# Gromov hyperbolicity in strong product graphs 

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#### Abstract

If X is a geodesic metric space and $x_{1}, x_{2}, x_{3} \in X$, a geodesic triangle $T=$ $\left\{x_{1}, x_{2}, x_{3}\right\}$ is the union of the three geodesics $\left[x_{1} x_{2}\right],\left[x_{2} x_{3}\right]$ and $\left[x_{3} x_{1}\right]$ in $X$. The space $X$ is $\delta$-hyperbolic (in the Gromov sense) if any side of $T$ is contained in a $\delta$-neighborhood of the union of the two other sides, for every geodesic triangle $T$ in $X$. If $X$ is hyperbolic, we denote by $\delta(X)$ the sharp hyperbolicity constant of $X$, i.e. $\delta(X)=\inf \{\delta \geqslant 0: X$ is $\delta$-hyperbolic $\}$. In this paper we characterize the strong product of two graphs $G_{1} \boxtimes G_{2}$ which are hyperbolic, in terms of $G_{1}$ and $G_{2}$ : the strong product graph $G_{1} \boxtimes G_{2}$ is hyperbolic if and only if one of the factors is hyperbolic and the other one is bounded. We also prove some sharp relations between $\delta\left(G_{1} \boxtimes G_{2}\right), \delta\left(G_{1}\right), \delta\left(G_{2}\right)$ and the diameters of $G_{1}$ and $G_{2}$ (and we find families of graphs for which the inequalities are attained). Furthermore, we obtain the exact values of the hyperbolicity constant for many strong product graphs.


Keywords: Strong Product Graphs; Geodesics; Gromov Hyperbolicity; Infinite Graphs

## 1 Introduction

The study of mathematical properties of Gromov hyperbolic spaces and its applications is a topic of recent and increasing interest in graph theory; see, for instance, $[5,6,9,11$, $12,13,14,15,16,18,19,27,28,29,30,31,33,34,35,38,39,42,43,45,48,49]$. It is well known that most networks can be modeled by a graph $G=(V, E)$, where $V$ is the set of mainly elements and $E$ is the set of communication links between them in the network. Different methods have been proposed for configuration processing and data generation. Some of them are structural models which can be seen as the product graph of two given graphs, known as factors or generators. Many properties of structural models can be obtained by considering the properties of their generators. The different kinds of products of graphs are an important research topic in Graph Theory. In particular, the strong product graph operation has been extensively investigated in relation to a wide range of subjects $[2,10,32,47]$. A fundamental principle for network design is extendability. That is to say, the possibility of building larger versions of a network preserving certain desirable properties. For designing large-scale interconnection networks, the strong p roduct is a useful method to obtain large graphs from smaller ones whose invariants can be easily calculated [10, 32, 47].

The theory of Gromov hyperbolic spaces was used initially for the study of finitely generated groups, where it was demonstrated to have an enormous practical importance. This theory was applied principally to the study of automatic groups (see [36]), which plays an important role in sciences of the computation. Another important application of these spaces is the secure transmission of information by internet. In particular, the hyperbolicity plays an important role in the spread of viruses through the network (see $[28,29])$. The hyperbolicity is also useful in the study of DNA data (see [9]).

Last years several researchers have been interested in showing that metrics used in geometric function theory are Gromov hyperbolic. For instance, the Gehring-Osgood $j$-metric is Gromov hyperbolic; and the Vuorinen $j$-metric is not Gromov hyperbolic except in the punctured space (see [21]). The study of Gromov hyperbolicity of the quasihyperbolic and the Poincaré metrics is the subject of $[1,3,7,22,23,24,25,39,40$, $43,44,45,49]$. In particular, the equivalence of the hyperbolicity of Riemannian manifolds and the hyperbolicity of a simple graph was proved in [39, 43, 45, 49], hence, it is useful to know hyperbolicity criteria for graphs.

Notations and terminology not explicitly given here can be found in [20]. We present now some basic facts about Gromov's spaces. Let $(X, d)$ be a metric space and let $\gamma$ : $[a, b] \longrightarrow X$ be a continuous function. We define the length of $\gamma$ as

$$
L(\gamma):=\sup \left\{\sum_{i=1}^{n} d\left(\gamma\left(t_{i-1}\right), \gamma\left(t_{i}\right)\right): a=t_{0}<t_{1}<\cdots<t_{n}=b\right\}
$$

We say that $\gamma$ is a geodesic if it is an isometry, i.e. $d(\gamma(t), \gamma(s))=s-t$ for every $t<s$. We say that $X$ is a geodesic metric space if for every $x, y \in X$ there exists a geodesic joining $x$ and $y$; we denote by $[x y]$ any of such geodesics (since we do not require uniqueness of geodesics, this notation is ambiguous, but it is convenient). It is clear that every geodesic
metric space is path-connected. If $X$ is a graph, we use the notation $[u, v]$ for the edge joining the vertices $u$ and $v$; in what follows, by $u \sim v$ we mean that $[u, v] \in E(X)$.

In order to consider a graph $G$ as a geodesic metric space, we must identify (by an isometry) any edge $[u, v] \in E(G)$ with a real interval with length $l:=L([u, v])$; therefore, an inner point of the edge $[u, v]$ is a point of $G$. A connected graph $G$ is naturally equipped with a distance defined on its points, induced by taking shortest paths in $G$. Then, $G$ can be seen as a metric graph.

Throughout this paper we just consider non-oriented (finite or infinite) connected graphs with edges of length 1 . These conditions guarantee that the graphs are geodesic metric spaces. We also consider simple graphs, that is without loops or multiple edges, which is not a restriction [6, Theorems 6 and 8 ].

If $X$ is a geodesic metric space and $J=\left\{J_{1}, J_{2}, \ldots, J_{n}\right\}$ is a polygon with sides $J_{j} \subseteq X$, we say that $J$ is $\delta$-thin if for every $x \in J_{i}$ we have that $d\left(x, \cup_{j \neq i} J_{j}\right) \leqslant \delta$. We denote by $\delta(J)$ the sharp thin constant of $J$, i.e., $\delta(J):=\inf \{\delta \geqslant 0: J$ is $\delta$-thin $\}$. If $x_{1}, x_{2}, x_{3} \in X$, a geodesic triangle $T=\left\{x_{1}, x_{2}, x_{3}\right\}$ is the union of the three geodesics $\left[x_{1} x_{2}\right],\left[x_{2} x_{3}\right]$ and $\left[x_{3} x_{1}\right]$ (sometimes we write $T=\left\{\left[x_{1} x_{2}\right],\left[x_{2} x_{3}\right],\left[x_{3} x_{1}\right]\right\}$ ). The space $X$ is $\delta$-hyperbolic (or satisfies the Rips condition with constant $\delta$ ) if every geodesic triangle in $X$ is $\delta$-thin. We denote by $\delta(X)$ the sharp hyperbolicity constant of $X$, i.e., $\delta(X):=\sup \{\delta(T):$ $T$ is a geodesic triangle in $X\}$. We say that $X$ is hyperbolic if $X$ is $\delta$-hyperbolic for some $\delta \geqslant 0$. If $X$ is hyperbolic, then $\delta(X)=\inf \{\delta \geqslant 0: X$ is $\delta$-hyperbolic $\}$. A geodesic bigon is a geodesic triangle $\left\{x_{1}, x_{2}, x_{3}\right\}$ with $x_{2}=x_{3}$. Therefore, every bigon in a $\delta$-hyperbolic geodesic metric space is $\delta$-thin.

There are several definitions of Gromov hyperbolicity. These different definitions are equivalent in the sense that if $X$ is $\delta$-hyperbolic with respect to the definition $A$, then it is $\delta^{\prime}$-hyperbolic with respect to the definition $B$ for some $\delta^{\prime}$ (see, e.g., [8, 20]). We have chosen this definition since it has a deep geometric meaning (see, e.g., [20]).

The following remarks are interesting examples of hyperbolic spaces. The real line $\mathbb{R}$ is 0 -hyperbolic due to any point of a geodesic triangle in the real line belongs to two sides of the triangle simultaneously. The Euclidean plane $\mathbb{R}^{2}$ is not hyperbolic since the equilateral triangles can be drawn with arbitrarily large diameter. This argument can be generalized in a similar way to higher dimensions: a normed vector space $E$ is hyperbolic if and only if $\operatorname{dim} E=1$. Every arbitrary length metric tree is 0-hyperbolic due to all points of a geodesic triangle in a tree belong simultaneously to two sides of the triangle. Every bounded metric space $X$ is $((\operatorname{diam} X) / 2)$-hyperbolic. Every simply connected complete Riemannian manifold with sectional curvature verifying $K \leqslant-k^{2}$, for some positive constant $k$, is hyperbolic. We refer to [8,20] for more background and further results.

Notice that the main examples of hyperbolic graphs are the trees. In fact, the hyperbolicity constant of a geodesic metric space can be viewed as a measure of how "tree-like" the space is, since those spaces $X$ with $\delta(X)=0$ are precisely the metric trees. This is an interesting subject since, in many applications, one finds that the borderline between tractable and intractable cases may be the tree-like degree of the structure to be dealt with (see, e.g., [17]).

If $D$ is a closed connected subset of $X$, we always consider in $D$ the inner metric obtained by the restriction of the metric in $X$, that is

$$
d_{D}(z, w):=\inf \left\{L_{X}(\gamma): \gamma \subset D \text { is a continuous curve joining } z \text { and } w\right\} \geqslant d_{X}(z, w)
$$

Consequently, $L_{D}(\gamma)=L_{X}(\gamma)$ for every curve $\gamma \subset D$.
Given a Cayley graph (of a presentation with solvable word problem) there is an algorithm which allows to decide if it is hyperbolic. However, the problem of deciding whether a general geodesic metric space is hyperbolic or not is usually very difficult. Note that, first of all, we have to consider an arbitrary geodesic triangle $T$, and calculate the minimum distance from an arbitrary point $P$ of $T$ to the union of the other two sides of the triangle to which $P$ does not belong to. Finally we have to take supremum over all the possible choices for $P$ and over all the possible choices for $T$. Without disregarding the difficulty to solve this minimax problem, notice that in general the main obstacle is that we do not know the location of geodesics in the space. Therefore, it is interesting to obtain inequalities involving the hyperbolicity constant and to study the hyperbolicity of a particular class of graphs.

The papers $[5,9,16,35,37,41,48]$ study the hyperbolicity of, respectively, complement of graphs, chordal graphs, line graphs, Cartesian product graphs, cubic graphs, short graphs and median graphs, respectively. Our aim in this work is to obtain interesting results about the hyperbolicity constant of strong product graphs.

The structure of this paper is as follows. First, in Section 2, we study several inequalities involving the distance in the strong product of graphs and we obtain the exact value of its diameter. Furthermore, we also study the relations between the geodesics of $G_{1} \boxtimes G_{2}$ and geodesics in $G_{1}$ and $G_{2}$; it is not a trivial issue as Example 7 will show.

In Section 3, we prove several lower and upper bounds for the hyperbolicity constant of $G_{1} \boxtimes G_{2}$, involving $\delta\left(G_{1}\right), \delta\left(G_{2}\right)$ and the diameters of $G_{1}$ and $G_{2}$. One of the main results of this work is Theorem 23, which characterizes the hyperbolic strong product graphs $G_{1} \boxtimes G_{2}$ in terms of $G_{1}$ and $G_{2}$. The graph $G_{1} \boxtimes G_{2}$ is hyperbolic if and only if one of its factors is hyperbolic and the other one is bounded. We also find families of graphs for which many of the inequalities of this section are attained. Another main result in this paper is Theorem 19 which provides the precise value of $\delta\left(G_{1} \boxtimes G_{2}\right)$ for a large class of graphs $G_{1}, G_{2}$; this kind of result is not usual at all in the theory of hyperbolic graphs.

We conclude this paper with Section 4 where the exact values of the hyperbolicity constant for many strong product graphs are calculated.

## 2 The distance in strong product graphs

In order to estimate the hyperbolicity constant of the strong product of two graphs $G_{1}$ and $G_{2}$, we must obtain lower and upper bound on the distances between any two arbitrary points in $G_{1} \boxtimes G_{2}$. The lemmas of this section provide these estimations. We will use the strong product definition given by Sabidussi in [46].

