Some advances in the research on Lie algebras

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Abstract: The main goal of this poster, which is written in the form of a survey and tries to show some aspects of the research of authors on Lie algebras, is to pay homage to the memory of **Pilar Pisón Casares**, who was firstly teacher of some of them and later colleague of all of them during different stages of her stay as a member of the Departamento de Álgebra, Computación, Geometría y Topología de la Universidad de Sevilla.

1 Introduction

This poster, dedicated to the memory of Pilar Pisón Casares, shows the achievements obtained by authors during the time in which they coincide with her at the Faculty of Mathematics in the University of Seville, when they did research on Lie algebras in general and on filiform Lie algebras, in particular. Such results, firstly got in the Ph. D. Thesis of each of them, were later lengthened and improved to be published in International Mathematical Reviews.

From now on, Lie algebras, solvable Lie algebras, nilpotent Lie algebras and filiform Lie algebras will be denoted by LA, SLA, NLA and FLA, respectively.

2 Filiform Lie algebras

At the best of our knowledge, the problem of classifying all the different types of Lie algebras is still unsolved. As it is already well-known, there exist three different types of LA: the semi-simple, the solvable and those which are neither semi-simple nor solvable. So, to know the classification of LA, in general, is equivalent to know the classification of each of these three types.

However, by the Levi-Maltsev theorem [18] in 1945, any finite-dimensional LA over a field of characteristic zero can be expressed as a semidirect sum of a semi-simple subalgebra and its radical (which is solvable). It reduces the task of classifying all LA to obtain the classification of semi-simple and of SLA.

The classification of semi-simple LA is completely solved, due to the works by Killing, Cartan and others (last decade of the 19th century).

With respect to the SLA, NLA and FLA, it is only known the explicit classification of them up to dimensions 12, 6 and 7, respectively. It was really M. Vergne, in 1966, who gave a great impulse to the study of filiform LA in her Ph. D. Thesis, later published in [21]. Indeed, Vergne showed that the variety of NLA of given dimension n has an irreducible component of dimension exceeding n^2, consisting entirely of LA of maximal class.

By continuing the works by Ancochea and Goze, who classified 8-dimensional FLA and 7-dimensional NLA, (see [1, 2], respectively), one of the first achievements of one of the authors was to introduce two invariants of FLA which allowed him to classify this type of algebras in dimensions $10, 11$ and 12. The main properties of such invariants, denoted for the first time by i and j, respectively, can be checked in [4], later extended and improved in [7].

3 c-Graded filiform Lie algebras

By considering the usual gradation of the second space of Chevalley-Eilemberg's cohomology of the model FLA of dimension n, P_n, given by $H^2(P_n,P_n) = \oplus_{c\geq 0} H^2_{c+1}(P_n,P_n)$ and by taking into account that every FLA is isomorphic to $(P_n)_\psi$, where $\psi \in H^2(P_n,P_n)$, it is possible to consider c-graded LA as algebras isomorphic to $(P_n)_{\psi_c}$, with $\psi_c \in H^2_{c+1}(P_n,P_n)$ (see [17]).

In [5] we obtain general results related c-graded FLA particularized to establish the classification of 3-graded FLA. The classification of c-graded LA allows, for one thing, to settle the classification of FLA of the type $(P_n)_{\psi_c+\psi_k}$ and, for another, to progress in the knowledge of the structure of $(P_n)_\psi$, because by considering the graduation, $\psi = \psi_c + \psi_{c+1} + \cdots + \psi_k$ then ψ_c and ψ_k have to be cocycles such that $(P_n)_{\psi_c}$ and $(P_n)_{\psi_k}$ are c-graded LA. Finally, we obtain that fixed $c \geq 2$, there exists $m = m(c) \in \mathbb{N}$ such that every c-graded LA of dimension $n \geq m$ is a derived algebra of a rigid SLA of dimension $n+1$.

For each dimension, it is possible to associate the space $H^2(P_n,P_n)$ with an affine algebraic variety. Polynomials defining this variety are preserved when the dimension increases, although the number of them certainly increases. In fact, all polynomials defining these varieties are homogeneous of degree two and when increasing the degree by one, the only polynomials which are added are linear in the new variables. This suggests that each component of the variety of FLA of dimension $n+1$ may be a bundle over the corresponding component of the variety of dimension n with affine fibres (see [17]).

In [6], we describe a constructive way to find a small set of polynomials which define this variety. It allows to improve previous results related with the cardinal of this set. We also describe an algorithm to determine in a computational way the variety of these LA. It is suitable for being used in the case of bigger dimensions and can be easily implemented in any symbolic computational package. Concretely, we have used MAPLE in our study.

4 Lie groups of filiform Lie algebras

Lie's Third Theorem sets a uniquely correspondence between simply connected Lie groups and their associated Lie algebras. The accustomed proof in the literature is based on Ado's Theorem, which sets that given any LA $\mathfrak{g}$, there is a linear LA isomorphic to it.

Recently (see [20]), Tuynman has given an elementary proof of Lie's Third Theorem by using the correspondence between Lie subalgebras and Lie subgroups and the fact that, for a simply connected Lie group G, one has $H^2(G) = 0$.

In our research we have improved Tuynman's work in a particular case, although vast enough: when dealing with filiform LA. To do this we give an explicit construction of a simply connected Lie group whose Lie algebra falls into a given FLA, say $\mathfrak{g}$.

By using the group of automorphisms of FLA of a finite dimension n, particularly the unipotent automorphisms, we find a kind of basis with respect to which these automorphisms can be represented by triangular matrices. The dimension of the associated algebra, which is a subalgebra of $Der\,\mathfrak{g}$, is, at most, $2n-3$.

