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A computational algebraic geometry approach to enumerate Malcev magma algebras over finite fields*

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The set $\mathcal{M}_n(\mathbb{K})$ of n-dimensional Malcev magma algebras over a finite field $\mathbb{K}$ can be identified with algebraic sets defined by zero-dimensional radical ideals for which the computation of their reduced Gröbner bases makes feasible their enumeration and distribution into isomorphism and isotopism classes. Based on this computation and the classification of Lie algebras over finite fields given by De Graaf and Strade, we determine the mentioned distribution for Malcev magma algebras of dimension $n \leq 4$. We also prove that every 3-dimensional Malcev algebra is isotopic to a Lie magma algebra. For $n = 4$, this assertion only holds when the characteristic of the base field $\mathbb{K}$ is distinct of two.

Keywords: Malcev algebra; magma algebra; finite field; polynomial ring

1. Introduction

Non-associative algebras have a special importance in terms of Mathematics, as well as for their multiple applications in Natural Sciences and Engineering [1, 2]. This paper deals in particular with the open problem of enumerating and classifying *Malcev magma algebras*, with possible application in quantum mechanics and string theory. A *magma* is a finite set endowed with a binary operation. Throughout the paper we suppose this set to be $[n] = \{1, \ldots, n\}$ and we denote the inner law as $\cdot$. An n-dimensional algebra A over a field $\mathbb{K}$ is said to be a *magma algebra* if there exists a basis $\{e_1, \ldots, e_n\}$ of the algebra and a magma $([n], \cdot)$ such that $e_i e_j = c_{ij} e_{i \cdot j}$ for each pair of elements $i, j \leq n$ and some structure constant $c_{ij} \in \mathbb{K}$. The algebra A is then said to be *based on* the magma $([n], \cdot)$. This is said to be a *Malcev magma algebra* if it also constitutes a *Malcev algebra* [3, 4], that is, $u^2 = 0$, for all $u \in A$, and

$$((uv)w)u + ((vw)u)u + ((wu)u)v = (uv)(uw), \text{ for all } u, v, w \in A. \quad (1)$$

Every Malcev algebra is, therefore, antisymmetric, that is, $uv = -vu$, for all $u, v \in A$. Besides, (1) is equivalent to the so-called *Malcev identity*

$$M(u, v, w) = J(u, v, w)u - J(u, v, uw) = 0, \text{ for all } u, v, w \in A, \quad (2)$$

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where J is the *Jacobian* defined as $J(u, v, w) = (uv)w + (vw)u + (wu)v$, for all $u, v, w \in A$. If the characteristic of the base field $\mathbb{K}$, which is denoted by $\mathrm{char}(\mathbb{K})$, is distinct of two, then both identities (1) and (2) are equivalent to the *Sagle identity*

$$S(u, v, w, y) = ((uv)w)y + ((vw)y)u + ((wy)u)v + ((yu)v)w - (uw)(vy) = 0, \text{ for all } u, v, w, y \in A. \tag{3}$$

Unlike Malcev identity, which is not linear in its first argument $u \in A$, Sagle identity is linear in the four arguments $u, v, w, y \in A$ and invariant under cyclic permutations of the variables.

If the *Jacobi identity*, $J(u, v, w) = 0$, is satisfied for any three vectors u, v and w of an algebra, then this is called a *Lie algebra* (a *Lie magma algebra* if this is also a magma algebra). Otherwise, the algebra is said to have a *Jacobi anomaly*. In quantum mechanics, the existence of Jacobi anomalies in the underlying non-associative algebraic structure related to the coordinates and momenta of a quantum non-Hamiltonian dissipative system was already claimed by Dirac [5] in the process of taking Poisson brackets. In string theory, for instance, one such an anomaly is involved by the non-associative algebraic structure that is defined by coordinates (x) and velocities or momenta (v) of an electron moving in the field of a constant magnetic charge distribution, at the position of the location of the magnetic monopole [6]. In particular, $J(v_1, v_2, v_3) = -\overrightarrow{\nabla} \cdot \overrightarrow{B}(x)$, where $\overrightarrow{\nabla} \cdot \overrightarrow{B}(x)$ denotes the divergence of the magnetic field $\overrightarrow{B}(x)$. The underlying algebraic structure constitutes a non-Lie Malcev algebra [7], with the commutation relations $[x_a, x_b] = 0$, $[x_a, v_b] = i\delta_{ab}$ and $[v_a, v_b] = i\epsilon_{abc}B_c(x)$, where $a, b, c \in \{1, 2, 3\}$, δ_{ab} denotes the Kronecker delta and ϵ_{abc} denotes the Levi-Civita symbol. If the magnetic field is proportional to the coordinates, the latter can be normalized and $B_c(x)$ can then be supposed to coincide with x_c. The resulting algebra is then called *magnetic* [8]. A generalization to electric charges has recently been considered [9] by defining the products $[x_a, x_b] = -i\epsilon_{abc}E_c(x, v)$, where the electric field E as well as the magnetic field B must depend not only on coordinates but also on velocities. Remark that both magnetic and electric algebras constitute magma algebras.

A main open problem in the theory of Malcev algebras is their enumeration and distribution into isomorphism classes [10, 11, 12]. Over finite fields, this problem has already been dealt with for Lie algebras of dimension up to six. Particularly, De Graaf [13] made use of Gröbner bases and computational algebraic geometry in order to determine the distribution of solvable Lie algebras of dimension up to four over any field, whereas Strade [14] obtained that of nonsolvable Lie algebras of dimension up to six over a finite field. The classification of nilpotent Lie algebras of dimension up to six over any field is also known [15]. The authors in this last reference indicated explicitly that some of their results were inspired by Gröbner basis computations.

In this paper, similarly to the methodology exposed by De Graaf [13], we make use of computational algebraic geometry as an approach to deal with the classification of the sets $\mathcal{L}_n(\mathbb{K})$ and $\mathcal{M}_n(\mathbb{K})$ of n-dimensional Lie magma algebras and Malcev magma algebras, respectively, over a finite field $\mathbb{K}$. We deal not only with the distribution into isomorphism classes, which is the usual criterion, but also with the more general distribution into isotopism classes, which constitutes a first attempt in this regard to the best of the authors knowledge. The concept of isotopism of algebras was introduced by Albert [16], who indicated that two algebras $(A, \cdot)$ and $(A', \circ)$ are *isotopic* if there exist three nonsingular linear maps f, g and h between A and A' such that

$$f(u) \circ g(v) = h(u \cdot v), \text{ for all } u, v \in A. \tag{4}$$

The triple (f, g, h) is said to be an *isotopism* between both algebras A and A', which is denoted as $A \simeq A'$. If $f = g = h$, then this constitutes an *isomorphism*, which is denoted as $A \cong A'$. To be isotopic is an equivalence relation among algebras of the same dimension and constitutes, therefore, a generalization of the concept of isomorphism. In his original paper, Albert dealt with isotopisms of division algebras, alternative algebras and Lie algebras. Shortly after, Bruck [17] dealt with isotopisms of quasigroup algebras, which constitute magma algebras in which division is possible. Since the manuscripts of Albert and Bruck, a wide amount of papers have dealt with isotopisms of distinct types of algebras as division algebras [18, 19, 20, 21], alternative algebras [22, 23], Lie algebras [24, 25], Jordan algebras [26, 27, 28, 29, 30, 31, 32], genetic algebras [33, 34], absolute valued algebras [35, 36], structural algebras [37] and real 2-dimensional commutative algebras [38]. Nevertheless, there does not exist any study on the distribution of Malcev algebras into isotopism classes.

