Lagrangian translators under mean curvature flow

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Abstract. We provide a new construction of Lagrangian surfaces in $\mathbb{C}^2$ in terms of two planar curves. When we take such curves as appropriate solutions of the curve shortening problem, including self-shrinking and self-expanding curves or spirals, we will obtain translating solitons which generalize the Joyce, Lee and Tsui ones in dimension two [6]. Finally, we characterize locally all examples in terms of an analytical condition on the Hermitian product of the position vector of the immersion and the translating vector that allows us *separation of varia-bles.* As a consequence we get the classification of the Hamiltonian stationary Lagrangian translating solitons for Lagrangian mean curvature flow in complex Euclidean plane. This work is based in [1].

1.   Introduction

The mean curvature flow (in short MCF) is one of the most important geometric evolution equations of submanifolds in Geometric Analysis. A family of smooth submanifolds $F = F(\cdot, t)$ evolves under the MCF if the speed $\frac{dF}{dt}$ at each point of the submanifold is given by the mean curvature vector at that point. Hence, the MCF is an evolution process under which a submanifold deforms in the direction of its mean curvature vector.

There are very interesting results on regularity, global existence and convergence of the MCF in several ambient spaces. When the ambient space is Euclidean, the MCF turns out to be the solution to a system of parabolic equations which can be considered as the heat equation for submanifolds. We first fix an immersion, which plays the role of the initial condition, and once the existence and uniqueness of solutions of the MCF are guaranteed in a maximal time interval $[0, T)$, the behavior of the MCF is studied by the evolution of the immersed submanifolds when $t \to T$. Unless the flow has an eternal solution (i.e., it is defined for all t), the MCF fails to exist after a finite time, giving rise to a singularity. This behavior appears, for instance, when the submanifold is compact in the Euclidean ambient space.

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A natural question is to understand the geometric and analytic nature of these singularities. As a first approximation, the singularities of the MCF are classified depending on the blow-up rate of the second fundamental form σ (see [5]). The so-called Type I singularities are those such that the second fundamental form blow-up is best controlled; the remaining singularities are known as Type II singularities. It is interesting to mention that there are many similarities between the Ricci flow singularities and the MCF singularities. In fact, in both flows the singularities are often modelled by soliton solutions.

Until the mid-nineties most authors studying MCF only considered hypersurfaces, whereas MCF in higher codimension did not play a fundamental role. Nevertheless, in the last few years, the MCF in higher codimension has attracted special attention, mainly when the initial submanifold is Lagrangian in complex Euclidean space $\mathbb{C}^n$. This is due to the fact that the Lagrangian condition is preserved by MCF (see [8]).

Wang [9] and Chen and Li [3] proved independently that there is no Type I singularity along the almost calibrated Lagrangian MCF. Therefore it is of great interest to understand dilations of the flow where the point at which we center the dilation changes with the scale, called Type II dilations, which converge to an eternal solution with second fundamental form uniformly bounded. One of the most important examples of Type II singularities is a class of eternal solutions known as translating solitons, which are surfaces which evolving by translations with constant velocity.

2.   Lagrangian translating solitons

An immersion $\phi : M \to \mathbb{R}^4$ is called a translating soliton for MCF if

$$H = \mathbf{e}^{\perp},$$

where $\mathbf{e}$ is a fixed nonzero vector which indicates the direction of the translation, $\mathbf{e}^{\perp}$ denotes the normal projection of the vector $\mathbf{e}$, and H is the mean curvature vector of ϕ. By scaling and choosing a suitable coordinate system in $\mathbb{R}^4 \equiv \mathbb{C}^2$, we can assume that $\mathbf{e} = (1, 0) \in \mathbb{C}^2$ without loss of generality.

The simplest examples of Lagrangian surfaces in $\mathbb{C}^2$ are usually found as product of planar curves. If we look for translating solitons for MCF in this family, we obtain the product of a grim-reaper curve and a straight line and the product of two grim-reaper curves. Recall that the grim-reaper curve is the graph of $-\log \cos y$.

