TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY, SERIES B
Volume 5, Pages 1–31 (January 19, 2018)
http://dx.doi.org/10.1090/btran/20

UNIQUENESS FOR SQG PATCH SOLUTIONS

ANTONIO CÓRDOBA, DIEGO CÓRDOBA, AND FRANCISCO GANCEDO

ABSTRACT. This paper is about the evolution of a temperature front governed by the surface quasi-geostrophic equation. The existence part of that program within the scale of Sobolev spaces was obtained by the third author (2008). Here we revisit that proof introducing some new tools and points of view which allow us to conclude the also needed uniqueness result.

1. INTRODUCTION

Among the more important partial differential equations of fluid dynamics we have the three dimensional Euler equation, modelling the evolution of an incompressible inviscid fluid, and the surface quasi-geostrophic (SQG) which describes the dynamics of atmospheric temperature [19]. SQG also has the extra mathematical interest of capturing the complexity of the 3D Euler equation but in a two dimensional scenario, as was described in the classical work [8].

This model reads

$$\theta_t + u \cdot \nabla\theta = 0,$$
$$u = (-R_2\theta, R_1\theta),$$

where $\theta(x,t)$ is the temperature of the 2D fluid with $(x,t) \in \mathbb{R}^2 \times [0,+\infty)$. The velocity u is related to the temperature through the Riezs transforms R_j given by

$$R_j(\theta)(x) = \frac{1}{\pi}\int_{\mathbb{R}^2} \frac{y_j}{|y|^3}\theta(x-y)dy.$$

Within the equation there is an underlying particle dynamic which preserves the value of θ, implying that the norms $\|\theta\|_{L^p}(t)$, $1 \le p \le \infty$, remain constants under the evolution.

In this paper we consider the patch problem, on which the temperature takes two constant values in two complementary domains and the solution of SQG has to be understood in a weak sense, namely:

$$\int_0^\infty \int_{\mathbb{R}^2} \theta(x,t)(\varphi_t(x,t) + u(x,t)\cdot\nabla\varphi(x,t))dxdt = \int_{\mathbb{R}^2}\theta_0(x)\varphi(x,0)dx, \quad (1)$$
$$u = (-R_2\theta, R_1\theta),$$

for every $\varphi \in C_c^\infty([0,\infty)\times\mathbb{R}^2)$. That is, the temperature reads

$$\theta(x,t) = \begin{cases} \theta^1, & x \in D^1(t), \\ \theta^2, & x \in D^2(t) = \mathbb{R}^2 \setminus D^1(t), \end{cases} \quad (2)$$

Received by the editors May 21, 2016 and, in revised form, February 7, 2017, May 30, 2017, July 14, 2017, and August 2, 2017.

2010 *Mathematics Subject Classification.* Primary 35Q35.

where $D^1(t)$ is a simply connected domain. It gives rise to a contour equation for the free boundary

$$\partial D^j(t) = \{x(\gamma,t) = (x_1(\gamma,t), x_2(\gamma,t)) : \gamma \in \mathbb{T}\}, \tag{3}$$

which is moving with the fluid and whose exact formulation can be found in [10]. It is then clear that the evolution of the patch is equivalent to that of its free boundary $\partial D^j(t)$. Therefore an important question for this problem is the propagation in time of the regularity of the interface $\partial D^j(t)$ or to the contrary the existence of finite time blow-up phenomena.

This problem was first considered by Resnick in his thesis [20]. Local-in-time existence and uniqueness was proven by Rodrigo [21] for C^∞ initial data using the Nash-Moser inverse function theorem. In [10] the third author proves local-in-time existence for the problem in Sobolev spaces, using energy estimates and properties of a particular parameterization of the contour. Namely, one such that the modulus of the tangent vector to the curve does not depend on the space variable, depending only on time [16] and giving us extra cancellations which allows to integrate the system.

In the distributional sense, the gradient of the temperature is given by

$$\nabla\theta(x,t) = (\theta^2 - \theta^1)\partial_\gamma^\perp x(\gamma,t)\delta(x(\gamma,t) = x)$$

for $x(\gamma,t)$ a given parameterization of the contour and

$$\partial_\gamma^\perp x(\gamma,t) = (-\partial_\gamma x_2(\gamma,t), \partial_\gamma x_1(\gamma,t)).$$

Then the Biot-Savart formula helps us to get the velocity field, outside the boundary, in terms of the geometry of the contour, that is,

$$u(x,t) = I_1(\nabla^\perp\theta)(x,t) = -\frac{\theta^2-\theta^1}{2\pi}\int_{-\pi}^{\pi}\frac{\partial_\gamma x(\gamma,t)}{|x - x(\gamma,t)|}d\gamma,$$

where I_1 is the Riesz potential of order 1, which on the Fourier side is multiplication by $|\xi|^{-1}$. The above integral diverges when x approaches the boundary but only on its tangential component, while its normal component is well defined. This fact is crucial to assign a normal velocity field to the boundary governing its evolution. Since the contribution of the tangential component amounts to a reparameterization of the boundary curve, we are free to add such a component satisfying both purposes: to be bounded and having a tangent vector with constant length. For a given parameterization $x(\gamma,t)$, approaching the boundary in both domains we obtain

$$u(x(\gamma,t),t)\cdot\partial_\gamma^\perp x(\gamma,t) = -\frac{\theta^2-\theta^1}{2\pi}\int_{-\pi}^{\pi}\frac{\partial_\gamma x(\eta,t)\cdot\partial_\gamma^\perp x(\gamma,t)}{|x(\gamma,t)-x(\eta,t)|}d\eta.$$

And we get the task of finding a good parameterization $x(\gamma,t)$ and a function λ so that

$$\begin{aligned} &u(x(\gamma,t),t)\cdot\partial_\gamma^\perp x(\gamma,t) \\ &\quad= \Big(\frac{\theta^2-\theta^1}{2\pi}\int_{-\pi}^{\pi}\frac{\partial_\gamma x(\gamma,t)-\partial_\gamma x(\eta,t)}{|x(\gamma,t)-x(\eta,t)|}d\eta + \lambda\partial_\gamma x(\gamma,t)\Big)\cdot\partial_\gamma^\perp x(\gamma,t), \end{aligned}$$

and the two purposes mentioned above are achieved.

Having the length of the vector $\partial_\gamma x(\gamma,t)$ as a function in the variable t only provides the following two identities:

$$\partial_\gamma^2 x(\gamma,t)\cdot\partial_\gamma x(\gamma,t) = 0 \quad \text{and} \quad \partial_\gamma^3 x(\gamma,t)\cdot\partial_\gamma x(\gamma,t) = -|\partial_\gamma^2 x(\gamma,t)|^2. \tag{4}$$

The first one gives extra cancellations while the second allows us to perform convenient integration by parts.

Although we cannot give justice to the many interesting contributions due to the different authors quoted in our references, let us say that, at the beginning, there was a conjecture about the formation of singularities in the evolution of a vortex patch for Euler equations in dimension two [2]. It was disproved by Chemin in a remarkable work [7] using paradifferential calculus, and later Bertozzi-Constantin [1] obtained a different proof taking advantage of an extra cancellation satisfied by singular integrals having even kernels.

Between the patch problem for 2D Euler and SQG there is a continuous set of interpolated equations given by

$$\begin{aligned} \theta_t + u\cdot\nabla\theta =&0, \\ u = (-R_2, R_1)(I_{1-\alpha}\theta), &\quad 0<\alpha<1. \end{aligned} \tag{5}$$

The case $\alpha = 0$ is the most regular, 2D Euler, while for $\alpha = 1$ one gets SQG. The patch problem for those equations was first studied in [9], where Córdoba, Fontelos, Mancho, and Rodrigo introduced a very interesting scenario for which they could show numerical evidence of singularity formation: two patches with different temperature approach each other in such a way that they collide at a point where the curvature blows-up. Let us mention that recently it has been shown analytically [11] that if the curvature is controlled, then pointwise collisions cannot happen in the patch problem for SQG. In [22, 23] a different finite time singularity scenario is shown where numerics point at a self-similar blow-up behaviour for SQG patches.

The system above can also be considered in more singular cases than SQG, replacing the last identity by the following one:

$$u = (-R_2, R_1)(\Lambda^{\beta}\theta), \quad 0<\beta<1,$$

where here $\Lambda = (-\Delta)^{1/2}$, whose Fourier symbol is $|\xi|$. See [6] for results on this equation with patch solutions.

A classical result in fluid dynamics is the existence for all time of vortex patches for the Euler equation which are rotating ellipses [2]. The patch problem for the system (5) and SQG present a more complex dynamic, as ellipses are not rotational solutions and some convex interfaces lose this property in finite time [5]. See [12] for a study of the growth of the patch support. Recently, in a remarkable series of papers and with an ingenious use of the Crandall-Rabinowitz mountain pass lemma, the authors have extended those global-in-time existence results to a more general class of geometrical shapes for the vortex patch problem [14, 15], the α-system (5) [13] and also to the SQG equation [3, 4].

There are two articles [17, 18] where the patch problem for the α-system is considered in a half plane with Dirichlet's condition. The system is proved to be well-posed for $0 < \alpha < \frac{1}{12}$ in the more singular scenario where the patch intersects the fixed boundary. In this framework, singularity formation is shown when two patches of different temperature approach each other.

In this paper we will take advantage of a special parameterization of the boundary in the following terms:

We say that a bounded simply connected domain $D \subset \mathbb{R}^2$ is $C^{2,\delta}(\mathbb{T})$ for $0 < \delta < 1$ if there exists a parameterization of the boundary

$$\partial D = \{x(\gamma) \in \mathbb{R}^2 : \gamma \in \mathbb{T},\, 2\pi\text{-periodic}\}$$

such that $x(\gamma) \in C^{2,\delta}(\mathbb{T})$. Specifically, a domain $\Omega \in C^{2,\delta}(\mathbb{T})$ given by

$$\partial\Omega = \{y(\xi) \in \mathbb{R}^2 : \xi \in \mathbb{T},\, 2\pi\text{-periodic}\}$$

is said to be equal to D if there exists a change of variable

$$\varphi : \mathbb{T} \to \mathbb{T}, \quad \text{biyective}, \quad \varphi'(\gamma) > 0, \quad \varphi(\gamma) - \gamma \quad 2\pi\text{-periodic}, \quad \varphi \in C^{2,\delta}(\mathbb{T}),$$

such that $x(\gamma) = y(\varphi(\gamma))$. Furthermore, a time dependent simply connected domain $D(t)$ belongs to $C([0,T]; C^{2,\delta}(\mathbb{T})) \cap C^1([0,T]; C^1(\mathbb{T}))$ if there exist parameterizations of the boundaries

$$\partial D(t) = \{x(\gamma, t) \in \mathbb{R}^2 : \gamma \in \mathbb{T},\, t \in [0,T],\, 2\pi\text{-periodic in } \gamma\}$$

such that $x(\gamma, t) \in C([0,T]; C^{2,\delta}(\mathbb{T})) \cap C^1([0,T]; C^1(\mathbb{T}))$. Throughout the paper we shall also deal with time dependent simply connected domains in the space $C([0,T]; H^k(\mathbb{T}))$, with H^k Sobolev spaces for $k \in \mathbb{N}$, meaning that its evolving boundary $x(\gamma, t)$ belongs to that time dependent space.

Another main character of this play is the so-called arc-chord condition which help to control the absence of self-intersections of the boundary curve. This is done through the following quantity:

$$F(x)(\gamma, \eta, t) = \frac{|\eta|}{|x(\gamma, t) - x(\gamma - \eta, t)|} \quad \forall\, \gamma, \eta \in [-\pi, \pi],$$

with

$$F(x)(\gamma, 0, t) = \frac{1}{|\partial_\gamma x(\gamma, t)|},$$

whose L^∞ norm has to be controlled in the evolution.

As was mentioned before, patch solutions for the SQG equation are understood in a weak sense. Any such solution with a free boundary given by a smooth parameterization $x(\gamma, t)$ has to satisfy the equation below

$$x_t(\gamma, t) \cdot \partial_\gamma^\perp x(\gamma, t) = -\int_{\mathbb{T}} \frac{\partial_\gamma^\perp x(\gamma, t) \cdot \partial_\gamma x(\gamma - \eta, t)}{|x(\gamma, t) - x(\gamma - \eta, t)|} d\eta, \tag{6}$$

where we have taken $\theta_2 - \theta_1 = \pi$ for the sake of simplicity. On the other hand, any smooth parameterization $x(\gamma, t)$ satisfying (6) provides a weak SQG solution with the temperature given by (2,3) (see [10] for more details).

It is easy to check that the equation above is a reparameterization invariance object, and that the following formula, introduced in [20] and [21], has a well-defined tangential velocity and identical normal component:

$$x_t(\gamma, t) = \int_{\mathbb{T}} \frac{\partial_\gamma x(\gamma, t) - \partial_\gamma x(\gamma - \eta, t)}{|x(\gamma, t) - x(\gamma - \eta, t)|} d\eta. \tag{7}$$

The local-in-time existence result was given in [10] for initial data satisfying (4) and evolving by

$$x_t(\gamma, t) = \Big(\int_{\mathbb{T}} \frac{\partial_\gamma x(\gamma, t) - \partial_\gamma x(\gamma - \eta, t)}{|x(\gamma, t) - x(\gamma - \eta, t)|} d\eta + \lambda(x)(\gamma, t) \partial_\gamma x(\gamma, t) \Big), \tag{8}$$

$$\begin{aligned}\lambda(x)(\gamma,t) = &\frac{\gamma+\pi}{2\pi}\int_{\mathbb{T}}\frac{\partial_\eta x(\eta,t)}{|\partial_\eta x(\eta,t)|^2}\cdot\partial_\eta\Big(\int_{\mathbb{T}}\frac{\partial_\eta x(\eta,t)-\partial_\eta x(\eta-\xi,t)}{|x(\eta,t)-x(\eta-\xi,t)|}d\xi\Big)d\eta\\ &-\int_{-\pi}^{\gamma}\frac{\partial_\eta x(\eta,t)}{|\partial_\eta x(\eta,t)|^2}\cdot\partial_\eta\Big(\int_{\mathbb{T}}\frac{\partial_\eta x(\eta,t)-\partial_\eta x(\eta-\xi,t)}{|x(\eta,t)-x(\eta-\xi,t)|}d\xi\Big)d\eta.\end{aligned}\tag{9}$$

We state the result here for completeness.

Theorem 1.1. *Let $x_0(\gamma)\in H^k(\mathbb{T})$ for $k\geq 3$ with $F(x_0)(\gamma,\eta)<\infty$ and $\partial_\gamma x_0(\gamma)\cdot\partial_\gamma^2 x_0(\gamma)=0$. Then there exists a time $T>0$ so that there is a solution to (8) in $C([0,T];H^k(\mathbb{T}))$ with $x(\gamma,0)=x_0(\gamma)$ and $\lambda(\gamma,t)$ given by (9).*

The main purpose of this paper is to show uniqueness for the patch problem for SQG which was until now an open problem. The following theorem provides this result:

Theorem 1.2. *Consider a solution of (1) with $\theta(x,t)$ given by a patch (2) and $D^j(t)$ time dependent simply connected domains whose moving boundary satisfies the arc-chord condition for any $t\in[0,T]$ and $C([0,T];C^{2,\delta}(\mathbb{T}))\cap C^1([0,T];C^1(\mathbb{T}))$ regularity. Furthermore, assume that the function $\bar\theta(x,t)$ given by*

$$\bar\theta(x,t)=\begin{cases}\theta^1, & x\in\bar D^1(t),\\ \theta^2, & x\in\bar D^2(t)=\mathbb{R}^2\setminus\bar D^1(t),\end{cases}$$

satisfies (1) with $\partial\bar D^j(t)\in C([0,T];C^{2,\delta}(\mathbb{T}))\cap C^1([0,T];C^1(\mathbb{T}))$ and $\theta(x,0)=\bar\theta(x,0)$. Then $\theta(x,t)=\bar\theta(x,t)$ for any $t\in[0,T]$.

This is an important part of the paper and it is proved in its section 2. In particular we show that any weak solutions of (1) identified by a patch (2), for a given parameterization (3) with a certain regularity, can be reparameterized satisfying (4). This property is preserved in time and, together with a new reparameterized curve, help us to fix the tangential velocity for a contour that evolves by (8,9) giving the patch solution. Then, one just needs to get uniqueness for the system (8,9). Next we check the evolution of the H^1 Sobolev norm of the difference among two different curves evolving by (8,9). We close the estimate revisiting the previous existence results and introducing new cancellation and tools to find uniqueness by Gronwall's lemma. However, in this process several different points of view with respect to the previous literature are introduced.

In the following we are going to show how it is possible to go from (8,9) to equation (7) through a convenient change of variable. This procedure is also valid to go from (8,9) to an SQG patch contour equation with a different and more convenient tangential term.

