XI Encuentro Andaluz de Geometría
IMUS (Universidad de Sevilla), 15 de mayo de 2015, págs. 23–28

Totally umbilic null hypersurfaces in generalized Robertson-Walker spaces

Manuel Gutiérrez • Benjamín Olea

Abstract. Using a correspondence between totally umbilic null hypersurfaces in generalized Robertson-Walker spaces and twisted decompositions of the fibre we show that nullcones are the unique totally umbilic null hypersurfaces in the closed Friedmann Cosmological model and a uniqueness result for dual pairs of totally umbilic null hypersurfaces in Reissner-Nordström and De Sitter-Schwarzschild spacetimes.

1. Introduction

A hypersurface L in a n-dimensional Lorentzian manifold is null if the induced metric tensor on it is degenerate. In this case, for each point in L there is a unique null direction which is orthogonal to L itself and, under time-orientability assumption, we can suppose that it is generated by a null vector field $\xi \in \mathfrak{X}(L)$. Moreover, tangent vectors to L non-proportional to ξ are spacelike.

The null second fundamental form of L is defined as $B(v,w) = -g(\nabla_v \xi, w)$ for all $v, w \in TL$ and the null mean curvature as $H = \sum_{i=1}^{n-3} B(e_i, e_i)$ being $\{e_1, \dots, e_{n-2}\}$ orthonormal spacelike vectors in TL. If $B = ag$, where necessarily $a = \frac{H}{n-2}$, then it is said that L is totally umbilic and if $B = 0$ it is totally geodesic. The null second fundamental form and the null mean curvature depend on the choosen null vector field, but being totally umbilic or geodesic is independent of any election.

If the ambient Lorentzian manifold is a generalized Robertson-Walker space $\left(I \times F, -dt^2 + f(t)^2 g_F\right)$, denoted $I \times_f F$, we choose $\xi \in \mathfrak{X}(L)$ such that $g(\xi, f\partial_t) = 1$. Since $f\partial_t$ is closed and conformal, it is easy to check that in this case ξ is geodesic and $\nabla_v \xi \in TL \cap \partial_t^{\perp}$ for all $v \in TL$. Using this fact, it can be proven the following.

Proposition 1.1. *Let γ be a null geodesic in a GRW space. Suppose that γ is contained in a totally umbilic nullcone. If there exists a conjugate point of $\gamma(0)$ along γ, then it has maximum multiplicity.*

Manuel Gutiérrez, *mgl@agt.cie.uma.es*
Departamento de Álgebra, Geometría y Topología, Universidad de Málaga.
Benjamín Olea, *benji@uma.es*
Departamento de Matemática Aplicada, Universidad de Málaga.

This result has been generelarized to any Lorentzian manifold using rigging techniques, [2].

Locally, any null hypersurface in a GRW space can be expressed as the graph of a function. Indeed, if we call $\pi : I \times_f F \to F$ and $T : I \times_f F \to I$ the canonical projections, then $\pi : L \to F$ is a local diffeomorphism and thus, locally, L coincides with the graph of the function given by $h = T \circ \pi^{-1} : \theta \to I$, where $\theta \subset F$. In the case of a nullcone, we can give an explicit description.

Proposition 1.2. *Let $I \times_f F$ be a GRW space and fix $p_* = (t_*, x_*) \in I \times F$. If Θ is a normal neighborhood of p_*, then the local nullcones at p_* are given by*

$$C^+_{p_*} = \{(t,x) \in \Theta : \int_{t_*}^{t} \frac{1}{f(r)} dr = d_F(x_*, x)\},$$

$$C^-_{p_*} = \{(t,x) \in \Theta : \int_{t}^{t_*} \frac{1}{f(r)} dr = d_F(x_*, x)\},$$

being d_F the Riemannian distance in F.

2. Main result

We will use the following result to deduce the applications to the closed Friedmann model, Reissner-Nordström and De Sitter-Schwarzschild spacetimes.

Theorem 2.1 ([1]). *Let $I \times_f F$ be a GRW space. If L is a totally umbilic null hypersurface, then for each $(t_0, x_0) \in L$ there exists a decomposition of F in a neighborhood of x_0 as a twisted product with one dimensional base*

$$\left(J \times S, ds^2 + \mu(s,z)^2 g_S\right),$$

where x_0 is identified with $(0, z_0)$ for some $z_0 \in S$ and L is given by

$$\{(t,s,z) \in I \times J \times S : s = \int_{t_0}^{t} \frac{1}{f(r)} dr\}.$$

Moreover, if H is the null mean curvature of L, then

$$\mu(s,z) = \frac{f(t_0)}{f(t)} exp\left(\int_0^s \frac{H(t,r,z) f(t)^2}{n-2} dr\right)$$

for all $(t,s,z) \in L$.

