Absolutely summing Carleson embeddings on Hardy spaces.

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On the occasion of the 60th birthday of A. Defant.

I will present some results obtained in collaboration with Pascal Lefèvre (Université d’Artois).

This is a work still in progress.

Composition operators

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ the open unit disk and $\phi \colon \mathbb{D} \to \mathbb{D}$ an holomorphic function.

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If $\mathcal{E}$ is a Banach space of analytic functions over the disk one tries to characterize the properties of the operator $C_\phi\colon \mathcal{E} \to \mathcal{E}$ in terms of the properties of the symbol ϕ.

In that way one can study when the operator is well defined (boundedness), when it is compact, weakly compact, q-summing, nuclear,...

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In this talk we will be dealing with the study of the q-summingness when $\mathcal{E}$ is a Hardy space H^p, $1 \leq p < +\infty$.

Suppose $1 \leq q < +\infty$ and let $T\colon X \to Y$ be bounded linear operator between two Banch spaces.
We say T is a q-summing operator if there exists $C > 0$ such that

$$\sum_{j=1}^{n} \|Tx_j\|^q \leq C \sup_{x^* \in B_{X^*}} \sum_{j=1}^{n} |\langle x^*, x_j\rangle|^q, \qquad (\clubsuit)$$

for every finite sequence $x_1, x_2, \ldots, x_n$ in X.

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The q-summing norm of T is

$$\pi_q(T) = \inf\{C^{1/q} : C > 0, C \text{ occurs in } (\clubsuit)\}.$$

1-summing operators are also called absolutely summing operators.

Hardy spaces

If $1 \leq p < +\infty$, the Hardy space $H^p = H^p(\mathbb{D})$ is formed by the holomorphic functions $f\colon \mathbb{D} \to \mathbb{C}$ such that

$$\|f\|_{H^p} = \sup_{0 \leq r < 1} \left(\frac{1}{2\pi} \int_0^{2\pi} |f(re^{it})|^p \, dt \right)^{1/p} < +\infty \, .$$

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$H^\infty(\mathbb{D})$ is the space of bounded analytic functions on $\mathbb{D}$.

Let $\mathbb{T} = \partial\mathbb{D} = \{z \in \mathbb{C} : |z| = 1\}$. On the torus $\mathbb{T}$ we consider the normalized arc–length measure m. Every $f \in H^p(\mathbb{D})$ has almost everywhere radial limit f^*

$$f^*(e^{it}) = \lim_{r \to 1^-} f(re^{it}) \, .$$

It is known that $f^* \in L^p(\mathbb{T}) = L^p(m)$ and $\|f\|_{H^p} = \|f^*\|_{L^p}$.

Boundedness and compactness

For every $\phi\colon \mathbb{D} \to \mathbb{D}$ and all p, the operator $C_\phi\colon H^p \to H^p$ is bounded (and well defined).

This is obvious for $p = \infty$. For $1 \leq p < +\infty$ it is a consequence of Littlewood's Subordination Principle.

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1) Using the Nevanlinna counting function (Shapiro).

2) Using vanishing Carleson measures (MacCluer).

Pullback measure

For $f \in H^p$ we have

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Let us denote μ_ϕ to the image measure of m by the map ϕ^*; that is, $\mu_\phi(B) = m(\{\phi^* \in B\})$, for all Borel set $B \subset \overline{\mathbb{D}}$. We have

$$\|C_\phi f\|_{H^p} = \|f\|_{L^p(\mu_\phi)} \, .$$

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This allows to see that the properties of the operator C_ϕ are the same that the properties of the inclusion operator

$$j_{\mu_\phi} : H^p \hookrightarrow L^p(\mu_\phi) \, .$$

Carleson windows

Let $0 < h < 1$. We define the window of center $\xi \in \mathbb{T}$ and radius h as $W(\xi, h) = \{z \in \overline{\mathbb{D}} : 1 - h < |z|, |\arg(\bar{\xi} z)| < h\}$.

