Lagrangian-Analytical Modelling of Damped Quintuple Pendulum System

M. C. Agarana, *IAENG Member*, O.O. Ajayi and M.E. Emetere

Abstract— **Quintuple pendulums are an extension of the chaotic double, triple and Quatertuple pendulums problems. In this paper, a planar compound quintuple pendulum was modelled with viscous damping forces. Using Lagrangian energy methods, we derive coupled ordinary differential equations of motion for the system and submit them to analytical manipulation to model the dynamics of the system. We obtain the simulated results. The inclusion of damping in the system has significant effect on the dynamics, highlighting the system's chaotic nature.**

***Index Terms*— Dynamic Modelling, Quintuple Pendulum, Lagrangian.**

I. INTRODUCTION

THE double pendulum is a classic system used in Dynamics courses everywhere. Through the span of the class, we have unravelled conditions of movement and recreated models of both straightforward (massless bars) and compound (bars with mass) planar twofold pendulums utilizing Newton's Second Law and Euler's Equations. In this venture, we would like to expand the twofold pendulum framework we have considered so well into a quintuple compound pendulum with damping at the joints. The trial framework we are attempting to demonstrate is appeared in figure1.[1] The bars of the pendulum have noteworthy mass, requiring the consideration of rotational flow in the framework. Besides, the framework has been seen to sodden fundamentally after some time. To understand these conditions of movement, we will investigate the utilization of Lagrangian Mechanics for non-traditionalist frameworks and will settle for conditions of movement. We will make a numerical reproduction for the framework so as to investigate our conditions of movement and will approve them by correlation with exploratory information from a genuine planar quintuple pendulum system.[1,2]

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II. THE MODEL

The bars of the pendulum have significant mass so it is modelled as a compound pendulum with the presence of damping [3,4]. Each bar l_1 is defined by a set of four parameters: Ii, the moment of inertia of the bar, mi, the mass of the bar, li, the length of the bar, and ki, the damping coefficient of the bar rotating about its upper joint. The position and velocity of the bars are defined by the ten system state variables: $\theta_1, \theta_2, \theta_3, \theta_4, \theta_5, \dot{\theta}_1, \dot{\theta}_2, \dot{\theta}_3, \dot{\theta}_4, \dot{\theta}_5$
An equation of motion of the frictionless ideal case was first derived. This allows for model validation by ensuring energy is conserved in the dynamics. Frictional Damping is later added, to observe changes in the dynamics[5,6]. Taking down as +y and right as +x, the positions of the centres of mass of the bars was written as functions of θ_i and the geometric parameters of the system as follows:

$$y_1 = \frac{l_1}{2}\cos\theta_1 \tag{1}$$

$$y_2 = l_1\cos\theta_1 + \frac{l_2}{2}\cos\theta_2 \tag{2}$$

$$y_3 = l_1\cos\theta_1 + l_2\cos\theta_2 + \frac{l_3}{2}\cos\theta_3 \tag{3}$$

$$y_4 = l_1\cos\theta_1 + l_2\cos\theta_2 + l_3\cos\theta_3 + \frac{l_4}{2}\cos\theta_4 \tag{4}$$

$$y_5 = l_1\cos\theta_1 + l_2\cos\theta_2 + l_3\cos\theta_3 + l_4\cos\theta_4 + \frac{l_5}{2}\cos\theta_5 \tag{5}$$

$$x_1 = \frac{l_1}{2}\sin\theta_1 \tag{6}$$

$$x_2 = l_1\sin\theta_1 + \frac{l_2}{2}\sin\theta_2 \tag{7}$$

$$x_3 = l_1\sin\theta_1 + l_2\sin\theta_2 + \frac{l_3}{2}\sin\theta_3 \tag{8}$$

$$x_4 = l_1\sin\theta_1 + l_2\sin\theta_2 + l_3\sin\theta_3 + \frac{l_4}{2}\sin\theta_4 \tag{9}$$

$$x_5 = l_1\sin\theta_1 + l_2\sin\theta_2 + l_3\sin\theta_3 + l_4\sin\theta_4 + \frac{l_5}{2}\sin\theta_5 \tag{10}$$