Definition 1. Let $G_{1}=\left(V\left(G_{1}\right), E\left(G_{1}\right)\right)$ and $G_{2}=\left(V\left(G_{2}\right), E\left(G_{2}\right)\right)$ two graphs. The strong product $G_{1} \boxtimes G_{2}$ of $G_{1}$ and $G_{2}$ has $V\left(G_{1}\right) \times V\left(G_{2}\right)$ as vertex set, so that two distinct vertices $\left(u_{1} ; v_{1}\right)$ and $\left(u_{2} ; v_{2}\right)$ of $G_{1} \boxtimes G_{2}$ are adjacent if either $u_{1}=u_{2}$ and $\left[v_{1}, v_{2}\right] \in E\left(G_{2}\right)$, or $\left[u_{1}, u_{2}\right] \in E\left(G_{1}\right)$ and $v_{1}=v_{2}$, or $\left[u_{1}, u_{2}\right] \in E\left(G_{1}\right)$ and $\left[v_{1}, v_{2}\right] \in E\left(G_{2}\right)$.

Note that the strong product of two graphs is commutative. We use the notation $(u ; v)$ instead of $(u, v)$ to the points of the graph $G_{1} \boxtimes G_{2}$. We consider that every edge of $G_{1} \boxtimes G_{2}$ has length 1 .

Next, we will bound the distances between any two different pair of points in the strong product graph. For this aim we must distinguish some cases depending on the situation of the considered points. Let $p \in G_{1}$ and $q \in G_{2}$ be two points of $G_{1}$ and $G_{2}$ respectively. The pair $(p ; q)$ is an inner point in $G_{1} \boxtimes G_{2}$, if $p \in G_{1} \backslash V\left(G_{1}\right)$ and $q \in V\left(G_{2}\right)$ or $p \in V\left(G_{1}\right)$ and $q \in G_{2} \backslash V\left(G_{2}\right)$ or $p \in G_{1} \backslash V\left(G_{1}\right)$ and $q \in G_{2} \backslash V\left(G_{2}\right)$ (i.e., $\left.(p ; q) \in G_{1} \boxtimes G_{2} \backslash V\left(G_{1} \boxtimes G_{2}\right)\right)$. Notice that the first and second cases of the inner points in $G_{1} \boxtimes G_{2}$ are contained in the Cartesian product graph $G_{1} \square G_{2} \subset G_{1} \boxtimes G_{2}$; so the first and second cases are the inner points of the Cartesian edges properly. In order to represent the inner points of the non Cartesian edges in $G_{1} \boxtimes G_{2}$ we will consider the following assumptions. Let $\left[A_{1}, A_{2}\right] \in E\left(G_{1}\right)$ and $\left[B_{1}, B_{2}\right] \in E\left(G_{2}\right)$ be edges in $G_{1}$ and $G_{2}$, respectively. Let $p \in\left[A_{1}, A_{2}\right]$ and $q \in\left[B_{1}, B_{2}\right]$ be inner points of theses fixed edges; we have $(p ; q) \in G_{1} \boxtimes G_{2} \backslash G_{1} \square G_{2}$ if $L\left(\left[p A_{1}\right]\right)=L\left(\left[q B_{1}\right]\right)$ or $L\left(\left[p A_{1}\right]\right)=L\left(\left[q B_{2}\right]\right)$.

Notice that there are different points on $G_{1} \boxtimes G_{2}$ with the same representation: the midpoints of $\left[\left(A_{1} ; B_{1}\right),\left(A_{2} ; B_{2}\right)\right]$ and $\left[\left(A_{1} ; B_{2}\right),\left(A_{2} ; B_{1}\right)\right]$. Then, this notation is ambiguous, but it is convenient.

The following lemmas provide bounds on the distance between any two pair of points in the strong product graph $\left(p_{1} ; q_{1}\right),\left(p_{2} ; q_{2}\right) \in G_{1} \boxtimes G_{2}$.

The first one is a well known property about distances between vertices in the strong product of graphs proved in [26].

Lemma 2 (Lemma 5.1 in [26]). Let $G_{1}, G_{2}$ be any graphs. If $p_{1}, p_{2} \in V\left(G_{1}\right)$ and $q_{1}, q_{2} \in$ $V\left(G_{2}\right)$, then

$$
d_{G_{1} \boxtimes G_{2}}\left(\left(p_{1} ; q_{1}\right),\left(p_{2} ; q_{2}\right)\right)=\max \left\{d_{G_{1}}\left(p_{1}, p_{2}\right), d_{G_{2}}\left(q_{1}, q_{2}\right)\right\} .
$$

Next, a lower bound on the distance between any two points in the strong product graph.

Proposition 3. Let $G_{1}, G_{2}$ be any graphs. For every $\left(p_{1} ; q_{1}\right),\left(p_{2} ; q_{2}\right) \in G_{1} \boxtimes G_{2}$ we have

$$
\begin{equation*}
d_{G_{1} \boxtimes G_{2}}\left(\left(p_{1} ; q_{1}\right),\left(p_{2} ; q_{2}\right)\right) \geqslant \max \left\{d_{G_{1}}\left(p_{1}, p_{2}\right), d_{G_{2}}\left(q_{1}, q_{2}\right)\right\} . \tag{1}
\end{equation*}
$$

Proof. By symmetry, it suffices to prove $d_{G_{1} \boxtimes G_{2}}\left(\left(p_{1} ; q_{1}\right),\left(p_{2} ; q_{2}\right)\right) \geqslant d_{G_{1}}\left(p_{1}, p_{2}\right)$. Seeking for a contradiction, assume that $d_{G_{1} \boxtimes G_{2}}\left(\left(p_{1} ; q_{1}\right),\left(p_{2} ; q_{2}\right)\right)<d_{G_{1}}\left(p_{1}, p_{2}\right)$.

Hence, there exist a geodesic $\Gamma$ joining $\left(p_{1} ; q_{1}\right)$ and $\left(p_{2} ; q_{2}\right)$ in $G_{1} \boxtimes G_{2}$ with $L(\Gamma)<$ $d_{G_{1}}\left(p_{1}, p_{2}\right)$. Denote by $\left(A_{1} ; B_{1}\right), \ldots,\left(A_{k} ; B_{k}\right)$ the vertices of $G_{1} \boxtimes G_{2}$ in $\Gamma$; without loss
of generality we can assume that $\Gamma$ meets $\left(A_{1} ; B_{1}\right), \ldots,\left(A_{k} ; B_{k}\right)$ in this order. Then, we have

$$
\Gamma:=\left[\left(p_{1} ; q_{1}\right)\left(A_{1} ; B_{1}\right)\right] \bigcup\left\{\bigcup_{j=1}^{k-1}\left[\left(A_{j} ; B_{j}\right),\left(A_{j+1} ; B_{j+1}\right)\right]\right\} \bigcup\left[\left(A_{k} ; B_{k}\right)\left(p_{2} ; q_{2}\right)\right] .
$$

By Definition 1, we obtain that

$$
\gamma:=\left[p_{1} A_{1}\right] \bigcup\left\{\bigcup_{j=1}^{k-1}\left[A_{j} A_{j+1}\right]\right\} \bigcup\left[A_{k} p_{2}\right]
$$

is a path joining $p_{1}$ and $p_{2}$ such that $L(\gamma) \leqslant L(\Gamma)<d_{G_{1}}\left(p_{1}, p_{2}\right)$. This is the contradiction we were looking for.

The following result provides an upper bound for the distance between a vertex and an inner point, as well as between two inner points in $G_{1} \boxtimes G_{2}$.

Proposition 4. Let $G_{1}, G_{2}$ be any graphs.
(i) If $(u ; v) \in V\left(G_{1} \boxtimes G_{2}\right)$ and $(p ; q) \in G_{1} \boxtimes G_{2} \backslash V\left(G_{1} \boxtimes G_{2}\right)$, then

$$
\begin{equation*}
d_{G_{1} \boxtimes G_{2}}((u ; v),(p ; q)) \leqslant \max \left\{d_{G_{1}}(u, p), d_{G_{2}}(v, q)\right\}+1 . \tag{2}
\end{equation*}
$$

(ii) If $\left(p_{1} ; q_{1}\right),\left(p_{2} ; q_{2}\right) \in G_{1} \boxtimes G_{2} \backslash V\left(G_{1} \boxtimes G_{2}\right)$, then

$$
\begin{equation*}
d_{G_{1} \boxtimes G_{2}}\left(\left(p_{1} ; q_{1}\right),\left(p_{2} ; q_{2}\right)\right) \leqslant \max \left\{d_{G_{1}}\left(p_{1}, p_{2}\right), d_{G_{2}}\left(q_{1}, q_{2}\right)\right\}+2 . \tag{3}
\end{equation*}
$$

Proof. In order to prove (i), let us consider $\left[\left(u_{1} ; v_{1}\right),\left(u_{2} ; v_{2}\right)\right] \in E\left(G_{1} \boxtimes G_{2}\right)$ such that $(p ; q) \in\left[\left(u_{1} ; v_{1}\right),\left(u_{2} ; v_{2}\right)\right]$. Let $\gamma$ be a geodesic in $G_{1} \boxtimes G_{2}$ joining $(u ; v)$ and $(p ; q)$. Without loss of generality we can assume that $\left(u_{1} ; v_{1}\right) \in \gamma$. Define $\varepsilon:=d_{G_{1} \boxtimes G_{2}}\left(\left(u_{1} ; v_{1}\right),(p ; q)\right)$. By Lemma 2, we have

$$
\begin{aligned}
d_{G_{1} \boxtimes G_{2}}((u ; v),(p ; q)) & =\max \left\{d_{G_{1}}\left(u, u_{1}\right), d_{G_{2}}\left(v, v_{1}\right)\right\}+\varepsilon \\
& \leqslant \max \left\{d_{G_{1}}(u, p)+d_{G_{1}}\left(p, u_{1}\right), d_{G_{2}}(v, q)+d_{G_{2}}\left(q, v_{1}\right)\right\}+\varepsilon \\
& \leqslant \max \left\{d_{G_{1}}(u, p), d_{G_{2}}(v, q)\right\}+2 \varepsilon .
\end{aligned}
$$

If $\varepsilon \leqslant 1 / 2$, then we have (2). If $\varepsilon>1 / 2$, then we have $\max \left\{d_{G_{1}}\left(u, u_{2}\right), d_{G_{2}}\left(v, v_{2}\right)\right\}$ $=\max \left\{d_{G_{1}}\left(u, u_{1}\right), d_{G_{2}}\left(v, v_{1}\right)\right\}+1 ;$ thus, $d_{G_{1} \boxtimes G_{2}}((u ; v),(p ; q))=\max \left\{d_{G_{1}}(u, p), d_{G_{2}}(v, q)\right\}$.

In order to proof (ii), notice that if $\left(p_{1} ; q_{1}\right),\left(p_{2} ; q_{2}\right)$ belong to the same edge of $G_{1} \boxtimes G_{2}$, then we have the result since $d_{G_{1} \boxtimes G_{2}}\left(\left(p_{1} ; q_{1}\right),\left(p_{2} ; q_{2}\right)\right)<1$. Assume now that $\left(p_{1} ; q_{1}\right),\left(p_{2} ; q_{2}\right)$ belong to different edges of $G_{1} \boxtimes G_{2}$. Let us consider $\left(u_{1} ; v_{1}\right),\left(u_{2} ; v_{2}\right),\left(u_{3} ; v_{3}\right),\left(u_{4} ; v_{4}\right) \in$ $V\left(G_{1} \boxtimes G_{2}\right)$ such that $\left(p_{1} ; q_{1}\right) \in\left[\left(u_{1} ; v_{1}\right),\left(u_{2} ; v_{2}\right)\right]$ and $\left(p_{2} ; q_{2}\right) \in\left[\left(u_{3} ; v_{3}\right),\left(u_{4} ; v_{4}\right)\right]$. Let $\gamma^{*}$ be a geodesic in $G_{1} \boxtimes G_{2}$ joining $\left(p_{1} ; q_{1}\right)$ and $\left(p_{2} ; q_{2}\right)$. Without loss of generality
we can assume that $\left(u_{2} ; v_{2}\right),\left(u_{3} ; v_{3}\right) \in \gamma^{*}$. Define $\varepsilon_{1}:=d_{G_{1} \boxtimes G_{2}}\left(\left(u_{2} ; v_{2}\right),\left(p_{1} ; q_{1}\right)\right)$ and $\varepsilon_{2}:=d_{G_{1} \boxtimes G_{2}}\left(\left(u_{3} ; v_{3}\right),\left(p_{2} ; q_{2}\right)\right)$. Then, we have

$$
\begin{aligned}
d_{G_{1} \boxtimes G_{2}}\left(\left(p_{1} ; q_{1}\right),\left(p_{2} ; q_{2}\right)\right) & =\varepsilon_{1}+\max \left\{d_{G_{1}}\left(u_{2}, u_{3}\right), d_{G_{2}}\left(v_{2}, v_{3}\right)\right\}+\varepsilon_{2} \\
& \leqslant 2 \varepsilon_{1}+\max \left\{d_{G_{1}}\left(p_{1}, p_{2}\right), d_{G_{2}}\left(q_{1}, q_{2}\right)\right\}+2 \varepsilon_{2} .
\end{aligned}
$$