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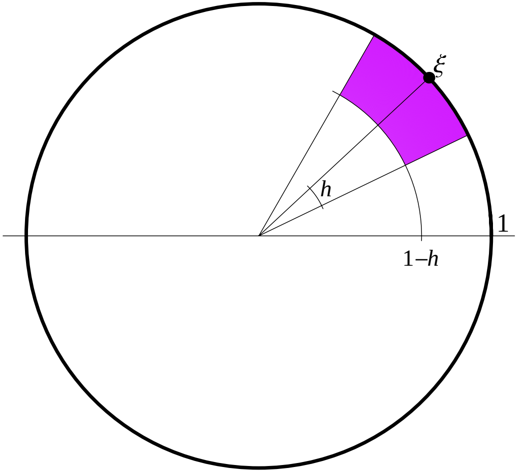

Carleson's Theorem

Theorem (Carleson, 1962)

Let μ be a finite measure on the Borel sets of $\overline{\mathbb{D}}$. For $1 \le p < \infty$, we have the inclusion $H^p(\mathbb{D}) \subset L^p(\mu)$ if and only if there exists $C > 0$ such that

$$\mu\big(W(\xi, h)\big) \le Ch\,, \quad \forall \xi \in \mathbb{T},\ \forall h \in (0,1)\,. \qquad (\clubsuit)$$

A measure satisfying ($\clubsuit$) is called a Carleson measure.

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Putting $\rho_\mu(h) = \sup_{\xi\in\mathbb{T}} \mu\big(W(\xi, h)\big)$, we have that μ is a Carleson measure if and only if

$$\frac{\rho_\mu(h)}{h} \quad \text{is bounded for } 0 < h < 1.$$

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Moreover we have

$$\big\|\, j_\mu : H^p \hookrightarrow L^p(\mu)\, \big\| \quad \approx \quad \left(\sup_{0<h<1} \frac{\rho_\mu(h)}{h} \right)^{1/p}.$$

MacCluer's Theorem

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MacCluer (1985)

The composition operator $C_\phi : H^p \to H^p$ is compact if and only if μ_ϕ is a vanishing Carleson measure.

Actually we have that, for any finite measure μ, the inclusion of $H^p(\mathbb{D})$ in $L^p(\mu)$ defines a compact operator if and only if μ is a vanishing Carleson measure.

Assume from now on that μ is concentrated in the open disk $\mathbb{D}$.

For μ a Carleson measure, our aim is to characterize when the Carleson embedding

$$j_\mu : H^p(\mathbb{D}) \hookrightarrow L^p(\mu)$$

is a q-summing operator.

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Observe that the conditions in Carleson's and MacCluer's theorems does not depend on p. So compactness and boundedness of Carleson embeddings do not depend on p.

We will see that this is not the case for q-summingness.

Known facts

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For $1 \leq p \leq 2$, H^p and L^p have cotype 2. For $p > 2$ they only have cotype p. So it is known that

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in the following cases:

- For $1 \leq p \leq 2$ and $q_1, q_2 \geq 1$.
- For $p > 2$, and $1 \leq q_1, q_2 < p'$, where p' is the conjugate exponent of p.

Known facts

Theorem (Shapiro-Taylor, 1973)

Let $p \geq 2$. The composition operator $C_\phi : H^p \to H^p$ is p-summing if and only if

$$\int_{\mathbb{T}} \frac{1}{1 - |\phi^*|} \, dm < +\infty \, .$$

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It is known that ($\spadesuit$) also implies $j_\mu : H^p \to L^p(\mu)$ is p-summing for $1 \leq p < 2$. But the converse is not true.

