



Abstract

Functionals of Harmonics Functions [†]

Niurka R. Quintero ^{1,2,*}, José A. Cuesta ^{3,4} and Renato Alvarez-Nodarse ⁵¹ Departamento de Física Aplicada I, E.P.S., Universidad de Sevilla, Virgen de África 7, 41011 Sevilla, Spain² Instituto Carlos I de Física Teórica y Computacional, Universidad de Granada, 18071 Granada, Spain³ Grupo Interdisciplinar de Sistemas Complejos (GISC), Departamento de Matemáticas, Universidad Carlos III de Madrid, Avda. de la Universidad 30, 28911 Leganes, Spain; cuesta.jose.a@gmail.com⁴ Instituto de Biocomputación y Física de Sistemas Complejos (BIFI), Universidad de Zaragoza, 50009 Zaragoza, Spain⁵ Departamento de Análisis Matemático, Universidad de Sevilla, Apdo 1160, 41080 Sevilla, Spain; ran@us.es* Correspondence: nrquintero@gmail.com[†] Presented at Symmetry 2017—The First International Conference on Symmetry, Barcelona, Spain, 16–18 October 2017.

Published: 9 January 2018

Let \mathcal{C}_T^s , with $T > 0$, be the set of continuous, T -periodic functions $\mathbf{f} : \mathbb{R} \rightarrow \mathbb{R}^s$, and let $\Gamma : \mathcal{C}_T^s \rightarrow \mathbb{R}$ be a real functional on \mathcal{C}_T^s . If Γ is n times Fréchet differentiable on \mathcal{C}_T^s , then it has an n -th order Taylor expansion around $\mathbf{0}$ (see e.g., [1]). Such a Taylor expansion can be obtained as the n -th order truncation of the series

$$\Gamma[\mathbf{f}] = \sum_{n_1=0}^{\infty} \cdots \sum_{n_s=0}^{\infty} \langle c_{\mathbf{n}}(t_{11}, \dots, t_{1n_1}, \dots, t_{s1}, \dots, t_{sn_s}) \times f_1(t_{11}) \cdots f_1(t_{1n_1}) \cdots f_s(t_{s1}) \cdots f_s(t_{sn_s}) \rangle, \quad (1)$$

where $\mathbf{n} = (n_1, \dots, n_s)$ and we have introduced the notation

$$\langle \Omega(t_1, \dots, t_r) \rangle = \frac{1}{T^r} \int_0^T dt_1 \cdots \int_0^T dt_r \Omega(t_1, \dots, t_r). \quad (2)$$

The kernels $c_{n_1, \dots, n_s}(t_{11}, \dots, t_{sn_s})$ are all real, T -periodic, and symmetric in all their arguments. In this contribution we will prove the following theorem.

Theorem 1. Let Γ be a functional with Taylor series (1), and take

$$\mathbf{f}(t) = (\epsilon_1 \cos(q_1 \omega t + \phi_1), \dots, \epsilon_s \cos(q_s \omega t + \phi_s)), \quad (3)$$

where $\mathbf{q} \equiv (q_1, \dots, q_s) \in \mathbb{N}^s$ is such that $\gcd(q_1, \dots, q_s) = 1$ and $\omega = 2\pi/T$. Then,

$$\Gamma[\mathbf{f}] = C_0(\epsilon) + \sum_{\mathbf{x} \in S_+} \epsilon_1^{|\mathbf{x}_1|} \cdots \epsilon_s^{|\mathbf{x}_s|} C_{\mathbf{x}}(\epsilon) \cos(\mathbf{x} \cdot \boldsymbol{\phi} + \theta_{\mathbf{x}}(\epsilon)), \quad (4)$$

where, $\boldsymbol{\phi} \equiv (\phi_1, \dots, \phi_s)$, $\boldsymbol{\epsilon} \equiv (\epsilon_1, \dots, \epsilon_s)$, and functions $C_{\mathbf{x}}(\epsilon)$ and $\theta_{\mathbf{x}}(\epsilon)$ do not depend on $\boldsymbol{\phi}$ and are even in each ϵ_i , $i = 1, \dots, s$, for every $\mathbf{x} \in S_+$. $\mathbf{x} \in S_+$ is the set of vectors \mathbf{x} whose leftmost nonzero component is positive.

In the special case when Γ is invariant under time-shift, i.e., $\Gamma[\mathbf{f}(t + \tau)] = \Gamma[\mathbf{f}(t)]$ for all $0 < \tau < T$, we recover the results in [2]

$$\Gamma[\mathbf{f}] = C_0(\epsilon) + \sum_{\mathbf{x} \in \mathcal{D}_+} \epsilon_1^{|\mathbf{x}_1|} \cdots \epsilon_s^{|\mathbf{x}_s|} C_{\mathbf{x}}(\epsilon) \cos(\mathbf{x} \cdot \boldsymbol{\phi} + \theta_{\mathbf{x}}(\epsilon)), \quad (5)$$

where \mathcal{D}_+ denote the set of nonzero solutions of the Diophantine equation $\mathbf{q} \cdot \mathbf{x} = q_1 x_1 + \dots + q_s x_s = 0$, whose leftmost nonzero component is positive.

Acknowledgments: We acknowledge financial support from the MINECO of Spain through FIS2014-54497-P (N.R.Q.) and MTM2015-65888-C4-1-P (R.A.N.).

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