The structure of the paper is the following. In Section 2 we expose some preliminary concepts and results on isotopisms of algebras, Malcev magma algebras and computational algebraic geometry. Section 3 deals with those ideals of polynomials whose algebraic sets are identified with the sets of n-dimensional Lie and Malcev magma algebras. We describe in particular two algorithms whose implementation enable us to determine the enumeration of Malcev magma algebras and their distribution into isomorphism and isotopism classes. This distribution is explicitly determined in Section 4 for dimension $n \in \{3, 4\}$.

2. Preliminary results

In this section we expose some basic concepts and results on isotopisms of algebras, Malcev algebras and computational algebraic geometry that are used throughout the paper. For more details about these topics we refer to the original articles of Albert [16], Malcev [3] and Sagle [4] and to the monographs of Cox, Little and O'Shea [39, 40].

2.1. Isotopisms of algebras

Let A be an algebra over a field $\mathbb{K}$. This algebra is called *abelian* if all its structure constants are zero. This constitutes a trivial magma algebra for which all the products of the corresponding magma are zeros.

Lemma 2.1 *The n-dimensional abelian algebra is not isotopic to any other n-dimensional algebra.*

Proof. Let (f, g, h) be an isotopism between an n-dimensional non-abelian algebra of basis $\{e_1, \ldots, e_n\}$ and the n-dimensional abelian algebra. Let $i, j \leq n$ be such that $e_ie_j \neq 0$. Then, $0 = f(e_i)g(e_j) = h(e_ie_j) \neq 0$, which is a contradiction. □

The *derived algebra* of the algebra A is the subalgebra A^2 formed by the set of elements uv such that $(u, v) \in A \times A$. The *derived series* of the algebra A is defined as

$$\mathcal{C}_1(A) = A \supseteq \mathcal{C}_2(A) = A^2 \supseteq \ldots \supseteq \mathcal{C}_k(A) = (\mathcal{C}_{k-1}(A))^2 \supseteq \ldots \tag{5}$$

The algebra A is said to be *solvable* if there exists a positive integer m such that $\mathcal{C}_{m+1}(A) \equiv 0$. The smallest such an integer is called the *solvability index* of the algebra. This index and the dimension of each vector subspace $\mathcal{C}_k(A)$ are preserved by isomorphisms. Nevertheless, this is not true in general for isotopisms, for which we can only assure the next result.

Lemma 2.2 *Let A and A' be two isotopic n-dimensional algebras. Then,* $\dim(A^2) = \dim(A'^2)$.

Proof. Let (f, g, h) be an isotopism between both algebras A and A'. The regularity of f and g involves that $f(A) = g(A) = A'$. Hence, $A'^2 = f(A)g(A) = h(A^2)$ and the result follows then from the regularity of h. □

Let us suppose now that A is a Lie algebra. The *centralizer* of a subset $S \subseteq A$ is the vector subspace $C_A(S) = \{u \in A \mid uv = 0, \text{ for all } v \in S\} \subseteq A$. The *center* of A is the set $Z(A) = C_A(A)$. The next results show the way in which isotopisms preserve centralizers and centers of Lie algebras.

Lemma 2.3 *Let (f, g, h) be an isotopism between two Lie algebras A and A'. Let S be a subset of L. Then,*

a) $f(C_A(S)) = C_{A'}(g(S))$.

b) $g(C_A(S)) = C_{A'}(f(S))$.

c) $\dim(C_A(S)) = \dim(C_{A'}(f(S))) = \dim(C_{A'}(g(S)))$.

Proof. Let $u \in C_A(S)$. Then, $f(u)v = h(ug^{-1}(v)) = h(0) = 0$, for all $v \in g(S)$ and hence, $f(C_A(S)) \subseteq C_{A'}(g(S))$. Reciprocally, if $u \in C_{A'}(g(S))$, then $f^{-1}(u)v = h^{-1}(ug(v)) = h^{-1}(0) = 0$ and hence, $C_{A'}(g(S)) \subseteq f(C_A(S))$. Thus, $f(C_A(S)) = C_{A'}(g(S))$. The regularity of f and g involves the same dimension of both vector subspaces. Item (b) and the fact of being $\dim(C_A(S)) = \dim(C_{A'}(f(S)))$ in item (c) hold similarly from the antisymmetry of L. □

Proposition 2.4 *Let A and A' be two isotopic Lie algebras. Then, $f(Z(A)) = Z(A')$ and $\dim(Z(A)) = \dim(Z(A'))$.*

Proof. The result is an immediate consequence of Lemma 2.3 from the definition of center of a Lie algebra. □

Let n be the dimension of the Lie algebra A. For each positive integer $m \leq n$, we define

$$d_m(A) := \min\{\dim C_A(S) \colon S \text{ is an } m\text{-dimensional vector subspace of } A\}. \tag{6}$$

$$D_m(A) := \max\{\dim C_A(S) \colon S \text{ is an } m\text{-dimensional vector subspace of } A\}. \tag{7}$$

We prove now that both values are preserved by isotopisms.

Proposition 2.5 *Let A and A' be two isotopic n-dimensional Lie algebras and let $m \leq n$ be a positive integer. Then, $d_m(A) = d_m(A')$ and $D_m(A) = D_m(A')$.*

Proof. Let (f, g, h) be an isotopism between A and A' and let S be an m-dimensional vector subspace of A such that $d_m(A) = \dim C_A(S)$. The regularity of g involves the set $g(S)$ to be m-dimensional. Besides, from Lemma 2.3, $C_{A'}(g(S)) = f(C_A(S))$. Hence, $d_m(A) \geq d_m(A')$. The equality follows similarly from the isotopism (f^{-1}, g^{-1}, h^{-1}) between A' and A. The invariance of D_m holds analogously. □

We finish with a remark on isotopisms of magma algebras. Let A and A' be two isotopic n-dimensional magma algebras over a field $\mathbb{K}$ and let $([n], \cdot)$ and $([n], *)$ be the respective magmas on which both algebras are based. If $\{e_1, \ldots, e_n\}$ and $\{e'_1, \ldots, e'_n\}$ are the respective bases of A and A', then every isotopism (f, g, h) between both algebras is characterized by three nonsingular matrices $F = (f_{ij})$, $G = (g_{ij})$ and $H = (h_{ij})$, with entries in the field $\mathbb{K}$, so that $\alpha(e_i) = \sum_{j=1}^{n} \alpha_{ij} e'_j$, for all $\alpha \in \{f, g, h\}$ and $i \in \{1, \ldots, n\}$. Then,

$$c_{ij} h_{(i \cdot j)k} = \sum_{s,t \leq n \mid s * t = k} f_{is} g_{jt} c'_{st}, \text{ for all } i, j, k \leq n, \tag{8}$$

where c_{ij} and c'_{st} are structure constants of A and A', respectively.

2.2. *Malcev magma algebras*

Every Malcev magma algebra in the current paper is described by means of a basis of vectors satisfying the conditions exposed in the next result.

