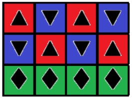
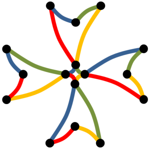
Autotopism stabilized colouring games on rook's graphs

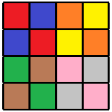
Raúl M. Falcón (University of Seville).

(Joint work with S. Dominique Andres).
(FernUniversität in Hagen).

Qwara. June 29, 2017.







CONTENTS

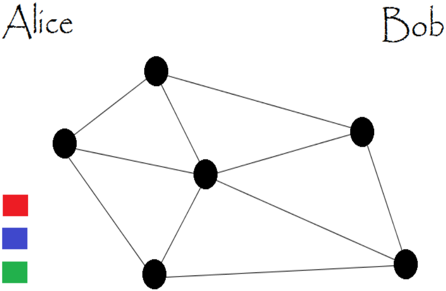
I. Colouring games and partial Latin rectangles.

Colouring games.

Steven J. Brams & Martin Gardner, 1981.

Hans L. Boblaender, 1991.

Based on the graph colouring problem, the game is played on a finite graph by two players, Alice (A) and Bob (B), with Alice playing first.



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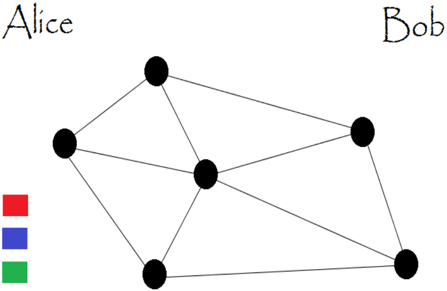
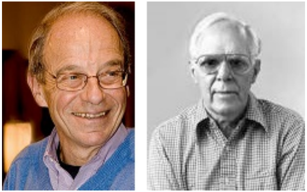
They alternately colour the vertices of the graph so that none two adjacent vertices are coloured with the same colour.





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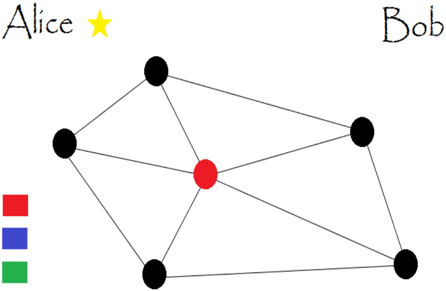

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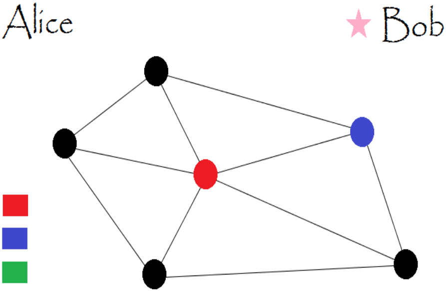


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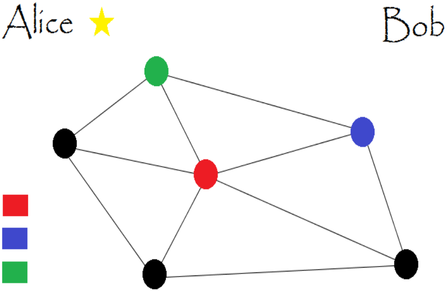

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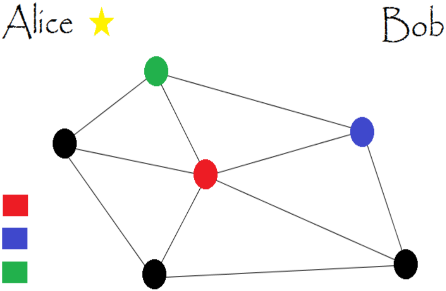
Alice wins if all vertices are coloured at the end of the game.

Otherwise, Bob wins.