We denote by $x(\gamma,t)\in C([0,T];H^3)$ a solution of (8,9) and let $\tilde x(\xi,t)$ be given by

$$\tilde x(\xi,t)=x(\phi^{-1}(\xi,t),t),\qquad\qquad \gamma=\phi^{-1}(\xi,t),$$

or equivalently

$$x(\gamma,t)=\tilde x(\phi(\gamma,t),t),\qquad \xi=\phi(\gamma,t),$$

where

$$\phi(\gamma,t):\mathbb{R}\times\mathbb{R}^+\to\mathbb{R},\quad \partial_\gamma\phi(\gamma,t)>0,\quad \phi(\gamma,t)-\gamma\ \ 2\pi\text{-periodic},\tag{10}$$

is a reparameterization in γ for any positive time. Here ϕ is a solution of the linear system

$$\phi_t(\gamma,t) = \int_{\mathbb{T}} \frac{\partial_\gamma\phi(\gamma,t) - \partial_\gamma\phi(\eta,t)}{|x(\gamma,t)-x(\eta,t)|} d\eta + \lambda(x)(\gamma,t)\partial_\gamma\phi(\gamma,t). \tag{11}$$

The existence and uniqueness for that system is given in the following proposition, for whose formulation we introduce the space:

$$H^{\frac{k}{\log}} \equiv \{f \in L^2(\mathbb{T}) : \sum_{n\in\mathbb{Z}} \frac{|n|^{2k}}{\log^2(|n|+e)} |\hat{f}(n)|^2 = \|f\|^2_{H^{\frac{k}{\log}}} < \infty\}.$$

Proposition 1.3. *Let* $\phi_0(\gamma) - \gamma \in H^{\frac{k}{\log}}$ *for* $k \geq 3$ *and* $x(\gamma,t) \in C([0,T]; H^k)$ *be a solution of* (8,9) *with* $F(x)(\gamma,\eta,t) \in L^\infty$ *and* $\partial_\gamma x(\gamma,0)\cdot\partial^2_\gamma x(\gamma,0) = 0$. *Then there exists a unique solution to* (11) *with* $\phi(\gamma,t) - \gamma \in C([0,T]; H^{\frac{k}{\log}})$ *such that* $\phi(\gamma,0) = \phi_0(\gamma)$. *In particular, if* $\partial_\gamma\phi_0(\gamma) > 0$, *then* $\partial_\gamma\phi(\gamma,t) > 0$ *holds for any* $t \in (0,t_p]$ *with* $t_p \in (0,T]$.

The proof of the proposition is given in section 3. The space $H^{\frac{k}{\log}}$ is needed because we can only assume that $\lambda(x) \in C([0,T]; H^{\frac{k}{\log}})$ for $x \in C([0,T]; H^k)$ (see the proof of Proposition 1.3). Observe that the logarithmic modification of Sobolev norms is not a problem in the proof of the existence theorem given in [10], because only control of the H^{k-1} norm of $\lambda(x)$ is needed, which is far from the $H^{\frac{k}{\log}}$ norm. In the energy estimates which provide local existence, one needs to consider the integral

$$\int \partial^k_\gamma x(\gamma,t)\cdot\partial^k_\gamma x_t(\gamma,t)d\gamma,$$

whose most singular term coming form $\lambda(x)$ is given by

$$I = \int \partial^k_\gamma \lambda(x)(\gamma,t)\partial^k_\gamma x(\gamma,t)\cdot\partial_\gamma x(\gamma,t)d\gamma.$$

Integration by parts yields

$$I = -\int \partial^{k-1}_\gamma \lambda(x)(\gamma,t)\partial_\gamma(\partial^k_\gamma x\cdot\partial_\gamma x)(\gamma,t)d\gamma,$$

and using identity (4) one gets the bound

$$I \leq \|\partial^{k-1}_\gamma\lambda(x)\|_{L^2}\|\partial_\gamma(\partial^k_\gamma x\cdot\partial_\gamma x)\|_{L^2} \leq \|\lambda(x)\|_{H^{k-1}}\|x\|^2_{H^k} \leq C\|x\|^p_{H^k}$$

with p and C constants depending on $k \geq 3$ (it is easy to observe that this extra cancellation cannot be used in the ϕ equation).

Next we shall show that $\tilde{x}(\xi,t)$ is a solution of (7). Here we consider ϕ regular enough ($\phi(\gamma,t) - \gamma \in C([0,T]; H^{\frac{k}{\log}})$ with $k \geq 3$) so that it is a bona fide reparameterization satisfying (10).

The chain rule implies

$$x_t(\gamma,t) = \tilde{x}_t(\phi(\gamma,t),t) + \phi_t(\gamma,t)\partial_\xi\tilde{x}(\phi(\gamma,t),t). \tag{12}$$

On the other hand, the equation for the evolution provides

$$\begin{aligned}x_t(\gamma,t) &= \int \frac{\partial_\gamma x(\gamma,t)-\partial_\gamma x(\eta,t)}{|x(\gamma,t)-x(\eta,t)|}d\eta + \lambda(x)(\gamma,t)\partial_\gamma x(\gamma,t)\\ &= \int \frac{\partial_\xi \tilde{x}(\phi(\gamma,t),t)\partial_\gamma\phi(\gamma,t)-\partial_\gamma x(\eta,t)}{|x(\gamma,t)-x(\eta,t)|}d\eta\\ &\quad+\lambda(x)(\gamma,t)\partial_\xi\tilde{x}(\phi(\gamma,t),t)\partial_\gamma\phi(\gamma,t),\end{aligned}$$

and therefore

$$\begin{aligned}x_t(\gamma,t) &= \partial_\xi\tilde{x}(\phi(\gamma,t),t)\int \frac{\partial_\gamma\phi(\gamma,t)-\partial_\gamma\phi(\eta,t)}{|x(\gamma,t)-x(\eta,t)|}d\eta\\ &\quad+\lambda(x)(\gamma,t)\partial_\xi\tilde{x}(\phi(\gamma,t),t)\partial_\gamma\phi(\gamma,t)\\ &\quad+\int \frac{\partial_\xi\tilde{x}(\phi(\gamma,t),t)\partial_\gamma\phi(\eta,t)-\partial_\gamma x(\eta,t)}{|x(\gamma,t)-x(\eta,t)|}d\eta.\end{aligned} \tag{13}$$

The fact that ϕ is a solution of (11) together with identities (12,13) allow us to get

$$\tilde{x}_t(\phi(\gamma,t),t) = \int \frac{\partial_\xi\tilde{x}(\phi(\gamma,t),t)-\partial_\xi\tilde{x}(\phi(\eta,t),t)}{|\tilde{x}(\phi(\gamma,t),t)-\tilde{x}(\phi(\eta,t),t)|}\partial_\gamma\phi(\eta,t)d\eta.$$

Introducing the change of variable $\phi(\eta,t)=\zeta$ in the integral above and taking $\gamma=\phi^{-1}(\xi,t)$ we obtain $\tilde{x}(\xi,t)$ as a solution of (7) replacing x by $\tilde{x}$, γ by ξ and η by ζ. Therefore, $\tilde{x}\in C([0,T];H^{\frac{k}{\log}})$ as a consequence of the Leibniz rule for derivatives of composite functions. An interesting feature in this process is the logarithm loss of derivative which affects the solutions of (7); nevertheless, we will show later how to take care of that.

Once at this point one can see clearly how this reparameterization process helps to solve the following system:

$$\tilde{x}_t(\xi,t) = \int \frac{\partial_\xi\tilde{x}(\xi,t)-\partial_\xi\tilde{x}(\zeta,t)}{|\tilde{x}(\xi,t)-\tilde{x}(\zeta,t)|}d\zeta + \tilde{\mu}(\xi,t)\partial_\xi\tilde{x}(\xi,t) \tag{14}$$

for any $\tilde{\mu}(\xi,t)$ having the same regularity as $\tilde{x}(\xi,t)$. We just have to repeat the argument but with the equation

$$\phi_t(\gamma,t) = \int_{\mathbb{T}} \frac{\partial_\gamma\phi(\gamma,t)-\partial_\gamma\phi(\eta,t)}{|x(\gamma,t)-x(\eta,t)|}d\eta + \lambda(x)(\gamma,t)\partial_\gamma\phi(\gamma,t)-\mu(\gamma,t),$$

where the function μ acts as a source term, and as long as ϕ and μ have the same regularity, the argument works. We then arrive at (14) with $\tilde{\mu}(\xi,t)=\mu(\phi^{-1}(\xi,t),t)$. This shows that the systems (14) or (7) come from the system (8,9) by a change of variable.

Theorem 1.1 together with Proposition 1.3 yield the existence of solutions for the system (7). Then Theorem 1.2 implies uniqueness:

Theorem 1.4. *Let $x_0(\gamma)\in H^k(\mathbb{T})$ for $k\geq 3$ with $F(x_0)(\gamma,\eta)\in L^\infty$. Then there exists a time $T>0$ so that there exists a unique solution to (7) in $C([0,T];H^{\frac{k}{\log}}(\mathbb{T}))$ with $x(\gamma,0)=x_0(\gamma)$.*

The uniqueness part of this theorem will be discussed in section 4. Its proof will not assume property (4) and it will be done controlling the evolution of the L^2 norm of the difference between any two given solutions.

An important linear operator in the study of patch solutions for SQG is given by

$$\mathcal{L}(f)(\gamma) = \int_{-\pi}^{\pi} \frac{f(\gamma) - f(\gamma - \eta)}{|\eta|} d\eta \tag{15}$$

for f 2π-periodic. Since $\mathcal{L}$ is a translations invariance (where we have extended $|\eta|^{-1}$ periodically), the operator is a Fourier multiplier given by

$$\widehat{\mathcal{L}(f)}(k) = O(\log(2|k|))\widehat{f}(k) \quad \text{for} \quad k \in \mathbb{Z} \setminus \{0\}, \quad \widehat{\mathcal{L}(f)}(0) = 0. \tag{16}$$

Uniqueness for the 2D Euler vortex patch problem was obtained in the classical Yudovich work [24]. The results presented in that paper hold in a more general setting but it is also valid for any 2D Euler weak solution with vorticity in $L^\infty(0, T; L^\infty \cap L^1)$. For the α-system, weak solutions given by patches have been shown to be unique in [18]. The uniqueness result in the present paper corresponds to the more singular and physically relevant case: $\alpha = 1$, but the arguments can be extended for $0 < \alpha < 1$. In those cases the equations for the reparameterization are more regular than (11) and there is no logarithm derivative loss in the change of variable process. Solutions for one of the contour evolution equations were shown to be unique in [10] for $0 < \alpha < 1$.

2. Uniqueness for the SQG patch problem

This section is devoted to showing the proof of uniqueness of SQG weak solutions given by patches: Theorem 1.2. As a consequence of its proof, the solutions found in [10] are unique:

Corollary 2.1. *Consider a solution of the system* (8,9) *given by Theorem* 1.1 *with* $x(\gamma, t) \in C([0, T]; H^3(\mathbb{T}))$*. Then* $x(\gamma, t)$ *is unique as a solution of* (8,9) *with initial data* $x(\gamma, 0)$*. Furthermore, it provides the unique weak solution of* (1,2,3) *with* $D^j(t)$ *a time dependent simply connected domain in* $C([0, T]; C^{2,\delta}(\mathbb{T})) \cap C^1([0, T]; C^1(\mathbb{T}))$, $0 < \delta \leq 1/2$.

Proof of Theorem 1.2. We consider a solution $\theta(x, t)$ satisfying the hypothesis in Theorem 1.2. Then, it is shown in [10], the parameterization of the free boundary has to fulfill equation (6) where, without loss of generality, we can assume that $\theta_2 - \theta_1 = \pi$. The length of the curve is

$$l(t) = \int_{\mathbb{T}} |\partial_\gamma x(\gamma, t)| d\gamma,$$

and we shall consider the following change of variable:

$$\phi(\cdot, t) : \mathbb{T} \to \mathbb{T}, \quad \phi(\gamma, t) = -\pi + \frac{2\pi}{l(t)} \int_{-\pi}^{\gamma} |\partial_\gamma x(\eta, t)| d\eta.$$

Consequently, we get the reparameterization

$$\tilde{x}(\xi, t) = x(\phi^{-1}(\xi, t), t), \qquad x(\gamma, t) = \tilde{x}(\phi(\gamma, t), t), \qquad \xi = \phi(\gamma, t),$$

satisfying property (4) ($|\partial_\xi \tilde{x}(\xi, t)| = (2\pi)^{-1} l(t)$) and having the same regularity ($\tilde{x}(\xi, t) \in C([0, T], C^{2,\delta}) \cap C^1([0, T]; C^1(\mathbb{T}))$). As we pointed out before, the curve $\tilde{x}(\xi, t)$ is a solution of (6) with the tilde notation. We mean by this that $\tilde{x}(\xi, t)$ is a solution of (6) replacing x by $\tilde{x}$ and γ by ξ.

For this new evolving curve $\tilde{x}$, the identity

$$\tilde{x}_t(\xi,t)=\tilde{x}_t(\xi,t)\cdot\frac{\partial_\xi\tilde{x}(\xi,t)}{|\partial_\xi\tilde{x}(\xi,t)|}\frac{\partial_\xi\tilde{x}(\xi,t)}{|\partial_\xi\tilde{x}(\xi,t)|}+\tilde{x}_t(\xi,t)\cdot\frac{\partial_\xi^\perp\tilde{x}(\xi,t)}{|\partial_\xi\tilde{x}(\xi,t)|}\frac{\partial_\xi^\perp\tilde{x}(\xi,t)}{|\partial_\xi\tilde{x}(\xi,t)|}$$

together with (6) provides

$$\tilde{x}_t(\xi,t)=\tilde{\mu}(\xi,t)\frac{\partial_\xi\tilde{x}(\xi,t)}{|\partial_\xi\tilde{x}(\xi,t)|}+\int_{\mathbb{T}}\frac{(\partial_\xi\tilde{x}(\xi,t)-\partial_\xi\tilde{x}(\xi-\zeta,t))}{|\tilde{x}(\xi,t)-\tilde{x}(\xi-\zeta,t)|}d\zeta\cdot\frac{\partial_\xi^\perp x(\xi,t)}{|\partial_\xi\tilde{x}(\xi,t)|}\frac{\partial_\xi^\perp\tilde{x}(\xi,t)}{|\partial_\xi\tilde{x}(\xi,t)|},$$

where we have defined $\tilde{\mu}(\xi,t)=\tilde{x}_t(\xi,t)\cdot\partial_\xi\tilde{x}(\xi,t)/|\partial_\xi\tilde{x}(\xi,t)|$. Taking

$$\tilde{\mu}(\xi,t)=\int_{\mathbb{T}}\frac{(\partial_\xi\tilde{x}(\xi,t)-\partial_\xi\tilde{x}(\xi-\zeta,t))}{|\tilde{x}(\xi,t)-\tilde{x}(\xi-\zeta,t)|}d\zeta\cdot\frac{\partial_\xi\tilde{x}(\xi,t)}{|\partial_\xi\tilde{x}(\xi,t)|}+\tilde{\lambda}(\tilde{x})(\xi,t)|\partial_\xi\tilde{x}(\xi,t)|,$$

it is easy to find that $\tilde{x}$ satisfies (8) with the tilde notation:

$$\tilde{x}_t(\xi,t)=\int_{\mathbb{T}}\frac{\partial_\xi\tilde{x}(\xi,t)-\partial_\xi\tilde{x}(\xi-\zeta,t)}{|\tilde{x}(\xi,t)-\tilde{x}(\xi-\zeta,t)|}d\zeta+\tilde{\lambda}(\tilde{x})(\xi,t)\partial_\xi\tilde{x}(\xi,t),$$

where

$$\tilde{\lambda}(\tilde{x})(\xi,t)=\Big(\tilde{x}_t(\xi,t)-\int_{\mathbb{T}}\frac{\partial_\xi\tilde{x}(\xi,t)-\partial_\xi\tilde{x}(\xi-\zeta,t)}{|\tilde{x}(\xi,t)-\tilde{x}(\xi-\zeta,t)|}d\zeta\Big)\cdot\frac{\partial_\xi\tilde{x}(\xi,t)}{|\partial_\xi\tilde{x}(\xi,t)|^2}.$$

The regularity of $\tilde{x}(\xi,t)$ yields $\tilde{\lambda}(\tilde{x})(\xi,t)\in C([0,T];C^1(\mathbb{T}))$. Then we can find a function $a\in C^1([0,\tilde{T}];\mathbb{R})$ as a unique solution of the o.d.e.

$$a'(t)=\tilde{\lambda}(\tilde{x})(-\pi-a(t),t),\quad a(0)=0,$$

where $0<\tilde{T}$ by the Picard-Lindelöf theorem. Since $\sup_{[0,T]}\|\tilde{\lambda}(\tilde{x})\|_{L^\infty}(t)\le C_m(x)$, for $C_m(x)$ depending on $\sup_{[0,T]}\|F(x)\|_{L^\infty}(t)$ and $\sup_{[0,T]}(\|x\|_{C^{2,\delta}}(t)+\|x_t\|_{L^\infty}(t))$, the function $a(t)$ can be extended to $[0,T]$ satisfying that $|a(t)|\le TC_m(x)$ for any $t\in[0,T]$.

The new curve given by $\tilde{x}(\xi,t)=\bar{x}(\xi+a(t),t)$ satisfies

$$\bar{x}_t(\alpha,t)=\int_{\mathbb{T}}\frac{\partial_\alpha\bar{x}(\alpha,t)-\partial_\alpha\bar{x}(\alpha-\beta,t)}{|\bar{x}(\alpha,t)-\bar{x}(\alpha-\beta,t)|}d\beta+\bar{\lambda}(\bar{x})(\alpha,t)\partial_\alpha\bar{x}(\alpha,t)$$

for $\alpha=\xi-a(t)$ and $\bar{\lambda}(\bar{x})(\alpha,t)=\tilde{\lambda}(\tilde{x})(\alpha-a(t),t)-\tilde{\lambda}(\tilde{x})(-\pi-a(t),t)$. Since $\partial_\alpha|\partial_\alpha\bar{x}(\alpha,t)|=0$ and $\bar{\lambda}(\bar{x})(-\pi,t)=0$, we proceed as in [10] (see pg. 2585) to find that $\bar{x}$ evolves according to equations (8,9) replacing x by $\bar{x}$ and γ by α. In particular it is easy to check that $\bar{x}(\alpha,t)$ has the same regularity as $\tilde{x}(\xi,t)$ and $\tilde{x}(\xi,0)=\bar{x}(\xi,0)$.