Lemma 2.6 *Let $n \geq 3$. Every n-dimensional non-abelian Malcev magma algebra based on a magma $([n], \cdot)$ holds, up to isomorphism, one of the next two non-isomorphic possibilities*

a) $e_1 e_2 = e_2$, *or*

b) $e_1 e_2 = e_3$ *and there does not exist a non-zero structure constant c_{ij} such that $i \cdot j \in \{i, j\}$.*

Proof. Since the algebra is not abelian, there exists at least one non-zero structure constant. If there exist two distinct positive integers $i, j \leq n$ such that $c_{ij} \neq 0$ and $i \cdot j = j$, then we consider the isomorphism that maps e_i and e_j to $c_{ij} e_1$ and e_2, respectively, and preserves the rest of basis vectors. We get in this way the product $e_1 e_2 = e_2$. Otherwise, we take a non-zero structure constant c_{uv} and the isomorphism that maps e_u, e_v and $e_{u \cdot v}$ to $c_{uv} e_1$, e_2 and e_3, respectively, in order to get the product $e_1 e_2 = e_3$. □

Every Malcev algebra is *binary-Lie*, that is, any two of its elements generate a Lie subalgebra. As a consequence, every Malcev algebra of dimension $n \leq 3$ is a Lie algebra. Particularly, the only 1-dimensional Malcev algebra is the abelian and the only 2-dimensional non-abelian Malcev algebra is, up to isomorphism, the magma algebra determined by the product of basis vectors $e_1 e_2 = e_2$. The next theorem deals with the distribution of 3-dimensional Malcev magma algebras over a finite field into isomorphism classes. In the statement of the result, each isomorphism class is labeled according to the notation given by De Graaf [13] and Strade [14] in their respective classifications of solvable and non-solvable Lie algebras, but we have chosen for its description a basis that follows the conditions exposed in Lemma 2.6.

Theorem 2.7 *Every 3-dimensional Malcev magma algebra over a finite field $\mathbb{K}$ is isomorphic to exactly one of the next algebras*

a) The 3-dimensional abelian Lie algebra L^1.

b) The solvable Lie algebras

- $L^2 \cong e_1e_2 = e_2$ *and* $e_1e_3 = e_3$.
- $L^3_0 \cong e_1e_2 = e_2$.
- $L^4_a \cong e_1e_2 = e_3$ *and* $e_1e_3 = ae_2$, *for all* $a \in \mathbb{K}$.
  Here, $L^4_a \cong L^4_b$ *if and only if there exists* $\alpha \in \mathbb{K} \setminus \{0\}$ *such that* $a = \alpha^2 b$.

c) The non-solvable Lie algebras

- $W(1;\underline{2})^{(1)} \cong e_1e_2 = e_2$, $e_1e_3 = e_3$ *and* $e_2e_3 = e_1$, *whenever* $\mathrm{char}(\mathbb{K}) = 2$.
- $\mathfrak{sl}(2,\mathbb{K}) \cong e_1e_2 = e_2$, $e_1e_3 = -e_3$ *and* $e_2e_3 = e_1$, *whenever* $\mathrm{char}(\mathbb{K}) \neq 2$.

Proof. The mentioned classifications of Lie algebras obtained by De Graaf and Strade enable us to assure that each Lie algebra of the list constitutes an isomorphism class in $\mathcal{M}_3(\mathbb{K})$. In order to assure that there does not exist any other isomorphism class, it is required to prove that none of the next 3-dimensional solvable Lie algebras is a magma algebra

$$L^3_a \cong e_1e_2 = e_2 + ae_3 \text{ and } e_1e_3 = e_2, \text{ with } a \in \mathbb{K} \setminus \{0\}. \tag{9}$$

We prove this fact in Proposition 4.1. □

For dimension $n = 4$, there exists, up to isomorphism, a unique non-Lie Malcev algebra [41] when $\mathrm{char}(\mathbb{K}) \notin \{2,3\}$. This coincides with the next solvable magma algebra

$$M^0 \cong e_1e_2 = e_2,\ e_1e_3 = -e_3,\ e_1e_4 = -e_4 \text{ and } e_3e_4 = -e_2.$$

The distribution of $\mathcal{M}_4(\mathbb{K})$ into isomorphism classes is exposed in the next result, which is again based on the classifications of De Graaf and Strade.

Theorem 2.8 *Let $\mathbb{K}$ be a finite field. Every Malcev magma algebra $A \in \mathcal{M}_4(\mathbb{K})$ is isomorphic to exactly one of the next algebras*

a) The non-Lie Malcev algebra M^0.

b) The 4-dimensional abelian Lie algebra M^1.

c) The solvable Lie algebras

- $M^2 \cong e_1e_2 = e_2$, $e_1e_3 = e_3$ *and* $e_1e_4 = e_4$.
- $M^3_{-1} \cong e_1e_2 = e_2$, $e_1e_3 = e_4$ *and* $e_1e_4 = e_3$.
- $M^3_0 \cong e_1e_2 = e_2$, $e_1e_3 = e_4$ *and* $e_1e_4 = e_4$.
- $M^4 \cong e_1e_2 = e_2$ *and* $e_1e_3 = e_2$.
- $M^5 \cong e_1e_2 = e_3$.
- $M^6_{0,0} \cong e_1e_2 = e_2$, $e_1e_3 = e_4$ *and* $e_1e_4 = e_2$.
- $M^7_{0,0} \cong e_1e_2 = e_3$ *and* $e_1e_3 = e_4$.
- $M^7_{a,0} \cong e_1e_2 = e_3$, $e_1e_3 = e_4$ *and* $e_1e_4 = ae_2$, *for all* $a \in \mathbb{K} \setminus \{0\}$.
  Here, $M^7_{a,0} \cong M^7_{b,0}$ *if and only if there exits* $\alpha \in \mathbb{K} \setminus \{0\}$ *such that* $a = \alpha^3 b$.
- $M^7_{0,a} \cong e_1e_2 = e_3$, $e_1e_3 = e_4$ *and* $e_1e_4 = ae_3$, *for all* $a \in \mathbb{K} \setminus \{0\}$.
  Here, $M^7_{0,a} \cong M^7_{0,b}$ *if and only if there exits* $\alpha \in \mathbb{K} \setminus \{0\}$ *such that* $a = \alpha^2 b$.
- $M^8 \cong e_1e_2 = e_2$ *and* $e_3e_4 = e_4$.
- $M^{11}_{1,0} \cong e_1e_2 = e_2$, $e_1e_4 = 1e_4$, $e_2e_4 = e_3$ *and* $e_3e_4 = e_2$, *whenever* $\mathrm{char}(\mathbb{K}) = 2$.
- $M^{12} \cong e_1e_2 = e_2$, $e_1e_3 = 2e_3$, $e_1e_4 = e_4$ *and* $e_2e_4 = -e_3$, *whenever* $\mathrm{char}(\mathbb{K}) \neq 2$.
- $M^{13}_0 \cong e_1e_2 = e_2$, $e_1e_3 = e_3$, $e_1e_4 = e_2$ *and* $e_2e_4 = -e_3$.

- $M_a^{14} \cong e_1e_2 = e_3,\ e_1e_3 = ae_2$ *and* $e_2e_3 = e_4$, *for all* $a \in \mathbb{K} \setminus \{0\}$.
  Here, $M_a^{14} \cong M_b^{14}$ *if and only if there exists* $\alpha \in \mathbb{K} \setminus \{0\}$ *such that* $a = \alpha^2 b$.

d) The non-solvable Lie algebras over a field of characteristic two

- $W(1;\underline{2}) \cong e_1e_2 = e_2,\ e_1e_3 = e_3,\ e_2e_3 = e_1$ *and* $e_3e_4 = e_2$.
- $W(1;\underline{2})^{(1)} \oplus Z(A) \cong e_1e_2 = e_2,\ e_1e_3 = e_3$ *and* $e_2e_3 = e_1$.

e) The non-solvable Lie algebra over a field of characteristic distinct of two

- $\mathfrak{gl}(2,\mathbb{K}) \cong e_1e_2 = e_2,\ e_1e_3 = -e_3,\ e_2e_3 = e_1,\ e_2e_4 = e_2$ *and* $e_3e_4 = -e_3$.

