic patterns of two distinct genotypes during a mitosis process.

Keywords. Line graph, evolution algebra, isomorphism.

1 Introduction

In order to simulate algebraically self-reproduction processes in Genetics, Tian and Vojtechovsky [8, 9] introduced the concept of *evolution algebra* on a set $\beta = \{e_1, \ldots, e_n\}$ of distinct genotypes with respect to a given phenotype of an asexual organism as an n-dimensional algebra over a field $\mathbb{K}$ having β as a *natural* basis such that $e_ie_j = 0$, whenever $i \neq j$. The tuple $(e_1e_1, \ldots, e_ne_n)$ is called the *genetic pattern* of the algebra. Evolution algebras present interesting connections with graph theory. In this regard, Tian [8] defined the evolution algebra related to a graph G with a finite set of vertices $\{v_1, \ldots, v_n\}$ as the algebra of basis $\{e_1, \ldots, e_n\}$ described as $e_i^2 = \sum_{k\in\Gamma(v_i)} e_k$ and $e_ie_j = 0$, for all $i, j \in \{1, \ldots, n\}$ such that $i \neq j$. Here, $\Gamma(v_i)$ denotes the set of neighbours of the vertex v_i in G. This definition enabled him to contemplate as a further work the development of known results on graph theory in the context of evolution algebras. More recently, Elduque and Labra [3, 4], and Cabrera et al. [1] dealt with the reverse problem. The former associated a weighted digraph to a given finite-dimensional evolution algebra and proved that the nonexistence of oriented cycles in such a graph is equivalent to the nilpotency of the corresponding algebra. The latter, in turn, described an alternative directed graph whatever the dimension of the evolution algebra is, from which its annihilator and irreducibility can be determined. Unlike Tian's graphs, whose isomorphisms are equivalent to those of the evolution algebras

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under consideration, both proposals depend on the selected basis of the algebra so that isomorphic algebras are not related in general to isomorphic graphs.

In order to ensure an affirmative statement in the last regard, this paper deals with a new proposal to establish a relationship between evolution algebras and graph theory. This is based on a recent work of the authors [7], who identify any algebra over a finite field with a pair of vertex-coloured graphs so that every isomorphism and every isotopism of the algebra under consideration gives rise to an isomorphism of the corresponding graph. We focus in particular on the study of isotopisms because of their importance to formulate algebraically the mutation of genotypes in the inheritance process [2]. Currently, it is known [5, 6] the distribution into isotopism classes of two- and three-dimensional evolution algebras over any base field, which determines in turn the spectrum of genetic patterns of two and three distinct genotypes during a mitosis process.

2 Mutation graphs

Let A be an algebra over a finite field $\mathbb{K}$ and let $\mathrm{Ann}(A) = \{u \in A \mid uv = 0, \text{ for all } v \in A\}$ denote its annihilator. Let us describe the step-by-step construction of the mutation graph of the algebra A. In order to illustrate this construction, we refer the reader to Figure 1.1, where we describe in detail how to obtain the mutation graph of the evolution algebra over $\mathbb{F}_2$ with genetic pattern (e_1, e_1). Further, Figure 1.2 shows the spectrum of mutation graphs related to non-trivial genetic patterns of two distinct genotypes during a mitosis process.

- **Step 1**: According to the description proposed by the authors in [7], we define the vertex-coloured graph $G_1(A)$ with the following four maximal monochromatic subsets of vertices
$$R_A = \{r_u \mid u \in A \setminus \mathrm{Ann}(A)\}, \qquad C_A = \{c_u \mid u \in A \setminus \mathrm{Ann}(A)\},$$
$$S_A = \{s_u \mid u \in A^2 \setminus \{0\}\} \quad \text{and} \quad T_A = \{t_{u,v} \mid u, v \in A, uv \neq 0\},$$
and set of edges $\{r_u t_{u,v}, c_v t_{u,v}, s_w t_{u,v} \mid u, v, w \in A, uv = w \neq 0\}$. Suppose the vertices of the four sets R_A, C_A, S_A and T_A to be respectively coloured with the colours red, blue, green and black.

- **Step 2**: We construct the edge-coloured line graph $G_2(A)$ associated to $G_1(A)$. The colours of its edges are inherited in natural way from those of the corresponding vertices in $G_1(A)$. Let $u, v, w \in A$ be such that $uv = w \neq 0$. Each triple $(r_u t_{u,v}, c_v t_{u,v}, s_w t_{u,v})$ of edges in $G_1(A)$ gives rise to a triangle in $G_2(A)$, which we call *structural triangle* in $G_2(A)$. Its edges are called *structural*, whereas the rest of edges in $G_2(A)$ are called *non-structural*.

Lemma 1. *The following results hold.*

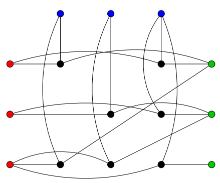
a) *Every structural edge in $G_2(A)$ is coloured in black.*
b) *The colours of every pair of non-structural edges that are incident to the same vertex in $G_2(A)$ coincide.*
c) *The colours of every pair of non-structural edges that are respectively incident to a pair of distinct vertices of the same structural triangle in $G_2(A)$ are distinct.*

- **Step 3**: We define the edge-coloured multigraph $G_3(A)$ that results after contracting the three vertices of each structural triangle in $G_2(A)$.

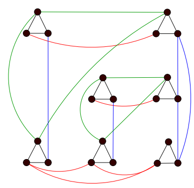
  Proposition 1. *The following results hold.*

  a) *The colours of every pair of parallel edges in $G_3(A)$ are distinct.*
  b) *There do not exist three parallel edges in $G_3(A)$.*
  c) *There always exists a green edge in any pair of parallel edges in $G_3(A)$.*

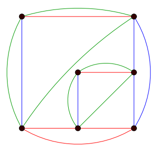
- **Step 4**: We define the edge-coloured graph $G_4(A)$ that results after contracting every pair of parallel edges in $G_3(A)$. The contraction of a pair of red-green parallel edges gives rise to a pink edge, whereas that related to a pair of blue-green parallel edges gives rise to an orange edge.



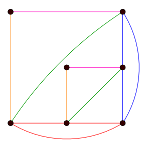
Step 1: Vertex-coloured graph.



Step 2: Line graph.



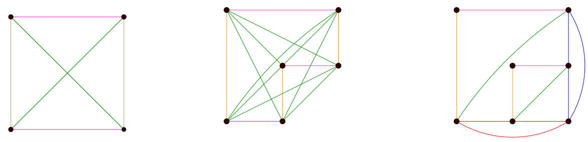
Step 3: Contracted multigraph.



Step 4: Mutation graph.

Figure 1.1: Step-by-step construction of the mutation graph related to the bi-dimensional evolution algebra over $\mathbb{F}_2$ of genetic pattern (e_1, e_2).

Theorem 1. *If two evolution algebras over a finite field are isomorphic, then their mutation graphs are isomorphic.*



$(e_1, 0)$ (e_1, e_1) (e_1, e_2)

Figure 1.2: Spectrum of non-trivial mutation graphs for two distinct genotypes.

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