We consider next another solution $\bar{\theta}(x,t)$, satisfying the hypothesis above with the free boundary parameterized by $y(\gamma,t)\in C([0,T];C^{2,\delta}(\mathbb{T}))\cap C^1([0,T];C^1(\mathbb{T}))$. As $\theta(x,0)=\bar{\theta}(x,0)$, we use a function $\varphi\in C^{2,\delta}(\mathbb{T})$ to define $\breve{y}(\gamma,t)=y(\varphi(\gamma),t)$ in such a way that $\breve{y}(\gamma,0)=x(\gamma,0)$. Therefore, it is easy to see that $\breve{y}$ has the same regularity as y and fulfills equation (6), providing the free boundary of the same patch solution $\bar{\theta}(x,t)$. Next, we reparameterize $\breve{y}(\gamma,t)$ as we did for $x(\gamma,t)$ to get $\tilde{y}(\xi,t)$ satisfying $\partial_\xi(|\partial_\xi\tilde{y}(\xi,t)|)=0$ and $\tilde{y}(\xi,0)=\tilde{x}(\xi,0)$. Then we obtain $\bar{y}(\alpha,t)$ similarly as before providing us a solution of equations (8,9) after replacing x by $\bar{y}$ and γ by α. In particular, all this reparameterization process provides $\bar{y}(\alpha,t)$ with the same kind of regularity and satisfying $\bar{x}(\alpha,0)=\bar{y}(\alpha,0)$.

From now on, we will drop the bars for simplicity, using the variables γ and η instead of α and β. As before we shall write $f=f(\gamma,t)$, $f'=f(\gamma-\eta,t)$, $f_-=f-f'$, and $\int=\int_{\mathbb{T}}$, when there is no danger of confusion in the writing of our double integrals in variables γ and η. During the time of existence $T>0$ one has the arc-chord condition $F(x)$ in $L^\infty(0,T;L^\infty)$. In the following C will denote a constant which may be different from inequality to inequality but depending only on $\sup_{[0,T]}\|x\|_{C^{2,\delta}}(t)$, $\sup_{[0,T]}\|y\|_{C^{2,\delta}}(t)$, $\sup_{[0,T]}\|F(x)\|_{L^\infty}(t)$, and $\sup_{[0,T]}\|F(y)\|_{L^\infty}(t)$.

Let us consider the function $z(\gamma,t)=x(\gamma,t)-y(\gamma,t)$. We have

$$\frac{1}{2}\frac{d}{dt}\|z\|^2_{L^2}=\int z\cdot z_t d\gamma = I_1+I_2,$$

where

$$I_1=\int z\cdot\int\Big(\frac{\partial_\gamma x_-}{|x_-|}-\frac{\partial_\gamma y_-}{|y_-|}\Big)d\eta d\gamma,\quad I_2=\int z\cdot(\lambda(x)\partial_\gamma x-\lambda(y)\partial_\gamma y)d\gamma.$$

Let us split I_1:

$$I_{1,1}=\int z\cdot\int\frac{\partial_\gamma z_-}{|x_-|}d\eta d\gamma,\quad I_{1,2}=\int z\cdot\int\partial_\gamma y_-\Big(\frac{1}{|x_-|}-\frac{1}{|y_-|}\Big)d\eta d\gamma.$$

Then with an adequate change of variables, we obtain

$$I_{1,1}=\int\int\frac{z(\gamma)\cdot(\partial_\gamma z(\gamma)-\partial_\gamma z(\eta))}{|x(\gamma)-x(\eta)|}d\eta d\gamma=-\int\int\frac{z(\eta)\cdot(\partial_\gamma z(\gamma)-\partial_\gamma z(\eta))}{|x(\gamma)-x(\eta)|}d\eta d\gamma$$

thus

$$I_{1,1}=\frac{1}{2}\int\int\frac{(z(\gamma)-z(\eta))\cdot(\partial_\gamma z(\gamma)-\partial_\gamma z(\eta))}{|x(\gamma)-x(\eta)|}d\eta d\gamma=\frac{1}{2}\int\int\frac{z_-\cdot\partial_\gamma z_-}{|x_-|}d\gamma d\eta.$$

Integration by parts provides

$$I_{1,1}=\frac{1}{4}\int\int|z_-|^2\frac{(x_-\cdot\partial_\gamma x_-)}{|x_-|^3}d\gamma d\eta=\frac{1}{4}\int\int|z_-|^2F(x)^3\frac{1}{|\eta|}\frac{(x_-\cdot\partial_\gamma x_-)}{\eta^2}d\gamma d\eta.$$

The inequality

$$|(x_-\cdot\partial_\gamma x_-)-\partial_\gamma x\cdot\partial_\gamma^2 x\eta^2|\leq 2\|x\|^2_{C^{2,\delta}}|\eta|^{2+\delta}, \tag{17}$$

together with the fact that $\partial_\gamma x\cdot\partial^2_\gamma x=0$ allows us to get

$$I_{1,1}\leq\|F(x)\|^3_{L^\infty}\|x\|^2_{C^{2,\delta}}\int|\eta|^{\delta-1}\int(|z|^2+|z'|^2)d\gamma d\eta\leq C\|z\|^2_{L^2}.$$

For $I_{1,2}$ one writes

$$I_{1,2}=\int z\cdot\int\partial_\gamma y_-\frac{|y_-|-|x_-|}{|x_-||y_-|}d\eta d\gamma\leq\int\int\frac{|z||\partial_\gamma y_-||z_-|}{|x_-||y_-|}d\eta d\gamma,$$

which yields

$$I_{1,2}\leq\int\int|z|\frac{|\partial_\gamma y_-|}{|\eta|}\frac{|z_-|}{|\eta|}F(x)F(y)d\gamma d\eta.$$

Then the identity

$$f_-=\eta\int_0^1\partial_\gamma f(\gamma+(s-1)\eta)ds \tag{18}$$

allows us to get the bound

$$I_{1,2}\leq\|F(x)\|_{L^\infty}\|F(y)\|_{L^\infty}\|\partial^2_\gamma y\|_{L^\infty}\int_0^1\int\int|z||\partial_\gamma z(\gamma+(s-1)\eta)|d\gamma d\eta ds,$$

which yields the desired control: $I_{1,2}\leq C\|z\|^2_{H^1}$.

Regarding I_2 we split further

$$I_{2,1}=\int z\cdot\partial_\gamma z\lambda(x)d\gamma,\quad I_{2,2}=\int z\cdot\partial_\gamma y(\lambda(x)-\lambda(y))d\gamma.$$

It is easy to get

$$I_{2,1}\leq\|z\|_{L^2}\|\partial_\gamma z\|_{L^2}\|\lambda(x)\|_{L^\infty}\leq C\|z\|^2_{H^1},$$

thus we are done with $I_{2,1}$.

For the reminder term we have

$$I_{2,2}\leq\|z\|_{L^2}\|\partial_\gamma y\|_{L^\infty}\|\lambda(x)-\lambda(y)\|_{L^2}\leq C\|z\|_{L^2}\|\lambda(x)-\lambda(y)\|_{L^2},$$

let us write $\lambda(x)-\lambda(y)=G_1+G_2$ where

$$G_1=\frac{\gamma+\pi}{2\pi}\int\Big[\frac{\partial_\gamma x}{|\partial_\gamma x|^2}\cdot\partial_\gamma\Big(\int\frac{\partial_\gamma x_-}{|x_-|}d\eta\Big)-\frac{\partial_\gamma y}{|\partial_\gamma y|^2}\cdot\partial_\gamma\Big(\int\frac{\partial_\gamma y_-}{|y_-|}d\eta\Big)\Big]d\gamma,$$

and

$$\begin{aligned}G_2=&-\int_{-\pi}^{\gamma}\frac{\partial_\gamma x(\eta,t)}{|\partial_\gamma x(\eta,t)|^2}\cdot\partial_\eta\Big(\int\frac{\partial_\gamma x(\eta,t)-\partial_\gamma x(\eta-\xi,t)}{|x(\eta,t)-x(\eta-\xi,t)|}d\xi\Big)d\eta\\&+\int_{-\pi}^{\gamma}\frac{\partial_\gamma y(\eta,t)}{|\partial_\gamma y(\eta,t)|^2}\cdot\partial_\eta\Big(\int\frac{\partial_\gamma y(\eta,t)-\partial_\gamma y(\eta-\xi,t)}{|y(\eta,t)-y(\eta-\xi,t)|}d\xi\Big)d\eta.\end{aligned}$$

Then we decompose further $G_1=G_{1,1}+G_{1,2}+G_{1,3}+G_{1,4}$:

$$G_{1,1}=\frac{\gamma+\pi}{2\pi}\int\frac{\partial_\gamma z}{|\partial_\gamma x|^2}\cdot\partial_\gamma\Big(\int\frac{\partial_\gamma x_-}{|x_-|}d\eta\Big)d\gamma,$$

$$G_{1,2}=\frac{\gamma+\pi}{2\pi}\int\big(\frac{1}{|\partial_\gamma x|^2}-\frac{1}{|\partial_\gamma y|^2}\big)\partial_\gamma y\cdot\partial_\gamma\Big(\int\frac{\partial_\gamma x_-}{|x_-|}d\eta\Big)d\gamma,$$

$$G_{1,3}=-\frac{\gamma+\pi}{2\pi}\int\frac{\partial_\gamma y}{|\partial_\gamma y|^2}\cdot\int\Big(\partial_\gamma x_-\frac{x_-\cdot\partial_\gamma x_-}{|x_-|^3}-\partial_\gamma y_-\frac{y_-\cdot\partial_\gamma y_-}{|y_-|^3}\Big)d\eta d\gamma,$$

and

$$G_{1,4}=\frac{\gamma+\pi}{2\pi}\int\frac{\partial_\gamma y}{|\partial_\gamma y|^2}\cdot\int\Big(\frac{\partial^2_\gamma x_-}{|x_-|}-\frac{\partial^2_\gamma y_-}{|y_-|}\Big)d\eta d\gamma.$$

We proceed as before

$$|G_{1,1}|\leq C(\|F(x)\|^3_{L^\infty}\|\partial^2_\gamma x\|_{C^\delta}+\|F(x)\|^4_{L^\infty}\|\partial^2_\gamma x\|^2_{L^\infty})\|\partial_\gamma z\|_{L^2}$$

and therefore $\|G_{1,1}\|_{L^2}\leq C\|z\|_{H^1}$. In a similar way we find $\|G_{1,2}\|_{L^2}\leq C\|z\|_{H^1}$. To estimate $G_{1,3}$ we write $G_{1,3}=G_{1,3,1}+G_{1,3,2}+G_{1,3,3}$ where $G_{1,3,1}$ and $G_{1,3,2}$ are the most singular terms:

$$G_{1,3,1}=-\frac{\gamma+\pi}{2\pi}\int\frac{\partial_\gamma y}{|\partial_\gamma y|^2}\cdot\int\partial_\gamma z_-\frac{x_-\cdot\partial_\gamma x_-}{|x_-|^3}d\eta d\gamma,$$

$$G_{1,3,2}=-\frac{\gamma+\pi}{2\pi}\int\frac{\partial_\gamma y}{|\partial_\gamma y|^2}\cdot\int\partial_\gamma y_-\frac{x_-\cdot\partial_\gamma z_-}{|x_-|^3}d\eta d\gamma,$$

because $G_{1,3,3}$ satisfies obviously the desired bound: $\|G_{1,3,3}\|_{L^2}\leq C\|z\|_{H^1}$. To control $I_{1,1}$, we use (17) and the fact that $\partial_\gamma x\cdot\partial^2_\gamma x=0$, that is:

$$|G_{1,3,1}|\leq\|F(y)\|_{L^\infty}\|F(x)\|^3_{L^\infty}\|x\|^2_{C^{2,\delta}}\int|\eta|^{\delta-1}\int(|\partial_\gamma z|+|\partial_\gamma z'|)d\gamma d\eta,$$

implying $\|G_{1,3,1}\|_{L^2}\leq C\|z\|_{H^1}$.

Inside the expression of $G_{1,3,2}$ we observe that

$$\partial_\gamma y\cdot\partial_\gamma y_-=\partial_\gamma y\cdot(\partial_\gamma y_--\eta\partial^2_\gamma y)$$

which together with the estimate

$$|\partial_\gamma y_- - \eta\partial_\gamma^2 y| \leq \|y\|_{C^{2,\delta}}|\eta|^{1+\delta}, \tag{19}$$

give us

$$|G_{1,3,2}| \leq \|F(y)\|_{L^\infty}\|F(x)\|^2_{L^\infty}\|y\|^2_{C^{2,\delta}}\int|\eta|^{\delta-1}\int(|\partial_\gamma z| + |\partial_\gamma z'|)d\gamma d\eta,$$

and $\|G_{1,3,2}\|_{L^2} \leq C\|z\|_{H^1}$.

Next let us write $G_{1,4} = G_{1,4,1} + G_{1,4,2}$, where

$$G_{1,4,1} = \frac{\gamma+\pi}{2\pi}\int\frac{\partial_\gamma y}{|\partial_\gamma y|^2}\cdot\int\partial_\gamma^2 x_-\Big(\frac{1}{|x_-|} - \frac{1}{|y_-|}\Big)d\eta d\gamma,$$

$$G_{1,4,2} = \frac{\gamma+\pi}{2\pi}\int\frac{\partial_\gamma y}{|\partial_\gamma y|^2}\cdot\int\frac{\partial_\gamma^2 z_-}{|y_-|}d\eta d\gamma.$$

Equality (18) allows us to obtain

$$|G_{1,4,1}| \leq \|F(y)\|^2_{L^\infty}\|F(x)\|_{L^\infty}\|\partial_\gamma^2 x\|_{C^\delta}\int_0^1\int|\eta|^{\delta-1}\int|\partial_\gamma z(\gamma+(r-1)\eta)|d\gamma d\eta dr,$$

and hence $\|G_{1,4,1}\|_{L^2} \leq C\|z\|_{H^1}$. Integration by parts allows us to decompose further $G_{1,4,2} = G^1_{1,4,2} + G^2_{1,4,2}$, where

$$G^1_{1,4,2} = \frac{\gamma+\pi}{2\pi}\int\frac{\partial_\gamma y}{|\partial_\gamma y|^2}\cdot\int\frac{\partial_\gamma z_-(y_-\cdot\partial_\gamma y_-)}{|y_-|^3}d\eta d\gamma,$$

$$G^2_{1,4,2} = -\frac{\gamma+\pi}{2\pi}\int\frac{\partial_\gamma^2 y}{|\partial_\gamma y|^2}\cdot\int\frac{\partial_\gamma z_-}{|y_-|}d\eta d\gamma.$$

The first term can be estimated as $G_{1,3,1}$:

$$|G^1_{1,4,2}| \leq C\|F(y)\|^4_{L^\infty}\|y\|^2_{C^{2,\delta}}\|z\|_{H^1} \leq C\|z\|_{H^1}.$$

We symmetrize $G^2_{1,4,2}$ as in $I_{1,1}$:

$$G^2_{1,4,2} = -\frac{\gamma+\pi}{4\pi|\partial_\gamma y|^2}\int\int\frac{\partial_\gamma^2 y_-\cdot\partial_\gamma z_-}{|y_-|}d\eta d\gamma$$

which yields the estimate:

$$|G^2_{1,4,2}| \leq C\|F(y)\|^3_{L^\infty}\|\partial_\gamma^2 y\|_{C^\delta}\|\partial_\gamma z\|_{L^2} \leq C\|z\|_{H^1},$$

implying that

$$|G_1| \leq C\|z\|_{H^1}.$$

For the sake of simplicity we exchange the variables in G_2 so that

$$G_2 = -\int_{-\pi}^{\xi}\frac{\partial_\gamma x}{|\partial_\gamma x|^2}\cdot\partial_\gamma\Big(\int\frac{\partial_\gamma x_-}{|x_-|}d\eta\Big)d\gamma + \int_{-\pi}^{\xi}\frac{\partial_\gamma y}{|\partial_\gamma y|^2}\cdot\partial_\gamma\Big(\int\frac{\partial_\gamma y_-}{|y_-|}d\eta\Big)d\gamma.$$

We claim that $\|G_2\|_{L^2} \leq C\|z\|_{H^1}$. To show that we decompose further $G_2 = G_{2,1} + G_{2,2}$, where

$$G_{2,1} = \int_{-\pi}^{\xi}\frac{\partial_\gamma x}{|\partial_\gamma x|^2}\cdot\int\partial_\gamma x_-\frac{x_-\cdot\partial_\gamma x_-}{|x_-|^3}d\eta d\gamma - \int_{-\pi}^{\xi}\frac{\partial_\gamma y}{|\partial_\gamma y|^2}\cdot\int\partial_\gamma y_-\frac{y_-\cdot\partial_\gamma y_-}{|y_-|^3}d\eta d\gamma$$

and

$$G_{2,2} = -\int_{-\pi}^{\xi}\frac{\partial_\gamma x}{|\partial_\gamma x|^2}\cdot\int\frac{\partial_\gamma^2 x_-}{|x_-|}d\eta d\gamma + \int_{-\pi}^{\xi}\frac{\partial_\gamma y}{|\partial_\gamma y|^2}\cdot\int\frac{\partial_\gamma^2 y_-}{|y_-|}d\eta d\gamma.$$

We deal with $G_{2,1}$ as with G_1, to obtain $|G_{2,1}|\leq C\|z\|_{H^1}$. The identities

$$\partial_\gamma x\cdot\partial_\gamma^2 x_-=-\partial_\gamma x\cdot\partial_\gamma^2 x'=-\partial_\gamma x_-\cdot\partial_\gamma^2 x'$$

allow us to obtain

$$G_{2,2}=\int_{-\pi}^{\xi}\int\frac{\partial_\gamma x_-\cdot\partial_\gamma^2 x'}{|\partial_\gamma x|^2|x_-|}d\eta d\gamma-\int_{-\pi}^{\xi}\int\frac{\partial_\gamma y_-\cdot\partial_\gamma^2 y'}{|\partial_\gamma y|^2|y_-|}d\eta d\gamma.$$