Proof. The classification given by De Graaf and Strade for solvable and non-solvable 4-dimensional Lie algebras over finite fields implies that all these algebras constitute distinct isomorphism classes. In order to assure that the distribution is exhaustive, it is required to prove that none of the next 4-dimensional solvable Lie algebras is isomorphic to a magma algebra

$$M_a^3 : e_1e_2 = e_2, e_1e_3 = e_4 \text{ and } e_1e_4 = -ae_3 + (a+1)e_4, \text{ with } a \notin \{0,-1\}. \tag{10}$$

$$M_{a,b}^6 : e_1e_2 = e_2 + ae_3 + be_4, e_1e_3 = e_4 \text{ and } e_1e_4 = e_2, \text{ for all } (a,b) \in \mathbb{K}^2 \setminus (0,0). \tag{11}$$

$$M_{a,a}^7 : e_1e_2 = e_3, e_1e_3 = e_4 \text{ and } e_1e_4 = ae_2 + ae_3, \text{ for all } a \in \mathbb{K} \setminus \{0\}. \tag{12}$$

$$M_{a_0}^9 : \begin{array}{c} e_1e_2 = e_2 + a_0e_3, e_1e_3 = e_2, e_2e_4 = e_2 \text{ and } e_3e_4 = e_3, \text{ whenever char}(\mathbb{K}) = 2 \text{ and} \\ a_0 \in \mathbb{K} \text{ is such that } T^2 - T - a_0 \text{ has not root in } \mathbb{K}. \end{array} \tag{13}$$

$$M_a^{13} : e_1e_2 = e_2 + ae_4, e_1e_3 = e_3, e_1e_4 = e_2 \text{ and } e_2e_4 = -e_3, \text{ for all } a \in \mathbb{K} \setminus \{0\}. \tag{14}$$

We prove this fact in Proposition 4.4. □

2.3. Computational algebraic geometry

Let X and $\mathbb{K}[X]$ be, respectively, the set of n variables $\{x_1,\ldots,x_n\}$ and the related multivariate polynomial ring over a field $\mathbb{K}$. The *algebraic set* defined by an ideal I of $\mathbb{K}[X]$ is the set $\mathcal{V}(I)$ of common zeros of all the polynomials in I. The ideal I is *zero-dimensional* if $\mathcal{V}(I)$ is finite. It is *radical* if every polynomial $f \in \mathbb{K}[X]$ belongs to I whenever there exists a natural number m such that $f^m \in I$. Let $<$ be a monomial term ordering on the set of monomials in $\mathbb{K}[X]$. The largest monomial of a polynomial f in I with respect to $<$ is its *leading monomial* $\mathrm{LM}(f)$, whose coefficient in f is the *leading coefficient* $\mathrm{LC}(f)$. The *leading term* of f is the product $\mathrm{LT}(f) = \mathrm{LC}(f)\cdot\mathrm{LM}(f)$. The ideal generated by all the leading monomials of I is the *initial ideal* $I_<$. Those monomials that are not in $I_<$ are called *standard monomials* of I. Regardless of the monomial term ordering, if the ideal I is zero-dimensional and radical, then the number of standard monomials in I coincides with the Krull dimension of the quotient ring $\mathbb{K}[X]/I$ and with the number of points of the algebraic set $\mathcal{V}(I)$.

A *Gröbner basis* of I with respect to $<$ is any subset G of polynomials in I whose leading monomials generate the initial ideal $I_<$. This is *reduced* if all its polynomials are monic and no monomial of a polynomial in G is generated by the leading monomials of the rest of polynomials in the basis. There exists only one reduced Gröbner basis of the ideal I, which becomes an optimal way to count their number of standard monomials. Its decomposition into finitely many disjoint subsets, each of them being formed by the polynomials of a triangular system of polynomial equations makes also possible to enumerate the elements of the algebraic set $\mathcal{V}(I)$ [42, 43, 44]. The reduced Gröbner basis of an ideal I can always be computed from Buchberger's algorithm [45]. Similar to the Gaussian elimination on linear systems of equations, this consists of a sequential multivariate division of polynomials, also called *reduction* or *normal form* computation, which is based on the construction of the so-called *S-polynomials* $\mathrm{S}(f,g) = \mathrm{lcm}(\mathrm{LM}(f),\mathrm{LM}(g))(f/\mathrm{LT}(f) - g/\mathrm{LT}(g))$, for all $f,g \in I$, where lcm denotes the least common multiple. In particular, a subset G of polynomials in I is a Gröbner basis if $\mathrm{S}(f,g)$ reduces to zero after division by the polynomials in G, for all $f,g \in G$. Buchberger's algorithm can be implemented on ideals defined over any base field, but its involved exact arithmetic becomes faster on finite fields. It is due to the large numbers that appear in general as intermediate coefficients during the computation of the reduced Gröbner basis and which constitute a major factor in the computational cost of the algorithm. Derived from Buchberger's algorithm, a pair of more efficient direct methods are the algorithms F_4 and F_5 [46, 47] and the algorithm *slimgb* [48]. The latter is based on F_4 and reduces the computation time and the memory usage by keeping small coefficients and short polynomials during the sequential division of polynomials. In any case, the computation of a reduced Gröbner basis is always extremely sensitive to the number of variables [49, 50].

3. Algebraic sets related to Malcev magma algebras

In this section we identify the sets $\mathcal{L}_n(\mathbb{K})$ and $\mathcal{M}_n(\mathbb{K})$ of n-dimensional Lie and Malcev magma algebras over a finite field $\mathbb{K}$ with the algebraic set of an ideal of polynomials. Let X and $\mathbb{K}[X]$ respectively be the set of n^3 variables $\{\mathfrak{c}_{ij}^k\colon\ i,j,k \leq n\}$ and its related multivariate polynomial ring over $\mathbb{K}$. Let $\mathfrak{A}$ be the n-dimensional algebra over $\mathbb{K}[X]$ with basis $\{e_1,\ldots,e_n\}$ so that

$$e_ie_j = \sum_{k=1}^{n} \mathfrak{c}_{ij}^k e_k, \tag{15}$$

for all $i,j \leq n$. Let us also consider

i. The coefficient $\mathrm{l}_{ijkl} \in \mathbb{K}[X]$ of e_l in the Jacobi identity $J(e_i,e_j,e_k)=0$, for all $i,j,k \leq n$.

ii. The coefficient $\mathrm{m}_{uijk} \in \mathbb{K}[X]$ of e_k in the Malcev identity $M(u,e_i,\ e_j)=0$, for all $u\in\mathfrak{A}$ and $i,j\leq n$.

iii. The coefficient $\mathrm{s}_{ijklm} \in \mathbb{K}[X]$ of e_m in the Sagle identity $S(e_i,e_j,e_k,e_l)=0$, for all $i,j,k,l\leq n$, whenever $\mathrm{char}(\mathbb{K}) \neq 2$.