The first results in this direction are due to Neves and Tian [7], who gave conditions that exclude the existence of nontrivial translating solitons to Lagrangian MCF. More precisely, they proved that translating solitons with a

L^2-bound on the mean curvature vector are planes, and almost calibrated translating solitons which are static are also planes. Nevertheless, Joyce, Lee and Tsui found out in [6] new surprising translating solitons for Lagrangian MCF with oscillation of the Lagrangian angle arbitrarily small. They play the same role as the cigar solitons in Ricci flow and are important in the study of the regularity of Lagrangian MCF.

2.1.   New examples of Lagrangian translating solitons

Let $\alpha = \alpha(t) \subset \mathbb{C} \setminus \{0\}$, $t \in I_1$, and $\omega = \omega(s) \subset \mathbb{C} \setminus \{0\}$, $s \in I_2$, be regular planar curves, where I_1 and I_2 are intervals of $\mathbb{R}$. For any $t_0 \in I_1$ and $s_0 \in I_2$, let us define

$$\Phi = \alpha * \omega : I_1 \times I_2 \subset \mathbb{R}^2 \to \mathbb{C}^2 = \mathbb{C} \times \mathbb{C},$$

$$\Phi(t,s) = \left(\int_{s_0}^{s} \dot\omega(y)\overline{\omega(y)}dy - \int_{t_0}^{t} \alpha'(x)\overline{\alpha(x)}dx \,,\, \alpha(t)\omega(s) \right), \qquad (1)$$

where $'$ and $\dot{}$ denote the derivatives respect to t and s, respectively. Then, Φ is a Lagrangian immersion (more information about this construction can be found in [2]).

Lemma 2.1. *Let α be a unit speed planar curve. Assume there exist $a, b \in \mathbb{R}$, not vanishing simultaneously, such that the curvature function κ_α of α satisfies*

$$\kappa_\alpha = a\langle \alpha, J\alpha' \rangle + b\langle \alpha, \alpha' \rangle$$

where $'$ denotes derivative with respect to the arc parameter of α. Then the family of curves $\alpha_t = \sqrt{2at+1}\, e^{i\frac{b}{2a}\log(2at+1)}\, \alpha$, with $2at+1 > 0$, is a solution to the curve shortening flow (CSF)

$$\left(\frac{\partial}{\partial t} \alpha_t \right)^{\perp} = \overrightarrow{\kappa_{\alpha_t}}$$

such that $\alpha_0 = \alpha$.

In the limit cases, $b = 0$ and $a \to 0$ we recover the well-known solutions to the CSF. If $b = 0$, we obtain that $\overrightarrow{\kappa_\alpha} = a\,\alpha^{\perp}$, that is, α is a *self-similar solution* to the CSF, *self-shrinking* or *self-expanding* according to $a < 0$ or $a > 0$, respectively. In particular, the flow $\alpha_t = \sqrt{2at+1}\,\alpha$ is given by dilations of α. When $a \to 0$, we get that $\overrightarrow{\kappa_\alpha} = b\,(J\alpha)^{\perp}$ so α is a spiral solution to the CSF with velocity $|b|$, under the flow $\alpha_t = e^{i\,bt}\alpha$ is given by rotations of α in this case. The properties of these curves have been studied in [4].

Taking the curves as in Lemma 2.1 with $a = \mp\cos\varphi$ and $b = \pm\sin\varphi$ for a given $\varphi \in [0, \pi)$ in (1), we obtain that $\alpha * \omega$ is a Lagrangian translating soliton for

mean curvature flow with translating vector $(1,0) \in \mathbb{C}^2$, whose induced metric is $(|\alpha|^2 + |\omega|^2)(dt^2 + ds^2)$ and its Lagrangian angle map is $\arg \alpha' + \arg \dot{\omega} + \pi + \varphi$.