A new decomposition yields $G_{2,2}=G_{2,2,1}+G_{2,2,2}+G_{2,2,3}$, where

$$G_{2,2,1}=\int_{-\pi}^{\xi}\int\frac{\partial_\gamma z_-\cdot\partial_\gamma^2 x'}{|\partial_\gamma x|^2|x_-|}d\eta d\gamma,\quad G_{2,2,2}=\int_{-\pi}^{\xi}\int\frac{\partial_\gamma y_-\cdot\partial_\gamma^2 z'}{|\partial_\gamma x|^2|x_-|}d\eta d\gamma,$$

and $G_{2,2,3}$ collect the lower order characters, which can be estimated as before: $|G_{2,2,3}|\leq C\|z\|_{H^1}$. One has

$$G_{2,2,1}=\int_{-\pi}^{\xi}\Big[\partial_\gamma\Big(\int\frac{z_-\cdot\partial_\gamma^2 x'}{|\partial_\gamma x|^2|x_-|}d\eta\Big)-\int z_-\cdot\partial_\gamma\Big(\frac{\partial_\gamma^2 x'}{|\partial_\gamma x|^2|x_-|}\Big)d\eta\Big]d\gamma,$$

which helps us to decompose as follows: $G_{2,2,1}=G_{2,2,1}^1+G_{2,2,1}^2+G_{2,2,1}^3$, where

$$G_{2,2,1}^1=\Big(\int\frac{z_-\cdot\partial_\gamma^2 x'}{|\partial_\gamma x|^2|x_-|}d\eta\Big)\Big|_{\gamma=-\pi}^{\gamma=\xi},\quad G_{2,2,1}^2=\int_{-\pi}^{\xi}\int z_-\cdot\frac{\partial_\gamma^3 x'}{|\partial_\gamma x|^2|x_-|}d\eta d\gamma,$$

and $G_{2,2,1}^3$ consists of the lower order terms. At this point it is easy to get the estimate $|G_{2,2,1}^3|\leq C\|z\|_{H^1}$ and

$$|G_{2,2,1}^1|\leq 2\|F(x)\|_{L^\infty}^3\|z\|_{C^{\frac12}}\|\partial_\gamma^2 x\|_{L^\infty}\int|\eta|^{-\frac12}d\eta\leq C\|z\|_{H^1}$$

as a consequence of Sobolev's embedding. Concerning $G_{2,2,1}^2$ we write

$$\partial_\gamma^3 x'=\partial_\eta\partial_\gamma^2 x_-$$

and integrate by parts to find

$$G_{2,2,1}^2=\int_{-\pi}^{\xi}\int z_-\cdot\partial_\gamma^2 x_-\frac{x_-\cdot\partial_\gamma x'}{|\partial_\gamma x|^2|x_-|^3}d\eta d\gamma-\int_{-\pi}^{\xi}\int\partial_\gamma z'\cdot\frac{\partial_\gamma^2 x_-}{|\partial_\gamma x|^2|x_-|}d\eta d\gamma.$$

Proceeding as before we obtain

$$\begin{aligned}|G_{2,2,1}^2|\leq&\|F(x)\|_{L^\infty}^3\|\partial_\gamma^2 x\|_{C^\delta}\int_0^1\int|\eta|^{\delta-1}\int|\partial_\gamma z(\gamma+(r-1)\eta)|d\gamma d\eta dr\\&+\|F(x)\|_{L^\infty}^3\|\partial_\gamma^2 x\|_{C^\delta}\int|\eta|^{\delta-1}\int|\partial_\gamma z'|d\gamma d\eta\leq C\|z\|_{H^1}.\end{aligned}$$

Gathering together the last three estimates we have $|G_{2,2,1}|\leq C\|z\|_{H^1}$.

Regarding $G_{2,2,2}$ identity $\partial_\gamma^2 z'=\partial_\eta\partial_\gamma z_-$ and integration by parts yield

$$G_{2,2,2}=-\int_{-\pi}^{\xi}\int\frac{\partial_\gamma^2 y'\cdot\partial_\gamma z_-}{|\partial_\gamma x|^2|x_-|}d\eta d\gamma+\int_{-\pi}^{\xi}\int\frac{(\partial_\gamma y_-\cdot\partial_\gamma z_-)(x_-\cdot\partial_\gamma x')}{|\partial_\gamma x|^2|x_-|^3}d\eta d\gamma.$$

In the formula above we find two terms analogous to those of $G_{2,2,1}$, so that a similar argument gives us $|G_{2,2,2}|\leq C\|z\|_{H^1}$. Thereby we have finally obtained $I_{2,2}\leq C\|z\|_{H^1}^2$.

A consequence of all those estimates is the differential inequalities:

$$\frac{d}{dt}\|z\|_{L^2}^2\leq C\|z\|_{H^1}^2.$$

The next step is to analyze

$$\frac{1}{2}\frac{d}{dt}\|\partial_\gamma z\|^2_{L^2} = \int \partial_\gamma z\cdot\partial_\gamma z_t d\gamma = I_3 + I_4,$$

where

$$I_3 = \int \partial_\gamma z\cdot\int\partial_\gamma\Big(\frac{\partial_\gamma x_-}{|x_-|} - \frac{\partial_\gamma y_-}{|y_-|}\Big)d\eta d\gamma,\quad I_4 = \int\partial_\gamma z\cdot\partial_\gamma(\lambda(x)\partial_\gamma x - \lambda(y)\partial_\gamma y)d\gamma.$$

We split further I_3 :

$$I_{3,1} = \int\partial_\gamma z\cdot\int\Big(\frac{\partial^2_\gamma x_-}{|x_-|} - \frac{\partial^2_\gamma y_-}{|y_-|}\Big)d\eta d\gamma,$$

$$I_{3,2} = \int\partial_\gamma z\cdot\int\Big(-\frac{\partial_\gamma x_-(x_-\cdot\partial_\gamma x_-)}{|x_-|^3} + \frac{\partial_\gamma y_-(y_-\cdot\partial_\gamma y_-)}{|y_-|^3}\Big)d\eta d\gamma.$$

Then we write $I_{3,1} = I_{3,1,1} + I_{3,1,2}$, where

$$I_{3,1,1} = \int\partial_\gamma z\cdot\int\frac{\partial^2_\gamma z_-}{|x_-|}d\eta d\gamma,\quad I_{3,1,2} = \int\partial_\gamma z\cdot\int\partial^2_\gamma y_-\frac{|y_-|-|x_-|}{|x_-||y_-|}d\eta d\gamma.$$

Replacing in $I_{1,1}$ z by $\partial_\gamma z$ we find $I_{3,1,1}$, and

$$I_{3,1,1} = \frac{1}{4}\int\int|\partial_\gamma z_-|^2\frac{x_-\cdot\partial_\gamma x_-}{|x_-|^3}d\eta d\gamma \le C\|\partial_\gamma z\|^2_{L^2}.$$

At this stage of the proof we can easily obtain the estimate

$$\begin{aligned}I_{3,1,2} &\le \|F(x)\|_{L^\infty}\|F(y)\|_{L^\infty}\|\partial^2_\gamma y\|_{C^\delta}\int_0^1\int|\eta|^{\delta-1}\int|\partial_\gamma z||\partial_\gamma z(\gamma+(s-1)\eta)|d\gamma d\eta ds\\ &\le C\|z\|^2_{H^1},\end{aligned}$$

and we are done with $I_{3,1}$.

For $I_{3,2}$ we split further: $I_{3,2} = I_{3,2,1} + I_{3,2,2} + I_{3,2,3} + I_{3,2,4}$, where

$$I_{3,2,1} = -\int\partial_\gamma z\cdot\int\frac{\partial_\gamma z_-(x_-\cdot\partial_\gamma x_-)}{|x_-|^3}d\eta d\gamma,$$

$$I_{3,2,2} = -\int\partial_\gamma z\cdot\int\frac{\partial_\gamma y_-(z_-\cdot\partial_\gamma x_-)}{|x_-|^3}d\eta d\gamma,$$

$$I_{3,2,3} = -\int\partial_\gamma z\cdot\int\partial_\gamma y_-(y_-\cdot\partial_\gamma x_-)(|x_-|^{-3} - |y_-|^{-3})d\eta d\gamma,$$

and

$$I_{3,2,4} = -\int\partial_\gamma z\cdot\int\frac{\partial_\gamma y_-(y_-\cdot\partial_\gamma z_-)}{|y_-|^3}d\eta d\gamma.$$

Inequality (17) yields $I_{3,2,1}\le C\|z\|^2_{H^1}$. No cancellation is needed to get

$$I_{3,2,2}\le C\|z\|^2_{H^1},\quad I_{3,2,3}\le C\|z\|^2_{H^1}.$$

On the other hand, we pay special attention to $I_{3,2,4}$. By identity (18) we split it further

$$I^1_{3,2,4} = -\int\int\partial_\gamma z\cdot\frac{\partial_\gamma y_-}{|y_-|^3}\int_0^1(y_- - \partial_\gamma y(\gamma+(r-1)\eta)\eta)\cdot\partial^2_\gamma z(\gamma+(r-1)\eta)\eta dr d\eta d\gamma,$$

$$I^2_{3,2,4} = -\int\int\partial_\gamma z\cdot\frac{\partial_\gamma y_-}{|y_-|^3}\eta^2\int_0^1\partial_\gamma y(\gamma+(r-1)\eta)\cdot\partial^2_\gamma z(\gamma+(r-1)\eta)dr d\eta d\gamma.$$

In $I^1_{3,2,4}$ we have $\partial^2_\gamma z(\gamma+(r-1)\eta)\eta = \frac{d}{dr}(\partial_\gamma z(\gamma+(r-1)\eta))$ and integration by parts in r provides $I^1_{3,2,4} = I^{1,1}_{3,2,4} + I^{1,2}_{3,2,4} + I^{1,3}_{3,2,4}$ where

$$I^{1,1}_{3,2,4} = -\int\int \partial_\gamma z \cdot \frac{\partial_\gamma y_-}{|y_-|^3}\int_0^1 \partial^2_\gamma y(\gamma+(r-1)\eta)\eta^2 \cdot \partial_\gamma z(\gamma+(r-1)\eta)drd\eta d\gamma,$$

$$I^{1,2}_{3,2,4} = -\iint \partial_\gamma z \cdot \frac{\partial_\gamma y_-}{|y_-|^3}(y_- - \partial_\gamma y\eta)\cdot \partial_\gamma z d\eta d\gamma,$$

$$I^{1,3}_{3,2,4} = \iint \partial_\gamma z \cdot \frac{\partial_\gamma y_-}{|y_-|^3}(y_- - \partial_\gamma y'\eta)\cdot \partial_\gamma z' d\eta d\gamma.$$

Proceeding as before, we obtain the estimate $I^{1,j}_{3,2,4} \le C\|F(y)\|^3_{L^\infty}\|\partial^2_\gamma y\|^2_{L^\infty}\|\partial_\gamma z\|^2_{L^2} \le C\|z\|^2_{H^1}$, for $1\le j\le 3$. To handle $I^2_{3,2,4}$ we observe that

$$\partial_\gamma y\cdot\partial^2_\gamma z = \partial_\gamma y\cdot \partial^2_\gamma x = -\partial_\gamma z\cdot\partial^2_\gamma x \tag{20}$$

to get

$$I^2_{3,2,4} = \int\int \partial_\gamma z\cdot \frac{\partial_\gamma y_-}{|y_-|^3}\eta^2\int_0^1 \partial_\gamma z(\gamma+(r-1)\eta)\cdot \partial^2_\gamma x(\gamma+(r-1)\eta)drd\eta d\gamma.$$

Finally we estimate this term $I^2_{3,2,4} \le C\|F(y)\|^3_{L^\infty}\|\partial^2_\gamma y\|_{L^\infty}\|\partial^2_\gamma x\|_{L^\infty}\|\partial_\gamma z\|^2_{L^2} \le C\|z\|^2_{H^1}$, which completes the control of I_3 .

Next we proceed with a last splitting: $I_4 = I_{4,1}+I_{4,2}+I_{4,3}+I_{4,4}$, where

$$I_{4,1} = \int \partial_\gamma z\cdot\lambda(x)\partial^2_\gamma z d\gamma, \quad I_{4,2} = \int \partial_\gamma z\cdot(\lambda(x)-\lambda(y))\partial^2_\gamma y d\gamma,$$

$$I_{4,3} = \int |\partial_\gamma z|^2\partial_\gamma\lambda(x)d\gamma, \quad I_{4,4} = \int \partial_\gamma z\cdot\partial_\gamma y(\partial_\gamma\lambda(x)-\partial_\gamma\lambda(y))d\gamma.$$

Integration by parts in $I_{4,1}$ yields: $I_{4,1} \le \frac{1}{2}\|\partial_\gamma z\|^2_{L^2}\|\partial_\gamma\lambda(x)\|_{L^\infty} \le C\|z\|^2_{H^1}$ using that

$$\|\partial_\gamma\lambda(x)\|_{L^\infty} \le 2\|F(x)\|^2_{L^\infty}\|\partial^2_\gamma x\|_{C^\delta} + 2\|F(x)\|^3_{L^\infty}\|\partial^2_\gamma x\|^2_{L^\infty} \le C.$$

We have

$$I_{4,2} \le C\|\partial^2_\gamma y\|_{L^\infty}\|\partial_\gamma z\|_{L^2}\|\lambda(x)-\lambda(y)\|_{L^2} \le C\|z\|^2_{H^1}$$

by similar arguments used for $I_{2,2}$. The control of $I_{4,3}$ follows as in $I_{4,1}$. Finally, integration by parts

$$I_{4,4} = -\int \partial_\gamma z\cdot\partial^2_\gamma y(\lambda(x)-\lambda(y))d\gamma - \int \partial^2_\gamma z\cdot\partial_\gamma y(\lambda(x)-\lambda(y))d\gamma,$$

and identity (20) allow us to get the estimate:

$$I_{4,4} \le (\|\partial^2_\gamma y\|_{L^\infty} + \|\partial^2_\gamma x\|_{L^\infty})\|\partial_\gamma z\|_{L^2}\|\lambda(x)-\lambda(y)\|_{L^2} \le C\|z\|^2_{H^1}.$$

Therefore we have obtained

$$\frac{d}{dt}\|z\|_{H^1} \le C\|z\|_{H^1},$$

which allows us the use of Gronwall's inequality to get uniqueness.

Remark. We have proven the equality $\bar{x}(\alpha,t) = \bar{y}(\alpha,t)$. Therefore, undoing the reparameterization process, the patch θ with a moving boundary given by $x(\gamma,t)$ is the same as the patch $\bar{\theta}$ described by $y(\gamma,t)$.

Proof of Corollary 2.1. Let us consider $x(\gamma,t)$ and $y(\gamma,t)$ two solutions of (8,9) given by Theorem 1.1 in $C([0,T];H^3)$ with $y(\gamma,0)=x(\gamma,0)$ and satisfying the arc-chord condition in $[0,T]$. Proceeding as in the proof above we get the estimate

$$\frac{d}{dt}\|z\|_{H^1}\leq C\|z\|_{H^1}$$

for $z(\gamma,t)=x(\gamma,t)-y(\gamma,t)$, and then Gronwall's inequality provides uniqueness.

We are left with the task of proving that the patch weak solution given by $x(\gamma,t)$ is unique. In order to obtain that result one just has to check that, for $\theta(x,t)$ given by a patch and $\partial D^j(t)$ parameterized by $x(\gamma,t)$, the regularity needed in Theorem 1.2 is achieved. The fact that $H^3\subset C^{2,\delta}$, $0<\delta\leq 1/2$, provides the appropriate regularity for $x(\gamma,t)$. Next we will show that $x_t(\gamma,t)\in C([0,T];H^{\frac{2}{\log}})$ and since $H^{\frac{2}{\log}}\subset H^{\frac{3}{2}+\delta}\subset C^1$, $0<\delta<1/2$, the regularity for $x_t(\gamma,t)$ follows.

At this point it is easy to check that $x_t(\gamma,t)\in C([0,T];L^2)$. Next we consider $\partial^2_\gamma x_t=G_1+G_2$, where

$$G_1=\partial^2_\gamma\Big(\int\frac{\partial_\gamma x_-}{|x_-|}d\eta\Big),\quad G_2=\partial^2_\gamma\Big(\lambda\partial_\gamma x\Big), \tag{21}$$

using the notation above.