Conversely, if F admits a twisted decomposition in a neighborhood of x_0 as above, then $L = \{(t,s,z) \in I \times J \times S : s = \int_{t_0}^{t} \frac{1}{f(r)} dr\}$ is a totally umbilic null hypersurface with null mean curvature

$$H = \frac{n-2}{f(t)^2}\left(f'(t) + \frac{\mu_s(s,z)}{\mu(s,z)}\right).$$

For each point $(t_0, x_0) \in L$ we can construct another totally umbilic null hypersurface, which we denote $\widetilde{L}_{(t_0,x_0)}$ and we call it the dual of L through (t_0, x_0), simply by changing the sign of the parameter in the base of the twisted decomposition of the fibre induced by L.

Specifically, if L induces a twisted decomposition $J \times_\mu S$ of F in a neighborhood of x_0 where L is $\{(t, s, z) \in I \times J \times S : s = \int_{t_0}^{t} \frac{1}{f(r)} dr\}$, then $\widetilde{L}_{(t_0,x_0)}$ is given by

$$\{(t, s, z) \in I \times J \times S : s = \int_{t}^{t_0} \frac{1}{f(r)} dr\}$$

and its null mean curvature is

$$\widetilde{H} = \frac{n-2}{f(t)^2} \left(f'(t) - \frac{\mu_s(s, z)}{\mu(s, z)} \right).$$

Example 2.1. *Consider* $\mathbb{R}^n_1 = \mathbb{R} \times \mathbb{R}^{n-1}$ *and* $C^+_{(0,0)}$ *the future nullcone with vertex* $(0, 0)$. *The dual hypersurface through* $(t_0, x_0) \in C^+_{(0,0)}$ *is a past null cone with vertex at* $(2t_0, 0)$.

Consider now $\mathbb{S}^n_1 = \mathbb{R} \times_{\cosh(t)} \mathbb{S}^{n-1}$ *and a future nullcone* $C^+_{(0,x_*)}$, *where* $x_* \in \mathbb{S}^{n-1}$. *Take* $t_c > 0$ *such that*

$$\int_0^{t_c} \frac{1}{\cosh(r)} dr = \int_{t_c}^{\infty} \frac{1}{\cosh(r)} dr = \frac{\pi}{4}.$$

The dual hypersurface through $(t_0, x_0) \in C^+_{(0,x_*)}$ *is a past null cone if* $t_0 < t_c$, *a future nullcone if* $t_c < t_0$ *and a totally geodesic hypersurface if* $t_0 = t_c$.

As an immediate corollary of Theorem 2.1, we can give the following obstruction to the existence of totally umbilic (geodesic) null hypersurfaces.

Corollary 2.1. *If the fibre of a GRW space does not admit any local decomposition as a twisted (warped) product with one-dimensional base, then it does not exist any totally umbilic (geodesic) null hypersurface.*

Example 2.2. *In a Riemannian twisted product manifold* $J \times_\mu S$, *the sectional curvature of any plane containing* ∂_s *is* $\frac{-1}{\mu} Hess_\mu(\partial_s, \partial_s)$. *Therefore,* $\mathbb{S}^2 \times \mathbb{S}^2$ *does not admit any local twisted product decomposition as above, since for any vector we can find two planes containing it with different sectional curvatures. Applying Corollary 2.1, in a GRW space* $I \times_f \left(\mathbb{S}^2 \times \mathbb{S}^2\right)$ *there are not totally umbilic null hypersurfaces.*

Example 2.3. *Consider the twisted product* $F = \mathbb{R} \times_\mu \mathbb{R}^n$, *where* $\mu(s, z) = e^s + |z|^2$. *A curvature analysis as before shows that it does not admit another* *local decomposition as a twisted nor warped product with a one dimensional base. Therefore, from Theorem 2.1, in a GRW space $I \times_f F$ there are exactly two totally umbilic null hypersurface through each point and using Corollary 2.1, it does not have any totally geodesic null hypersurface.*

3. Applications

Nullcones in a Robertson-Walker space are totally umbilic. We can prove that in fact they are the unique totally umbilic null hypersurfaces under suitable hypothesis.

Theorem 3.1 ([1]). *Any totally umbilic null hypersurface in a Robertson-Walker space $I \times_f \mathbb{S}^{n-1}$ $(n > 3)$ with*

$$\int_I \frac{1}{f(r)} dr > \pi \tag{1}$$

is an open set of a nullcone. In particular, it cannot exist totally geodesic null hypersurfaces.

The condition (1) can not be sharpened. For example, in $\mathbb{S}_1^n = \mathbb{R} \times_{\cosh(t)} \mathbb{S}^{n-1}$ there are totally geodesic null hypersurfaces which evidently are not contained in a nullcone.

We can also get the following immediate corollaries.