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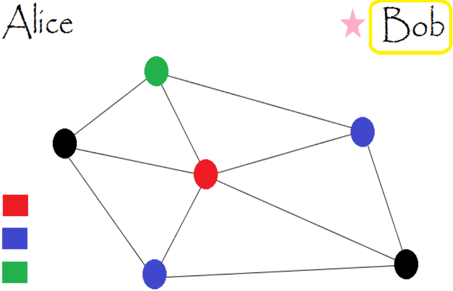

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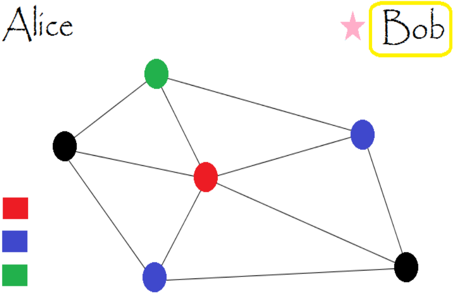


Does there exist a winning strategy for the players?

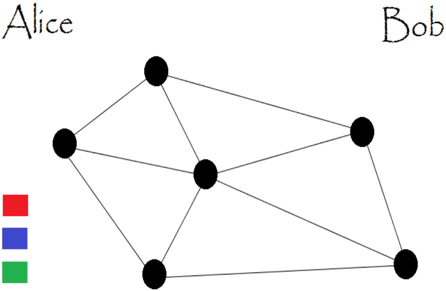
Colouring games.

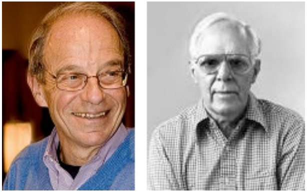
 

Steven J. Brams & Martin Gardner, 1981.

Hans L. Boblaender, 1991.

$\chi_g(G)$: **Game chromatic number**.
Least integer k s. t. A has a winning strategy on a graph G and k colours.





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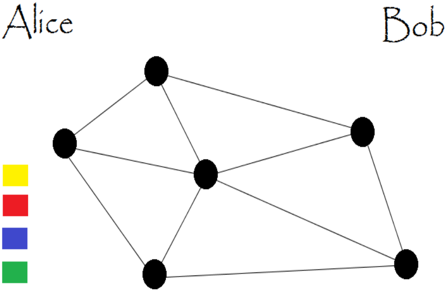
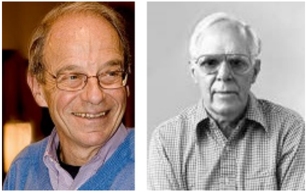


$$\chi_g(G) = 4.$$

Colouring games.



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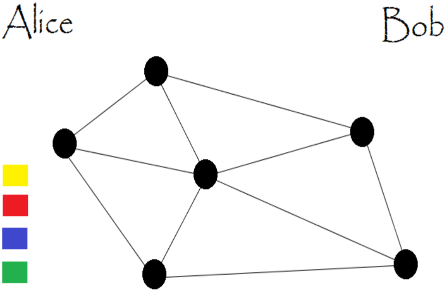
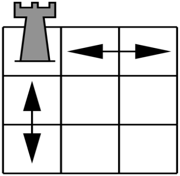
$\chi_{g_A}(G)$: Alice starts.

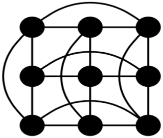
$\chi_{g_B}(G)$: Bob starts.



$$\chi_{g_A}(G) = \chi_{g_B}(G) = 4.$$

Maximilian Schlund, 2011.

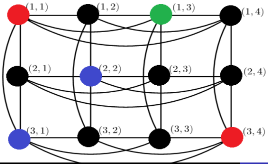




Rook's graph $\equiv$ Hamming graph: $\mathcal{R}_{r,s} = \mathcal{H}_{r,s} = K_r \square K_s$.

Lemma

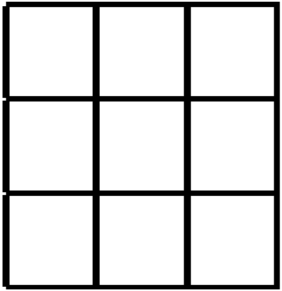
Every $r \times s$ partial Latin rectangle based on n symbols is uniquely identified with a partial n-colouring of a vertex-labeled rook's graph $\mathcal{R}_{r,s}$.



1		4	
	2		
2			1

Colouring games on rook's graphs.