The term G_1 is decomposed further $G_1=G_{1,1}+G_{1,2}+G_{1,3}+G_{1,4}$, where

$$G_{1,1}=\int\frac{\partial^3_\gamma x_-}{|x_-|}d\eta,$$
$$G_{1,2}=-2\int\frac{\partial^2_\gamma x_-}{|x_-|^3}x_-\cdot\partial_\gamma x_- d\eta,$$
$$G_{1,3}=-\int\frac{\partial_\gamma x_-}{|x_-|^3}x_-\cdot\partial^2_\gamma x_- d\eta,$$

gathering in $G_{1,4}$ the terms in which only derivatives of order lower than 2 are involved. As before, the splitting easily yields

$$\sup_{[0,T]}\|G_{1,4}\|_{L^2}\leq\sqrt{2\pi}\sup_{[0,T]}\|G_{1,4}\|_{L^\infty}\leq C,$$

and, furthermore,

$$\sup_{[0,T]}(\|G_{1,2}\|_{L^2}+\|G_{1,3}\|_{L^2})$$
$$\leq\sup_{[0,T]}(3\sqrt{2\pi}\|F(x)\|^2_{L^\infty}\|\partial^2_\gamma x\|_{L^\infty}\|\partial^2_\gamma x\|_{C^\delta}\int|\eta|^{1-\delta}d\eta)\leq C.$$

The most singular term can be decomposed one more time as $G_{1,1}=G_{1,1,1}+G_{1,1,2}$, where

$$G_{1,1,1}=\frac{1}{|\partial_\gamma x|}\mathcal{L}(\partial^3_\gamma x),\quad G_{1,1,2}=\int\partial^3_\gamma x_-\Big(\frac{1}{|x_-|}-\frac{1}{|\partial_\gamma x||\eta|}\Big)d\eta.$$

It yields

$$|G_{1,1,2}|\leq(\|F(x)\|^2_{L^\infty}\|\partial^2_\gamma x\|_{L^\infty}))(2\pi|\partial^3_\gamma x|+\int|\partial^3_\gamma x'|d\eta),$$

and, therefore,

$$\sup_{[0,T]}\|G_{1,1,2}\|_{L^2}\leq C\sup_{[0,T]}\|x\|_{H^3}.$$

It remains to deal with $G_{1,1,1}$, which cannot be placed in $C([0,T],L^2)$. However, since less regularity is needed for $x_t(\gamma,t)$, we have

$$\sup_{[0,T]}\sum_{n\in\mathbb{Z}}\frac{|\widehat{G_{1,1,1}}(n)|^2}{\log^2(|n|+e)}\leq\sup_{[0,T]}\Big(\|F(x)\|_{L^\infty}\sum_{n\in\mathbb{Z}}\frac{|\widehat{\mathcal{L}(\partial_\gamma^3 x)}(n)|^2}{\log^2(|n|+e)}\Big)\leq C\sup_{[0,T]}\|x\|_{H^3},$$

where we have used (16). Hence we are done with $G_{1,1,1}$ and consequently with G_1. Let us observe that we have obtained a better regularity for G_2 due to the fact that $\lambda(x)\in C([0,T];H^{\frac{3}{\log}})$ (see (37) and below in the next section). That is, $G_2\in C([0,T];L^2)$ and, therefore, $x_t\in C([0,T];H^{\frac{2}{\log}})$, as desired.

3. Existence of an appropriate parameterization and commutator estimate

First let us define the operators used within the proofs, namely $\partial_{\log}$ and $I_{\log}$, a derivative and potential operator, respectively, as the following Fourier multipliers:

$$\widehat{\partial_{\log}f}(j)=\frac{j}{\log(|j|+e)}\hat{f}(j),\quad \widehat{I_{\log}f}(j)=\frac{1}{\log(|j|+e)}\hat{f}(j) \tag{22}$$

for $f\in L^2(\mathbb{T})$. Clearly we have that $f\in L^2(\mathbb{T})$ belongs to $H^{\frac{k}{\log}}$ if

$$\partial_{\log}\partial_\gamma^{k-1}f\in L^2\quad\text{or}\quad\partial_\gamma^k I_{\log}f\in L^2.$$

Next we show a commutator estimate needed in the existence and uniqueness proofs.

Lemma 3.1. *Let l^1 be the space of an absolutely convergence series. Then*

$$\|\partial_{\log}\partial_\gamma(gf)-g\partial_{\log}\partial_\gamma f\|_{L^2}\leq C(\|\widehat{\partial_\gamma g}\|_{l^1}\|\partial_{\log}f\|_{L^2}+\|\partial_{\log}\partial_\gamma g\|_{L^2}\|\widehat{f}\|_{l^1}), \tag{23}$$

where C is a universal constant. In particular, Sobolev's embedding implies that for any $\epsilon>0$ there is a constant $C_\epsilon>0$ such that

$$\|\partial_{\log}\partial_\gamma(gf)-g\partial_{\log}\partial_\gamma f\|_{L^2}\leq C_\varepsilon(\|g\|_{H^{3/2+\epsilon}}\|\partial_{\log}f\|_{L^2}+\|\partial_{\log}\partial_\gamma g\|_{L^2}\|f\|_{H^{1/2+\epsilon}}). \tag{24}$$

Proof. We have that

$$|(\partial_{\log}\partial_\gamma(gf)-g\partial_{\log}\partial_\gamma f)\hat{\ }(j)|\leq\sum_l\Big|\frac{j^2}{\log(|j|+e)}-\frac{(j-l)^2}{\log(|j-l|+e)}\Big||\hat{f}(j-l)||\hat{g}(l)|,$$

and the function $h(j)=j^2/\log(|j|+e)$ satisfies

$$h(j)-h(j-l)=\int_0^1\frac{d}{dr}h((j-l)+rl)dr=l\int_0^1h'(rl+(j-l))dr,$$

and, therefore,

$$|h(j)-h(j-l)|\leq|l|\int_0^1|h'(rl+(j-l))|dr\leq\frac{3(|l|+|j-l|)}{\log(|l|+|j-l|+e)}|l|.$$

It yields

$$|h(j)-h(j-l)|\leq\frac{3|l|^2}{\log(|l|+e)}+\frac{3|l||j-l|}{\log(|j-l|+e)},$$

and finally

$$\begin{aligned}&|(\partial_{\log}\partial_\gamma(gf)-g\partial_{\log}\partial_\gamma f)\hat{\ }(j)|\\&\leq\sum_l\frac{3|l|^2}{\log(|l|+e)}|\hat{f}(j-l)||\hat{g}(l)|+\sum_l\frac{3|l||j-l|}{\log(|j-l|+e)}|\hat{f}(j-l)||\hat{g}(l)|.\end{aligned}$$

Then Parseval's theorem gives

$$\|\partial_{\log}\partial_\gamma(gf)-g\partial_{\log}\partial_\gamma f\|_{L^2}\leq\Big(\sum_j\Big(\sum_l\frac{3|l|^2}{\log(|l|+e)}|\hat{f}(j-l)||\hat{g}(l)|\Big)^2\Big)^{1/2}$$
$$+\Big(\sum_j\Big(\sum_l\frac{3|l||j-l|}{\log(|j-l|+e)}|\hat{f}(j-l)||\hat{g}(l)|\Big)^2\Big)^{1/2}.$$

The Minkowski inequality provides (23). The proof ends by Sobolev's embedding in dimension one.

Proof of Proposition 1.3. Without loss of generality we may consider the case $k=3$, because the extension to $k>3$ is just a straightforward exercise once we know how to handle $k=3$. Also, in order to be concise we will show only the main part of the proof. That is, we will deal with the more dangerous terms in the needed estimates, leaving as an exercise to the reader the treatment to all the other more benevolent characters. In the main core of the proof are energy estimates; from them and with recent well-known mollifying arguments one can apply the classical Picard to conclude existence. The whole strategy can be found in [2, Chapter 3].

Often, in the following we will have to write double integrals in variables, say γ and η, and differences $f(\gamma)-f(\gamma-\eta)$. To simplify notation we shall write $f=f(\gamma,t)$, $f'=f(\gamma-\eta,t)$, and $f-f'=f_-$ when there is no danger of confusion. Furthermore, we shall write $\int=\int_{\mathbb{T}}$ and denote id as the identity, $C(t)$ will be a polynomial function in $\|F(x)\|_{L^\infty}$ and $\|x\|_{H^3}$ so that $C(t)\in C([0,T])$. As was mentioned before, most of the time we will show how to estimate the most singular terms: those in which the derivative of higher order is involved by the use of Leibnitz's derivative rule. The rest of the terms are denoted by $l.o.t.$ standing for lower order terms. Writing $l.o.t.\in X$ means that the lower order terms belong to the space X.

First we consider the evolution of the L^2 norm:

$$\frac{1}{2}\frac{d}{dt}\|\phi-id\|^2_{L^2}=\int(\phi-id)\phi_t d\gamma=I_1+I_2,$$

where

$$I_1=\int(\phi-id)\int\frac{\partial_\gamma\phi_-}{|x_-|}d\eta d\gamma,\quad I_2=\int\lambda(x)(\phi-id)\partial_\gamma\phi d\gamma.$$

For I_1 we find

$$\begin{aligned}I_1&=\iint(\phi(\gamma,t)-\gamma)\frac{\partial_\gamma\phi(\gamma,t)-\partial_\eta\phi(\eta,t)}{|x(\gamma,t)-x(\eta,t)|}d\eta d\gamma\\&=-\iint(\phi(\eta,t)-\eta)\frac{\partial_\gamma\phi(\gamma,t)-\partial_\eta\phi(\eta,t)}{|x(\gamma,t)-x(\eta,t)|}d\eta d\gamma\\&=\frac{1}{2}\iint(\phi(\gamma,t)-\gamma-(\phi(\eta,t)-\eta))\frac{\partial_\gamma\phi(\gamma,t)-\partial_\eta\phi(\eta,t)}{|x(\gamma,t)-x(\eta,t)|}d\eta d\gamma,\end{aligned}$$

hence

$$I_1=\frac{1}{2}\iint(\phi-id)_-\frac{\partial_\gamma((\phi-id)_-)}{|x_-|}d\eta d\gamma.$$

Integration by parts yields

$$I_1=\frac{1}{4}\iint|(\phi-id)_-|^2\frac{x_-\cdot\partial_\gamma x_-}{|x_-|^3}d\eta d\gamma.$$

Now we use (4) to rewrite

$$\frac{x_-\cdot\partial_\gamma x_-}{|x_-|^3}=\frac{x_-\cdot\partial_\gamma x_- -\partial_\gamma x\cdot\partial_\gamma^2 x\eta^2}{|x_-|^3},$$

and obtain

$$\Big|\frac{x_-\cdot\partial_\gamma x_-}{|x_-|^3}\Big|\leq\frac{2\|x\|^2_{C^{2,\frac12}}|\eta|^{2+\frac12}}{|x_-|^3}\leq 2\|F(x)\|^3_{L^\infty}\|x\|^2_{C^{2,\frac12}}|\eta|^{-\frac12}. \tag{25}$$

This yields

$$\begin{aligned}I_1\leq&\frac12\|F(x)\|^3_{L^\infty}\|x\|^2_{C^{2,\frac12}}\int|\eta|^{-\frac12}\int|(\phi-id)_-|^2d\gamma d\eta\\ \leq&\|F(x)\|^3_{L^\infty}\|x\|^2_{C^{2,\frac12}}\int|\eta|^{-\frac12}\int(|\phi-id|^2+|(\phi-id)'|^2)d\gamma d\eta\\ \leq&2\|F(x)\|^3_{L^\infty}\|x\|^2_{C^{2,\frac12}}\|\phi-id\|^2_{L^2}(t)\leq C(t)\|\phi-id\|^2_{L^2}(t).\end{aligned}$$

The term I_2 can be rewritten as follows:

$$I_2=\int\lambda(x)(\phi-id)(\partial_\gamma\phi-1)d\gamma+\int\lambda(x)(\phi-id)d\gamma.$$

The first term above can be handled by integration by parts. In the second the Cauchy-Schwarz inequality yields

$$I_2\leq C(t)\|\phi-id\|^2_{L^2}(t)+\frac12\|\lambda(x)\|^2_{L^2}.$$

The bounds for $\lambda(x)$ (below we show that $\lambda\in H^{\frac{3}{\log}}$) finally provide

$$\frac{d}{dt}\|\phi-id\|^2_{L^2}(t)\leq C(t)\|\phi-id\|^2_{L^2}(t)+C(t). \tag{26}$$

Next, we consider the evolution of the higher order norm

$$\begin{aligned}\frac12\frac{d}{dt}\|\partial_{\log}\partial^2_\gamma\phi\|^2_{L^2}=&\int\partial_{\log}\partial^2_\gamma\phi\partial_{\log}\partial^2_\gamma\phi_t d\gamma\\ =&\int\partial_{\log}\partial^2_\gamma\phi\partial_{\log}\Big(\partial^2_\gamma\Big(\int\frac{\partial_\gamma\phi_-}{|x_-|}d\eta\Big)\Big)d\gamma+\int\partial_{\log}\partial^2_\gamma\phi\partial_{\log}\Big(\partial^2_\gamma\Big(\lambda(x)\partial_\gamma\phi\Big)\Big)d\gamma\\ =&J+K,\end{aligned} \tag{27}$$

to bound the J and K terms.

With J we split further $J=J_1+J_2+J_3$, where

$$J_1=\int\partial_{\log}\partial^2_\gamma\phi\partial_{\log}\Big(\int\frac{\partial^3_\gamma\phi_-}{|x_-|}d\eta\Big)d\gamma,$$

$$J_2=-2\int\partial_{\log}\partial^2_\gamma\phi\partial_{\log}\Big(\int\frac{\partial^2_\gamma\phi_- x_-\cdot\partial_\gamma x_-}{|x_-|^3}d\eta\Big)d\gamma,$$

and

$$J_3=-\int\partial_{\log}\partial^2_\gamma\phi\partial_{\log}\Big(\int\partial_\gamma\phi_-\partial_\gamma\Big(\frac{x_-\cdot\partial_\gamma x_-}{|x_-|^3}\Big)d\eta\Big)d\gamma.$$

The fact that $|\partial_\gamma x|$ does not depend on γ gives

$$\int\partial_{\log}\partial^2_\gamma\phi\partial_{\log}\Big(\int\frac{\partial^3_\gamma\phi_-}{|\partial_\gamma x||\eta|}\Big)d\eta\Big)d\gamma=\frac{1}{|\partial_\gamma x|}\int\partial_{\log}\partial^2_\gamma\phi\partial_{\log}\mathcal{L}(\partial^3_\gamma\phi)d\gamma=0,$$

where $\mathcal{L}$ was defined in (15) and has properties (16). Therefore, one obtains

$$J_1 = \int \partial_{\log}\partial_\gamma^2\phi \partial_{\log}\Big(\int \partial_\gamma^3\phi_-\Big(\frac{1}{|x_-|}-\frac{1}{|\partial_\gamma x||\eta|}\Big)d\eta\Big)d\gamma.$$

This extra cancellation suggests the further splitting $J_1 = J_{1,1}+J_{1,2}$, where

$$J_{1,1} = \int \partial_{\log}\partial_\gamma^2\phi \partial_{\log}\Big(\partial_\gamma^3\phi \int \Big(\frac{1}{|x_-|}-\frac{1}{|\partial_\gamma x||\eta|}\Big)d\eta\Big)d\gamma,$$

$$J_{1,2} = -\int \partial_{\log}\partial_\gamma^2\phi \partial_{\log}\Big(\int \partial_\gamma^3\phi'\Big(\frac{1}{|x_-|}-\frac{1}{|\partial_\gamma x||\eta|}\Big)d\eta\Big)d\gamma,$$

and $J_{1,1} = J_{1,1}^1 + J_{1,1}^2 + J_{1,1}^3$, where

$$J_{1,1}^1 = \int \partial_{\log}\partial_\gamma^2\phi[\partial_{\log}\partial_\gamma(A\partial_\gamma^2\phi) - A\partial_{\log}\partial_\gamma(\partial_\gamma^2\phi)]d\gamma,$$

$$J_{1,1}^2 = \int \partial_{\log}\partial_\gamma^2\phi A\partial_{\log}\partial_\gamma(\partial_\gamma^2\phi)d\gamma, \quad J_{1,1}^3 = -\int \partial_{\log}\partial_\gamma^2\phi\partial_{\log}(\partial_\gamma A\partial_\gamma^2\phi)d\gamma,$$

with

$$A = \int \Big(\frac{1}{|x_-|}-\frac{1}{|\partial_\gamma x||\eta|}\Big)d\eta. \tag{28}$$

In $J_{1,1}^1$ we use the commutator estimate (24) to find

$$\begin{aligned} J_{1,1}^1 &\leq C\|\partial_{\log}\partial_\gamma^2\phi\|_{L^2}(\|A\|_{H^2}\|\partial_{\log}\partial_\gamma^2\phi\|_{L^2} + \|\partial_{\log}\partial_\gamma A\|_{L^2}\|\partial_\gamma^2\phi\|_{H^{\frac{1}{\log}}}) \\ &\leq C\|A\|_{H^2}\|\partial_{\log}\partial_\gamma^2\phi\|_{L^2}^2. \end{aligned}$$

Furthermore, we have

$$\partial_\gamma A = -\int \frac{x_-\cdot\partial_\gamma x_- - \partial_\gamma x\cdot\partial_\gamma^2 x\eta^2}{|x_-|^3}d\eta, \tag{29}$$

and, therefore,

$$\partial_\gamma^2 A = -\int \frac{x_-\cdot\partial_\gamma^2 x_- - \partial_\gamma x\cdot\partial_\gamma^3 x\eta^2}{|x_-|^3}d\eta + l.o.t.,$$

where $\|l.o.t.\|_{L^2}\leq C(t)$. Identity (4) yields

$$\begin{aligned} x_-\cdot\partial_\gamma^2 x_- - \partial_\gamma x\cdot\partial_\gamma^3 x\eta^2 &= (x_- - \partial_\gamma x\eta)\cdot\partial_\gamma^2 x_- + \eta\partial_\gamma x\cdot\partial_\gamma^2 x_- - \partial_\gamma x\cdot\partial_\gamma^3 x\eta^2 \\ &= (x_- - \partial_\gamma x\eta)\cdot\partial_\gamma^2 x_- - \eta\partial_\gamma x_-\cdot\partial_\gamma^2 x' + |\partial_\gamma^2 x|^2\eta^2, \end{aligned}$$

implying

$$x_-\cdot\partial_\gamma^2 x_- - \partial_\gamma x\cdot\partial_\gamma^3 x\eta^2 = (x_- - \partial_\gamma x\eta)\cdot\partial_\gamma^2 x_- - \eta(\partial_\gamma x_- - \partial_\gamma^2 x\eta)\cdot\partial_\gamma^2 x' + \eta^2\partial_\gamma^2 x\cdot\partial_\gamma^2 x_-.$$