Finally, let us define the ideal

$$I = \langle\, \mathfrak{c}_{ii}^j \mid\ i,j \leq n \,\rangle + \langle\, \mathfrak{c}_{ij}^k - \mathfrak{c}_{ji}^k \mid\ i,j,k \leq n \,\rangle + \langle\, \mathfrak{c}_{ij}^k \mathfrak{c}_{ij}^{k'} \mid\ i,j,k,k' \leq n \text{ such that } k<k' \,\rangle \subset \mathbb{K}[X]. \tag{16}$$

Theorem 3.1 *The sets $\mathcal{L}_n(\mathbb{K})$ and $\mathcal{M}_n(\mathbb{K})$ are respectively identified with the algebraic sets defined by the zero-dimensional radical ideals in $\mathbb{K}[X]$*

$$I_L = \langle\, \mathrm{l}_{ijkl}\colon\ i,j,k,l \leq n \,\rangle + I \qquad \text{and} \qquad I_M = \langle\, \mathrm{m}_{uijk}\colon\ u\in\mathfrak{A}, i,j,k\leq n \,\rangle + I.$$

If $\mathrm{char}(\mathbb{K}) \neq 2$, then the set $\mathcal{M}_n(\mathbb{K})$ is also identified with the algebraic set defined by the zero-dimensional radical ideal in $\mathbb{K}[X]$

$$I_S = \langle\, \mathrm{s}_{ijklm}\colon\ i,j,k,l,m \leq n \,\rangle + I.$$

Besides, $|\mathcal{L}_n(\mathbb{K})| = \dim_{\mathbb{K}}(\mathbb{K}[X]/I_L)$ and $|\mathcal{M}_n(\mathbb{K})| = \dim_{\mathbb{K}}(\mathbb{K}[X]/I_M)$ $(= \dim_{\mathbb{K}}(\mathbb{K}[X]/I_S)$ if $\mathrm{char}(\mathbb{K}) \neq 2)$.

Proof. We prove the theorem for Malcev magma algebras and the ideal I_M. The other cases follow similarly. Observe that each zero $(c_{11}^1,\ldots,c_{nn}^n)\in\mathcal{V}(I_M)$ constitutes the structure constants of an n-dimensional algebra A with basis $\{e_1,\ldots,e_n\}$ such that $e_ie_j=\sum_{k=1}^n c_{ij}^k e_k$ and $M(u,e_i,e_j)=0$, for all $i,j\leq n$ and $u\in A$. The generators of $I\subset I_M$ involve A to be a magma algebra such that $u^2=0$, for all $u\in A$. The linearity of $M(u,v,w)$ in the two last arguments involves A to be a Malcev algebra.

Since the base field $\mathbb{K}$ is finite, the ideal I_M is zero-dimensional and the affine variety $\mathcal{V}(I_M)$ is a finite subset of $\mathbb{K}^{n^3}$. If $\mathbb{K}$ is isomorphic to a Galois field $\mathrm{GF}(q)$, then, from Proposition 2.7 of [39], the ideal I_M is also radical, because the unique monic generator of $I_M\cap\mathbb{K}[\mathfrak{c}_{ij}^k]$ is the polynomial $(\mathfrak{c}_{ij}^k)^q-\mathfrak{c}_{ij}^k$, which is intrinsically included in each one of the ideals and is square-free. As a consequence, the number of zeros in the algebraic set $\mathcal{V}(I_M)$ coincides with the Krull dimension of the quotient ring $\mathbb{K}[X]/I_M$ over $\mathbb{K}$. □

Corollary 3.2 *Let $\mathbb{K}$ be a finite field isomorphic to the Galois field* GF(q)*. The run time that is required by the Buchberger's algorithm in order to compute the reduced Gröbner bases of the ideals I_L, I_M and I_S in Theorem 3.1 are, respectively,*

$$q^{O(n^3)} + O(n^8), \qquad \max\{3, q\}^{O(n^3)} + O(q^{2n}) \qquad \textit{and} \qquad q^{O(n^3)} + O(n^{10}).$$

Proof. The result follows straightforward from Proposition 4.1.1 exposed by Gao in [51], once we observe that all the generators of the three ideals in Theorem 3.1 are sparse in $\mathbb{K}[X]$. According to that result, the run time that is required by the Buchberger's algorithm in order to compute each reduced Gröbner basis is $q^{O(v)} + O(g^2 l)$, where v, g and l are, respectively, the number of variables, the number of generators of the corresponding ideal that are not of the form $(\mathfrak{c}_{ij}^k)^q - \mathfrak{c}_{ij}^k$ and the longest length of these generators. □

The previous result discards the use of the ideal I_M whenever char$(\mathbb{K}) \neq 2$. We have made use of the open computer algebra system for polynomial computations Singular [52] and the algorithm *slimgb* [48] to compute the reduced Gröbner bases of the ideals in Theorem 3.1 and hence, their respective algebraic sets and Krull dimensions. To this end, we have implemented four subprocedures called *Prod*, *JacobiId*, *MalcevId* and *SageId* and a main procedure called *MalcevAlg*. All of them have been included in the library *malcev.lib*, which is available online at `http://personales.us.es/raufalgan/LS/malcev.lib`. The subprocedure *Prod* outputs the list of polynomials corresponding to the coefficient of each basis vector in the product between two arbitrary vectors of the algebra A. Similar lists of polynomials are output by the rest of subprocedures. Their respective pseudocodes are exposed in Algorithms 1-4.

Algorithm 1 Polynomials related to the product of two vectors in A.

```
procedure Prod(u, v)
    for k ← 1, n do
        for i ← 1, n do
            for j ← i + 1, n do
                L_k ← L_k + (u_i v_j − u_j v_i) c_ij^k;
            end for
        end for
    end for
    return {L_1, ..., L_n}
end procedure
```

Algorithm 2 Polynomials related to the Jacobi identity in A.

```
procedure JacobiId(u, v, w)
    L1 ← Prod(Prod(u, v), w);
    L2 ← Prod(Prod(v, w), u);
    L3 ← Prod(Prod(w, u), v);
    for i ← 1, n do
        L1_i ← L1_i + L2_i + L3_i;
    end for
    return L1
end procedure
```

Algorithm 3 Polynomials related to the Malcev identity in A.

```
procedure MalcevId(u, v, w)
    L1 ← Prod(JacobiId(u, v, w), u);
    L2 ← JacobiId(u, v, Prod(u, w));
    for i ← 1, n do
        L1_i ← L1_i − L2_i;
    end for
    return L1
end procedure
```

Algorithm 4 Polynomials related to the Sagle identity in A.

procedure SagleId(u, v, w, y)
    $L1 \leftarrow Prod(Prod(Prod(u, v), w), y)$;
    $L2 \leftarrow Prod(Prod(Prod(v, w), y), u)$;
    $L3 \leftarrow Prod(Prod(Prod(w, y), u), v)$;
    $L4 \leftarrow Prod(Prod(Prod(y, u), v), w)$;
    $L5 \leftarrow Prod(Prod(u, w), Prod(v, y))$;
    for $i \leftarrow 1, n$ **do**
        $L1_i \leftarrow L1_i + L2_i + L3_i + L4_i - L5_i$;
    end for
    return $L1$
end procedure

In case of being interested in computing an algebra containing a specific structure constant $c_{ij}^k \in \mathbb{K}$, we include the polynomial $\mathfrak{c}_{ij}^k - c_{ij}^k$ in the corresponding ideal of Theorem 3.1. This aspect has been incorporated in the main procedure *MalcevAlg*, whose pseudocode is exposed in Algorithm 5. This procedure receives as input

1. The dimension n of the algebra.
2. The order q of the base field $\mathbb{K}$.
3. A list C of tuples of the form (i, j, k, c_{ij}^k), where $i, j, k \leq n$, $i < j$ and $c_{ij}^k \in \mathbb{K}$.
4. A positive integer opt1 ≤ 2 so that we use the ideal I_L if opt1 $= 1$. Otherwise, we use the ideals I_M or I_S if $\mathrm{char}(\mathbb{K}) = 2$ or $\mathrm{char}(\mathbb{K}) \neq 2$, respectively.
5. A positive integer opt2 ≤ 2 that determines as output the Krull dimension of the ideal if opt2 $= 1$. Otherwise, it determines the algebraic set defined by the ideal.