Notice that if $\varepsilon_{1}, \varepsilon_{2} \leqslant 1 / 2$, then (3) holds directly. If $\varepsilon_{1}>1 / 2$ (the case $\varepsilon_{2}>1 / 2$ is analogous), then $\max \left\{d_{G_{1}}\left(u_{1}, u_{3}\right), d_{G_{2}}\left(v_{1}, v_{3}\right)\right\}=\max \left\{d_{G_{1}}\left(u_{2}, u_{3}\right), d_{G_{2}}\left(v_{2}, v_{3}\right)\right\}+1$; thus, $d_{G_{1} \boxtimes G_{2}}\left(\left(p_{1} ; q_{1}\right),\left(u_{3} ; v_{3}\right)\right)=\max \left\{d_{G_{1}}\left(p_{1}, u_{3}\right), d_{G_{2}}\left(q_{1}, v_{3}\right)\right\}$. Hence, we have

$$
\begin{aligned}
d_{G_{1} \boxtimes G_{2}}\left(\left(p_{1} ; q_{1}\right),\left(p_{2} ; q_{2}\right)\right) & =\max \left\{d_{G_{1}}\left(p_{1}, u_{3}\right), d_{G_{2}}\left(q_{1}, v_{3}\right)\right\}+\varepsilon_{2} \\
& \leqslant \max \left\{d_{G_{1}}\left(p_{1}, p_{2}\right)+d_{G_{1}}\left(p_{2}, u_{3}\right), d_{G_{2}}\left(q_{1}, q_{2}\right)+d_{G_{2}}\left(q_{2}, v_{3}\right)\right\}+\varepsilon_{2} \\
& \leqslant \max \left\{d_{G_{1}}\left(p_{1}, p_{2}\right), d_{G_{2}}\left(q_{1}, q_{2}\right)\right\}+2 \varepsilon_{2} .
\end{aligned}
$$

This finishes the proof.
The previous lemmas let us announce the following general result on the distances in the strong product of two graphs.

Theorem 5. For all graphs $G_{1}, G_{2}$ we have:
a) $d_{G_{1} \boxtimes G_{2}}\left(\left(p_{1} ; q_{1}\right),\left(p_{2} ; q_{2}\right)\right)=\max \left\{d_{G_{1}}\left(p_{1}, p_{2}\right), d_{G_{2}}\left(q_{1}, q_{2}\right)\right\}$, for every $\left(p_{1} ; q_{1}\right),\left(p_{2} ; q_{2}\right) \in$ $V\left(G_{1} \boxtimes G_{2}\right)$,
b) $\max \left\{d_{G_{1}}\left(p_{1}, p_{2}\right), d_{G_{2}}\left(q_{1}, q_{2}\right)\right\} \leqslant d_{G_{1} \boxtimes G_{2}}\left(\left(p_{1} ; q_{1}\right),\left(p_{2} ; q_{2}\right)\right) \leqslant \max \left\{d_{G_{1}}\left(p_{1}, p_{2}\right), d_{G_{2}}\left(q_{1}, q_{2}\right)\right\}$ +1 , for every $\left(p_{1} ; q_{1}\right) \in V\left(G_{1} \boxtimes G_{2}\right)$ and $\left(p_{2} ; q_{2}\right) \in G_{1} \boxtimes G_{2}$,
c) $\max \left\{d_{G_{1}}\left(p_{1}, p_{2}\right), d_{G_{2}}\left(q_{1}, q_{2}\right)\right\} \leqslant d_{G_{1} \boxtimes G_{2}}\left(\left(p_{1} ; q_{1}\right),\left(p_{2} ; q_{2}\right)\right) \leqslant \max \left\{d_{G_{1}}\left(p_{1}, p_{2}\right), d_{G_{2}}\left(q_{1}, q_{2}\right)\right\}$ +2 , for every $\left(p_{1} ; q_{1}\right),\left(p_{2} ; q_{2}\right) \in G_{1} \boxtimes G_{2}$.

Let us consider the projection $P_{k}: G_{1} \boxtimes G_{2} \longrightarrow G_{k}$ for $k \in\{1,2\}$.
Corollary 6. Let $\{i, j\}$ be a permutation of $\{1,2\}$. Then, for every $x, y$ in $G_{1} \boxtimes G_{2}$,

$$
\begin{equation*}
d_{G_{i}}\left(P_{i}(x), P_{i}(y)\right) \leqslant d_{G_{1} \boxtimes G_{2}}(x, y) \leqslant d_{G_{i}}\left(P_{i}(x), P_{i}(y)\right)+\operatorname{diam} G_{j}+2 \tag{4}
\end{equation*}
$$

These results provide information about the geodesics in $G_{1} \boxtimes G_{2}$. Notice that, if $\gamma$ is a geodesic joining $x$ and $y$ in $G_{1} \boxtimes G_{2}$, then it is possible that $P_{j}(\gamma)$ does not contain a geodesic joining $P_{j}(x)$ and $P_{j}(y)$ in $G_{j}$, as the following example shows.

Example 7. Consider a cycle graph $G_{1}$ with vertices $\left\{v_{1}, \ldots, v_{n}\right\}$ such that $v_{i} \sim v_{i+1}$ for every $i \in\{1, \ldots, n-1\}$ and a path graph $G_{2}$ with vertices $\left\{w_{1}, \ldots, w_{n}\right\}$ such that $w_{i} \sim$ $w_{i+1}$ for every $i \in\{1, \ldots, n-1\}$. By Lemma 2, we have that $\gamma:=\cup_{i=1}^{n-1}\left[\left(v_{i} ; w_{i}\right),\left(v_{i+1} ; w_{i+1}\right)\right]$ is a geodesic joining $\left(v_{1} ; w_{1}\right)$ and $\left(v_{n} ; w_{n}\right)$ in $G_{1} \boxtimes G_{2}$, but $P_{1}(\gamma)=\cup_{i=1}^{n-1}\left[v_{i}, v_{i+1}\right]$ does not contain the geodesic joining $v_{1}$ and $v_{n}$ in $G_{1}$ (the edge $\left[v_{1}, v_{n}\right]$ ).

The following result allows to compute the diameter of the strong product of two graphs. We denote by $E_{1}$ the graph with just a single vertex.

Theorem 8. Let $G_{1}, G_{2}$ be any graphs. Then we have
$\operatorname{diam} G_{1} \boxtimes G_{2}=\left\{\begin{array}{l}\max \left\{\operatorname{diam} G_{1}, \operatorname{diam} G_{2}\right\}, \quad \text { if } G_{1} \text { or } G_{2} \text { is an isomorphic graph to } E_{1}, \\ \max \left\{\operatorname{diam} V\left(G_{1}\right), \operatorname{diam} V\left(G_{2}\right)\right\}+1, \quad \text { otherwise } .\end{array}\right.$

Proof. Since for any graph $G, E_{1} \boxtimes G$ is isomorphic to $G$ we have the first equality. By Lemma 2, we have $\operatorname{diam} V\left(G_{1} \boxtimes G_{2}\right)=\max \left\{\operatorname{diam} V\left(G_{1}\right)\right.$, $\left.\operatorname{diam} V\left(G_{2}\right)\right\}$; hence,
$\max \left\{\operatorname{diam} V\left(G_{1}\right), \operatorname{diam} V\left(G_{2}\right)\right\} \leqslant \operatorname{diam} G_{1} \boxtimes G_{2} \leqslant \max \left\{\operatorname{diam} V\left(G_{1}\right), \operatorname{diam} V\left(G_{2}\right)\right\}+1$.
Without loss of generality we can assume that $\operatorname{diam} V\left(G_{1}\right) \leqslant \operatorname{diam} V\left(G_{2}\right)$. If diam $V\left(G_{2}\right)$ $=\infty$, then the inequality holds. Hence, we can assume that $G_{1}$ and $G_{2}$ are bounded. Let $B_{1}, B_{2}$ be vertices of $G_{2}$ such that $d_{G_{2}}\left(B_{1}, B_{2}\right)=\operatorname{diam} V\left(G_{2}\right)$, and let $A_{1}, A_{2}$ be two adjacent vertices of $G_{1}$. Let $M_{1}$ (respectively, $M_{2}$ ) be the midpoint of $\left[\left(A_{1} ; B_{1}\right),\left(A_{2} ; B_{1}\right)\right]$ (respectively, $\left.\left[\left(A_{1} ; B_{2}\right),\left(A_{2} ; B_{2}\right)\right]\right)$. One can check that $d_{G_{1} \boxtimes G_{2}}\left(M_{1}, M_{2}\right)=\operatorname{diam} V\left(G_{2}\right)+1$.

This finish the proof.
Note that, in particular, $\operatorname{diam} G_{1} \boxtimes G_{2}=\operatorname{diam} V\left(G_{1} \boxtimes G_{2}\right)+1$ if $G_{1}$ and $G_{2}$ are not isomorphic to $E_{1}$.

We say that a subgraph $\Gamma$ of $G$ is isometric if $d_{\Gamma}(x, y)=d_{G}(x, y)$ for every $x, y \in \Gamma$.
We can deduce several results from Theorem 8. The first one says that $\max \left\{\operatorname{diam} G_{1}\right.$, $\left.\operatorname{diam} G_{2}\right\}$ is a good approximation of the diameter of $G_{1} \boxtimes G_{2}$.

Corollary 9. For all graphs $G_{1}, G_{2}$ we have

$$
\max \left\{\operatorname{diam} G_{1}, \operatorname{diam} G_{2}\right\} \leqslant \operatorname{diam} G_{1} \boxtimes G_{2} \leqslant \max \left\{\operatorname{diam} G_{1}, \operatorname{diam} G_{2}\right\}+1
$$

Proof. If $V$ is a vertex of $G_{1}$ (respectively, $G_{2}$ ), then, by Proposition 3, we have that $\{V\} \boxtimes G_{2}$ (respectively, $\left.G_{1} \boxtimes\{V\}\right)$ is an isometric subgraph of $G_{1} \boxtimes G_{2}$. Hence, we obtain the first inequality. The second one is a consequence of Theorem 8 and the inequality $\operatorname{diam} V(G) \leqslant \operatorname{diam} G$.

Furthermore, we characterize the graphs with $\operatorname{diam} G_{1} \boxtimes G_{2}=\max \left\{\operatorname{diam} G_{1}, \operatorname{diam} G_{2}\right\}$.
Corollary 10. The equality $\operatorname{diam} G_{1} \boxtimes G_{2}=\max \left\{\operatorname{diam} G_{1}\right.$, $\left.\operatorname{diam} G_{2}\right\}$ holds if and only if $G_{1}$ or $G_{2}$ is isomorphic to $E_{1}$, or $\operatorname{diam} G=\operatorname{diam} V(G)+1$ for $G \in\left\{G_{1}, G_{2}\right\}$ with $\operatorname{diam} G=\max \left\{\operatorname{diam} G_{1}, \operatorname{diam} G_{2}\right\}$.

## 3 Bounds for the hyperbolicity constant.

Some bounds for the hyperbolicity constant of the strong product of two graphs are studied in this section. These bounds allow to prove Theorem 23, which characterizes the hyperbolic strong product graphs. The next well-known result will be useful.

Theorem 11 (Theorem 8 in [42]). In any graph $G$ the inequality $\delta(G) \leqslant \frac{1}{2} \operatorname{diam} G$ holds and it is sharp.

Thanks to the Theorems 8 and 11 we obtain the following consequence.
Corollary 12. For all graphs $G_{1}, G_{2}$, we have

$$
\delta\left(G_{1} \boxtimes G_{2}\right) \leqslant \frac{\max \left\{\operatorname{diam} V\left(G_{1}\right), \operatorname{diam} V\left(G_{2}\right)\right\}+1}{2},
$$

and the inequality is sharp.
Theorems 32,34 and 35 are families of examples for which the equality in the previous corollary is attained.