The above configuration provides

$$|x_-\cdot\partial_\gamma^2 x_- - \partial_\gamma x\cdot\partial_\gamma^3 x\eta^2| \leq 3\|x\|_{C^2}\|x\|_{C^{2,\frac{1}{2}}}|\eta|^{2+\frac{1}{2}},$$

and, therefore,

$$\Big|-\int \frac{(x_- - \partial_\gamma x\eta)\cdot\partial_\gamma^2 x_- - \partial_\gamma x\cdot\partial_\gamma^3 x\eta^2}{|x_-|^3}d\eta\Big| \leq 3\|x\|_{C^2}\|x\|_{C^{2,\frac{1}{2}}}\|F(x)\|_{L^\infty}^3,$$

implying that $\partial_\gamma^2 A\in C([0,T],L^2)$ and the estimate

$$J_{1,1}^1 \leq C(t)\|\partial_{\log}\partial_\gamma^2\phi\|_{L^2}^2.$$

Then, integration by parts yields

$$J_{1,1}^2 \leq \|\partial_\gamma A\|_{L^\infty}\|\partial_{\log}\partial_\gamma^2\phi\|_{L^2}^2 \leq \|A\|_{H^2}\|\partial_{\log}\partial_\gamma^2\phi\|_{L^2}^2 \leq C(t)\|\partial_{\log}\partial_\gamma^2\phi\|_{L^2}^2.$$

In order to estimate $J_{1,1}^3$ we use the following inequalities:

$$\begin{aligned}J_{1,1}^3 &\leq \|\partial_{\log}\partial_\gamma^2\phi\|_{L^2}\|\partial_{\log}(\partial_\gamma A\partial_\gamma^2\phi)\|_{L^2}\\ &\leq C\|\partial_{\log}(\partial_\gamma A)\|_{L^2}\|\partial_{\log}\partial_\gamma^2\phi\|_{L^2}^2 \leq C\|A\|_{H^2}\|\partial_{\log}\partial_\gamma^2\phi\|_{L^2}^2\\ &\leq C(t)\|\partial_{\log}\partial_\gamma^2\phi\|_{L^2}^2.\end{aligned}$$

Hence

$$J_{1,1} \leq C(t)\|\partial_{\log}\partial_\gamma^2\phi\|_{L^2}^2.$$

It remains to control $J_{1,2}$. We rewrite $J_{1,2} = J_{1,2}^1 + J_{1,2}^2$ given by

$$J_{1,2}^1 = \int \partial_{\log}\partial_\gamma^2\phi\partial_{\log}\Big(\int \partial_\gamma^3\phi'\Omega_1 d\eta\Big)d\gamma, \quad J_{1,2}^2 = \int \partial_{\log}\partial_\gamma^2\phi\partial_{\log}\Big(\int \partial_\gamma^3\phi'\Omega_2 d\eta\Big)d\gamma,$$

where $|\partial_\gamma x|^{-1}|\eta|^{-1} - |x_-|^{-1} = \Omega_1 + \Omega_2$ with

$$\Omega_1 = \frac{\big|\frac{x_-}{\eta} - \partial_\gamma x\big|^2}{|\partial_\gamma x|\big(|\partial_\gamma x| + |\frac{x_-}{\eta}|\big)|x_-|}, \quad \Omega_2 = 2\frac{\big(\frac{x_-}{\eta} - \partial_\gamma x\big)\cdot\partial_\gamma x}{|\partial_\gamma x|\big(|\partial_\gamma x| + |\frac{x_-}{\eta}|\big)|x_-|}. \tag{30}$$

Next we will show how to deal with $J_{1,2}^2$ and since the kernel Ω_2 is more singular than Ω_1, we leave to the reader the analogous details for $J_{1,2}^1$.

Identity (4) allows us to rewrite

$$\Omega_2 = 2\frac{\big(\frac{x_-}{\eta} - \partial_\gamma x + \frac{1}{2}\partial_\gamma^2 x\eta\big)\cdot\partial_\gamma x}{|\partial_\gamma x|\big(|\partial_\gamma x| + |\frac{x_-}{\eta}|\big)|x_-|} \tag{31}$$

and the splitting $J_{1,2}^2 = J_{1,2}^{2,1} + J_{1,2}^{2,2}$, where

$$\begin{aligned}J_{1,2}^{2,1} &= \int \partial_{\log}\partial_\gamma^2\phi\partial_{\log}\Big(\int \partial_\gamma^2\phi'\partial_\eta\Omega_2 d\eta\Big)d\gamma,\\ J_{1,2}^{2,2} &= -\int \partial_{\log}\partial_\gamma^2\phi\partial_{\log}\Big(\partial_\gamma^2\phi'\Omega_2\Big|_{\eta=-\pi}^{\eta=\pi}\Big)d\gamma.\end{aligned} \tag{32}$$

In the case of $J_{1,2}^{2,2}$ let us observe that the functions $\Omega_2(\gamma, \pm\pi)$ are regular enough to obtain

$$\begin{aligned}J_{1,2}^{2,2} &\leq \|\partial_{\log}\partial_\gamma^2\phi\|_{L^2}\|\partial_{\log}(\partial_\gamma^2\phi'\Omega_2\Big|_{\eta=-\pi}^{\eta=\pi})\|_{L^2} \leq C\|\partial_{\log}\partial_\gamma^2\phi\|_{L^2}^2\sum_{\pm}\|\partial_{\log}\Omega_2\,_{|\eta=\pm\pi}\|_{L^2}\\ &\leq C\|\partial_{\log}\partial_\gamma^2\phi\|_{L^2}^2\sum_{\pm}\|\partial_\gamma\Omega_2\,_{|\eta=\pm\pi}\|_{L^2} \leq C(t)\|\partial_{\log}\partial_\gamma^2\phi\|_{L^2}^2.\end{aligned}$$

Regarding $J_{1,2}^{2,1}$, we proceed as follows:

$$\begin{aligned}J_{1,2}^{2,1} &\leq \|\partial_{\log}\partial_\gamma^2\phi\|_{L^2}\|\partial_{\log}\Big(\int \partial_\gamma^2\phi'\partial_\eta\Omega_2 d\eta\Big)\|_{L^2}\\ &\leq \|\partial_{\log}\partial_\gamma^2\phi\|_{L^2}\|\partial_\gamma\Big(\int \partial_\gamma^2\phi'\partial_\eta\Omega_2 d\eta\Big)\|_{L^2}\\ &\leq \|\partial_{\log}\partial_\gamma^2\phi\|_{L^2}\Big(\|\int \partial_\gamma^2\phi'\partial_\gamma\partial_\eta\Omega_2 d\eta\|_{L^2} + \|\int \partial_\gamma^3\phi'\partial_\eta\Omega_2 d\eta\|_{L^2}\Big),\end{aligned} \tag{33}$$

and two new terms appear that have to be controlled in L^2:

$$B = \int \partial_\gamma^2 \phi' \partial_\gamma \partial_\eta \Omega_2 d\eta, \quad D = \int \partial_\gamma^3 \phi' \partial_\eta \Omega_2 d\eta. \tag{34}$$

In order to do that first we will prove the bound $\|\partial_\gamma \partial_\eta \Omega_2\|_{L^2} \leq C(t)$ to obtain

$$\|B\|_{L^2} \leq \|\partial_\gamma^2 \phi\|_{L^1} \|\partial_\gamma \partial_\eta \Omega_2\|_{L^2} \leq C(t) \|\partial_{\log} \partial_\gamma^2 \phi\|_{L^2}. \tag{35}$$

With the help of formula (31) we split $\partial_\eta \Omega_2 = \partial\Omega_{2,1} + \partial\Omega_{2,2} + \partial\Omega_{2,3} + \partial\Omega_{2,4}$, where

$$\partial\Omega_{2,1} = \frac{-2}{\eta^2} \frac{(x_- - \partial_\gamma x \eta + \frac{1}{2}\partial_\gamma^2 x \eta^2) \cdot \partial_\gamma x}{|\partial_\gamma x|\big(|\partial_\gamma x| + |\frac{x_-}{\eta}|\big)|x_-|}, \quad \partial\Omega_{2,2} = \frac{2}{\eta} \frac{(\partial_\gamma x' - \partial_\gamma x + \partial_\gamma^2 x \eta) \cdot \partial_\gamma x}{|\partial_\gamma x|\big(|\partial_\gamma x| + |\frac{x_-}{\eta}|\big)|x_-|},$$

$$\partial\Omega_{2,3} = \frac{-2}{\eta} \frac{(x_- - \partial_\gamma x \eta + \frac{1}{2}\partial_\gamma^2 x \eta^2) \cdot \partial_\gamma x}{|\partial_\gamma x|\big(|\partial_\gamma x| + |\frac{x_-}{\eta}|\big)|x_-|^2} \frac{x_- \cdot \partial_\gamma x'}{|x_-|}, \tag{36}$$

and

$$\partial\Omega_{2,4} = \frac{-2}{\eta} \frac{(x_- - \partial_\gamma x \eta + \frac{1}{2}\partial_\gamma^2 x \eta^2) \cdot \partial_\gamma x}{|\partial_\gamma x|\big(|\partial_\gamma x| + |\frac{x_-}{\eta}|\big)^2|x_-|} \frac{\frac{x_-}{\eta} \cdot \frac{\partial_\gamma x' \eta - x_-}{\eta^2}}{|\frac{x_-}{\eta}|}.$$

Next we will show how to deal with $\partial_\gamma \partial\Omega_{2,1}$ and since the other kernels are similar or even easier to handle we will skip the details.

We have

$$\partial_\gamma \partial\Omega_{2,1} = \frac{-2}{\eta^2} \frac{(\partial_\gamma x_- - \partial_\gamma^2 x \eta + \frac{1}{2}\partial_\gamma^3 x \eta^2) \cdot \partial_\gamma x}{|\partial_\gamma x|\big(|\partial_\gamma x| + |\frac{x_-}{\eta}|\big)|x_-|} + l.o.t.,$$

where $\|l.o.t.\|_{L^2} \leq C(t)$. The identity

$$\partial_\gamma x_- - \partial_\gamma^2 x \eta + \frac{1}{2}\partial_\gamma^3 x \eta^2 = \eta^2 \int_0^1 r(\partial_\gamma^3 x(\gamma) - \partial_\gamma^3 x(\gamma + (r-1)\eta))dr$$

allows us to write

$$\begin{aligned}(\partial_\gamma x_- &- \partial_\gamma^2 x \eta + \frac{1}{2}\partial_\gamma^3 x \eta^2) \cdot \partial_\gamma x \\ &= \eta^2 \int_0^1 r\partial_\gamma^3 x(\gamma+(r-1)\eta)) \cdot (\partial_\gamma x(\gamma+(r-1)\eta) - \partial_\gamma x(\gamma))dr \\ &\quad + \eta^2 \int_0^1 r(\partial_\gamma^3 x(\gamma) \cdot \partial_\gamma x(\gamma) - \partial_\gamma^3 x(\gamma+(r-1)\eta) \cdot \partial_\gamma x(\gamma+(r-1)\eta))dr.\end{aligned}$$

The use of equality (4) and integration by parts in r yield

$$\begin{aligned}(\partial_\gamma x_- &- \partial_\gamma^2 x \eta + \frac{1}{2}\partial_\gamma^3 x \eta^2) \cdot \partial_\gamma x \\ &= \eta^2 \int_0^1 r\partial_\gamma^3 x(\gamma+(r-1)\eta)) \cdot (\partial_\gamma x(\gamma+(r-1)\eta) - \partial_\gamma x(\gamma))dr \\ &\quad - \eta^3 \int_0^1 r^2 \partial_\gamma^2 x(\gamma+(r-1)\eta) \cdot \partial_\gamma^3 x(\gamma+(r-1)\eta)dr,\end{aligned}$$

and, therefore,

$$|(\partial_\gamma x_- - \partial_\gamma^2 x \eta + \frac{1}{2}\partial_\gamma^3 x \eta^2) \cdot \partial_\gamma x| \leq 2|\eta|^3 \|x\|_{C^2} \int_0^1 |\partial_\gamma^3 x(\gamma+(r-1)\eta)|dr.$$

Hence

$$|\partial_\gamma \partial\Omega_{2,1}| \leq 4\|F(x)\|^3_{L^\infty}\|x\|_{C^2}\int_0^1 |\partial^3_\gamma x(\gamma+(r-1)\eta)|dr + |l.o.t.|.$$

Finally an integration in γ gives the desired property: $\|\partial_\gamma\partial\Omega_{2,1}\|_{L^2} \leq C(t)$. Analogously we have $\|\partial_\gamma\partial\Omega_{2,j}\|_{L^2} \leq C(t)$ for $j = 2, 3, 4$ and therefore the same bound holds for $\partial_\gamma\partial_\eta\Omega_2$:

$$\|\partial_\gamma\partial_\eta\Omega_2\|_{L^2} \leq C(t).$$

We achieve the desired estimate (35).

Regarding D, we first integrate by parts and then split

$$D = \int \partial_\eta(\partial^2_\gamma\phi_-)\partial_\eta\Omega_2 d\eta = -\int \partial^2_\gamma\phi_-\partial^2_\eta\Omega_2 d\eta + \partial^2_\gamma\phi_-\partial_\eta\Omega_2\Big|_{\eta=-\pi}^{\eta=\pi} = D_1 + D_2.$$

Then formulas (36) show that the functions $\partial_\eta\Omega_2(\gamma, \pm\pi)$ are regular enough to get an appropriate bound for D_2:

$$\|D_2\|_{L^2} \leq C(t)\|\partial^2_\gamma\phi\|_{L^2} \leq C(t)\|\partial_{\log}\partial^2_\gamma\phi\|_{L^2}.$$

Following the decomposition for $\partial_\eta\Omega_2$ in (36), let us introduce $\partial_\eta\partial\Omega_{2,1} = \partial^2\Omega^1_{2,1} + \partial^2\Omega^2_{2,1} + \partial^2\Omega^3_{2,1} + \partial^2\Omega^4_{2,1}$, where

$$\partial^2\Omega^1_{2,1} = \frac{4}{\eta^3}\frac{(x_- - \partial_\gamma x\eta + \frac{1}{2}\partial^2_\gamma x\eta^2)\cdot\partial_\gamma x}{|\partial_\gamma x|\big(|\partial_\gamma x| + |\frac{x_-}{\eta}|\big)|x_-|}, \quad \partial^2\Omega^2_{2,1} = \frac{-2}{\eta^2}\frac{(-\partial_\gamma x_- + \partial^2_\gamma x\eta)\cdot\partial_\gamma x}{|\partial_\gamma x|\big(|\partial_\gamma x| + |\frac{x_-}{\eta}|\big)|x_-|},$$

$$\partial^2\Omega^3_{2,1} = \frac{2}{\eta^2}\frac{(x_- - \partial_\gamma x\eta + \frac{1}{2}\partial^2_\gamma x\eta^2)\cdot\partial_\gamma x}{|\partial_\gamma x|\big(|\partial_\gamma x| + |\frac{x_-}{\eta}|\big)|x_-|^2}\frac{x_-\cdot\partial_\gamma x'}{|x_-|},$$

and

$$\partial^2\Omega^4_{2,1} = \frac{2}{\eta^2}\frac{(x_- - \partial_\gamma x\eta + \frac{1}{2}\partial^2_\gamma x\eta^2)\cdot\partial_\gamma x}{|\partial_\gamma x|\big(|\partial_\gamma x| + |\frac{x_-}{\eta}|\big)^2|x_-|}\frac{\frac{x_-}{\eta}\cdot\frac{\partial_\gamma x'\eta - x_-}{\eta^2}}{|\frac{x_-}{\eta}|}.$$

As was shown before, we have

$$x_- - \partial_\gamma x\eta + \frac{1}{2}\partial^2_\gamma x\eta^2 = \frac{1}{2}\eta^3\int_0^1 r^2\partial^3_\gamma x(\gamma + (r-1)\eta)dr$$

and, therefore,

$$|\partial^2\Omega^1_{2,1}| \leq \frac{2}{|\eta|}\|F(x)\|^2_{L^\infty}\int_0^1 |\partial^3_\gamma x(\gamma + (r-1)\eta)|dr.$$

Analogously, we obtain

$$\sum_{j=2}^4 |\partial^2\Omega^j_{2,1}|$$
$$\leq \frac{2}{|\eta|}(\|F(x)\|^2_{L^\infty} + \|F(x)\|^3_{L^\infty}\|x\|_{C^1} + \|F(x)\|^2_{L^\infty}\|x\|_{C^1})\int_0^1 |\partial^3_\gamma x(\gamma+(r-1)\eta)|dr,$$

implying

$$|\partial_\eta\partial\Omega_{2,1}| \leq \frac{1}{|\eta|}C(t)\int_0^1 |\partial^3_\gamma x(\gamma + (r-1)\eta)|dr.$$

The same approach for $\partial_\eta \partial \Omega_{2,j}$ with $j = 2, 3, 4$ yields

$$|\partial^2_\eta \Omega_2| \leq \frac{1}{|\eta|} C(t) \int_0^1 |\partial^3_\gamma x(\gamma + (r-1)\eta)| dr.$$

Therefore, we get the estimate

$$|D_1| \leq C(t) \|\partial^2_\gamma \phi\|_{C^{\frac{1}{3}}} \int \frac{1}{|\eta|^{2/3}} \int_0^1 |\partial^3_\gamma x(\gamma + (r-1)\eta)| dr d\eta,$$

and consequently

$$\|D_1\|_{L^2} \leq C(t) \|\partial^2_\gamma \phi\|_{C^{\frac{1}{3}}} \|\partial^3_\gamma x\|_{L^2} \leq C(t) \|\partial^2_\gamma \phi\|_{H^{\frac{11}{12}}} \leq C(t) \|\partial_{\log} \partial^2_\gamma \phi\|_{L^2}$$

by Sobolev embedding. Putting all those estimates together we obtain

$$\|D\|_{L^2} \leq C(t) \|\partial_{\log} \partial^2_\gamma \phi\|_{L^2},$$

which together with (35) allows us to get finally the needed estimate for $J^{2,1}_{1,2}$ in (32) using (33). We are then done with $J^2_{1,2}$.