1 2 3



1 2 3

1		

1 2 3

1		
	2	

Colouring games on rook's graphs.

1 2 3

1	⚠	⚠
⚠	2	⚠
⚠	⚠	⚠

Colouring games on rook's graphs.

1 2 3

1		
	2	
		3

Colouring games on rook's graphs.

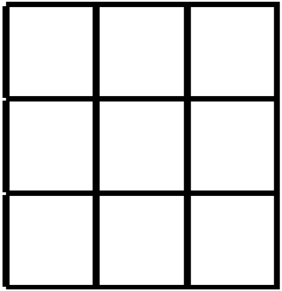
1 2 3

1		
	2	
		3

$$\chi_{g_A} = 3.$$

Colouring games on rook's graphs.

1 2 3



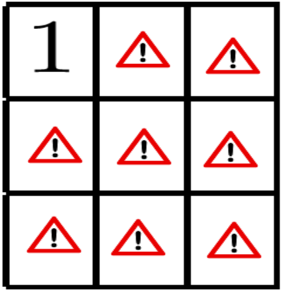
Colouring games on rook's graphs.

1 2 3

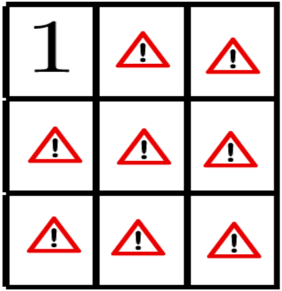
1		

Colouring games on rook's graphs.

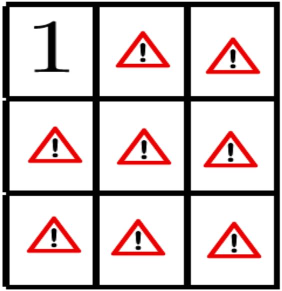
1 2 3



1 2 3



1 2 3



$$\chi_{g_B} > 3.$$

Partial Latin rectangles ($\mathcal{PLR}_{r,s,n}$).

- **Partial Latin rectangle**: $r \times s$ array where
  1. each cell is empty or contains one symbol of $[n] := \{1, \ldots, n\}$.
  2. each symbol occurs at most once in each row and each column.

$P =$

1		3	
	2		4
2	3	4	1

- **Entry set**: $\{(\mathrm{row}, \mathrm{column}, \mathrm{symbol})\}$.

$$\mathrm{Ent}(P) = \begin{cases} (1,1,1), (1,3,3), (2,2,2), (2,4,4), \\ (3,1,2), (3,2,3), (3,3,4), (3,4,1). \end{cases}$$

- $\mathfrak{I}_{r,s,n} := S_r \times S_s \times S_n \Rightarrow$ **Isotopism**: $(\Theta = (\alpha, \beta, \gamma) \in \mathfrak{I}_{r,s,n})$.

$$\mathrm{Ent}(P^{\Theta}) = \{(\alpha(i), \beta(j), \gamma(k)) \mid (i,j,k) \in \mathrm{Ent}(P)\}.$$

- **Principal isotopism**: $\gamma = \mathrm{Id}$.
- **Autotopism**: $P^{\Theta} = P$.

$$\Theta = ((12), (1234), (1234)) \in \mathfrak{I}_{3,4,4}.$$

Partial Latin rectangles ($\mathcal{PLR}_{r,s,n}$).

- **Conjugate**: $\Theta_1 = \Theta\Theta_2\Theta^{-1} \equiv$ Equivalence relation in $\mathfrak{I}_{r,s,n}$.
- Classes characterized by the **cycle structure** of their elements.
- **Cycle structure** of $\pi \in S_m$: $z_\pi := m^{\lambda_m^\pi} \ldots 1^{\lambda_1^\pi}$.

  $\lambda_i^\pi := \#\{i\text{-cycles in the unique cycle decomposition of } \pi\}$.

- **Cycle structure** of $(\alpha, \beta, \gamma) \in \mathfrak{I}_{r,s,n}$: $z_\Theta = (z_\alpha, z_\beta, z_\gamma)$.

  $((1234), (12)(3)(45), (12)(345)(6)) \in S_4 \times S_5 \times S_6 \to (4, 2^2 1, 321)$.