Taking into account that $E_{1} \boxtimes G$ is an isomorphic graph to $G$, we have the following result.

Corollary 13. For every graph $G$ we have

$$
\delta\left(G \boxtimes E_{1}\right)=\delta\left(E_{1} \boxtimes G\right)=\delta(G)
$$

The next result will be useful.
Lemma 14 (Lemma 5 in [42]). If $\Gamma$ is an isometric subgraph of $G$, then $\delta(\Gamma) \leqslant \delta(G)$.
All the previous results allow us to present the following theorem which provides some lower bounds for $\delta\left(G_{1} \boxtimes G_{2}\right)$.

Theorem 15. For all graphs $G_{1}, G_{2}$ we have:
(a) $\delta\left(G_{1} \boxtimes G_{2}\right) \geqslant \max \left\{\delta\left(G_{1}\right), \delta\left(G_{2}\right)\right\}$,
(b) $\delta\left(G_{1} \boxtimes G_{2}\right) \geqslant \frac{1}{2} \min \left\{\operatorname{diam} V\left(G_{1}\right), \operatorname{diam} V\left(G_{2}\right)\right\}$,
(c) $\delta\left(G_{1} \boxtimes G_{2}\right) \geqslant \frac{1}{2}\left(\operatorname{diam} V\left(G_{1}\right)+1\right)$, if $0<\operatorname{diam} V\left(G_{1}\right)<\operatorname{diam} V\left(G_{2}\right)$,
(d) $\delta\left(G_{1} \boxtimes G_{2}\right) \geqslant \frac{1}{4} \min \left\{\operatorname{diam} V\left(G_{1}\right)+2 \delta\left(G_{2}\right), \operatorname{diam} V\left(G_{2}\right)+2 \delta\left(G_{1}\right)\right\}$.

Proof. Part $(a)$ is immediate due to $G_{1} \boxtimes\{v\}$ and $\{u\} \boxtimes G_{2}$ are isometric subgraphs of $G_{1} \boxtimes G_{2}$ for every $(u ; v) \in V\left(G_{1} \boxtimes G_{2}\right)$. Then Lemma 14 gives that $\delta\left(G_{1} \boxtimes G_{2}\right) \geqslant \delta\left(G_{1} \boxtimes\right.$ $\{v\})=\delta\left(G_{1}\right)$ and $\delta\left(G_{1} \boxtimes G_{2}\right) \geqslant \delta\left(\{u\} \boxtimes G_{2}\right)=\delta\left(G_{2}\right)$. Hence, we obtain $\delta\left(G_{1} \boxtimes G_{2}\right) \geqslant$ $\max \left\{\delta\left(G_{1}\right), \delta\left(G_{2}\right)\right\}$.

Let $D:=\min \left\{\operatorname{diam} V\left(G_{1}\right), \operatorname{diam} V\left(G_{2}\right)\right\}$.
Let us prove (b). If $D=0$, then (b) holds; so, we just consider $D>0$. If $D<\infty$, let us consider a geodesic square $K:=\left\{\gamma_{1}, \gamma_{2}, \gamma_{3}, \gamma_{4}\right\}$ in $G_{1} \square G_{2} \subset G_{1} \boxtimes G_{2}$ with sides of length $D$; then $T:=\left\{\gamma_{1}, \gamma_{2}, \gamma\right\}$ is a geodesic triangle in $G_{1} \boxtimes G_{2}$, where $\gamma$ is a "diagonal" geodesic joining the endpoints of $\gamma_{1} \cup \gamma_{2}$. It is clear that the midpoint $p$ of $\gamma$ satisfies $d_{G_{1} \boxtimes G_{2}}\left(p, \gamma_{1} \cup \gamma_{2}\right)=D / 2$; therefore $\delta(T) \geqslant D / 2$ and, consequently, $\delta\left(G_{1} \boxtimes G_{2}\right) \geqslant D / 2$. If $D=\infty$, we can repeat the same argument for any integer $N$ instead of $D$, and we obtain $\delta\left(G_{1} \boxtimes G_{2}\right) \geqslant N / 2$, for every $N$ : hence, $\delta\left(G_{1} \boxtimes G_{2}\right)=\infty=D / 2$.

In order to prove $(c)$, note that $D<\infty$. Let us consider a geodesic rectangle $R:=$ $\left\{\sigma_{1}, \sigma_{2}, \sigma_{3}, \sigma_{4}\right\}$ in $G_{1} \square G_{2} \subset G_{1} \boxtimes G_{2}$ with $L\left(\sigma_{1}\right)=L\left(\sigma_{3}\right)=\operatorname{diam} V\left(G_{1}\right)$ and $L\left(\sigma_{2}\right)=$ $L\left(\sigma_{4}\right)=\operatorname{diam} V\left(G_{1}\right)+1$. Denote by $\gamma$ a geodesic in $G_{1} \boxtimes G_{2}$ joining the endpoints of $\sigma_{1} \cup \sigma_{2}$ which contains the edge in $\sigma_{4}$ incident to $\sigma_{1} \cap \sigma_{4}$; we may choose $\gamma$ such that it contains a diagonal of a geodesic square in $G_{1} \boxtimes G_{2}$. Then $B:=\left\{\sigma_{1}, \sigma_{2}, \gamma\right\}$ is a geodesic triangle in $G_{1} \boxtimes G_{2}$. If $p$ is the midpoint of $\gamma$, then

$$
d_{G_{1} \boxtimes G_{2}}\left(p, \sigma_{1} \cup \sigma_{2}\right)=\frac{\operatorname{diam} V\left(G_{1}\right)+1}{2} .
$$

Consequently, $\delta\left(G_{1} \boxtimes G_{2}\right) \geqslant \delta(B) \geqslant\left(\operatorname{diam} V\left(G_{1}\right)+1\right) / 2$.
Finally, $(d)$. Let $E:=\max \left\{\delta\left(G_{1}\right), \delta\left(G_{2}\right)\right\}$. Then from parts $(a)$ and $(b)$, we have

$$
\begin{aligned}
\delta\left(G_{1} \boxtimes G_{2}\right) & \geqslant \max \left\{\frac{D}{2}, E\right\} \geqslant \frac{1}{2}\left(\frac{D}{2}+E\right) \\
& =\frac{1}{4} \min \left\{\operatorname{diam} V\left(G_{1}\right)+2 E, \operatorname{diam} V\left(G_{2}\right)+2 E\right\} \\
& \geqslant \frac{1}{4} \min \left\{\operatorname{diam} V\left(G_{1}\right)+2 \delta\left(G_{2}\right), \operatorname{diam} V\left(G_{2}\right)+2 \delta\left(G_{1}\right)\right\} .
\end{aligned}
$$

Theorems 34 and 35 provide a family of examples for which the equality in Theorem 15 (a) is attained.

Corollary 12 and Theorem 15 provide lower and upper bounds for $\delta\left(G_{1} \boxtimes G_{2}\right)$ just in terms of distances in $G_{1}$ and $G_{2}$.

Corollary 16. For all graphs $G_{1}, G_{2}$, we have
$\frac{1}{2} \min \left\{\operatorname{diam} V\left(G_{1}\right), \operatorname{diam} V\left(G_{2}\right)\right\} \leqslant \delta\left(G_{1} \boxtimes G_{2}\right) \leqslant \frac{1}{2}\left(\max \left\{\operatorname{diam} V\left(G_{1}\right), \operatorname{diam} V\left(G_{2}\right)\right\}+1\right)$.
From Theorem 15 we have obtained several interesting consequences. The following one is a qualitative result about the hyperbolicity of $G_{1} \boxtimes G_{2}$.

Theorem 17. If $G_{1}$ and $G_{2}$ are infinite graphs, then $G_{1} \boxtimes G_{2}$ is not hyperbolic.
Theorem 18. Let $G_{1}, G_{2}$ be graphs with at least two vertices. Let $m$ and $M$ be the minimum and the maximum between $\operatorname{diam} V\left(G_{1}\right)$ and $\operatorname{diam} V\left(G_{2}\right)$, respectively. Then we have

$$
\begin{equation*}
\delta\left(G_{1} \boxtimes G_{2}\right) \geqslant \min \left\{m+\frac{1}{2}, \frac{M}{2}\right\} . \tag{5}
\end{equation*}
$$

Proof. First of all, we prove

$$
\begin{equation*}
\delta\left(G_{1} \boxtimes G_{2}\right) \geqslant \min \left\{m, \frac{M}{2}\right\} . \tag{6}
\end{equation*}
$$

In order to prove this inequality, assume first that $2 m \leqslant M$. If $m<\infty$, then let us consider a geodesic rectangle $R:=\gamma_{1} \cup \gamma_{2} \cup \gamma_{3} \cup \gamma_{4}$ in $G_{1} \square G_{2} \subset G_{1} \boxtimes G_{2}$ with $L\left(\gamma_{1}\right)=L\left(\gamma_{3}\right)=2 m$ and $L\left(\gamma_{2}\right)=L\left(\gamma_{4}\right)=m$, and consider a geodesic $\gamma$ joining the endpoints of $\gamma_{1}$ and containing the midpoint of $\gamma_{3}$; then $B:=\left\{\gamma_{1}, \gamma\right\}$ is a geodesic bigon in $G_{1} \boxtimes G_{2}$. If $p$ is the midpoint of $\gamma_{3}$, then $d_{G_{1} \boxtimes G_{2}}\left(p, \gamma_{1}\right)=m$; therefore $\delta(B) \geqslant m$, and consequently $\delta\left(G_{1} \boxtimes G_{2}\right) \geqslant m$. If $m=\infty$, then we can repeat the same argument for any integer $N$ instead of $m$, and we obtain $\delta\left(G_{1} \boxtimes G_{2}\right) \geqslant N$, for every $N$; hence, $\delta\left(G_{1} \boxtimes G_{2}\right)=\infty=m$.

If $2 m>M$, then $M<\infty$ and we can repeat the previous argument with $\lfloor M / 2\rfloor$ instead of $m$, and we obtain the result when $M$ is even. If $M$ is odd, let us consider a geodesic rectangle $R:=\gamma_{1} \cup \gamma_{2} \cup \gamma_{3} \cup \gamma_{4}$ in $G_{1} \square G_{2} \subset G_{1} \boxtimes G_{2}$ with $L\left(\gamma_{1}\right)=L\left(\gamma_{3}\right)=$ $2\lfloor M / 2\rfloor+1=M$ and $L\left(\gamma_{2}\right)=L\left(\gamma_{4}\right)=\lfloor M / 2\rfloor$; let $p_{1}, p_{2}$ be points on $\gamma_{3}$ such that $d_{G_{1} \boxtimes G_{2}}\left(p_{1}, \gamma_{4}\right)=\lfloor M / 2\rfloor$ and $d_{G_{1} \boxtimes G_{2}}\left(p_{2}, \gamma_{2}\right)=\lfloor M / 2\rfloor$; consider a geodesic $\gamma$ joining the endpoints of $\gamma_{1}$ and containing $p_{1}$ and $p_{2}$; then $B:=\left\{\gamma_{1}, \gamma\right\}$ is a geodesic bigon in $G_{1} \boxtimes G_{2}$. Denote by $p$ the midpoint of $\left[p_{1} p_{2}\right] \subset \gamma_{3}$; so, $d_{G_{1} \boxtimes G_{2}}\left(p, \gamma_{1}\right)=M / 2$; therefore, $\delta\left(G_{1} \boxtimes G_{2}\right) \geqslant \delta(B) \geqslant M / 2$.

Since we have proved (6), in order to obtain (5), we can assume that $0<2 m<M$; then we have $m<\infty$. If we replace $\lfloor M / 2\rfloor$ by $m$ in the previous argument, we obtain $\delta\left(G_{1} \boxtimes G_{2}\right) \geqslant m+1 / 2$.

Corollary 33 and Theorems 34 and 35 show that the inequality in Theorem 18 is sharp.

Theorem 19. Let $G_{1}, G_{2}$ be any graphs. Let $m$ and $M$ be the minimum and the maximum between $\operatorname{diam} V\left(G_{1}\right)$ and $\operatorname{diam} V\left(G_{2}\right)$, respectively. If $2 m \geqslant M$, then

$$
\begin{equation*}
\frac{M}{2} \leqslant \delta\left(G_{1} \boxtimes G_{2}\right) \leqslant \frac{M+1}{2} . \tag{7}
\end{equation*}
$$

Furthermore, if $2 m>M>0$, then

$$
\begin{equation*}
\delta\left(G_{1} \boxtimes G_{2}\right)=\frac{M+1}{2} . \tag{8}
\end{equation*}
$$

Proof. If $M=0$, then $\delta\left(G_{1} \boxtimes G_{2}\right)=0$ and (7) holds. If $M>0$, then, by Corollary 12 and Theorem 18, the inequalities in (7) hold directly.