Algorithm 5 Computation of Lie and Malcev magma algebras.

procedure MalcevAlg$(n, q, C, \mathrm{opt1}, \mathrm{opt2}))$
    Depending on the argument opt1, use the subprocedures *JacobiId*, *MalcevId* or *SagleId* to initialize an ideal I as I_L, I_M or I_S, respectively.
    for $i \leftarrow 1, size(C)$ **do**
        $I = I + (\mathfrak{c}_{C_{i1}C_{i2}}^{C_{i3}} - C_{i4})$;
    end for
    $I = slimgb(I)$;
    if opt1 $= 1$ **then**
        return $|\mathcal{V}(I)|$
    else
        return $\mathcal{V}(I)$
    end if
end procedure

The correctness and termination of this main procedure is based on those of the algorithm *slimgb* [48] and hence, on those of Buchberguer's algorithm [45]. This procedure has been implemented in a system with an *Intel Core i7-2600, with a 3.4 GHz processor and 16 GB of RAM*. Its effectiveness has been checked by computing the cardinalities that are exposed in Tables 1 and 2. The run time and memory usage are explicitly indicated in both tables. Both measures of computation efficiency fit positive exponential models even for dimension $n = 4$. Nevertheless, we are not interested in the computation of all Malcev magma algebras, but only in their distribution into isomorphism and isotopism classes. We indicate now how computational algebraic geometry can also be used to determine this distribution.

Let us consider the sets of variables

$$X = \{\mathfrak{f}_{11}, \ldots, \mathfrak{f}_{nn}\} \qquad \text{and} \qquad Y = X \cup \{\mathfrak{g}_{11}, \ldots, \mathfrak{g}_{nn}, \mathfrak{h}_{11}, \ldots, \mathfrak{h}_{nn}\}.$$

q	$\lvert\mathcal{M}_3(\mathrm{GF}(q))\rvert$	Run time I_L	Run time $I_{M/S}$	Used memory I_L	Used memory $I_{M/S}$
2	32	0 s	0 s	0 MB	0 MB
3	123	0 s	0 s	0 MB	0 MB
5	581	0 s	0 s	0 MB	0 MB
7	1,567	0 s	0 s	0 MB	0 MB
11	5,891	0 s	0 s	0 MB	0 MB
13	9,613	0 s	0 s	0 MB	0 MB
17	21,137	0 s	0 s	0 MB	0 MB
⋮	⋮	⋮	⋮	⋮	⋮
499	498,492,523	0 s	0 s	0 MB	0 MB

Table 1. Computation of $|\mathcal{M}_3(\mathrm{GF}(q))|$.

q	$\lvert\mathcal{L}_4(\mathrm{GF}(q))\rvert$	Run time	Used memory	$\lvert\mathcal{M}_4(\mathrm{GF}(q))\rvert$	Run time	Used memory
2	853	0 s	0 MB	897	6 s	0 MB
3	7,073	1 s	0 MB	7,073	8 s	0 MB
5	89,185	11 s	0 MB	89,377	41 s	0 MB
7	445,537	20 s	3 MB	445,969	55 s	2 MB
11	3,803,041	91 s	17 MB	3,804,241	154 s	427 MB
13	8,412,193	183 s	676 MB	8,413,921	258 s	859 MB
17	30,247,297	595 s	1,2 GB	30,250,369	752 s	1,5 GB

Table 2. Computation of $|\mathcal{L}_4(\mathrm{GF}(q))|$ and $|\mathcal{M}_4(\mathrm{GF}(q))|$.

These variables play the role of the entries in the nonsingular matrices F, G and H related to a possible isotopism (f, g, h) between two Malcev magma algebras of respective bases $\{e_1, \ldots, e_n\}$ and $\{e'_1, \ldots, e'_n\}$. Let $\mathfrak{p}_{ijk}$ and $\mathfrak{q}_{ijk}$ be the respective polynomials in $\mathbb{K}[X]$ and $\mathbb{K}[Y]$ that constitutes the coefficients of e_k in the expressions $f(e_i)f(e_j) - f(e_ie_j)$ and $f(e_i)g(e_j) - h(e_ie_j)$, for all $i, j, k \leq n$. The next result holds analogously to Theorem 3.1.

Theorem 3.3 *Let $\mathbb{K}$ be a finite field isomorphic to the Galois field $\mathrm{GF}(q)$ and let A and A' be two n-dimensional Malcev magma algebras over $\mathbb{K}$. The sets of isomorphisms and isotopisms between both algebras are respectively identified with the algebraic sets defined by the zero-dimensional radical ideals*

$$I_{A,A'} = \langle\, \mathfrak{p}_{ijk} : \ i, j, k \leq n \,\rangle + \langle\, \det(F)^{q-1} - 1 \,\rangle \subset \mathbb{K}[X].$$

$$J_{A,A'} = \langle\, \mathfrak{q}_{ijk} : \ i, j, k \leq n \,\rangle + \langle\, \det(F)^{q-1} - 1, \det(G)^{q-1} - 1, \det(H)^{q-1} - 1 \,\rangle \subset \mathbb{K}[Y].$$

Besides, $|\mathcal{V}(I_{A,A'})| = \dim_{\mathbb{K}}(\mathbb{K}[X]/I_{A,A'})$ $|\mathcal{V}(J_{A,A'})| = \dim_{\mathbb{K}}(\mathbb{K}[Y]/J_{A,A'})$. □

We have included the procedure *isoMalcev* in the previously mentioned library *malcev.lib* in order to compute the algebraic sets and the Krull dimensions in Theorem 3.3. Having as output the number of isomorphisms or that of isotopisms between two Malcev magma algebras A and A', this procedure receives as input

1. The dimension n of both algebras.
2. The order q of the base field.
3. A list $L1$ formed by tuples (i, j, k, c_{ij}^k) that indicates the non-zero structure constants of the Malcev algebra A.
4. A list $L2$ formed by tuples (i, j, k, c'^k_{ij}) that indicates the non-zero structure constants of the Malcev algebra A'.
5. A positive integer opt ≤ 2 that enable us to use the ideal $I_{A,A'}$ if opt $= 1$, or the ideal $J_{A,A'}$ if opt $= 2$.

Algorithm 6 shows how the procedure *isoMalcev* is implemented to determine the distribution of a set of Malcev magma algebras into isomorphism and isotopism classes.

Algorithm 6 Computation of the isomorphism (isotopism, resp.) classes of a set of Malcev algebras.

Require: A set M of Malcev magma algebras.
Ensure: S, the set of isomorphism (isotopism, resp.) classes of S

$S = \emptyset$.
while $M \neq \emptyset$ **do**
  Take $A \in M$.
  $M := M \setminus \{A\}$.
  $S := S \cup \{A\}$.
  for $A' \in M$ **do**
    if $|\mathcal{V}(I_{A,A'})| > 0$ ($|\mathcal{V}(J_{A,A'})| > 0$, resp.) **then**
      $M := M \setminus \{A'\}$.
    end if
  end for
end while
return S.

4. Classification of Malcev magma algebras of small dimensions

We have implemented the procedures and algorithms of the previous section to determine the distribution of 3- and 4-dimensional Malcev magma algebras into isomorphism and isotopism classes.

4.1. 3-dimensional Malcev magma algebras

The next result finishes the proof of Theorem 2.7 about the distribution of $\mathcal{M}_3(\mathbb{K})$ into isomorphism classes.

Proposition 4.1 *None of the algebras that are described in (9) is isomorphic to a Lie magma algebra.*

Proof. Let $a \in \mathbb{K} \setminus \{0\}$. Let f be an isomorphism between the algebra L^3_a and a Malcev magma algebra A and let $F = (f_{ij})$ be the nonsingular matrix related to f. This algebra A should have a 2-dimensional derived algebra and its solvability index should be 2. The implementation of Algorithm 5 into a case study that takes into account Lemma 2.6 and the Jacobi identity involves A to be isomorphic to one of the next two magma algebras

a) $e_1e_2 = e_3$ and $e_1e_3 = \alpha e_2$.

b) $e_1e_2 = e_2$ and $e_1e_3 = \alpha e_3$.