Luecking rectangles

Decompose the disk $\mathbb{D}$ into the family of annulus $\{\Gamma_n\}_{n\geq 0}$ where

$$\Gamma_n = \{z \in \mathbb{D} : 1 - 2^{-n} \leq |z| < 1 - 2^{-n-1}\} \quad n = 0, 1, 2, \ldots$$

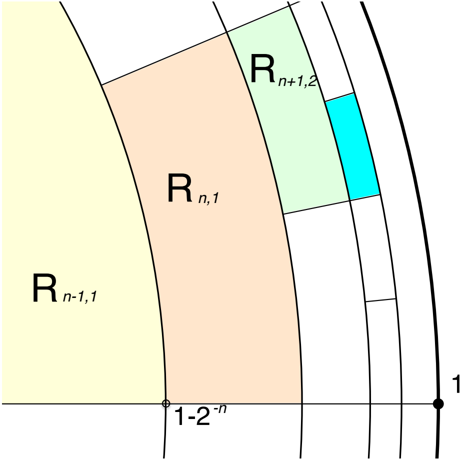
Then decompose each annulus into 2^n equal pieces with the shape of "round" rectangles. We will call them Luecking rectangles.

$$R_{n,j} = \{z = re^{i\theta} : 1-2^{-n} \leq r < 1-2^{-n-1}, 2\pi(j-1)/2^n \leq \theta < 2\pi j/2^n\}$$

with $n = 0, 1, 2, 3, \ldots$ and $1 \leq j \leq 2^n$.

Luecking rectangles

These sets $R_{n,j}$ were used by D. Luecking to characterize the membership of composition operators on H^2 to the Schatten classes.



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Now consider, for $n \geq 0$, the 2^n-dimensional subspace X_n of $H^p(\mathbb{D})$ generated by the monomials z^k, with $2^n \leq k < 2^{n+1}$. We have, the decomposition

$$H_0^p(\mathbb{D}) = \{f \in H^p(\mathbb{D}) : f(0) = 0\} = \bigoplus_{n \geq 0} X_n$$

which is an orthogonal decomposition in the case of H^2.

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Finally let α_n be the restriction of j_n to X_n.

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3. $\pi_q(D_{\mathbf{a}})$, where $D_{\mathbf{a}} \colon \ell_p^{2^n} \to \ell_p^{2^n}$ is the diagonal operator

$$\mathbf{x} = (x_j)_j \mapsto D_{\mathbf{a}}(\mathbf{x}) = (a_j x_j)_j,$$

with $a_j = \big(2^n \mu(R_{n,j})\big)^{1/p}, \quad j = 1, 2, \ldots, 2^n$.

First results

In consequence we have:

$$1 < p \leq 2: \quad \pi_q(j_n) \approx \Big(\sum_{j=1}^{2^n} \big[2^n \mu(R_{n,j})\big]^{2/p}\Big)^{1/2}.$$

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Theorem

In the case $p \geq 2$ and $q \geq p$ we have:

$$\pi_q(j_\mu) \approx \Big(\sum_n [\pi_q(j_n)]^p\Big)^{1/p} \approx \Big(\sum_{n,j} [2^n \mu(R_{n,j})]\Big)^{1/p}$$

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For $p > 2$, the case $1 \leq q < 2$ is still open.

The case $p \leq 2$.

Littlewood-Paley theorem says that, if $f_n \in X_n$, $n = 0, 1, \ldots$ we have

$$\Big\| \sum_n f_n \Big\|_{H^p} \approx \Big\| \Big(\sum_n |f_n^*|^2 \Big)^{1/2} \Big\|_{L^p(\mathbb{T})}$$

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This can be used to prove

$$\Big(\sum_n \pi_2(j_n)^2\Big)^{1/2} \lessapprox \pi_2(j_\mu) \lessapprox \Big(\sum_n \pi_2(j_n)^p\Big)^{1/p}$$

But none of these two estimates is the correct one.

The case $p \leq 2$.