Corollary 3.1. *Nullcones are the unique totally umbilic null hypersurfaces in the closed Friedmann Cosmological model.*

Corollary 3.2. *Any totally umbilic null hypersurface in $\mathbb{R} \times \mathbb{S}^{n-1}$ $(n > 3)$ is contained in a nullcone.*

A standard static Lorentzian manifold $(F \times I, g_F - \phi^2 dt^2)$ is conformal to a direct product, which, in particular, is a GRW space. Since totally umbilic null hypersurfaces are preserved under conformal changes, we can adapt Theorem 2.1 to the case of static spaces.

Theorem 3.2 ([1]). *Let $F \times_\phi I$ be a n-dimensional standard static space. If L is a totally umbilic null hypersurface, then for each $(x_0, t_0) \in L$ there exists a local decomposition of $\left(F, \frac{1}{\phi^2} g_F\right)$ in a neighborhood of x_0 as a twisted product with one-dimensional base*

$$\left(J \times S, ds^2 + \mu(s, z)^2 g_S\right),$$

where x_0 *is identified with* $(0, z_0)$ *for some* $z_0 \in S$ *and* L *is given by*

$$\{(s, z, s + t_0) \in J \times S \times I\}.$$

Moreover, if H *is the null mean curvature of* L*, then*

$$\mu(s, z) = \frac{\phi(0, z)}{\phi(s, z)} \exp\left(\int_0^s \frac{H(r, z, r + t_0)}{n - 2} dr\right).$$

Conversely, if $\left(F, \frac{1}{\phi^2} g_F\right)$ *admits a twisted decomposition in a neighborhood of* x_0 *as above, then*

$$L = \{(s, z, s + t_0) \in J \times S \times I\}$$

is a totally umbilic null hypersurface with null mean curvature

$$H = (n - 2)\frac{d}{ds} \ln(\mu\phi). \tag{2}$$

As in the case of GRW spaces, we can construct the dual null hypersurface $\widetilde{L}_{(x_0,t_0)}$ through $(x_0, t_0) \in L$. In fact, if L induces a twisted decomposition of $\left(F, \frac{1}{\phi^2} g_F\right)$ in a neighborhood of x_0 where L is given by $\{(s, z, s + t_0)\}$, then $\widetilde{L}_{(x_0,t_0)}$ is given by $\{(s, z, -s + t_0)\}$. Moreover, its null mean curvature is

$$\widetilde{H} = (n - 2)\frac{d}{ds} \ln\left(\frac{\phi}{\mu}\right).$$

Since simply connected Lorentzian manifold of constant sectional curvature can be decomposed as GRW spaces (at least locally in the case of negative curvature), we can use Theorem 2.1 and Proposition 1.2 to prove the following.

Theorem 3.3. *Any totally umbilic null hypersurface in a complete space of constant curvature and dimension greater than three is totally geodesic or is contained in a nullcone.*

Now, we consider the family of standard static spacetimes given by

$$\left(I \times \mathbb{S}^2 \times \mathbb{R}, \frac{1}{h(r)} dr^2 + r^2 g_0 - h(r) dt^2\right),$$

where $I \subset \mathbb{R}$, $h \in C^\infty(I)$ is a positive function and g_0 is the canonical metric on $\mathbb{S}^2$. This family includes important examples of spacetimes. If $h(r) = 1 - \frac{m^2}{r} + \frac{c^2}{r^2}$ for certain constant m and c, then we get the Reissner-Nordström spacetime (the Schwarzschild exterior in the case $c = 0$) and if $h(r) = 1 - \frac{m^2}{r} + kr^2$, then we obtain the De Sitter-Schwarzschild spacetime (Minkowski, De Sitter or anti-De Sitter if $m = 0$ and $k = 0$, $k > 0$ or $k < 0$ respectively).

Theorem 3.4 ([1]). *If $h \in C^\infty(I)$ is a positive function such that $h''' \not\equiv 0$ in any open subset of I, then the spacetime given by (3) has exactly two totally umbilic null hypersurface through each point.*

Corollary 3.3. *In a De Sitter-Schwarzschild with $m \neq 0$ and in a Reissner-Nordström spacetime (in particular in the Schwarzschild exterior) there are exactly two totally umbilic non-totally geodesic null hypersurface through each point.*

If we consider the Schwarzschild exterior embedded in the Kruskal spacetime $Q \times_r \mathbb{S}^2$, the totally umbilic null hypersurfaces claimed in the above corollary are given by

$$\{(u, v, x) \in Q \times \mathbb{S}^2 : u = u_0\}$$

and

$$\{(u, v, x) \in Q \times \mathbb{S}^2 : v = v_0\}.$$

Acknowledgment

This work was supported in part by Grupo Junta de Andalucía FQM-324. The first author was supported in part by MEYC-FEDER Grant MTM2013-41768-P.

References

[1] M. Gutiérrez and B. Olea, *Totally umbilic null hypersurfaces in generalized Robertson-Walker spaces* Differ. Geom. Appl. **42** (2015), no. 1, 15–30.

[2] M. Gutiérrez and B. Olea, *Induced Riemannian structures on a null hypersurface* Math. Nachr. **289** (2016), 1219-1236.