In order to prove (8), without loss of generality we can assume that diam $V\left(G_{1}\right)=m$ and $\operatorname{diam} V\left(G_{2}\right)=M$. Assume first that $M$ is an even number. Since $m>M / 2$, let us consider $A_{0}, A_{1}, \ldots, A_{M / 2+1} \in V\left(G_{1}\right)$ and $B_{0}, B_{1}, \ldots, B_{M} \in V\left(G_{2}\right)$ with $\gamma_{1}:=$ $A_{0} A_{1} \ldots A_{M / 2+1}$ is a geodesic in $G_{1}$ and $\gamma_{2}:=B_{0} B_{1} \ldots B_{M}$ is a geodesic in $G_{2}$. Denote by $X$ (respectively, $Y$ ) the midpoint of $\left[\left(A_{0} ; B_{0}\right),\left(A_{1} ; B_{0}\right)\right]$ (respectively, $\left[\left(A_{0} ; B_{M}\right),\left(A_{1} ; B_{M}\right)\right]$ ). Let us consider

$$
\Gamma^{*}:=\left[X\left(A_{0} ; B_{0}\right)\right] \bigcup\left\{\bigcup_{i=1}^{M}\left[\left(A_{0} ; B_{i-1}\right),\left(A_{0} ; B_{i}\right)\right]\right\} \bigcup\left[\left(A_{0} ; B_{M}\right) Y\right]
$$

and

$$
\begin{aligned}
\Gamma^{\prime}:= & {\left[X\left(A_{1} ; B_{0}\right)\right] \bigcup\left\{\bigcup_{i=1}^{M / 2}\left[\left(A_{i} ; B_{i-1}\right),\left(A_{i+1} ; B_{i}\right)\right]\right\} \bigcup } \\
& \bigcup\left\{\bigcup_{j=M / 2+1}^{M}\left[\left(A_{M+2-j} ; B_{j-1}\right),\left(A_{M+1-j} ; B_{j}\right)\right]\right\} \bigcup\left[\left(A_{1} ; B_{M}\right) Y\right] .
\end{aligned}
$$

Then $B:=\left\{\Gamma^{*}, \Gamma^{\prime}\right\}$ is a geodesic bigon in $G_{1} \boxtimes G_{2}$. If $p$ is the midpoint of $\Gamma^{\prime}$, then $d_{G_{1} \boxtimes G_{2}}\left(p, \Gamma^{*}\right)=(M+1) / 2$; therefore, $\delta\left(G_{1} \boxtimes G_{2}\right) \geqslant \delta(B) \geqslant(M+1) / 2$. Then, Corollary 12 gives the equality.

Assume now that $M$ is an odd number. Since $m \geqslant(M+1) / 2$, let us consider $A_{0}, A_{1}, \ldots, A_{(M+1) / 2} \in V\left(G_{1}\right)$ and $B_{0}, B_{1}, \ldots, B_{M} \in V\left(G_{2}\right)$ with $\gamma_{1}:=A_{0} A_{1} \ldots A_{(M+1) / 2}$ is a geodesic in $G_{1}$ and $\gamma_{2}:=B_{0} B_{1} \ldots B_{M}$ is a geodesic in $G_{2}$. Denote by $X$ (respectively, $Y$ ) the midpoint of $\left[\left(A_{0} ; B_{0}\right),\left(A_{1} ; B_{0}\right)\right]$ (respectively, $\left[\left(A_{0} ; B_{M}\right),\left(A_{1} ; B_{M}\right)\right]$ ). Let us consider

$$
\Gamma^{*}:=\left[X\left(A_{0} ; B_{0}\right)\right] \bigcup\left\{\bigcup_{i=1}^{M}\left[\left(A_{0} ; B_{i-1}\right),\left(A_{0} ; B_{i}\right)\right]\right\} \bigcup\left[\left(A_{0} ; B_{M}\right) Y\right]
$$

and

$$
\begin{aligned}
\Gamma^{\prime}:= & {\left[X\left(A_{1} ; B_{0}\right)\right] \bigcup\left\{\bigcup_{i=1}^{(M-1) / 2}\left[\left(A_{i} ; B_{i-1}\right),\left(A_{i+1} ; B_{i}\right)\right]\right\} \bigcup } \\
& \bigcup\left[\left(A_{(M+1) / 2} ; B_{(M-1) / 2}\right),\left(A_{(M+1) / 2} ; B_{(M+1) / 2}\right)\right] \bigcup \\
& \bigcup\left\{\bigcup_{j=(M+1) / 2}^{M}\left[\left(A_{M+1-j} ; B_{j-1}\right),\left(A_{M-j} ; B_{j}\right)\right]\right\} \bigcup\left[\left(A_{1} ; B_{M}\right) Y\right] .
\end{aligned}
$$

Then $B:=\left\{\Gamma^{*}, \Gamma^{\prime}\right\}$ is a geodesic bigon in $G_{1} \boxtimes G_{2}$. If $p$ is the midpoint of $\Gamma^{\prime}$, then $d_{G_{1} \boxtimes G_{2}}\left(p, \Gamma^{*}\right)=(M+1) / 2$; therefore, $\delta\left(G_{1} \boxtimes G_{2}\right) \geqslant \delta(B) \geqslant(M+1) / 2$. Finally, Corollary 12 gives the equality.

Theorems 34 and 35 show that the first inequality in Theorem 19 is attained.
Let $X$ be a metric space, $Y$ a non-empty subset of $X$ and $\varepsilon$ a positive number. We call $\varepsilon$-neighborhood of $Y$ in $X$, denoted by $V_{\varepsilon}(Y)$ to the set $\left\{x \in X: d_{X}(x, Y) \leqslant \varepsilon\right\}$.

The next result will be useful in order to prove the upper bound for $\delta\left(G_{1} \boxtimes G_{2}\right)$ in Theorem 21 below.
Theorem 20 (Theorem 2.9 in [41]). Let $X$ be a $\delta$-hyperbolic geodesic metric space, $u, v \in$ $X, b$ a non-negative constant, $h$ a curve joining $u$ and $v$ with $L(h) \leqslant d(u, v)+b$, and $g=[u v]$. Then,

$$
h \subseteq V_{8 \delta+b / 2}(g), \quad g \subseteq V_{16 \delta+b}(h)
$$

Theorem 21. Let $G_{1}, G_{2}$ be any graphs. Then, we have

$$
\begin{equation*}
\delta\left(G_{1} \boxtimes G_{2}\right) \leqslant \frac{5}{2} \operatorname{diam} G_{1}+25 \delta\left(G_{2}\right)+5 \tag{9}
\end{equation*}
$$

Proof. It suffices to prove (9) if $G_{1}$ is bounded and $G_{2}$ is hyperbolic, since otherwise the inequality $\delta\left(G_{1} \boxtimes G_{2}\right) \leqslant \infty$ holds. Let us consider any fixed geodesic triangle $T=\{x, y, z\}$ in $G_{1} \boxtimes G_{2}$ and $\alpha \in T$. In order to bound $\delta(T)$, without loss of generality we can assume that $\alpha \in[x y]$. Consider the projection $P_{2}: G_{1} \boxtimes G_{2} \longrightarrow G_{2}$ and any geodesic $\gamma:=[u v]$ in $G_{1} \boxtimes G_{2}$. By Corollary 6, we obtain

$$
L\left(P_{2}(\gamma)\right) \leqslant L(\gamma)=d_{G_{1} \boxtimes G_{2}}(u, v) \leqslant d_{G_{2}}\left(P_{2}(v), P_{2}(v)\right)+b, \quad \text { with } b=\operatorname{diam} G_{1}+2
$$

Then, by Theorem 20, there is $\alpha^{\prime} \in\left[P_{2}(x) P_{2}(y)\right]$ such that

$$
\begin{equation*}
d_{G_{2}}\left(P_{2}(\alpha), \alpha^{\prime}\right) \leqslant 8 \delta\left(G_{2}\right)+\frac{b}{2} \tag{10}
\end{equation*}
$$

Since $G_{2}$ is hyperbolic, there is $\beta^{\prime} \in\left[P_{2}(y) P_{2}(z)\right] \cup\left[P_{2}(z) P_{2}(x)\right]$ such that

$$
\begin{equation*}
d_{G_{2}}\left(\alpha^{\prime}, \beta^{\prime}\right) \leqslant \delta\left(G_{2}\right) \tag{11}
\end{equation*}
$$

By Theorem 20, there is $\beta^{\prime \prime} \in P_{2}([y z] \cup[z x])$ such that

$$
\begin{equation*}
d_{G_{2}}\left(\beta^{\prime}, \beta^{\prime \prime}\right) \leqslant 16 \delta\left(G_{2}\right)+b \tag{12}
\end{equation*}
$$

Consequently, by (10), (11) and (12) we obtain

$$
\begin{equation*}
d_{G_{2}}\left(P_{2}(\alpha), P_{2}([y z] \cup[z x])\right) \leqslant d_{G_{2}}\left(P_{2}(\alpha), \beta^{\prime \prime}\right) \leqslant 25 \delta\left(G_{2}\right)+\frac{3 b}{2} \tag{13}
\end{equation*}
$$

Finally, by Corollary 6 and (13) we obtain

$$
d_{G_{1} \boxtimes G_{2}}(\alpha,[y z] \cup[z x]) \leqslant d_{G_{2}}\left(P_{2}(\alpha), P_{2}([y z] \cup[z x])\right)+b \leqslant 25 \delta\left(G_{2}\right)+\frac{5 b}{2} .
$$

This finishes the proof.

Theorems 15 and 21 provide lower and upper bounds of $\delta\left(G_{1} \boxtimes G_{2}\right)$ in terms of linear combinations of hyperbolicity constants and diameters of its generator graphs, as the following result shows.

Corollary 22. For all graphs $G_{1}, G_{2}$, we have

$$
\begin{gathered}
\frac{1}{4} \min \left\{2 \delta\left(G_{1}\right)+\operatorname{diam} V\left(G_{2}\right), 2 \delta\left(G_{2}\right)+\operatorname{diam} V\left(G_{1}\right)\right\} \leqslant \delta\left(G_{1} \boxtimes G_{2}\right) \\
\quad \leqslant \frac{5}{2} \min \left\{\operatorname{diam} G_{1}+10 \delta\left(G_{2}\right), \operatorname{diam} G_{2}+10 \delta\left(G_{1}\right)\right\}+5
\end{gathered}
$$

Corollary 22 allows to obtain the main result of this work: the characterization of the hyperbolic graphs $G_{1} \boxtimes G_{2}$.

Theorem 23. For all graphs $G_{1}, G_{2}$ we have that $G_{1} \boxtimes G_{2}$ is hyperbolic if and only if $G_{1}$ is hyperbolic and $G_{2}$ is bounded or $G_{2}$ is hyperbolic and $G_{1}$ is bounded.

Many parameters $\gamma$ of graphs satisfy the inequality $\gamma\left(G_{1} \boxtimes G_{2}\right) \geqslant \gamma\left(G_{1}\right)+\gamma\left(G_{2}\right)$. Therefore, one could think that the inequality $\delta\left(G_{1} \boxtimes G_{2}\right) \geqslant \delta\left(G_{1}\right)+\delta\left(G_{2}\right)$ holds for all graphs $G_{1}, G_{2}$. However, this is false, as the following example shows:

Example 24. $\delta\left(P \boxtimes C_{4}\right)<\delta(P)+\delta\left(C_{4}\right)$, where $P$ is the Petersen graph.
We have that $\operatorname{diam} V(P)=2$, $\operatorname{diam} V\left(C_{4}\right)=2$. Besides, Theorem 11 in [42] gives that $\delta(P)=3 / 2$ and $\delta\left(C_{4}\right)=1$. By Theorem 19, we obtain $\delta\left(P \boxtimes C_{4}\right)=3 / 2<3 / 2+1=$ $\delta(P)+\delta\left(C_{4}\right)$.

The inequality $\delta\left(G_{1} \boxtimes G_{2}\right) \leqslant \delta\left(G_{1}\right)+\delta\left(G_{2}\right)$ is also false, since $\delta\left(P_{2} \boxtimes P_{2}\right)=\delta\left(K_{4}\right)=$ $1>2 \delta\left(P_{2}\right)=0$.