Theorem A

For $1 < p \leq 2$, the Carleson embedding $j_\mu\colon H^p(\mathbb{D}) \to L^p(\mu)$ is absolutely summing if and only if the space $H^1(\mathbb{D})$ is included in $L^r(\nu)$, where

$$r = 1 - \frac{p}{2}$$

and

$$d\nu(z) = \frac{d\mu(z)}{(1 - |z|)^{p/2}}$$

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Applying a result of Blasco and Jarchow, we obtain:

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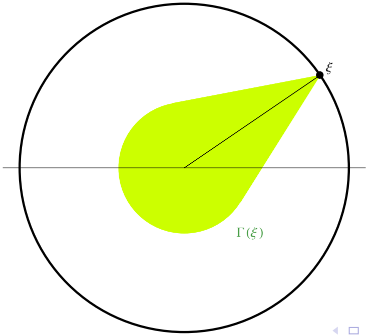
Theorem A'

For $1 < p \leq 2$, the Carleson embedding $j_\mu : H^p(\mathbb{D}) \to L^p(\mu)$ is absolutely summing if and only if

$$\int_{\mathbb{T}} \left(\int_{\Gamma(\xi)} \frac{d\mu(z)}{(1-|z|)^{1+p/2}} \right)^{2/p} dm(\xi) < +\infty$$



Proof of Theorem A.

Proposition 1

Suppose $1 < p \leq 2$. The necessary and sufficient condition for the natural injection $j\colon H^p(\mathbb{D}) \to L^2(\mu)$ to be a 2-summing operator is that

$$\int_{\mathbb{T}} \Big(\int_{\mathbb{D}} \frac{1}{|z-w|^2}\, d\mu(z) \Big)^{p'/2} dm(w) < +\infty\,,$$

In fact we have

$$\pi_2(j\colon H^p(\mathbb{D}) \to L^2(\mu)) \approx \Big(\int_{\mathbb{T}} \Big(\int_{\mathbb{D}} \frac{d\mu(z)}{|z-w|^2} \Big)^{q/2} dm(w) \Big)^{1/q}.$$

Proof of Theorem A.

Proposition 2

Suppose $1 < p < 2$ and let $r > 1$ be such that $1/r + 1/2 = 1/p$. Let X be a Banach space, and $T\colon X \to L^p(\mu)$ a bounded operator. The necessary and sufficient condition for T to be a 2-summing operator is that there exists $F \in L^r(\mu)$, with $F > 0$ μ-a.e., such that $T\colon X \to L^2(\nu)$ is well defined and 2-summing, where ν is the measure defined by

$$d\nu(z) = \frac{1}{F(z)^2}\, d\mu(z)\,.$$

Moreover, we have

$$\pi_2\big(T\colon X \to L^p(\mu)\big) \approx \inf\Big\{\pi_2\big(T\colon X \to L^2(\nu)\big) :$$
$$d\nu = d\mu/F^2, F \geq 0, \int F^r\, d\mu \leq 1\Big\}.$$

Proof of Theorem A.

$j_\mu : H^p(\mathbb{D}) \to L^p(\mu)$ is 2-summing $\iff$ the following is finite:

$$inf\left\{ \int_{\mathbb{T}} \left(\int_{\mathbb{D}} \frac{d\mu(z)}{|z-w|^2 \cdot F(z)^2} \right)^{p'/2} dm(w) : F \geq 0, \int F^r \, d\mu \leq 1 \right\}$$

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$\iff$ the following is finite:

$$\inf_{F \in B^+_{L^t(\mathbb{T})}} \sup_{g \in B^+_{L^{r/2}(\mu)}} \int_{\mathbb{T}} \int_{\mathbb{D}} \frac{g(w)}{|z-w|^2 \cdot F(z)} d\mu(z) \, dm(w), \qquad (\clubsuit)$$

where t is the conjugate exponent of $p'/2$, and $1/r + 1/2 = 1/p$.

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if and only if $H^t(\mathbb{D}) \subset L^{p/2}(\nu)$.