For the less singular kernel Ω_1 in (30) a similar analysis yields

$$J^1_{1,2} \leq C(t) \|\partial_{\log} \partial^2_\gamma \phi\|^2_{L^2}.$$

Hence the same estimate is achieved for $J_{1,2}$ and accordingly for J_1:

$$J_1 \leq C(t) \|\partial_{\log} \partial^2_\gamma \phi\|^2_{L^2}.$$

Next we estimate $J_2 = J_{2,1} + J_{2,2}$ given by

$$J_{2,1} = -2 \int \partial_{\log} \partial^2_\gamma \phi \partial_{\log} \Big(\partial^2_\gamma \phi \int \frac{x_- \cdot \partial_\gamma x_-}{|x_-|^3} d\eta \Big) d\gamma = 2 \int \partial_{\log} \partial^2_\gamma \phi \partial_{\log} \Big(\partial^2_\gamma \phi \partial_\gamma A \Big) d\gamma,$$

and

$$\begin{aligned} J_{2,2} &= 2 \int \partial_{\log} \partial^2_\gamma \phi \partial_{\log} \Big(\int \partial^2_\gamma \phi' \frac{x_- \cdot \partial_\gamma x_-}{|x_-|^3} d\eta \Big) d\gamma \\ &= 2 \int \partial_{\log} \partial^2_\gamma \phi \partial_{\log} \Big(\int \partial^2_\gamma \phi' \Omega_3 d\eta \Big) d\gamma, \end{aligned}$$

where $\partial_\gamma A$ was introduced in (29) and the kernel Ω_3 can be rewritten as

$$\Omega_3 = \frac{x_- \cdot \partial_\gamma x_- - \partial_\gamma x \cdot \partial^2_\gamma x \eta^2}{|x_-|^3}.$$

Observe that $J_{2,1} = 2J^3_{1,1}$ and, therefore, we already know the estimate of that term. The other $J_{2,2}$ is similar to $J^{2,1}_{1,2}$ because the kernel Ω_3 is of degree 0 as $\partial_\eta \Omega_2$, and has the same loss of regularity in the tangential direction. Then, as before we obtain

$$|\partial_\gamma \Omega_3| \leq C(t) \int_0^1 |\partial^3_\gamma x(\gamma + (r-1)\eta)| dr, \quad |\partial_\eta \Omega_3| \leq \frac{1}{|\eta|} C(t) \int_0^1 |\partial^3_\gamma x(\gamma + (r-1)\eta)| dr,$$

helping to estimate $J_{2,2}$, and

$$J_2 \leq C(t) \|\partial_{\log} \partial^2_\gamma \phi\|^2_{L^2}.$$

Finally, to deal with J_3, we proceed as follows:

$$J_3 \le \|\partial_{\log}\partial_\gamma^2\phi\|_{L^2}\|\partial_{\log}\Big(\int \partial_\gamma\phi_-\partial_\gamma\Big(\frac{x_-\cdot\partial_\gamma x_-}{|x_-|^3}\Big)d\eta\Big)\|_{L^2}$$
$$\le \|\partial_{\log}\partial_\gamma^2\phi\|_{L^2}\|\partial_\gamma\Big(\int \partial_\gamma\phi_-\partial_\gamma\Big(\frac{x_-\cdot\partial_\gamma x_-}{|x_-|^3}\Big)d\eta\Big)\|_{L^2},$$

that is,

$$J_3 \le \|\partial_{\log}\partial_\gamma^2\phi\|_{L^2}\Big(\|\int \partial_\gamma^2\phi_-\partial_\gamma\Big(\frac{x_-\cdot\partial_\gamma x_-}{|x_-|^3}\Big)d\eta\|_{L^2}+\|\int \partial_\gamma\phi_-\partial_\gamma^2\Big(\frac{x_-\cdot\partial_\gamma x_-}{|x_-|^3}\Big)d\eta\|_{L^2}\Big).$$

Next let us observe that the two inequalities

$$\Big|\partial_\gamma^2\phi_-\partial_\gamma\Big(\frac{x_-\cdot\partial_\gamma x_-}{|x_-|^3}\Big)\Big|$$
$$\le \frac{\|\partial_\gamma^2\phi\|_{C^{\frac13}}}{|\eta|^{\frac23}}\Big(\|F(x)\|_{L^\infty}^2\int_0^1|\partial_\gamma^3x(\gamma+(r-1)\eta)|dr+4\|F(x)\|_{L^\infty}^3\|x\|_{C^2}^2\Big),$$
$$\Big|\partial_\gamma\phi_-\partial_\gamma^2\Big(\frac{x_-\cdot\partial_\gamma x_-}{|x_-|^3}\Big)\Big|$$
$$\le C(t)\|\partial_\gamma^2\phi\|_{L^\infty}\Big(|\partial_\gamma^3x|+|\partial_\gamma^3x'|+\int_0^1|\partial_\gamma^3x(\gamma+(r-1)\eta)|dr+1\Big),$$

together with Sobolev embedding yield

$$J_3 \le C(t)\|\partial_{\log}\partial_\gamma^2\phi\|_{L^2}(\|\partial_\gamma^2\phi\|_{C^{\frac13}}+\|\partial_\gamma^2\phi\|_{L^\infty}) \le C(t)\|\partial_{\log}\partial_\gamma^2\phi\|_{L^2}^2,$$

giving us the control:

$$J \le C(t)\|\partial_{\log}\partial_\gamma^2\phi\|_{L^2}^2.$$

To finish, it remains to deal with K. First we will show the regularity of

$$\lambda(x)\in C([0,T];H^{\frac{3}{\log}}) \quad \text{for } x\in C([0,T];H^3). \tag{37}$$

To do that we begin observing that $\lambda(x)\in C([0,T];L^2)$. Next we continue showing that $I_{\log}(\partial_\gamma^3\lambda(x)) \in C([0,T];L^2)$ with $I_{\log}$ given in (22). We use the following decomposition $\partial_\gamma^3\lambda(x) = E_1+E_2+E_3$, where

$$E_1 = -\frac{\partial_\gamma^3 x}{|\partial_\gamma x|^2}\cdot\partial_\gamma\Big(\int\frac{\partial_\gamma x_-}{|x_-|}d\eta\Big), \quad E_2 = -2\frac{\partial_\gamma^2 x}{|\partial_\gamma x|^2}\cdot\partial_\gamma^2\Big(\int\frac{\partial_\gamma x_-}{|x_-|}d\eta\Big),$$

and

$$E_3 = -\frac{\partial_\gamma x}{|\partial_\gamma x|^2}\cdot\partial_\gamma^3\Big(\int\frac{\partial_\gamma x_-}{|x_-|}d\eta\Big).$$

The inequality

$$E_1 \le (\|F(x)\|_{L^\infty}^3\|\partial_\gamma^2x\|_{C^{\frac12}}+\|F(x)\|_{L^\infty}^4\|\partial_\gamma^2x\|_{L^\infty}^2)|\partial_\gamma^3x|$$

gives $E_1\in C([0,T];L^2)$. For E_2 we consider $E_2 = E_{2,1}+E_{2,2}+E_{2,3}$, where

$$E_{2,1} = -2\frac{\partial_\gamma^2 x}{|\partial_\gamma x|^2}\cdot\Big(\int\frac{\partial_\gamma^3 x_-}{|x_-|}d\eta\Big), \quad E_{2,2} = -2\frac{\partial_\gamma^2 x}{|\partial_\gamma x|^2}\cdot\Big(\int\frac{\partial_\gamma^2 x_-}{|x_-|^3}x_-\cdot\partial_\gamma x_- d\eta\Big),$$

and

$$E_{2,3} = -2\frac{\partial_\gamma^2 x}{|\partial_\gamma x|^2}\cdot\Big(\int\partial_\gamma^2 x_-\partial_\gamma\Big(\frac{x_-\cdot\partial_\gamma x_-}{|x_-|^3}\Big)d\eta\Big).$$

A similar approach provides $E_{2,2}$ and $E_{2,3}$ in $C([0,T];L^2)$. As usual we will focus our attention on the most singular term $E_{2,1}$, which can be decomposed as $E_{2,1} = E^1_{2,1} + E^2_{2,1} + E^3_{2,1}$, where

$$E^1_{2,1} = -2\frac{\partial^2_\gamma x}{|\partial_\gamma x|^2}\cdot\Big(\int \partial^3_\gamma x_-\Big(\frac{1}{|x_-|} - \frac{1}{|\partial_\gamma x||\eta|}\Big)d\eta\Big),$$

$$E^2_{2,1} = \frac{2}{|\partial_\gamma x|^3}\Big(\int \frac{\partial^2_\gamma x_- \cdot \partial^3_\gamma x'}{|\eta|}d\eta\Big),$$

and

$$E^3_{2,1} = \frac{-2}{|\partial_\gamma x|^3}\int \frac{(\partial^2_\gamma x\cdot\partial_\gamma x^3)_-}{|\eta|}d\eta = \frac{-2}{|\partial_\gamma x|^3}\mathcal{L}(\partial^2_\gamma x\cdot\partial^3_\gamma x). \tag{38}$$

As before one finds

$$|E^1_{2,1}| + |E^2_{2,1}| \le C(\|F(x)\|^4_{L^\infty}\|x\|^2_{C^{2,\frac12}} + 1)(|\partial^3_\gamma x| + \int |\eta|^{-\frac12}|\partial^3_\gamma x'|d\eta),$$

and consequently $E^1_{2,1}, E^2_{2,1} \in C([0,T];L^2)$. It remains then to deal with $E^3_{2,1}$, which is the most singular term not belonging to $C([0,T];L^2)$. Nevertheless, one has

$$\|I_{\log}(E^3_{2,1})\|_{L^2} \le 2\|F(x)\|^3_{L^\infty}\|I_{\log}(\mathcal{L}(\partial^2_\gamma x\cdot\partial^3_\gamma x))\|_{L^2} \le C\|F(x)\|^3_{L^\infty}\|\partial^2_\gamma x\cdot\partial^3_\gamma x\|_{L^2}$$

as a consequence of properties (16), from where we reach the desired estimate

$$\|I_{\log}(E^3_{2,1})\|_{L^2} \le C\|F(x)\|^3_{L^\infty}\|\partial^2_\gamma x\|_{L^\infty}\|\partial^3_\gamma x\|_{L^2} \le C(t).$$

In the following, we show that all the remaining terms (except one) are integrable in $C([0,T];L^2)$. This singular term is a constant times $E^3_{2,1}$. We are done with $E_{2,1}$ and consequently with E_2.

Regarding E_3, we introduce the splitting $E_3 = E_{3,1} + E_{3,2} + E_{3,3} + E_{3,4}$, where

$$E_{3,1} = -\frac{\partial_\gamma x}{|\partial_\gamma x|^2}\cdot\int\frac{\partial^4_\gamma x_-}{|x_-|}d\eta, E_{3,2} = 3\frac{\partial_\gamma x}{|\partial_\gamma x|^2}\cdot\int\frac{\partial^3_\gamma x_-}{|x_-|^3}x_-\cdot\partial_\gamma x_- d\eta,$$

and

$$E_{3,3} = 3\frac{\partial_\gamma x}{|\partial_\gamma x|^2}\cdot\int \partial^2_\gamma x_-\partial_\gamma\Big(\frac{x_-\cdot\partial_\gamma x_-}{|x_-|^3}\Big)d\eta,$$

$$E_{3,4} = \frac{\partial_\gamma x}{|\partial_\gamma x|^2}\cdot\int \partial_\gamma x_-\partial^2_\gamma\Big(\frac{x_-\cdot\partial_\gamma x_-}{|x_-|^3}\Big)d\eta.$$

Using (25), $E_{3,2}$ has the following estimate:

$$|E_{3,2}| \le 6\|F(x)\|^4_{L^\infty}\|x\|^2_{C^{2,\frac12}}(|\partial^3_\gamma x| + \int |\eta|^{-\frac12}|\partial^3_\gamma x'|d\eta),$$

proving that $E_{3,2} \in C([0,T];L^2)$. The lower order term $E_{3,3}$ can be estimated similarly and it is also in the same space. Next we continue rewriting

$$E_{3,4} = \frac{\partial_\gamma x}{|\partial_\gamma x|^2}\cdot\int(\partial_\gamma x_- - \partial^2_\gamma x\eta)\partial^2_\gamma\Big(\frac{x_-\cdot\partial_\gamma x_-}{|x_-|^3}\Big)d\eta,$$

from where we obtain with the same methods the bound

$$|E_{3,4}| \le 2\|F(x)\|^3_{L^\infty}\|x\|_{C^{2,\frac12}}(|\partial^3_\gamma x| + \int |\eta|^{-\frac12}|\partial^3_\gamma x'|d\eta) + C(t).$$

It remains to estimate $E_{3,1}$ which can be rewritten as follows:

$$E_{3,1} = \frac{1}{|\partial_\gamma x|^2}\int \frac{\partial_\gamma x_- \cdot \partial_\gamma^4 x'}{|x_-|}d\eta - \frac{1}{|\partial_\gamma x|^2}\int \frac{(\partial_\gamma x\cdot \partial_\gamma^4 x)_-}{|x_-|}d\eta,$$

suggesting the splitting $E_{3,1} = E_{3,1}^1 + E_{3,1}^2 + E_{3,1}^3 + E_{3,1}^4$, where

$$E_{3,1}^1 = \frac{1}{|\partial_\gamma x|^2}\int \Big(\frac{\partial_\gamma x_-}{|x_-|} - \frac{\partial_\gamma^2 x}{|\partial_\gamma x|}\frac{\eta}{|\eta|}\Big)\cdot \partial_\gamma^4 x' d\eta, \quad E_{3,1}^2 = \frac{\partial_\gamma^2 x}{|\partial_\gamma x|^3}\cdot \int \partial_\gamma^4 x' \frac{\eta}{|\eta|}d\eta,$$

and

$$E_{3,1}^3 = \frac{-1}{|\partial_\gamma x|^2}\int (\partial_\gamma x\cdot \partial_\gamma^4 x)_-\Big(\frac{1}{|x_-|} - \frac{1}{|\partial_\gamma x||\eta|}\Big)d\eta,$$

$$E_{3,1}^4 = \frac{-1}{|\partial_\gamma x|^3}\int \frac{(\partial_\gamma x\cdot \partial_\gamma^4 x)_-}{|\eta|}d\eta.$$

We have the kernel:

$$\Omega_4 = \frac{\partial_\gamma x_-}{|x_-|} - \frac{\partial_\gamma^2 x}{|\partial_\gamma x|}\frac{\eta}{|\eta|} = \frac{\partial_\gamma x_- - \partial_\gamma^2 x\eta}{|x_-|} + \partial_\gamma^2 x\eta\Big(\frac{1}{|x_-|} - \frac{1}{|\partial_\gamma x||\eta|}\Big)$$

and

$$E_{3,1}^1 = \frac{-1}{|\partial_\gamma x|^2}\int \partial_\eta\Omega_4\cdot \partial_\gamma^3 x' d\eta + \Omega_4 \partial_\gamma^3 x'\Big|_{\eta=-\pi}^{\eta=\pi}.$$

Dealing with $\partial_\eta \Omega_4$ in a similar manner as we did before, we get the estimate $|\partial_\eta\Omega_4| \leq C(t)|\eta|^{-\frac{1}{2}}$, implying that $E_{3,1}^1 \in C([0,T];L^2)$.

A convenient integration yields

$$E_{3,1}^2 = \frac{\partial_\gamma^2 x}{|\partial_\gamma x|^3}\cdot(-2\partial_\gamma^3 x + \partial_\gamma^3 x(\gamma+\pi) + \partial_\gamma^3 x(\gamma-\pi)),$$

from where the appropriate estimate for $E_{3,1}^2$ follows. Identity (4) allows us to obtain

$$E_{3,1}^3 = \frac{3}{|\partial_\gamma x|^2}\int (\partial_\gamma^2 x\cdot \partial_\gamma^3 x)_-\Big(\frac{1}{|x_-|} - \frac{1}{|\partial_\gamma x||\eta|}\Big)d\eta,$$

and, therefore,

$$|E_{3,1}^3| \leq C\|F(x)\|_{L^\infty}^4\|\partial_\gamma^2 x\|_{L^\infty}^2\int (|\partial_\gamma^3 x| + |\partial_\gamma^3 x'|)d\eta.$$

Finally, using (4) one more time we get

$$E_{3,1}^4 = \frac{3}{|\partial_\gamma x|^3}\int \frac{(\partial_\gamma^2 x\cdot\partial_\gamma^3 x)_-}{|\eta|}d\eta = \frac{-2}{3}E_{2,1}^3,$$

where $E_{2,1}^3$ is given in (38). Then, $E_{3,1}^4$ can also be estimated as before. We are done with $E_{3,4}$ and, therefore, with E_3. It gives $\lambda(x)\in C([0,T];H^{\frac{3}{\log}})$, as desired.