## 4 Computation of the hyperbolicity constant for some product graphs

This last section present the value of the hyperbolicity constant for many product of graphs.

The following results in [4] will be useful. Denote by $J(G)$ the set of vertices and midpoints of edges in $G$. As usual, by cycle we mean a simple closed curve, i.e., a path with different vertices, unless the last one, which is equal to the first vertex.

First, remark some previous results of [4] which will be useful.
Theorem 25 (Theorem 2.6 in [4]). For every hyperbolic graph $G, \delta(G)$ is a multiple of $1 / 4$.

Theorem 26 (Theorem 2.7 in [4]). For any hyperbolic graph $G$, there exists a geodesic triangle $T=\{x, y, z\}$ that is a cycle with $x, y, z \in J(G)$ and $\delta(T)=\delta(G)$.

Remark 27. By Theorems 25 and 26, in order to compute the hyperbolicity constant of a graph $G$ it suffices to consider $d_{G}(p,[x z] \cup[y z])$ where $T=\{x, y, z\}$ is a geodesic triangle that is a cycle with $x, y, z \in J(G)$ and $p \in[x y]$ satisfies $d_{G}(p, V(G)) \in\{0,1 / 4,1 / 2\}$.

The following results characterize the hyperbolicity constant of the strong product of trees and certain graphs. These results are interesting by themselves and, furthermore, they will be useful in order to prove the last theorems of this paper.

Theorem 28. Let $T$ be any tree and $G$ any graph with $0<\operatorname{diam} V(G)<\operatorname{diam} T / 2$. Then, we have

$$
\delta(G \boxtimes T)=\operatorname{diam} V(G)+\frac{1}{2}
$$

Proof. On the one hand, Theorem 18 gives $\delta(G \boxtimes T) \geqslant \operatorname{diam} V(G)+1 / 2$. On the other hand, by Theorem 26 it suffices to consider geodesic triangles $\triangle=\{x, y, z\}$ in $G \boxtimes T$ which are cycles with $x, y, z \in J(G \boxtimes T)$. Let $(v ; w)$ be a vertex in [xy]. If $d_{G \boxtimes T}((v ; w),\{x, y\}) \leqslant \operatorname{diam} V(G)$, then $d_{G \boxtimes T}((v ; w),[y z] \cup[z x]) \leqslant \operatorname{diam} V(G)$. Assume that $d_{G \boxtimes T}((v ; w),\{x, y\})>\operatorname{diam} V(G)$. Let $V_{x}$ (respectively, $\left.V_{y}\right)$ be the closest vertex to $x($ respectively, $y)$ in $[x y]$. Note that $d_{G \boxtimes T}\left(V_{x}, V_{y}\right)=d_{G \boxtimes T}\left(V_{x},(v ; w)\right)+d_{G \boxtimes T}\left((v ; w), V_{y}\right) \geqslant$ 2 diam $V(G)$. Consider the projection $P_{T}$ on $T$. By Lemma 2 we have $d_{G \boxtimes T}\left(V_{x}, V_{y}\right)=$ $d_{T}\left(P_{T}\left(V_{x}\right), P_{T}\left(V_{y}\right)\right)$. Due to $d_{T}\left(P_{T}\left(V_{x}\right), P_{T}\left(V_{y}\right)\right) \leqslant d_{T}\left(P_{T}\left(V_{x}\right), w\right)+d_{T}\left(w, P_{T}\left(V_{y}\right)\right)$, we have $d_{G \boxtimes T}\left(V_{x},(v ; w)\right)=d_{T}\left(P_{T}\left(V_{x}\right), w\right)$ and $d_{G \boxtimes T}\left((v ; w), V_{y}\right)=d_{T}\left(w, P_{T}\left(V_{y}\right)\right)$. Then, $w \in$ $\left[P_{T}(x) P_{T}(y)\right]=P_{T}([x y])$. Since $T$ is a tree, $w \in P_{T}([y z] \cup[z x])$. Then, $([y z] \cup[z x]) \cap(G \boxtimes$ $\{w\}) \neq \emptyset$ and $d_{G \boxtimes T}((v ; w),[y z] \cup[z x]) \leqslant \operatorname{diam} V(G)$. So, we have $d_{G \boxtimes T}((v ; w),[y z] \cup[z x]) \leqslant$ $\operatorname{diam} V(G)$ for every vertex $(v ; w)$ in $[x y]$. Since $x, y \in J(G \boxtimes T), d_{G \boxtimes T}(p,[y z] \cup[z x]) \leqslant$ $\operatorname{diam} V(G)+1 / 2$ for every $p \in[x y]$. Hence, $\delta(\triangle) \leqslant \operatorname{diam} V(G)+1 / 2$, and we obtain $\delta(G \boxtimes T) \leqslant \operatorname{diam} V(G)+1 / 2$.

Theorem 29. Let $T$ be any tree and $G$ any graph with $0<\operatorname{diam} V(G)=\operatorname{diam} T / 2$. Then, we have

$$
\delta(G \boxtimes T)=\operatorname{diam} V(G)+\frac{1}{4}
$$

Proof. By Theorem 19, we have that $\operatorname{diam} V(G) \leqslant \delta(G \boxtimes T) \leqslant \operatorname{diam} V(G)+1 / 2$.
Now we show a geodesic bigon $B$ in $G \boxtimes T$ with $\delta(B)=\operatorname{diam} V(G)+1 / 4$. Define by $n:=\operatorname{diam} V(G)$ and consider $v_{1}, \ldots, v_{n+1} \in V(G)$ with $v_{i} \sim v_{i+1}$ for $i=1, \ldots, n$ and $d_{G}\left(v_{1}, v_{n+1}\right)=n$. Also, consider $w_{1}, \ldots, w_{2 n+1} \in V(T)$ with $w_{i} \sim w_{i+1}$ for $i=$ $1, \ldots, 2 n$ and $d_{T}\left(w_{1}, w_{2 n+1}\right)=\operatorname{diam} T=2 n$. Denote by $a$ (respectively, $b$ ) the midpoint of $\left[\left(v_{1} ; w_{1}\right),\left(v_{2} ; w_{1}\right)\right]$ (respectively, $\left.\left[\left(v_{1} ; w_{2 n+1}\right),\left(v_{2} ; w_{2 n+1}\right)\right]\right)$. Let us consider

$$
\gamma^{*}:=\left[a\left(v_{1} ; w_{1}\right)\right] \bigcup\left\{\bigcup_{i=1}^{2 n}\left[\left(v_{1} ; w_{i}\right),\left(v_{1} ; w_{i+1}\right)\right]\right\} \bigcup\left[\left(v_{1} ; w_{2 n+1}\right) b\right]
$$

and

$$
\begin{aligned}
\gamma^{\prime}:= & {\left[a\left(v_{2} ; w_{1}\right)\right] \bigcup\left\{\bigcup_{i=1}^{n-1}\left[\left(v_{i+1} ; w_{i}\right),\left(v_{i+2} ; w_{i+1}\right)\right]\right\} \bigcup\left[\left(v_{n+1} ; w_{n}\right),\left(v_{n+1} ; w_{n+1}\right)\right] \bigcup } \\
& \bigcup\left[\left(v_{n+1} ; w_{n+1}\right),\left(v_{n+1} ; w_{n+2}\right)\right] \bigcup\left\{\bigcup_{j=1}^{n-1}\left[\left(v_{n+2-j} ; w_{n+1+j}\right),\left(v_{n+1-j} ; w_{n+2+j}\right)\right]\right\} \bigcup \\
& \bigcup\left[\left(v_{2} ; w_{2 n+1}\right) b\right] .
\end{aligned}
$$

Consider the geodesic bigon $B:=\left\{\gamma^{*}, \gamma^{\prime}\right\}$ in $G \boxtimes T$. Let $p$ be the midpoint of $\gamma^{\prime}$ and let $p_{0}$ be a point in $\gamma^{\prime}$ with $d_{G \boxtimes T}\left(p_{0}, p\right)=1 / 4$; then $d_{G \boxtimes T}\left(p_{0}, \gamma^{*}\right)=n+1 / 4$ and $\delta(G \boxtimes T) \geqslant$ $\delta(B) \geqslant n+1 / 4$.

Hence, by Theorem 25 we have $\delta(G \boxtimes T) \in\{n+1 / 4, n+1 / 2\}$. Seeking for a contradiction assume that $\delta(G \boxtimes T)=n+1 / 2$. Then there are a geodesic triangle $\triangle=$ $\{x, y, z\}$ in $G \boxtimes T$ and $p \in[x y]$ with $d_{G \boxtimes T}(p,[y z] \cup[z x])=n+1 / 2$. By Theorem 26 we can assume that $\triangle$ is a cycle with $x, y, z \in J(G \boxtimes T)$. By Theorem $8, \operatorname{diam}(G \boxtimes T)=2 n+1$ and we conclude that $L([x y])=2 n+1$ and $p$ is the midpoint of $[x y]$. Since diam $V(G \boxtimes T)=2 n$, we have that $x, y$ are midpoints of edges in $G \boxtimes T$, and so, $p$ is a vertex of $G \boxtimes T$. We can write $[x y] \cap V(G \boxtimes T)=\left\{\left(a_{1} ; b_{1}\right),\left(a_{2} ; b_{2}\right), \ldots,\left(a_{2 n+1} ; b_{2 n+1}\right)\right\}$ with $a_{1}, \ldots, a_{2 n+1} \in V(G)$, $\left(a_{i} ; b_{i}\right) \sim\left(a_{i+1} ; b_{i+1}\right)$ for $i=1, \ldots, 2 n$ and $d_{T}\left(b_{1}, b_{2 n+1}\right)=2 n$. Thus, $p=\left(a_{n+1} ; b_{n+1}\right)$ and $p \in V\left(G \boxtimes\left\{b_{n+1}\right\}\right)$. Since $T$ is a tree we have that $([y z] \cup[z x]) \cap\left(G \boxtimes\left\{b_{n+1}\right\}\right) \neq \emptyset$; in particular, $d_{G \boxtimes T}(p,[y z] \cup[z x]) \leqslant \operatorname{diam} V(G)$. This is the contradiction we were looking for, and then $\delta(G \boxtimes T)=\operatorname{diam} V(G)+1 / 4$.

The following lemma will be useful.
Lemma 30. Let $C_{m}$ be a cycle graph and $G$ any graph with $\operatorname{diam} V(G)<\operatorname{diam} V\left(C_{m}\right)$. Let $\gamma=[x y]$ be a geodesic in $G \boxtimes C_{m}$ such that $x, y \in J\left(G \boxtimes C_{m}\right)$. Then, $L\left(P_{C_{m}}(\gamma)\right) \leqslant m / 2$ where $P_{C_{m}}$ is the projection on $C_{m}$.

Proof. If $\operatorname{diam} V(G)=0$, it is a trivial case. Assume now that $\operatorname{diam} V(G)>0$.
If $L(\gamma) \leqslant m / 2$, then we have the result since $L\left(P_{C_{m}}(\gamma)\right) \leqslant L(\gamma)$. Assume that $L(\gamma)>m / 2$. Seeking for a contradiction, assume that $L\left(P_{C_{m}}(\gamma)\right)>m / 2$.

Assume that $m$ is even (the case $m$ odd is similar). Since $x, y \in J\left(G \boxtimes C_{m}\right)$ and $L\left(P_{C_{m}}(\gamma)\right)>m / 2$, there are $x^{\prime}, y^{\prime} \in \gamma \cap J\left(G \boxtimes C_{m}\right)$ such that $d_{C_{m}}\left(P_{C_{m}}\left(x^{\prime}\right), P_{C_{m}}\left(y^{\prime}\right)\right)=$ $(m+1) / 2$. Without loss of generality we can assume that $x^{\prime} \in V\left(G \boxtimes C_{m}\right)$ and $y^{\prime} \notin$ $V\left(G \boxtimes C_{m}\right)$. Let $A, A_{1}, A_{2} \in V(G)$ and $B, B_{1}, B_{2} \in V\left(C_{m}\right)$ such that $x^{\prime}=(A ; B)$ and $y^{\prime} \in\left[\left(A_{1} ; B_{1}\right),\left(A_{2} ; B_{2}\right)\right]$. Since $d_{C_{m}}\left(P_{C_{m}}\left(x^{\prime}\right), P_{C_{m}}\left(y^{\prime}\right)\right)=(m+1) / 2$, without loss of generality we can assume that $d_{C_{m}}\left(B, B_{1}\right)+1=d_{C_{m}}\left(B, B_{2}\right)=m / 2$. Since diam $V\left(C_{m}\right)>$ $\operatorname{diam} V(G)$, by Lemma 2 we have $d_{G \boxtimes C_{m}}\left((A ; B),\left(A_{1} ; B_{1}\right)\right)=m / 2-1$; thus, $d_{G \boxtimes C_{m}}\left(x^{\prime}, y^{\prime}\right)$ $\leqslant(m-1) / 2$. This is the contradiction we were looking for.