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$$\dot{y}_1 = -\frac{l_1}{2}\sin\theta_1 \quad (11)$$

$$\dot{y}_2 = -l_1\sin\theta_1 - \frac{l_2}{2}\sin\theta_2 \quad (12)$$

$$\dot{y}_3 = -l_1\sin\theta_1 - l_2\sin\theta_2 - \frac{l_3}{2}\sin\theta_3 \quad (13)$$

$$\dot{y}_4 = -l_1\sin\theta_1 - l_2\sin\theta_2 - l_3\sin\theta_3 - \frac{l_4}{2}\sin\theta_4 \quad (14)$$

$$\dot{y}_5 = -l_1\sin\theta_1 - l_2\sin\theta_2 - l_3\sin\theta_3 - l_4\sin\theta_4 - \frac{l_5}{2}\sin\theta_5 \quad (15)$$

$$\dot{x}_1 = \frac{l_1}{2}\cos\theta_1 \quad (16)$$

$$\dot{x}_2 = l_1\cos\theta_1 + \frac{l_2}{2}\cos\theta_2 \quad (17)$$

$$\dot{x}_3 = l_1\cos\theta_1 + l_2\cos\theta_2 + \frac{l_3}{2}\cos\theta_3 \quad (18)$$

$$\dot{x}_4 = l_1\cos\theta_1 + l_2\cos\theta_2 + l_3\cos\theta_3 + \frac{l_4}{2}\cos\theta_4 \quad (19)$$

$$\dot{x}_5 = l_1\cos\theta_1 + l_2\cos\theta_2 + l_3\cos\theta_3 + l_4\cos\theta_4 + \frac{l_5}{2}\cos\theta_5 \quad (20)$$

These positions are then differentiated with respect to time to find the x and y components of the velocities as functions of angles and angular velocities[5,6,7]. They will not be shown here for brevity.

The magnitude of the velocity of each bar is given as:

$$v_i = \sqrt{\dot{x}_i^2 + \dot{y}_i^2} \quad (11)$$

The translational, rotational kinetic energy and the gravitational potential energy (TKE, RKE and GPE) of each bar, are given as:

$$TKE_i = 0.5 m_i v_i^2 \quad (12)$$

$$\mathrm{RKE}_i = 0.5\mathrm{I}_i\dot{\theta}_i \quad (13)$$

$$\mathrm{GPE}_i = \mathrm{m}_i \mathrm{g} y_i \quad (14)$$

The Lagrangian of the system can be written as:

$$L = T - V$$
$$= \sum_{i=1}^{5}(TKE_i + RKE_i - GPE_i) \quad (15)$$

Lagrange's equation

$$\frac{d}{dt}\left(\frac{dL}{\dot{q}_i}\right) = \frac{dL}{dq_i} \quad (16)$$

taking θ_i as the generalised coordinate q_i becomes

$$\frac{d}{dt}\left(\frac{dL}{d\dot{\theta}_i}\right) = \frac{dL}{d\theta_i} \quad (17)$$

With the addition of damping to the system, modelled with a viscous drag, caused by the angular velocity of the joints changed the lagrangian equation in equation (17). This viscous form of drag can be modelled in Lagrangian mechanics with the Reyleigh Distribution Function:

$$D = 0.5\left(\sum_{i=1}^{5} k_i\dot{\theta}_i^2\right) \quad (18)$$

The lagrangian equation becomes:

$$\frac{d}{dt}\left(\frac{dL}{\dot{\theta}_i}\right) = \frac{dL}{d\theta_i} - \frac{dD}{d\dot{\theta}_i} \quad (19)$$

This modified form of Lagrange's equation produces a system of five equations which contain the angular acceleration terms, just as in the un-damped case. This is omitted for brevity. The solution of this system of five equations and five unknowns yields the expression for the angular velocities. The angular velocities were then numerically integrated to produce the path of the pendulum. Solving for the angular velocity terms produces the equations of motion

III. ANALYSIS

For a point mas, force is equal to mass times acceleration, according to Newton's second law of motion,

$$F = ma = m\ddot{x} = m\frac{d\dot{x}}{dt} \quad (20)$$

Integrating both sides of equation (20) gives

$$\int F.dx = \int m\ddot{x}.dx \quad (21)$$

But

$$\ddot{x}dx = \left(\frac{d\dot{x}}{dt}\right)dx = d\dot{x}\left(\frac{dx}{dt}\right) = \dot{x}d\dot{x} \quad (22)$$

Therefore equation (21) can be written as

$$\int F.dx = \int m\dot{x}.d\dot{x} \quad (23)$$

Equation (23) represents work done.