- $\Theta \in \mathfrak{I}_{r,s,n} \Rightarrow \mathcal{PLR}_\Theta = \{P \in \mathcal{PLR}_{r,s,n} \mid P^\Theta = P\}$.

Lemma

The cardinality of $\mathcal{PLR}_\Theta$ only depends on the cycle structure of Θ.

Partial Latin rectangles ($\mathcal{PLR}_{r,s,n}$).

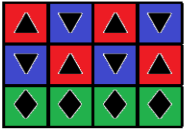
Lemma

Every isotopism $\Theta \in \mathfrak{I}_{r,s,n}$ determines a unique partition of the cells of a partial Latin rectangle in $\mathcal{PLR}_{r,s,n}$.

- $\lambda_{\pi,i}$: Length of the cycle in $\pi \in S_m$ containing $i \in [m]$.
- **Cell orbit** of $(i,j) \in [r]\times[s]$ under the action of $\Theta = (\alpha, \beta, \gamma) \in \mathfrak{I}_{r,s,n}$.

$$\mathfrak{o}_\Theta((\mathbf{i},\mathbf{j})) = \{(\alpha^m(i), \beta^m(j)) \mid m \leq \mathrm{lcm}(\lambda_{\alpha,i}, \lambda_{\beta,j})\}.$$

$$\Theta = ((12), (1234), (1234)) \in S_3 \times S_4 \times S_4.$$



Partial Latin rectangles ($\mathcal{PLR}_{r,s,n}$).

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1		3	
	2		
2	3	4	1

Marked orbit.

Symbol-free orbit.

Complete orbit.

Partial Latin rectangles ($\mathcal{PLR}_{r,s,n}$).

Proposition

$\Theta = (\alpha, \beta, \gamma) \in \mathfrak{I}_{r,s,n}$. $\quad P = (p_{ij}) \in \mathcal{PLR}_{r,s,n}$.

$P \in \mathcal{PLR}_\Theta$ if and only if the next two conditions hold.

a) *Every cell orbit in P is complete or symbol-free.*

b) *If a cell (i,j) is non-empty, then its cell orbit is complete and formed by the entries*

$$(\alpha^m(i), \beta^m(j), \gamma^m(p_{ij})).$$

$\Theta = ((12), (1234), (1234)) \in S_3 \times S_4 \times S_4$.

1		3	
	2		4
2	3	4	1

II. Compatibility and feasibility.

Compatibility.

A triple of positive integers (i, j, k) is **lcm-compatible** if

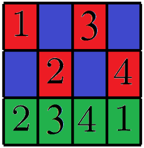
$$\mathrm{lcm}(i,j,k) = \mathrm{lcm}(i,j) = \mathrm{lcm}(i,k) = \mathrm{lcm}(j,k).$$

Proposition

$\Theta = (\alpha, \beta, \gamma) \in \mathfrak{I}_{r,s,n}$. $P \in \mathcal{PLR}_{\Theta}$.

$(i,j,k) \in \mathrm{Ent}(P) \Rightarrow (l_{\alpha,i}, l_{\beta,j}, l_{\gamma,k})$ *is lcm-compatible.*

$$\Theta = ((12),(1234),(1234)) \in S_3 \times S_4 \times S_4.$$



Compatibility.

$\Theta = (\alpha, \beta, \gamma) \in \mathfrak{I}_{r,s,n}$.

$P \in \mathcal{PLR}_{r,s,n}$ is **Θ-compatible** if, for each $(i, j, k) \in \mathrm{Ent}(P)$,

C.1) $(l_{\alpha,i}, l_{\beta,j}, l_{\gamma,k})$ is lcm-compatible.

C.2) For each $m \geq 0$,

- Either the cell $(\alpha^m(i), \beta^m(j))$ is empty or
- $(\alpha^m(i), \beta^m(j), \gamma^m(k)) \in \mathrm{Ent}(P)$.

$\Theta = ((12), (1234), (1234)) \in S_3 \times S_4 \times S_4$.