The following theorem provides the exact value of the hyperbolicity constant of the strong product of a cycle $C_{m}$ and any graph $G$ with diam $V(G) \leqslant \operatorname{diam} V\left(C_{m}\right) / 2$. This
result is interesting by itself and, furthermore, it will be useful in order to prove the last theorems of this paper.

Theorem 31. Let $C_{m}$ be a cycle graph and $G$ any graph with $\operatorname{diam} V(G) \leqslant \operatorname{diam} V\left(C_{m}\right) / 2$. Then, we have

$$
\delta\left(G \boxtimes C_{m}\right)= \begin{cases}\lfloor m / 2\rfloor / 2+1 / 4, & \text { if } \operatorname{diam} V(G)=\operatorname{diam} V\left(C_{m}\right) / 2,  \tag{14}\\ m / 4, & \text { if } \operatorname{diam} V(G)<\operatorname{diam} V\left(C_{m}\right) / 2 .\end{cases}
$$

Proof. If diam $V(G)=0$, then the equality is trivial. Assume now that diam $V(G)>0$. Let $V\left(C_{m}\right)=\left\{w_{1}, \ldots, w_{m}\right\}$ where $w_{i} \sim w_{i+1}$ for $i=1, \ldots, m-1$. Let $P_{C_{m}}$ be the projection on $C_{m}$.

First, we prove that $\delta\left(G \boxtimes C_{m}\right)<(\lfloor m / 2\rfloor+1) / 2$. Seeking for a contradiction, assume that there are a geodesic triangle $T=\{x, y, z\}$ in $G \boxtimes C_{m}$ and a point $p \in \gamma:=[x y]$ with $d_{G \boxtimes C_{m}}(p,[y z] \cup[z x])=(\lfloor m / 2\rfloor+1) / 2=\operatorname{diam}\left(G \boxtimes C_{m}\right) / 2$. Then $L(\gamma)=\operatorname{diam}\left(G \boxtimes C_{m}\right)$ and $d_{G \boxtimes C_{m}}(p,[y z] \cup[z x])=\operatorname{diam}\left(G \boxtimes C_{m}\right) / 2$, and we conclude that $p$ is the midpoint of $\gamma$. By Theorem 26, we can assume that $T$ is a cycle with $x, y, z \in J\left(G \boxtimes C_{m}\right)$. Since diam $V(G \boxtimes$ $\left.C_{m}\right)=\operatorname{diam}\left(G \boxtimes C_{m}\right)-1$, by Theorem 8 we have that $x, y$ are midpoints of edges in $G \boxtimes C_{m}$. Let $V_{x}$ (respectively, $V_{y}$ ) be the closest vertex to $x$ (respectively, $y$ ) in $\gamma$. Let $V_{x}^{\prime}$ (respectively, $V_{y}^{\prime}$ ) be the closest vertex to $x$ (respectively, $y$ ) in $[x z]$ (respectively, $[y z]$ ). By Lemma 2, we have $d_{G \boxtimes C_{m}}\left(V_{x}, V_{y}\right)=d_{C_{m}}\left(P_{C_{m}}\left(V_{x}\right), P_{C_{m}}\left(V_{y}\right)\right)=\lfloor m / 2\rfloor$. Therefore, since $\operatorname{diam} V(G) \leqslant \operatorname{diam} V\left(C_{m}\right) / 2$ we have $d_{C_{m}}\left(P_{C_{m}}\left(V_{x}\right), P_{C_{m}}(p)\right)=d_{C_{m}}\left(P_{C_{m}}(p), P_{C_{m}}\left(V_{y}\right)\right)=$ $\lfloor m / 2\rfloor / 2$. By Lemma 30 we have $L\left(P_{C_{m}}(\gamma)\right) \leqslant m / 2$; since $2(\lfloor m / 2\rfloor / 2+1 / 2)>m / 2$ we have either $P_{C_{m}}\left(V_{x}\right)=P_{C_{m}}(x)=P_{C_{m}}\left(V_{x}^{\prime}\right)$ or $P_{C_{m}}\left(V_{y}\right)=P_{C_{m}}(y)=P_{C_{m}}\left(V_{y}^{\prime}\right)$. So, we have

$$
d_{G \boxtimes C_{m}}(p,[x z] \cup[y z]) \leqslant d_{G \boxtimes C_{m}}\left(p,\left\{V_{x}^{\prime}, V_{y}^{\prime}\right\}\right) \leqslant\lfloor m / 2\rfloor / 2 \leqslant m / 4
$$

This is the contradiction we were looking for, and we have $\delta\left(G \boxtimes C_{m}\right)<(\lfloor m / 2\rfloor+1) / 2$. So, by Theorem 25 we have $\delta\left(G \boxtimes C_{m}\right) \leqslant\lfloor m / 2\rfloor / 2+1 / 4$.

Assume now that $\lfloor m / 2\rfloor=2 \operatorname{diam} V(G)$. If $m$ is odd (i.e., $m=4 k+1$ ), then Theorem 15 (a) gives $\delta\left(G \boxtimes C_{m}\right) \geqslant m / 4=\lfloor m / 2\rfloor / 2+1 / 4$. So, (14) holds. Assume that $m$ in even (i.e., $m=4 k$ ). Now we show a geodesic bigon $B$ in $G \boxtimes C_{m}$ with $\delta(B)=\lfloor m / 2\rfloor / 2+1 / 4=$ $k+1 / 4$. Note that $k=\operatorname{diam} V(G)$ and consider $v_{1}, \ldots, v_{k+1} \in V(G)$ with $v_{i} \sim v_{i+1}$ for $i=1, \ldots, k$ and $d_{G}\left(v_{1}, v_{k+1}\right)=k$. Denote by $a$ (respectively, $b$ ) the midpoint of $\left[\left(v_{1} ; w_{1}\right),\left(v_{2} ; w_{1}\right)\right]$ (respectively, $\left.\left[\left(v_{1} ; w_{2 k+1}\right),\left(v_{2} ; w_{2 k+1}\right)\right]\right)$. Let us consider

$$
\gamma^{*}:=\left[a\left(v_{1} ; w_{1}\right)\right] \bigcup\left\{\bigcup_{i=1}^{2 k}\left[\left(v_{1} ; w_{i}\right),\left(v_{1} ; w_{i+1}\right)\right]\right\} \bigcup\left[\left(v_{1} ; w_{2 k+1}\right) b\right]
$$

and

$$
\begin{aligned}
\gamma^{\prime}:= & {\left[a\left(v_{2} ; w_{1}\right)\right] \bigcup\left\{\bigcup_{i=1}^{k-1}\left[\left(v_{i+1} ; w_{i}\right),\left(v_{i+2} ; w_{i+1}\right)\right]\right\} \bigcup\left[\left(v_{k+1} ; w_{k}\right),\left(v_{k+1} ; w_{k+1}\right)\right] \bigcup } \\
& \bigcup\left[\left(v_{k+1} ; w_{k+1}\right),\left(v_{k+1} ; w_{k+2}\right)\right] \bigcup\left\{\bigcup_{j=1}^{k-1}\left[\left(v_{k+2-j} ; w_{k+1+j}\right),\left(v_{k+1-j} ; w_{k+2+j}\right)\right]\right\} \bigcup \\
& \bigcup\left[\left(v_{2} ; w_{2 k+1}\right) b\right] .
\end{aligned}
$$

Then $B:=\left\{\gamma^{*}, \gamma^{\prime}\right\}$ is a geodesic bigon in $G \boxtimes C_{m}$ with $\delta(B)=k+1 / 4=\lfloor m / 2\rfloor / 2+1 / 4$.
Finally, assume that $\lfloor m / 2\rfloor>2$ diam $V(G)$. By Theorem 15 (a) it suffices to prove $\delta\left(G \boxtimes C_{m}\right) \leqslant m / 4$. If $m$ is odd, then $\lfloor m / 2\rfloor / 2+1 / 4=m / 4$ and (14) holds.

Assume that $m$ is even, then $\operatorname{diam} V(G) \leqslant m / 4-1 / 2$. Fix any geodesic triangle $T=\{x, y, z\}$ in $G \boxtimes C_{m}$ and $p \in[x y]$. By Remark 27, we can assume that $T$ is a cycle, $x, y, z \in J\left(G \boxtimes C_{m}\right)$ and $p$ satisfies $d_{G}(p, V(G)) \in\{0,1 / 4,1 / 2\}$. If $d_{G \boxtimes C_{m}}(p,\{x, y\}) \leqslant m / 4$, then $d_{G \boxtimes C_{m}}(p,[y z] \cup[z x]) \leqslant m / 4$. Assume that $d_{G \boxtimes C_{m}}(p,\{x, y\})>m / 4$; since $x, y \in$ $J\left(G \boxtimes C_{m}\right)$ and $d_{G}(p, V(G)) \in\{0,1 / 4,1 / 2\}$, we have $d_{G \boxtimes C_{m}}(p,\{x, y\}) \geqslant m / 4+1 / 4$. We have $L([x y])>m / 2$. Let $V_{x}$ (respectively, $V_{y}$ ) be the closest vertex to $x$ (respectively, $y)$ in $[x y]$; then $d_{G \boxtimes C_{m}}\left(p,\left\{V_{x}, V_{y}\right\}\right) \geqslant m / 4-1 / 4$. Let $V_{x}^{\prime}$ (respectively, $V_{y}^{\prime}$ ) be the closest vertex to $x$ (respectively, $y$ ) in $[x z]$ (respectively, $[y z]$ ). Since $m$ is even and $x, y \in J\left(G \boxtimes C_{m}\right)$ we have $d_{G \boxtimes C_{m}}\left(V_{x}, V_{y}\right) \geqslant m / 2$ and we conclude $d_{G \boxtimes C_{m}}\left(V_{x}, V_{y}\right)=m / 2$. By Lemma 2 we have $d_{G \boxtimes C_{m}}\left(V_{x}, V_{y}\right)=d_{C_{m}}\left(P_{C_{m}}\left(V_{x}\right), P_{C_{m}}\left(V_{y}\right)\right)=m / 2$; by Lemma 30 we conclude $L\left(P_{C_{m}}([x y])\right)=m / 2$. Since $m / 2=\lfloor m / 2\rfloor>\operatorname{diam} V(G)$, we have $P_{C_{m}}\left(V_{x}\right)=$ $P_{C_{m}}(x)=P_{C_{m}}\left(V_{x}^{\prime}\right)$ and $P_{C_{m}}\left(V_{y}\right)=P_{C_{m}}(y)=P_{C_{m}}\left(V_{y}^{\prime}\right)$. Since $d_{G \boxtimes C_{m}}\left(p,\left\{V_{x}, V_{y}\right\}\right) \leqslant$ $d_{G \boxtimes C_{m}}\left(V_{x}, V_{y}\right) / 2=m / 4$, without loss of generality we can assume that $d_{G \boxtimes C_{m}}\left(p,\left\{V_{x}, V_{y}\right\}\right)$ $=d_{G \boxtimes C_{m}}\left(p, V_{x}\right) \leqslant m / 4$. Let $V_{p}$ be the closest vertex to $p$ in $[x p]$. Since $d_{G \boxtimes C_{m}}\left(p, V_{x}\right) \geqslant$ $m / 4-1 / 4>m / 4-1 / 2 \geqslant \operatorname{diam} V(G)$, we have $\operatorname{diam} V(G) \geqslant d_{G \boxtimes C_{m}}\left(V_{p}, V_{x}\right)=d_{C_{m}}\left(P_{C_{m}}\left(V_{p}\right)\right.$, $\left.P_{C_{m}}\left(V_{x}\right)\right)=d_{C_{m}}\left(P_{C_{m}}\left(V_{p}\right), P_{C_{m}}\left(V_{x}^{\prime}\right)\right)$ and we conclude $d_{G \boxtimes C_{m}}\left(V_{p}, V_{x}\right)=d_{G \boxtimes C_{m}}\left(V_{p}, V_{x}^{\prime}\right)$ and $d_{G \boxtimes C_{m}}(p,[x z] \cup[y z]) \leqslant d_{G \boxtimes C_{m}}\left(p, V_{x}^{\prime}\right) \leqslant d_{G \boxtimes C_{m}}\left(p, V_{x}\right) \leqslant m / 4$. Then $\delta\left(G \boxtimes C_{m}\right) \leqslant m / 4$.