In both cases, $\alpha \in \mathbb{K} \setminus \{0\}$. The implementation of the procedure *isoMalcev* enable us to compute for each case the corresponding reduced Gröbner basis in Theorem 3.3. In (a), the normal form of the polynomial that is related to the determinant of the corresponding matrix F modulo this reduced Gröbner basis is zero whatever the number α and the characteristic of the base field are. This means that F is a singular matrix, which is a contradiction with being f an isomorphism. Hence, L^3_a is not isomorphic to a Malcev magma algebra of type (a). On the other hand, in (b), the previous normal form is the polynomial $(1-\alpha)f_{11}f_{32}f_{33}$. The regularity of the matrix F involves that $\alpha \neq 1$. Besides, the reduced Gröbner basis involves the identity $a\alpha = a$. Since $\alpha \neq 1$, it must be $a = 0$, what is a contradiction with the hypothesis. Hence, L^3_a is not isomorphic either to a Malcev magma algebra of type (b). □

Even if not every 3-dimensional Malcev algebra is isomorphic to a Lie magma algebra, the next result shows that this statement is true in case of dealing with isotopisms instead of isomorphisms.

Proposition 4.2 *Every 3-dimensional Malcev algebra is isotopic to a Lie magma algebra.*

Proof. The result holds from the fact that, for all $a \in \mathbb{K} \setminus \{0\}$, the Lie algebras L^3_a and L^2 are isotopic by means of the isotopism $(\mathrm{Id}, \mathrm{Id}, h)$, where $h(e_1) = e_1$, $h(e_2) = e_3$ and $h(e_3) = -\frac{1}{a}(e_3 - e_2)$. □

The next result indicates the distribution of 3-dimensional Malcev magma algebras over a finite field into isotopism classes. In the proof and hereafter, any linear map between two n-dimensional algebras that is linearly defined from a permutation of the indices of their basis vectors is denoted by the corresponding permutation of the symmetric group S_n. Thus, for instance, given two 3-dimensional algebras of respective bases $\{e_1, e_2, e_3\}$ and $\{e'_1, e'_2, e'_3\}$, the permutation $(23) \in S_3$ maps e_1 to e'_1, e_2 to e'_3 and e_3 to e'_2. Besides, Id denotes the trivial linear map between algebras of the same dimension.

Theorem 4.3 *Let $\mathbb{K}$ be a finite field. There exist four isotopism classes of $\mathcal{M}_3(\mathbb{K})$. They correspond to the abelian algebra and the algebras L^2, L^3_0 and*

- $W(1;\underline{2})^{(1)}$ *if* $\mathrm{char}(\mathbb{K}) = 2$.
- $\mathfrak{sl}(2, \mathbb{K})$, *otherwise.*

Proof. From Proposition 4.2, it is enough to study the distribution into isotopism classes of the Lie algebras in Theorem 2.7. Particularly, from Lemma 2.2, the algebras $W(1;\underline{2})^{(1)}$ and $\mathfrak{sl}(2, \mathbb{K})$, with a 3-dimensional derived algebra, are not isotopic to any other Malcev magma algebra of a distinct isomorphism class. Similarly, the algebra L^2, with a 2-dimensional derived algebra, can only be isotopic to L^4_a, with $a \in \mathbb{K} \setminus \{0\}$; whereas the algebra L^3_0, with an 1-dimensional derived algebra, can only be isotopic to L^4_0. Specifically, it is straightforward verified that the triple $(\mathrm{Id}, \mathrm{Id}, h)$ such that $h(e_2) = e_3$ and $h(e_3) = ae_2$ is an isotopism between L^2 and L^4_a, whereas the triple $(\mathrm{Id}, \mathrm{Id}, (23))$ is an isotopism between L^3_0 and L^4_0. □

4.2. *4-dimensional Malcev magma algebras*

The next result concludes the proof of Theorem 2.8 about the distribution of 4-dimensional Malcev magma algebras into isomorphism classes.

Proposition 4.4 *None of the 4-dimensional Lie algebras that are described in (10–14) is isomorphic to a Malcev magma algebra.*

Proof. The implementation of Algorithm 5 into a case study that takes into account the derived series and centers of a Malcev algebra, together with the Jacobi identity, enable us to assure that any possible 4-dimensional Malcev magma algebra with a 2-dimensional derived algebra and solvability index 2 is isomorphic to exactly one of the algebras M^3_0, $M^6_{0,0}$, $M^7_{0,a}$ (with $a \in \mathbb{K}$) or M^{13}_0. Hence, the Lie algebras $M^6_{0,b}$ (with $b \in \mathbb{K} \setminus \{0\}$) and $M^9_{a_0}$ are not isomorphic to any Malcev magma algebra. Further, a similar case study enable us to assure that none of the algebras M^3_a, $M^6_{a,b}$, $M^7_{a,a}$ and M^{13}_a, with $a, b \in \mathbb{K}$ such that $a \neq 0$, is isomorphic to a Malcev magma algebra. This is due to the fact that any possible 4-dimensional Malcev magma algebra with a 3-dimensional derived algebra is isomorphic to one of the algebras M^2, M^3_{-1} or $M^7_{a,0}$ if its solvability index is 2 or to $M^{11}_{1,0}$, M^{12} or M^{14}_a if its solvability index is 3. In both cases, $a \in \mathbb{K} \setminus \{0\}$. □

We finish our study by focusing on the distribution of 4-dimensional Malcev magma algebras into isotopism classes.

Proposition 4.5 *Every 4-dimensional Malcev algebra over a finite field $\mathbb{K}$ of characteristic distinct of two is isotopic to a Malcev magma algebra. If $\mathrm{char}(\mathbb{K}) = 2$, then this assertion holds except for those Malcev algebras that are isomorphic to the Malcev algebra $M^9_{a_0}$.*

Proof. Let $\mathbb{K}$ be a finite field. The implementation of the procedure *isoMalcev* in Algorithm 6 gives us the next isotopisms among the Malcev magma algebras described in Theorem 2.8 and those in (10–14). For each pair of isotopic algebras we show an isotopism (f, f, h), which is described by means of those basis vectors that are not preserved by the transformations f and h.

a) $M^2 \simeq M^3_a \simeq M^6_{b,c}$, for all $a, b, c \in \mathbb{K}$ such that $a \notin \{0, -1\}$ and $b \neq 0$. Here,

- The triple $(\mathrm{Id}, \mathrm{Id}, h)$ such that $h(e_3) = e_4$ and $h(e_4) = -ae_3 + (a + 1)e_4$ is an isotopism between M^2 and M^3_a, for all $a \in \mathbb{K} \setminus \{0, -1\}$.

- The triple $(\mathrm{Id}, \mathrm{Id}, h)$ such that $h(e_2) = e_2 + be_3 + ce_4$, $h(e_3) = e_4$ and $h(e_4) = e_2$ is an isotopism between M^2 and $M^6_{b,c}$, for all $b, c \in \mathbb{K}$ such that $b \neq 0$.

b) $M^0 \simeq M^{13}_a$, for all $a \in \mathbb{K} \setminus \{0\}$. Here, the triple (f, f, h) such that $f(e_2) = h(e_2) = e_3$, $f(e_3) = -e_4$, $f(e_4) = e_2 - e_4$, $h(e_3) = e_2$ and $h(e_4) = -ae_4$ is an isotopism between M^0 and M^{13}_a, for all $a \in \mathbb{K} \setminus \{0\}$.