Now in the Lagrangian L = T – V, T does not depend on position and V does not depend on velocity, so

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$$\frac{\partial L}{\partial \dot{x}_i} = \frac{\partial T}{\partial \dot{x}_i} \tag{24}$$

$$\frac{\partial L}{\partial x_i} = \frac{\partial V}{\partial x_i} \tag{25}$$

Inputting equation (15) into (19) gives

$$\frac{d}{dt}\left(\frac{d(T-V)}{\dot{\theta}_i}\right) = \frac{d(T-V)}{d\theta_i} - \frac{dD}{d\dot{\theta}_i} \tag{26}$$

$$\frac{d}{dt}\left(\frac{d[\sum_{i=1}^{5}(TKE_i + RKE_i - GPE_i)]}{\dot{\theta}_i}\right)$$

$$= \frac{d[\sum_{i=1}^{5}(TKE_i + RKE_i - GPE_i)]}{d\theta_i}$$

$$-\frac{d(0.5\left(\sum_{i=1}^{5} k_i \dot{\theta}_i^2\right))}{d\dot{\theta}_i} \tag{27}$$

$$\frac{d}{dt}\left(\frac{d[\sum_{i=1}^{5} 0.5 m_i v_i^3 + 0.5 I_i \dot{\theta}_i - m_i g y_i)]}{d\dot{\theta}_i}\right)$$

$$= \frac{d(\sum_{i=1}^{5} 0.5 m_i v_i^3 + 0.5 I_i \dot{\theta}_i - m_i g y_i)}{d\theta_i}$$

$$-\frac{d(0.5\left(\sum_{i=1}^{5} k_i \dot{\theta}_i^2\right))}{d\dot{\theta}_i} \tag{28}$$

But $v_i = \sqrt{\dot{x}_i^2 + \dot{y}_i^2}$

$$v^3{}_i = \left(\dot{x}_i^2 + \dot{y}_i^2\right)^{\frac{3}{2}} \tag{29}$$

Substituting equation (29) into (28) gives

$$\frac{d}{dt}\left(\frac{d(\sum_{i=1}^{5} 0.5 m_i \{\left(\dot{x}_i^2 + \dot{y}_i^2\right)^{\frac{3}{2}}\} + 0.5 I_i \dot{\theta}_i - m_i g y_i)}{d\dot{\theta}_i}\right)$$

$$= \frac{d(\sum_{i=1}^{5} 0.5 m_i \{\left(\dot{x}_i^2 + \dot{y}_i^2\right)^{\frac{3}{2}}\} + 0.5 I_i \dot{\theta}_i - m_i g y_i)}{d\theta_i}$$

$$-\frac{d(0.5\left(\sum_{i=1}^{5} k_i \dot{\theta}_i^2\right))}{d\dot{\theta}_i} \tag{30}$$

$$i = 1, 2, 3, 4.5$$

For the sake of brevity, only the analysis for i = 1 is Presented in this paper.

Now for i=1, equation (30) becomes:

$$\frac{d}{dt}\left(\frac{0.5 m_1 d(\{\left(\dot{x}_1^2 + \dot{y}_1^2\right)^{\frac{3}{2}}\} + 0.5 I_1 \dot{\theta}_1 - m_1 g y_1)}{d\dot{\theta}_1}\right)$$

$$= \frac{0.5 m_1 d\{\left(\dot{x}_1^2 + \dot{y}_1^2\right)^{\frac{3}{2}}\} + 0.5 I_1 \dot{\theta}_1 - m_1 g y_1)}{d\theta_1}$$

$$-\frac{0.5 k_1 d\dot{\theta}_1^2}{d\dot{\theta}_1} \tag{31}$$

From equations (11) and (16)