As a consequence of Theorems 19, 28, 29 and 31 we obtain the precise values of the hyperbolicity constants of the following families of graphs.

Theorem 32. Let $T_{1}, T_{2}$ be two trees with $\operatorname{diam} T_{1} \leqslant \operatorname{diam} T_{2}$. Then

$$
\delta\left(T_{1} \boxtimes T_{2}\right)= \begin{cases}0, & \text { if } \operatorname{diam} T_{1}=0, \\ \operatorname{diam} T_{1}+1 / 2, & \text { if } 0<\operatorname{diam} T_{1}<\left(\operatorname{diam} T_{2}\right) / 2, \\ \operatorname{diam} T_{1}+1 / 4, & \text { if } 0<\operatorname{diam} T_{1}=\left(\operatorname{diam} T_{2}\right) / 2, \\ \left(\operatorname{diam} T_{2}+1\right) / 2, & \text { if } \operatorname{diam} T_{1}>\left(\operatorname{diam} T_{2}\right) / 2\end{cases}
$$

Corollary 33. Let $P_{n}, P_{m}$ be two path graphs with $2 \leqslant n \leqslant m$. Then

$$
\delta\left(P_{n} \boxtimes P_{m}\right)= \begin{cases}m / 2, & \text { if } m-1<2(n-1), \\ n-3 / 4, & \text { if } m-1=2(n-1), \\ n-1 / 2, & \text { if } m-1>2(n-1) .\end{cases}
$$

Theorem 34. Let $C_{n}, C_{m}$ be two cycle graphs with $3 \leqslant n \leqslant m$. Then

$$
\delta\left(C_{n} \boxtimes C_{m}\right)= \begin{cases}\lfloor m / 2\rfloor / 2+1 / 2, & \text { if }\lfloor m / 2\rfloor<2\lfloor n / 2\rfloor, \\ \lfloor m / 2\rfloor / 2+1 / 4, & \text { if }\lfloor m / 2\rfloor=2\lfloor n / 2\rfloor, \\ m / 4, & \text { if }\lfloor m / 2\rfloor>2\lfloor n / 2\rfloor .\end{cases}
$$

Theorem 35. For every $m \geqslant 2, n \geqslant 3$,

$$
\delta\left(C_{n} \boxtimes P_{m}\right)= \begin{cases}\lfloor n / 2\rfloor+1 / 2, & \text { if }\lfloor n / 2\rfloor<(m-1) / 2, \\ \lfloor n / 2\rfloor+1 / 4, & \text { if }\lfloor n / 2\rfloor=(m-1) / 2, \\ m / 2, & \text { if }(m-1) / 2<\lfloor n / 2\rfloor \leqslant(m-1), \\ (\lfloor n / 2\rfloor+1) / 2, & \text { if } m-1<\lfloor n / 2\rfloor<2(m-1), \\ \lfloor n / 2\rfloor / 2+1 / 4, & \text { if }\lfloor n / 2\rfloor=2(m-1), \\ n / 4, & \text { if }\lfloor n / 2\rfloor>2(m-1) .\end{cases}
$$

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## References

[1] Alvarez, V., Portilla, A., Rodríguez, J. M. and Tourís, E., Gromov hyperbolicity of Denjoy domains, Geom. Dedicata 121 (2006), 221-245.
[2] Abajo, E., Casablanca, R. M., Diánez, A., García-Vázquez, P. The Menger number of the strong product of graphs, Discrete Math 313 (2013), 1490-1495.
[3] Balogh, Z. M. and Buckley, S. M., Geometric characterizations of Gromov hyperbolicity, Invent. Math. 153 (2003), 261-301.
[4] Bermudo, S., Rodríguez, J. M. and Sigarreta, J. M., Computing the hyperbolicity constant, Comp. Math. Appl. 62 (2011), 4592-4595.
[5] Bermudo, S., Rodríguez, J. M., Sigarreta, J. M. and Tourís, E., Hyperbolicity and complement of graphs, Appl. Math. Letters 24 (2011), 1882-1887.
[6] Bermudo, S., Rodríguez, J. M., Sigarreta, J. M. and Vilaire, J.-M., Gromov hyperbolic graphs, Discrete Math. 313 (2013), 1575-1585.
[7] Bonk, M., Heinonen, J. and Koskela, P., Uniformizing Gromov hyperbolic spaces. Astérisque 270 (2001).
[8] Bowditch, B. H., Notes on Gromov's hyperobolicity criterion for path-metric spaces. Group theory from a geometrical viewpoint, Trieste, 1990 (ed. E. Ghys, A. Haefliger and A. Verjovsky; World Scientific, River Edge, NJ, 1991), 64-167.
[9] Brinkmann, G., Koolen J. and Moulton, V., On the hyperbolicity of chordal graphs, Ann. Comb. 5 (2001), 61-69.
[10] Cáceres, J., Hernando, C., Mora, M., Pelayo, I. M., Puertas, M. L., On the geodetic and the hull numbers in strong product graphs, Comput. Math. Appl. 60 (2010), 3020-3031.
[11] Cantón, A., Granados, A., Pestana, D., Rodríguez, J. M., Gromov hyperbolicity of periodic planar graphs, to appear in Acta Math. Sinica.
[12] Cantón, A., Granados, A., Pestana, D., Rodríguez, J. M., Gromov hyperbolicity of planar graphs, to appear in Central Europ. J. Math.
[13] Carballosa, W., Pestana, D., Rodríguez, J. M. and Sigarreta, J. M., Distortion of the hyperbolicity constant of a graph, Electr. J. Comb. 19 (2012), \#P67.
[14] Carballosa, W., Portilla, A., Rodríguez, J. M. and Sigarreta, J. M., Gromov hyperbolicity of planar graphs and CW complexes. Submitted.
[15] Carballosa, W., Rodríguez, J. M. and Sigarreta, J. M., New inequalities on the hyperbolity constant of line graphs, to appear in Ars Combinatoria.
[16] Carballosa, W., Rodríguez, J. M., Sigarreta, J. M. and Villeta, M., On the Hyperbolicity Constant of Line Graphs, Electr. J. Comb. 18 (2011), \#P210.
[17] Chen, B., Yau, S.-T. and Yeh, Y.-N., Graph homotopy and Graham homotopy, Discrete Math. 241 (2001), 153-170.
[18] Chepoi, V., Dragan, F. F., Estellon, B., Habib, M. and Vaxes Y., Notes on diameters, centers, and approximating trees of $\delta$-hyperbolic geodesic spaces and graphs, Electr. Notes Discrete Math. 31 (2008), 231-234.
[19] Frigerio, R. and Sisto, A., Characterizing hyperbolic spaces and real trees, Geom. Dedicata 142 (2009), 139-149.
[20] Ghys, E. and de la Harpe, P., Sur les Groupes Hyperboliques d'après Mikhael Gromov. Progress in Mathematics 83, Birkhäuser Boston Inc., Boston, MA, 1990.
[21] Hästö, P. A., Gromov hyperbolicity of the $j_{G}$ and $\tilde{\jmath}_{G}$ metrics, Proc. Amer. Math. Soc. 134 (2006), 1137-1142.
[22] Hästö, P. A., Lindén, H., Portilla, A., Rodríguez, J. M. and Tourís, E., Gromov hyperbolicity of Denjoy domains with hyperbolic and quasihyperbolic metrics, $J$. Math. Soc. Japan 64 (2012), 247-261.
[23] Hästö, P. A., Portilla, A., Rodríguez, J. M. and Tourís, E., Gromov hyperbolic equivalence of the hyperbolic and quasihyperbolic metrics in Denjoy domains, Bull. London Math. Soc. 42 (2010), 282-294.
[24] Hästö, P. A., Portilla, A., Rodríguez, J. M. and Tourís, E., Comparative Gromov hyperbolicity results for the hyperbolic and quasihyperbolic metrics, Complex Var. Ellip. Eq. 55 (2010), 127-135.
[25] Hästö, P. A., Portilla, A., Rodríguez, J. M. and Tourís, E., Uniformly separated sets and Gromov hyperbolicity of domains with the quasihyperbolic metric, Medit. J. Math. 8 (2011), 47-65.
[26] Imrich, W. and Klavžar, S., Product Graphs: Structure and Recognition, Wiley Series in Discrete Mathematics and Optimization, (2000).
[27] Jonckheere, E. and Lohsoonthorn, P., A hyperbolic geometry approach to multipath routing, Proceedings of the 10th Mediterranean Conference on Control and Automation (MED 2002), Lisbon, Portugal, July 2002. FA5-1.
[28] Jonckheere, E. A., Controle du trafic sur les reseaux a geometrie hyperbolique-Une approche mathematique a la securite de l'acheminement de l'information, J. Europ. de Syst. Autom. 37 (2003), 145-159.
[29] Jonckheere, E. A. and Lohsoonthorn, P., Geometry of network security, American Control Conference ACC (2004), 111-151.
[30] Jonckheere, E. A., Lohsoonthorn, P. and Ariaesi, F, Upper bound on scaled Gromovhyperbolic delta, Applied Mathematics and Computation 192 (2007), 191-204.
[31] Jonckheere, E. A., Lohsoonthorn, P. and Bonahon, F., Scaled Gromov hyperbolic graphs, J. Graph Theory 2 (2007), 157-180.
[32] Kaveh, A., Koohestani, K., Graph products for configuration processing of space structures, Comput. Struct. 86 (2008), 1219-1231.
[33] Koolen, J. H. and Moulton, V., Hyperbolic Bridged Graphs, Europ. J. Comb. 23 (2002), 683-699.
[34] Michel, J., Rodríguez, J. M., Sigarreta, J. M. and Villeta, M., Hyperbolicity and parameters of graphs, Ars Combinatoria 100 (2011), 43-63.
[35] Michel, J., Rodríguez, J. M., Sigarreta, J. M. and Villeta, M., Gromov hyperbolicity in Cartesian product graphs, Proc. Indian Acad. Sci. Math. Sci. 120 (2010), 1-17.
[36] Oshika, K., Discrete groups, AMS Bookstore, 2002.
[37] Pestana, D., Rodríguez, J. M., Sigarreta, J. M. and Villeta, M., Gromov hyperbolic cubic graphs, Central Europ. J. Math. 10(3) (2012), 1141-1151.
[38] Portilla, A., Rodríguez, J. M., Sigarreta, J. M. and Vilaire, J.-M., Gromov hyperbolic tessellation graphs, to appear in Utilitas Math.
[39] Portilla, A., Rodríguez, J. M. and Tourís, E., Gromov hyperbolicity through decomposition of metric spaces II, J. Geom. Anal. 14 (2004), 123-149.
[40] Portilla, A. and Tourís, E., A characterization of Gromov hyperbolicity of surfaces with variable negative curvature, Publ. Mat. 53 (2009), 83-110.
[41] Rodríguez, J. M., Characterization of Gromov hyperbolic short graphs, to appear in Acta Math. Sinica.
[42] Rodríguez, J. M., Sigarreta, J. M., Vilaire, J.-M. and Villeta, M., On the hyperbolicity constant in graphs, Discrete Math. 311 (2011), 211-219.
[43] Rodríguez, J. M. and Tourís, E., Gromov hyperbolicity through decomposition of metric spaces, Acta Math. Hung. 103 (2004), 53-84.
[44] Rodríguez, J. M. and Tourís, E., A new characterization of Gromov hyperbolicity for Riemann surfaces, Publ. Mat. 50 (2006), 249-278.
[45] Rodríguez, J. M. and Tourís, E., Gromov hyperbolicity of Riemann surfaces, Acta Math. Sinica 23 (2007), 209-228.
[46] Sabidussi, G., Graph multiplication, Math. Z. 72 (1960), 446-457.
[47] Špacapan, S., Connectivity of Strong Products of Graphs, Graphs Combin. 26 (2010), 457-467.
[48] Sigarreta, J. M. Hyperbolicity in median graphs, to appear in Proc. Indian Acad. Sci. Math. Sci.
[49] Tourís, E., Graphs and Gromov hyperbolicity of non-constant negatively curved surfaces. J. Math. Anal. Appl. 380 (2011), 865-881.