1		3	
	2		
2	3	4	1

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- Either the cell $(\alpha^m(i), \beta^m(j))$ is empty or
- $(\alpha^m(i), \beta^m(j), \gamma^m(k)) \in \mathrm{Ent}(P)$.

Theorem

$\mathcal{PLR}_\Theta \neq \emptyset$ if and only if there exists a Θ-compatible $P \in \mathcal{PLR}_{r,s,n}$ for which every marked orbit is complete.

Feasibility.

$\Theta = (\alpha, \beta, \gamma) \in \mathfrak{I}_{r,s,n}$.

- **Feasible**: $\lambda_i^{\alpha}\lambda_j^{\beta}\lambda_k^{\gamma} > 0 \Rightarrow (i, j, k)$ lcm-compatible.

  $((12)(34)(56), (123)(456), (123456))$.

- **Extension of size $n' \geq n$**:
  Any $(\alpha, \beta, \gamma') \in \mathfrak{I}_{r,s,n'}$, s. t. $\gamma'(s) = \gamma(s), \forall s \leq n$.
- **Natural extension**: $\gamma'(s) = s$, $\forall\, n < s$.
- **Extendable isotopism**: All its natural extensions are feasible.

Proposition

- *Extendability $\Rightarrow$ Feasibility.*
- *If a natural extension of Θ is feasible, then Θ is feasible.*
- *Every feasible principal isotopism is extendable.*

Feasibility.

$\Theta = (\alpha, \beta, \gamma) \in \mathfrak{I}_{r,s,n}$.

- **Feasible**: $\lambda_i^{\alpha}\lambda_j^{\beta}\lambda_k^{\gamma} > 0 \Rightarrow (i,j,k)$ lcm-compatible.

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- **Extension of size $n' \geq n$**:
  Any $(\alpha, \beta, \gamma') \in \mathfrak{I}_{r,s,n'}$, s. t. $\gamma'(s) = \gamma(s), \forall s \leq n$.
- **Natural extension**: $\gamma'(s) = s$, $\forall\, n < s$.
- **Extendable isotopism**: All its natural extensions are feasible.

Theorem

(α, β, γ) is extendable if and only if

- *it is feasible, and*
- *all the cycles of α and β have the same length.*

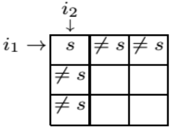
III. The autotopism stabilized colouring game.

- **Initial board**: Empty $r \times s$ array: $P = (p_{ij})$.
- **Key**: $\Theta = (\alpha, \beta, \gamma) \in \mathfrak{I}_{r,s,n}$.

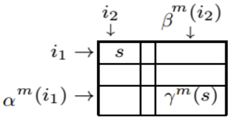
Basic rules:

Alternately, the players colour an empty cell $(i, j) \in [r] \times [s]$ in the board with a colour $\mathfrak{c} \in [n]$ by respecting the following rules:

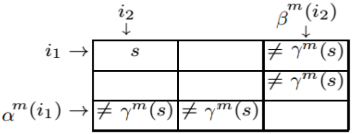
- **Rule 1**) The Latin array condition must always hold.
- **Rule 2**) The Θ-compatibility condition must always hold.
- **Rule 3**) For each $m < \mathrm{lcm}(l_{\alpha,i}, l_{\beta,j})$ and each coloured collinear cell (i', j') of $(\alpha^m(i), \beta^m(j))$, it must be $p_{i'j'} \neq \gamma^m(\mathfrak{c})$.



Rule 1

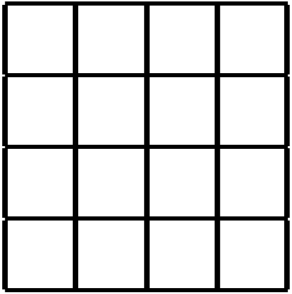


Rule 2



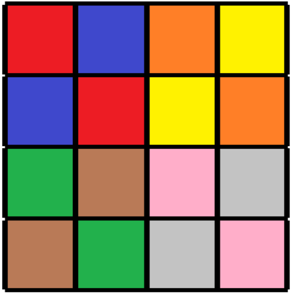
Rule 3

1 2 3 4



$$\Theta = ((12)(34), (12)(34), \mathrm{Id})$$

1 2 3 4



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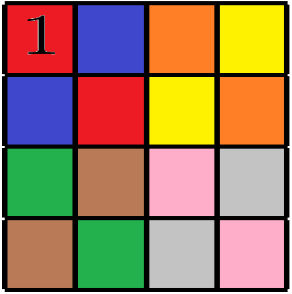
1 2 3 4