If we focus on the case $\varphi = 0$, we obtain the following result:

Corollary 2.1. *Let α and ω be self-similar solutions for the CSF satisfying $\overrightarrow{\kappa_\alpha} = -\alpha^\perp$ and $\overrightarrow{\kappa_\omega} = \omega^\perp$. Then $\alpha * \omega : I_1 \times I_2 \subset \mathbb{R}^2 \to \mathbb{C}^2$ given by*

$$(\alpha * \omega)(t,s) = \left(\frac{|\omega(s)|^2 - |\alpha(t)|^2}{2} - i(\arg \alpha'(t) + \arg \dot{\omega}(s)), \alpha(t)\omega(s) \right) \qquad (2)$$

is a Lagrangian translating soliton for the MCF with translating vector $(1,0) \in \mathbb{C}^2$. By considering the straight lines $\alpha_0(t) = t$ and $\omega_0(s) = s$, the circle $\alpha_1(t) = e^{it}$, self-shrinking curves $\alpha_{\mathcal{S}}$ and self-expanding curves $\omega_{\mathcal{E}}$, we obtain up the following particular examples:

(i)

$$(\alpha_0 * \omega_{\mathcal{E}})(t,s) = \left(\frac{|\omega_{\mathcal{E}}(s)|^2}{2} - i \arg \dot{\omega}_{\mathcal{E}}(s) - \frac{t^2}{2}, \, t\,\omega_{\mathcal{E}}(s) \right),$$

which correspond to the Joyce, Lee and Tsui examples (see [6]);

(ii)

$$(\alpha_1 * \omega_{\mathcal{E}})(t,s) = \left(\frac{|\omega_{\mathcal{E}}(s)|^2}{2} - i \arg \dot{\omega}_{\mathcal{E}}(s) - it, \, e^{it}\omega_{\mathcal{E}}(s) \right),$$

for which ∂_t is a Killing vector field;

(iii)

$$(\alpha_{\mathcal{S}} * \omega_0)(t,s) = \left(\frac{s^2}{2} - \frac{|\alpha_{\mathcal{S}}(t)|^2}{2} - i \arg \alpha'_{\mathcal{S}}(t), \, \alpha_{\mathcal{S}}(t)s \right),$$

which satisfies that its Lagrangian angle map is the angle that the tangent vector $\alpha'_{\mathcal{S}}(t)$ makes with a fixed direction.

2.2. Classification of Separable Lagrangian Translating Solitons

In this section, we characterize locally all examples in terms of an analytical condition on the Hermitian product of the position vector of the immersion and the translating vector that allows us *separation of variables* (see [1, Theorem 4.1]).

Theorem 2.1. *Let $\phi : M^2 \to \mathbb{C}^2$ be a Lagrangian translating soliton to the MCF with translating vector $\mathbf{e}$. Assume that there exists a local isothermal coordinate $z = x + iy$ such that the smooth complex function $(\phi, \mathbf{e})$ satisfies $\frac{\partial^2}{\partial x \partial y}(\phi, \mathbf{e}) = 0$. Then ϕ is –up to dilations– locally congruent to some of the following:*

(i) *the product of a grim-reaper curve and a straight line;*

(ii) *the product of two grim-reaper curves;*

(iii) *the example $\alpha_1 * \alpha_2$ for some $\varphi \in [0, \pi)$.*

As a consequence, we get that if $\phi : M \to \mathbb{C}^2$ is a Hamiltonian stationary (non totally geodesic) Lagrangian translating soliton for mean curvature flow (note that *Hamiltonian stationary* Lagrangian means that ϕ is a critical point of the area functional among all Hamiltonian deformations), then $\phi(M)$ is –up to dilations– an open subset of the Lagrangian

$$\mathcal{M} := \left\{ (z, w) \in \mathbb{C}^2 : w^2 = 2 \operatorname{re} z \, e^{-2\,i\,\operatorname{im} z}, \ \operatorname{re} z \geq 0 \right\}.$$

It corresponds to the simplest nontrivial choice election of α_1 (the circle $\alpha_1(t) = e^{it}$) and α_2 (the line $\alpha_2(s) = s$) in the particular case $\varphi = 0$.

Acknowledgment

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