A case study based on the same implementation of the procedure *isoMalcev* also enables us to assure that, if $\mathrm{char}(\mathbb{K}) = 2$, then the Malcev algebra $M^9_{a_0}$ is not isotopic to any Malcev magma algebra with a 2-dimensional derived algebra and solvability index 3. □

The proof of Proposition 4.5 enables us to assure that isotopisms do not preserve Jacobi anomalies. Specifically, we have proved that the non-Lie Malcev algebra M^0 is isotopic to any Lie algebra M^{13}_a, with $a \in \mathbb{K} \setminus \{0\}$. The next result holds straightforward.

Theorem 4.6 *Any 4-dimensional Malcev algebra is isotopic to a Lie algebra.* □

The distribution of 4-dimensional Malcev algebras into isotopism classes is exposed in the next final result.

Theorem 4.7 *Let $\mathbb{K}$ be a finite field. There exist eight isotopism classes of $\mathcal{M}_3(\mathbb{K})$. They correspond to the abelian algebra and the algebras M^0, M^2, M^3_0, M^4, M^8, M^{13}_0 and M^{14}_1.*

Proof. From Proposition 4.5, it is enough to study the distribution into isotopism classes of the Lie magma algebras in Theorem 2.8. This distribution has been obtained again from the implementation of the procedure *isoMalcev* in Algorithm 6 together with the results exposed in Subsection 2.1. For each pair of isotopic algebras we show an isotopism (f, f, h), which is described by means of those basis vectors that are not preserved by the transformations f and h. Besides, each class is described according to the dimension of their derived algebras and centers and the isotopism invariants described in (6-7).

a) $\dim(\mathcal{C}_2(M)) = 1$: $M^4 \simeq M^5$. Here, the triple $(f, f, (23))$ such that $f(e_3) = e_2 + e_3$ is an isotopism between both algebras.

b) $\dim(\mathcal{C}_2(M)) = 2$:

- $\dim(Z(M)) = 0$:
  - $d_3(M) = 0$: M^{13}_0.
  - $d_3(M) = 1$: M^8.
- $\dim(Z(M)) = 1$: $M^3_0 \simeq M^6_{0,a} \simeq M^7_{0,b}$, for all $a, b \in \mathbb{K}$ such that $b \neq 0$. Here,
  - The triple (f, f, h) such that $f(e_4) = -e_2 + e_3 + e_4$ and $h(e_2) = e_2 - ae_4$ is an isotopism between M^3_0 and $M^6_{0,a}$, for all $a \in \mathbb{K}$.
  - The triple $(f, f, (23))$ such that $f(e_4) = e_3 + e_4$ is an isotopism between M^3_0 and $M^7_{0,b}$, for all $b \in \mathbb{K} \setminus \{0\}$.

c) $\dim(\mathcal{C}_2(M)) = 3$:

- $\dim(Z(M)) = 0$:
  - $D_3(M) = 1$: If $\mathrm{char}(\mathbb{K}) = 2$, then $M^0 \simeq M^{11}_{1,0} \simeq W(1; \underline{2})$. Otherwise, $M^0 \simeq M^{12}$. Here,
    - If $\mathrm{char}(\mathbb{K}) = 2$, then the triple $((14)(23), (14)(23), \mathrm{Id})$ is an isotopism between M^0 and $M^{11}_{1,0}$, whereas the triple $((1324), (1324), (143))$ is an isotopism between M^0 and $W(1; \underline{2})$.
    - If $\mathrm{char}(\mathbb{K}) \neq 2$, then the triple (f, f, h) such that $f(e_2) = e_3$, $f(e_3) = -2e_4$, $f(e_4) = e_2 + e_4$, $h(e_2) = 2e_3$, $h(e_3) = 2e_4$ and $h(e_4) = -e_2 - e_4$ is an isotopism between M^0 and M^{12}.
  - $D_3(M) = 3$: $M^2 \simeq M^3_{-1} \simeq M^7_{a,a} \simeq M^7_{c,0}$, for all $a, c \in \mathbb{K} \setminus \{0\}$. Here,
    - The triple $(\mathrm{Id}, \mathrm{Id}, (34))$ is an isotopism between M^2 and M^3_{-1}.
    - The triple $(\mathrm{Id}, \mathrm{Id}, h)$ such that $h(e_2) = e_3$, $h(e_3) = e_4$ and $h(e_4) = ae_2 + ae_3$ is an isotopism between M^2 and $M^7_{a,a}$, for all $a \in \mathbb{K} \setminus \{0\}$.
    - The triple $(\mathrm{Id}, \mathrm{Id}, h)$ such that $h(e_2) = e_3$, $h(e_3) = e_4$ and $h(e_4) = ce_2$ is an isotopism between M^2 and $M^7_{c,0}$, for all $c \in \mathbb{K} \setminus \{0\}$.

- $\dim(Z(M)) = 1$: $M_1^{14} \simeq M_a^{14}$, for all $a \in \mathbb{K} \setminus \{0\}$. If $\mathrm{char}(\mathbb{K}) = 2$, then $M_1^{14} \simeq W(1;\underline{2})^{(1)} \oplus Z(L)$. Otherwise, $M_1^{14} \simeq \mathfrak{gl}(2,\mathbb{K})$. Here,
  - The triple $(\mathrm{Id}, \mathrm{Id}, h)$ such that $h(e_2) = ae_2$ is an isotopism between M_1^{14} and M_a^{14}, for all $\in \mathbb{K} \setminus \{0\}$.
  - If $\mathrm{char}(\mathbb{K}) = 2$, then the triple $(\mathrm{Id}, \mathrm{Id}, (1432))$ is an isotopism between M_1^{14} and $W(1;\underline{2})^{(1)} \oplus Z(L)$.
  - If $\mathrm{char}(\mathbb{K}) \neq 2$, then the triple (f, f, h) such that $f(e_4) = e_1 + e_4$, $h(e_1) = e_4$, $h(e_2) = -e_3$, $h(e_3) = e_2$ and $h(e_4) = e_1$ is an isotopism between M_1^{14} and $\mathfrak{gl}(2,\mathbb{K})$. □

5. Conclusions and further studies

We have defined in this paper distinct zero-dimensional radical ideals whose related algebraic sets are uniquely identified with the set $\mathcal{M}_n(\mathbb{K})$ of n-dimensional Malcev magma algebras over a finite field $\mathbb{K}$. The computation of their reduced Gröbner bases together with the classification of Lie algebras over finite fields given by De Graaf and Strade have enabled us to determine the distribution of $\mathcal{M}_3(\mathbb{K})$ and $\mathcal{M}_4(\mathbb{K})$ not only into isomorphism classes, which is the usual criterion, but also into isotopism classes. Particularly, we have proved the existence of four isotopism classes in $\mathcal{M}_3(\mathbb{K})$ and eight isotopism classes in $\mathcal{M}_4(\mathbb{K})$. Besides, we have proved that every 3-dimensional Malcev algebra over any finite field and every 4-dimensional Malcev algebra over a finite field of characteristic distinct of two is isotopic to a Lie magma algebra. Keeping in mind the obtained results, the study of magma algebras by means of computational algebraic geometry constitutes a good first approach to the distribution of Malcev algebras over finite fields into isomorphism and isotopism classes. In this regard, the study of the sets $\mathcal{M}_5(\mathbb{K})$ and $\mathcal{M}_6(\mathbb{K})$ is established as a further work that complements the already known classification of 5- and 6-dimensional Malcev algebras in case of non-solvability [14] and nilpotency [15].

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