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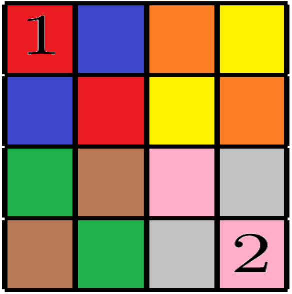
The game.

1 2 3 4



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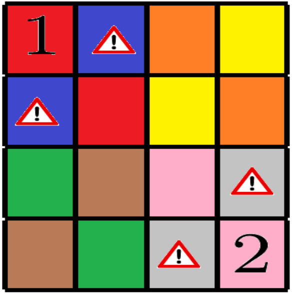
1 2 3 4



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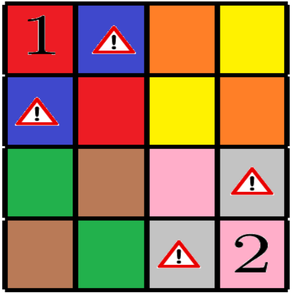
The game.

1 2 3 4



$\chi^{\ominus}_{g_A} > 4.$

The game.

$\Theta \in \mathfrak{I}_{r,s,n}$.

Lemma

Alice always wins the Θ-stabilized colouring game when a player colours any cell of the last symbol-free cell orbit of the board.

Proposition (Well-defined game)

Let Θ be extendable and let P be Θ-compatible, satisfying Rule 3 and with at least one symbol-free cell orbit. Then,

a) *any empty cell in P can be coloured.*
b) *If a symbol $s > n$ of an extension of Θ does not appear in any cell of P, then there exists at least one empty cell in P that can be coloured with the colour s by obeying Rules 1–3.*

The game.

$\Theta \in \mathfrak{I}_{r,s,n}$

- **Θ-stabilized game chromatic number** (χ_g^{Θ}): Smallest $n' > n$ for the natural extension of Θ for which Alice has a winning strategy.

Proposition

Let Θ_1 and Θ_2 be two extendable isotopisms with the same cycle structure. Then,

$$\chi_g^{\Theta_1} = \chi_g^{\Theta_2}.$$

Lemma

If Θ is extendable and $n = \max\{r, s\}$, then

$$n \leq \chi_g^{\Theta} \leq |\mathfrak{o}(\Theta)| + n - 1.$$

The game.

- If $\Theta = (\mathrm{Id}, \mathrm{Id}, \mathrm{Id})$, then $\chi_g^{\Theta} = \chi_g$.

Lemma

Let $G = (V, E)$ be a graph with $|E| \neq \emptyset$. Then,

$$\chi_{g_B}(K_2 \Box G) \leq \Delta(K_2 \Box G)) \leq \Delta(G) + 1.$$

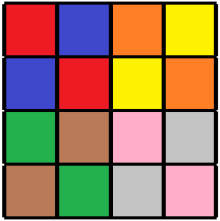
Theorem

- $\chi_{g_A}(\mathcal{R}_{1,n}) = \chi_{g_B}(\mathcal{R}_{1,n}) = n$.
- $\chi_{g_A}(\mathcal{R}_{2,n}) = n + 1$.
- $\chi_{g_B}(\mathcal{R}_{2,n}) = n$, *for* $n \geq 2$.
- $\chi_{g_A}(\mathcal{R}_{3,3}) = 3$.
- $\chi_{g_B}(\mathcal{R}_{3,3}) = 4$.

The game.

Theorem

- $\chi_{g_A}^{((12)(34),(12)(34),\mathrm{Id})}(\mathcal{R}_{4,4}) = 5.$
- $\chi_{g_B}^{((12)(34),(12)(34),\mathrm{Id})}(\mathcal{R}_{4,4}) = 4.$



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Autotopism stabilized colouring games on rook's graphs

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