Regarding K in (27), we have $K = K_1 + K_2 + K_3$, where

$$K_1 = \int \partial_{\log}\partial_\gamma^2\phi\partial_{\log}(\partial_\gamma^2\lambda(x)\partial_\gamma\phi)d\gamma, \quad K_2 = \int \partial_{\log}\partial_\gamma^2\phi\partial_{\log}(\partial_\gamma\lambda(x)\partial_\gamma^2\phi)d\gamma,$$

and

$$K_3 = \int \partial_{\log}\partial_\gamma^2\phi\partial_{\log}\partial_\gamma(\lambda(x)\partial_\gamma^2\phi)d\gamma.$$

At this point it is easy to get

$$K_1 \leq C\|\partial_{\log}\partial_\gamma^2\phi\|_{L^2}\|\partial_{\log}\partial_\gamma^2\lambda(x)\|_{L^2}\|\partial_{\log}\partial_\gamma\phi\|_{L^2} \leq C(t)\|\partial_{\log}\partial_\gamma^2\phi\|_{L^2}^2$$

and

$$K_2 \leq C\|\partial_{\log}\partial_\gamma^2\phi\|_{L^2}^2\|\partial_{\log}\partial_\gamma\lambda(x)\|_{L^2} \leq C(t)\|\partial_{\log}\partial_\gamma^2\phi\|_{L^2}^2.$$

For K_3 the commutator estimate (24) allows us to get

$$\begin{aligned} K_3 =& \int \partial_{\log}\partial_\gamma^2\phi(\partial_{\log}\partial_\gamma(\lambda(x)\partial_\gamma^2\phi)-\lambda(x)\partial_{\log}\partial_\gamma\partial_\gamma^2\phi)d\gamma + \int \lambda(x)\partial_{\log}\partial_\gamma^2\phi\partial_\gamma\partial_{\log}\partial_\gamma^2\phi d\gamma \\ \leq& C\|\partial_{\log}\partial_\gamma^2\phi\|_{L^2}(\|\lambda(x)\|_{H^2}\|\partial_{\log}\partial_\gamma^2\phi\|_{L^2} \\ &+ \|\partial_{\log}\partial_\gamma\lambda(x)\|_{L^2}\|\partial_\gamma^2\phi\|_{H^{\frac{5}{6}}}) - \frac{1}{2}\int \partial_\gamma\lambda(x)|\partial_{\log}\partial_\gamma^2\phi|^2 d\gamma \end{aligned}$$

to obtain finally

$$K_3 \leq C\|\lambda(x)\|_{H^2}\|\partial_{\log}\partial_\gamma^2\phi\|_{L^2}^2 \leq C(t)\|\partial_{\log}\partial_\gamma^2\phi\|_{L^2}^2.$$

Having such good estimates for K and J we can go back to (27) and obtain

$$\frac{d}{dt}\|\partial_{\log}\partial_\gamma^2\phi\|_{L^2}^2 \leq C(t)\|\partial_{\log}\partial_\gamma^2\phi\|_{L^2}^2,$$

which together with (26) yields

$$\frac{d}{dt}\|\phi - id\|_{H^{\frac{3}{\log}}}^2 \leq C(t)\|\phi - id\|_{H^{\frac{3}{\log}}}^2 + C(t),$$

and then the Gronwall lemma gives existence so long as $\int_0^t C(s)ds < \infty$.

Uniqueness then follows similarly because we have

$$\frac{d}{dt}\|\phi^2 - \phi^1\|_{L^2}^2 \leq C(t)\|\phi^2 - \phi^1\|_{L^2}^2,$$

where ϕ^2 and ϕ^1 are two solutions of the equation and $\phi^2(x,0) = \phi^1(x,0)$, and because the above inequality can be obtained with the method described before.

It remains to show that $\partial_\gamma\phi(\gamma,t) > 0$ for some positive time. This is done with the observation

$$\partial_\gamma\phi(\gamma,t) = \partial_\gamma\phi(\gamma,0) + \int_0^t \partial_\gamma\phi_t(\gamma,s)ds \geq \min_\gamma \partial_\gamma\phi(\gamma,0) - \int_0^t |\partial_\gamma\phi_t(\gamma,s)|ds.$$

The fact that $|\partial_\gamma\phi_t(\gamma,s)| \leq C(t)\|\phi\|_{H^{\frac{3}{\log}}}$ implies that ϕ remains as a legitimate change of variable so long as

$$\min_\gamma \partial_\gamma\phi(\gamma,0) > \int_0^t C(s)\|\phi\|_{H^{\frac{3}{\log}}}(s)ds.$$

4. Uniqueness for the system (7)

This section is devoted to showing uniqueness for the system (7). The argument shown below is straight, dealing with the system (7) without any change of parameterization. As before, to simplify notation we shall write $f = f(\gamma,t)$, $f' = f(\gamma-\eta)$ and $f - f' = f_-$ when there is no danger of confusion.

We consider two solutions for the system (7):

$$x_t(\gamma,t) = \int_{\mathbb{T}} \frac{\partial_\gamma x(\gamma,t) - \partial_\gamma x(\gamma-\eta,t)}{|x(\gamma,t) - x(\gamma-\eta,t)|}d\eta,$$

given by $x(\gamma,t)$ and $y(\gamma,t)$ in the space $C([0,T]; H^{\frac{3}{\log}}(\mathbb{T}))$ with the same initial data. During the time of existence $T > 0$ one finds $F(x)$ and $F(y)$ in $C([0,T]; L^\infty(\mathbb{T}\times\mathbb{T}))$. Here C denotes a constant which may be different from inequality to inequality but only depends on $\sup_{[0,T]}\|x\|_{H^{\frac{3}{\log}}}(t)$, $\sup_{[0,T]}\|y\|_{H^{\frac{3}{\log}}}(t)$, $\sup_{[0,T]}\|F(x)\|_{L^\infty}(t)$, and $\sup_{[0,T]}\|F(y)\|_{L^\infty}(t)$.

Let us consider the function $z(\gamma,t)=x(\gamma,t)-y(\gamma,t)$. One finds

$$\frac{1}{2}\frac{d}{dt}\|z\|_{L^2}^2=\int z\cdot z_t d\gamma=I_1+I_2,$$

where

$$I_1=\int z\cdot\int\frac{\partial_\gamma z_-}{|x_-|}d\eta d\gamma,\quad I_2=\int z\cdot\int\partial_\gamma y_-\Big(\frac{1}{|x_-|}-\frac{1}{|y_-|}\Big)d\eta d\gamma.$$

Next we symmetrize I_1 and integrate by parts to get

$$I_1=\frac{1}{2}\int\int z_-\cdot\frac{\partial_\gamma z_-}{|x_-|}d\eta d\gamma=\frac{1}{4}\int\int|z_-|^2\frac{x_-\cdot\partial_\gamma x_-}{|x_-|^3}d\gamma d\eta.$$

We have the splitting: $I_1=I_{1,1}+I_{1,2}$, where

$$I_{1,1}=\frac{1}{4}\int\int z\cdot z_-\frac{x_-\cdot\partial_\gamma x_-}{|x_-|^3}d\eta d\gamma\quad\text{and}\quad I_{1,2}=-\frac{1}{4}\int\int z'\cdot z_-\frac{x_-\cdot\partial_\gamma x_-}{|x_-|^3}d\eta d\gamma.$$

Then a simple exchange of variables yields $I_{1,1}=I_{1,2}$. We have:

$$I_{1,1}=\frac{1}{4}\int\int z\cdot z_-\Big(\frac{x_-\cdot\partial_\gamma x_-}{|x_-|^3}-\frac{\partial_\gamma x\cdot\partial_\gamma^2 x}{|\partial_\gamma x|^3|\eta|}\Big)d\eta d\gamma+\frac{1}{4}\int z\frac{\partial_\gamma x\cdot\partial_\gamma^2 x}{|\partial_\gamma x|^3}\mathcal{L}(z)d\gamma,$$

hence

$$|I_1|\leq 2|I_{1,1}|\leq C\|z\|_{L^2}^2+C\|z\|_{L^2}\|\mathcal{L}(z)\|_{L^2}. \tag{39}$$

It remains an estimate for I_2. We rewrite

$$I_2=-\int z\cdot\int\partial_\gamma y_-\frac{(x_-+y_-)\cdot z_-}{|x_-||y_-|(|x_-|+|y_-|)}d\eta d\gamma$$

and decompose $I_2=I_{2,1}+I_{2,2}$, where

$$I_{2,1}=-\int z\cdot\int\Big(\frac{\partial_\gamma y_-(x_-+y_-)\cdot z_-}{|x_-||y_-|(|x_-|+|y_-|)}-\frac{\partial_\gamma^2 y(\partial_\gamma x+\partial_\gamma y)\cdot z_-}{|\partial_\gamma x||\partial_\gamma y|(|\partial_\gamma x|+|\partial_\gamma y|)|\eta|}\Big)d\eta d\gamma$$

and

$$I_{2,2}=-\int z\cdot\partial_\gamma^2 y\frac{\partial_\gamma x+\partial_\gamma y}{|\partial_\gamma x||\partial_\gamma y|(|\partial_\gamma x|+|\partial_\gamma y|)}\cdot\mathcal{L}(z)d\gamma.$$

As before, we control $I_{2,1}$ and $I_{2,2}$ in the following manner:

$$I_{2,1}\leq C\int|\eta|^{-\frac{2}{3}}\int|z|(|z|+|z'|)d\gamma d\eta\leq C\|z\|_{L^2}^2,\quad I_{2,2}\leq C\|z\|_{L^2}\|\mathcal{L}(z)\|_{L^2}.$$

Adding both estimates we obtain the bound for I_2, which together with (39) yield

$$\frac{d}{dt}\|z\|_{L^2}\leq C(\|z\|_{L^2}+\|\mathcal{L}(z)\|_{L^2}). \tag{40}$$

Next we show that

$$\|\mathcal{L}(f)\|_{L^2}\leq pC\|f\|_{L^2}^{1-\frac{1}{p}}\|\partial_\gamma f\|_{L^2}^{1/p}. \tag{41}$$

We have

$$\|\mathcal{L}(z)\|_{L^2}^2\leq C\sum_{k\neq0}\ln^2(2|k|)|\widehat{f}(k)|^2\leq C\Big(\sum_{k\neq0}|\widehat{f}(k)|^2\Big)^{1-\frac{1}{p}}\Big(\sum_{k\neq0}\ln^{2p}(2|k|)|\widehat{f}(k)|^2\Big)^{\frac{1}{p}}$$

for $1 \leq p < \infty$ and, therefore, inequality $\ln^p |k| \leq p^p |k|$ with $|k| \geq 1$ gives (41). Introducing that estimate in (40) we obtain

$$\frac{d}{dt}\|z\|_{L^2} \leq Cp\|z\|_{L^2}^{1-\frac{1}{p}}$$

for $p \geq 1$. Since $\|z\|_{L^2}(0) = 0$, we can conclude that the maximal solution of this inequality satisfies

$$\|z\|_{L^2}(t) \leq (Ct)^p$$

for $p \geq 1$. Therefore, choosing $t \leq (2C)^{-1}$ and taking the limit as $p \to +\infty$ we prove uniqueness.

Acknowledgments

The first author was partially supported by the grant MTM2014-56350-P (Spain). The first and second authors were partially supported by the ICMAT Severo Ochoa project SEV-2015-556. The second and third authors were partially supported by the grant MTM2014-59488-P (Spain). The third author was partially supported by the Ramón y Cajal program RyC-2010-07094, the grant P12-FQM-2466 from Junta de Andalucía (Spain), and the ERC Starting Grant 639227.

References

[1] A. L. Bertozzi and P. Constantin, *Global regularity for vortex patches*, Comm. Math. Phys. **152** (1993), no. 1, 19–28. MR1207667

[2] Andrew J. Majda and Andrea L. Bertozzi, *Vorticity and incompressible flow*, Cambridge Texts in Applied Mathematics, vol. 27, Cambridge University Press, Cambridge, 2002. MR1867882

[3] Angel Castro, Diego Córdoba, and Javier Gómez-Serrano, *Existence and regularity of rotating global solutions for the generalized surface quasi-geostrophic equations*, Duke Math. J. **165** (2016), no. 5, 935–984, DOI 10.1215/00127094-3449673. MR3482335

[4] Angel Castro, Diego Córdoba, and Javier Gómez-Serrano, *Uniformly rotating analytic global patch solutions for active scalars*, Ann. PDE **2** (2016), no. 1, Art. 1, 34, DOI 10.1007/s40818-016-0007-3. MR3462104

[5] Angel Castro, Diego Córdoba, Javier Gómez-Serrano, and Alberto Martín Zamora, *Remarks on geometric properties of SQG sharp fronts and α-patches*, Discrete Contin. Dyn. Syst. **34** (2014), no. 12, 5045–5059, DOI 10.3934/dcds.2014.34.5045. MR3223861

[6] Dongho Chae, Peter Constantin, Diego Córdoba, Francisco Gancedo, and Jiahong Wu, *Generalized surface quasi-geostrophic equations with singular velocities*, Comm. Pure Appl. Math. **65** (2012), no. 8, 1037–1066, DOI 10.1002/cpa.21390. MR2928091

[7] Jean-Yves Chemin, *Persistance de structures géométriques dans les fluides incompressibles bidimensionnels* (French, with English and French summaries), Ann. Sci. École Norm. Sup. (4) **26** (1993), no. 4, 517–542. MR1235440

[8] Peter Constantin, Andrew J. Majda, and Esteban Tabak, *Formation of strong fronts in the 2-D quasigeostrophic thermal active scalar*, Nonlinearity **7** (1994), no. 6, 1495–1533. MR1304437

[9] Diego Córdoba, Marco A. Fontelos, Ana M. Mancho, and Jose L. Rodrigo, *Evidence of singularities for a family of contour dynamics equations*, Proc. Natl. Acad. Sci. USA **102** (2005), no. 17, 5949–5952, DOI 10.1073/pnas.0501977102. MR2141918

[10] Francisco Gancedo, *Existence for the α-patch model and the QG sharp front in Sobolev spaces*, Adv. Math. **217** (2008), no. 6, 2569–2598, DOI 10.1016/j.aim.2007.10.010. MR2397460

[11] Francisco Gancedo and Robert M. Strain, *Absence of splash singularities for surface quasi-geostrophic sharp fronts and the Muskat problem*, Proc. Natl. Acad. Sci. USA **111** (2014), no. 2, 635–639, DOI 10.1073/pnas.1320554111. MR3181769

[12] R. Garra, *Confinement of a hot temperature patch in the modified SQG model*, Arxiv:1604.03284, 2016.

[13] Zineb Hassainia and Taoufik Hmidi, *On the V-states for the generalized quasi-geostrophic equations*, Comm. Math. Phys. **337** (2015), no. 1, 321–377, DOI 10.1007/s00220-015-2300-5. MR3324164

[14] Taoufik Hmidi, Joan Mateu, and Joan Verdera, *Boundary regularity of rotating vortex patches*, Arch. Ration. Mech. Anal. **209** (2013), no. 1, 171–208, DOI 10.1007/s00205-013-0618-8. MR3054601

[15] Taoufik Hmidi, Joan Mateu, and Joan Verdera, *On rotating doubly connected vortices*, J. Differential Equations **258** (2015), no. 4, 1395–1429, DOI 10.1016/j.jde.2014.10.021. MR3294351

[16] T. Y. Hou, J. S. Lowengrub, and M. J. Shelley, *The long-time motion of vortex sheets with surface tension*, Phys. Fluids **9** (1997), no. 7, 1933–1954, DOI 10.1063/1.869313. MR1455083

[17] Alexander Kiselev, Lenya Ryzhik, Yao Yao, and Andrej Zlatoš, *Finite time singularity for the modified SQG patch equation*, Ann. of Math. (2) **184** (2016), no. 3, 909–948, DOI 10.4007/annals.2016.184.3.7. MR3549626

[18] A. Kiselev, Y. Yao, and A. Zlatos, *Local regularity for the modified SQG patch equation*, to appear in Comm. Pure and Appl. Math. ArXiv:1508.07611, 2015.

[19] J. Pedlosky, *Geophysical fluid dynamics*, Springer-Verlag, New York, 1987.

[20] S. Resnick, *Dynamical problems in nonlinear advective partial differential equations*, Ph.D. thesis University of Chicago, Chicago, 1995.

[21] José Luis Rodrigo, *On the evolution of sharp fronts for the quasi-geostrophic equation*, Comm. Pure Appl. Math. **58** (2005), no. 6, 821–866, DOI 10.1002/cpa.20059. MR2142632

[22] R. K. Scott, *A scenario for finite-time singularity in the quasigeostrophic model*, Journal of Fluid Mechanics, 687:492-502, 11, 2011.

[23] R. K. Scott and D. G. Dritschel, *Numerical simulation of a self-similar cascade of filament instabilities in the surface quasigeostrophic system*, Phys. Rev. Lett., 112:144505, 2014.

[24] V. I. Yudovich, *Non-stationary flows of an ideal incompressible fluid* (Russian), Ž. Vyčisl. Mat. i Mat. Fiz. **3** (1963), 1032–1066. MR0158189

Instituto de Ciencias Matemáticas (ICMAT) & Departamento de Matemáticas, Facultad de Ciencias, Universidad Autónoma de Madrid, Crta. Colmenar Viejo km. 15, 28049 Madrid, Spain

Email address: antonio.cordoba@uam.es

Instituto de Ciencias Matemáticas (ICMAT), Consejo Superior de Investigaciones Científicas, C/ Nicolás Cabrera, 13-15, Campus Cantoblanco UAM, 28049 Madrid, Spain

Email address: dcg@icmat.es

Departamento de Análisis Matemático & IMUS, Universidad de Sevilla, C/ Tarfia, s/n, Campus Reina Mercedes, 41012, Sevilla, Spain

Email address: fgancedo@us.es