Starting from the model FLA of each dimension, its algebra of derivations contains as subalgebras those FLA such that $[\mathfrak{g},\mathfrak{g}]$ is abelian. Note that one of them is the own initial model algebra. These subalgebras determine, in a unique way, the corresponding subgroups from the initial group of automorphisms which constitute, as it is pretended, a matrix representation of the simply connected Lie group corresponding to each of such algebras.

5 Nilpotent Lie algebras of maximal rank

Let $\mathfrak{L}$ be a finite-dimensional nilpotent Lie algebra, $Der\mathfrak{L}$ its derivation algebra, $Aut\mathfrak{L}$ its automorphism group. A *torus* on $\mathfrak{L}$ is a commutative subalgebra of $Der\mathfrak{L}$ whose elements are semi-simple. The *rank* r of $\mathfrak{L}$ is the dimension of a maximal torus. The Lie algebra $\mathfrak{L}$ is of *maximal rank* if $r = \mathfrak{L}/[\mathfrak{L},\mathfrak{L}]$.

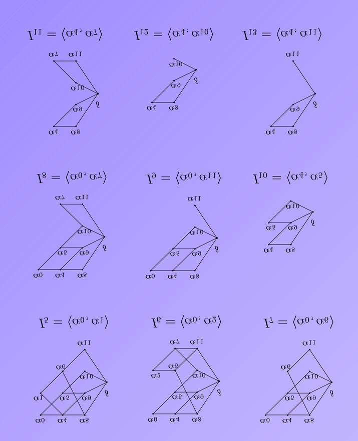
In [19] Santharoubane associated canonically a Kac-Moody algebra $\mathfrak{g}(A)$ with each NLA $\mathfrak{L}$ of maximal rank, where A is a generalized Cartan matrix. The LA $\mathfrak{L}$ is isomorphic to a quotient of the positive part $\mathfrak{n}_+$ of the Kac-Moody algebra of type A.

This classification problem is reduced to finding some ideals of the positive part $\mathfrak{n}_+$ of the Kac-Moody algebra $\mathfrak{g}(A)$, up to the action of the automorphism group G of the Dynkin diagram of A.

There are 3 families of Kac-Moody algebras: finite, affine and indefinite, and the second family is divided in two subfamilies: non-twisted affine and twisted affine.

The study of NLA of maximal rank associated with the finite Kac-Moody algebras (i.e. the finite-dimensional simple Lie algebras) was already done.

In [15] we used the graphs as a tool to study NLA. If A is affine, we can associate $\mathfrak{n}_+$ with a directed graph (in fact, a digraph) and we can also associate a subgraph of this digraph with every isomorphism class of NLA of maximal rank and of type A. We proposed an algorithm which computes these subgraphs and also groups them in isomorphism classes.



At present, several authors are studying the NLA of maximal rank associated with non-twisted affine Kac-Moody algebras. We studied the NLA of maximal rank and types $F_4^{(1)}$ and $E_6^{(1)}$ (see [12] and [14]). The main results are that up to isomorphism there are: 1095 (respectively 2087) infinite series with discrete parameters and 28 (respectively, 126) infinite series with continuous parameters of nilpotent Lie algebras of maximal rank and of Kac-Moody type $F_4^{(1)}$ (respectively, $E_6^{(1)}$).

We studied in [16] the NLA associated with the twisted affine Kac-Moody algebra $\mathfrak{g}(D_4^{(3)})$. The main result is that there are 88 infinite series (up to isomorphism) with discrete parameters and 1 infinite series with continuous parameter for this type.

In [13] 2-step nilpotent Lie algebras of maximal rank and of type A are classified, where A is of finite or affine type. We proved that $\mathfrak{L}$ is isomorphic to $\mathfrak{n}_+/C^3\mathfrak{n}_+$, so that a metabelian Lie algebra of maximal rank is uniquely determined by its type.

6 Isotopic Lie algebras

Bruck [3] defined a *quasigroup* as a nonempty set G endowed with a product $\cdot$, such that if any two of the three symbols a, b, c in the equation $a \cdot b = c$ are given as elements of G, the third one is uniquely determined as an element of G. Moreover, two quasigroups $(G,\cdot)$ and $(\widehat{G},\widehat{\cdot})$ are *isotopic* if there are three bijections α,β,γ from G to $\widehat{G}$, such that $\gamma(a\cdot b) = \alpha(a)\widehat{\cdot}\beta(b)$, for all $a, b \in G$. The triple $\Theta = (\alpha,\beta,\gamma)$ is called an *isotopism* from $(G,\cdot)$ to $(\widehat{G},\widehat{\cdot})$. Bruck's isotopisms of quasigroups can be used to define some different isotopies of Lie algebras. Specifically, starting from a Lie algebra of dimension n, we used in [8, 9] a family of isotopisms, called Santilli's isotopisms, in order to define Lie algebras of dimension greater than n.

Since the Caley's table of a quasigroup of n elements is a *Latin square* of order n, that is to say, a $n\times n$ array $L = (l_{i,j})$ with elements chosen from the set $[n] = \{1, 2, \ldots, n\}$, such that each symbol occurs precisely once in each row and each column, the previous study has allowed us to research and develop new results on design theory. So, for example, we established a bijective relation between the graphs relating the different autotopisms of the autotopy group of a Latin square [10]. We also gave an upper bound of the size of the smallest set of triples equivalent to Θ [11].

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