$$\dot{x}_1^2 + \dot{y}_1^2 = \left(\frac{l_1}{2}\cos\theta_1\right)^2 + \left(-\frac{l_1}{2}\sin\theta_1\right)^2$$

$$= \frac{l_1^2}{4}\cos^2\theta_1 + \frac{l_1^2}{4}\sin^2\theta_1$$

$$= \frac{l_1^2}{4}(\cos^2\theta_1 + \sin^2\theta_1)$$

$$= \frac{l_1^2}{4}$$

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$$\frac{d}{dt}\left(\frac{0.5m_1 d(\{\left(\frac{l_1^2}{4}\right)^{\frac{3}{2}}\}+0.5I_1\dot{\theta}_1-m_1 g\frac{l_1}{2}\cos\theta_1)}{d\dot{\theta}_1}\right)$$

$$=\frac{0.5m_1 d\{\left(\frac{l_1^2}{4}\right)^{\frac{3}{2}}\}+0.5I_1\dot{\theta}_1-m_1 g\frac{l_1}{2}\cos\theta_1)}{d\theta_1}-\ 2k_1\dot{\theta}_1 \quad (32)$$

$$\frac{d}{dt}\left(0.5m_1(0.5I_1\dot{\theta}_1-m_1 g\frac{l_1}{2}\frac{d}{d\dot{\theta}_1}\cos\theta_1)\right)$$
$$=0.5m_1[0.5I_1\frac{d}{d\theta_1}\dot{\theta}_1+m_1 g\frac{l_1}{2}\sin\theta_1]-\ 2k_1\dot{\theta}_1 \quad (33)$$

$$0.25m_1I_1\ddot{\theta}_1-m_1 g\frac{l_1}{2}\frac{d^2}{dtd\dot{\theta}_1}\cos\theta_1$$
$$=m_1^2 g l_1\sin\theta_1-\ 2k_1\dot{\theta}_1 \quad (34)$$

$$0.25m_1I_1\ddot{\theta}_1-m_1 g\frac{l_1}{2}\frac{d^2}{dtd\dot{\theta}_1}\cos\theta_1$$
$$=m_1^2 g l_1\sin\theta_1-\ 2k_1\dot{\theta}_1 \quad (35)$$

$$0.25m_1I_1\ddot{\theta}_1-m_1 g\frac{l_1}{2}\sin\theta_1-m_1^2 g l_1\sin\theta+2k_1\dot{\theta}_1\ =0 \quad (36)$$

$$0.25m_1I_1\ddot{\theta}_1+2k_1\dot{\theta}_1-m_1 g\frac{l_1}{2}(1+2m_1)\sin\theta_1\ =0 \quad (37)$$

For small values of θ_1, $\theta_1\approx\sin\theta_1$,
So equation () gives

$$0.25m_1I_1\ddot{\theta}_1+2k_1\dot{\theta}_1-m_1 g\frac{l_1}{2}(1+2m_1)\theta_1\ =0 \quad (38)$$

$$0.5m_1I_1\ddot{\theta}_1+4k_1\dot{\theta}_1-m_1 g l_1(1+2m_1)\theta_1\ =0 \quad (39)$$

$$\ddot{\theta}_1+\frac{8k_1}{m_1I_1}\dot{\theta}_1-\frac{2gl_1(1+2m_1)}{I_1}\theta_1\ =0 \quad (40)$$

At this stage, we carefully choose the values of t and p such that:

$$\frac{8k_1}{m_1I_1}=-2t \text{ and } \frac{2gl_1(1+2m_1)}{I_1}=-2p \quad (41)$$

where t is time and p is a parameter

Substituting equation (41) into equation (40) gives

$$\ddot{\theta}_1-2t\dot{\theta}_1+2p\theta_1\ =0 \quad (42)$$

Equation (42) is Hermitite equation [7]
Therefore,

$$\theta=\sum_{n=0}^{\infty}a_n(t-t_0)^n \quad (43)$$

$$\theta=\sum_{n=0}^{\infty}a_n t^n,\ \text{since } t_0=0 \quad (44)$$

Differentiating term by term gives

$$\dot{\theta}=\sum_{n=1}^{\infty}na_n t^{n-1} \quad (45)$$

$$\ddot{\theta}=\sum_{n=2}^{\infty}n(n-1)a_n t^{n-2} \quad (46)$$

Replacing *n* by *m+2* in Equation (46), gives

$$\ddot{\theta}=\sum_{m+2=2}^{\infty}m+2(m+2-1)a_n t^{m+2-2} \quad (47)$$

$$\ddot{\theta}=\sum_{m=0}^{\infty}(m+2)(m+1)a_{m+2}t^m \quad (48)$$

Solving equation (48) analytically gives [7]:

$$\theta=1+t-\frac{2p}{2!}t^2-\frac{2(p-1)}{3!}t^3+\frac{2^2p(p-2)}{4!}t^4$$
$$+\frac{2^2p(p-1)(p-2)}{5!}t^5-\frac{2^3p(p-2)(p-4)}{6!}t^6$$
$$-\frac{2^3(p-1)(p-3)(p-5)}{7!}t^7+\ldots \quad (49)$$

IV. NUMERICAL RESULTS AND DISCUSSION

For the purpose of numerical example, different values of m_1, l_1, I_1, k_1 were considered in order to get the values of θ. g was taken to be equal to 9.81.

From equation (41) :

$$t=\frac{-4k_1}{m_1I_1} \quad (50)$$

$$p=\frac{gl_1(1+2m_1)}{I_1} \quad (51)$$

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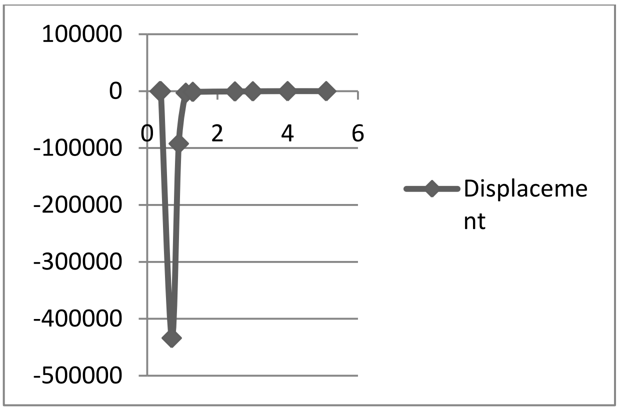

Fig.1. Angular displacement of first segment at different masses (m_1)

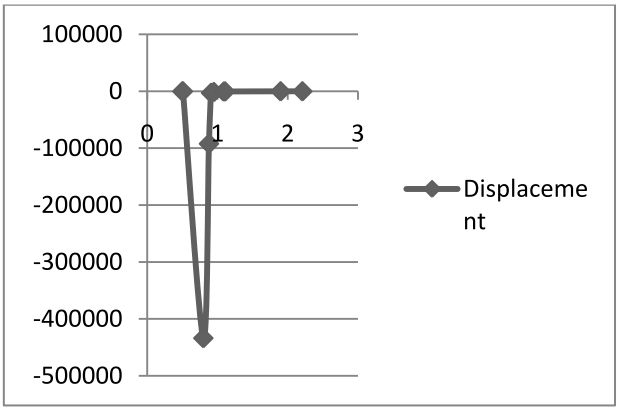

Fig.2. Angular displacement of first segment at different lengths (l_1)

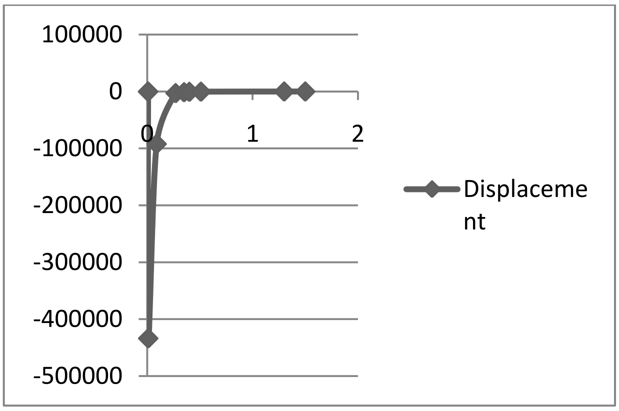

Fig.3. Angular displacement of first segment at different Inertia (I_1)

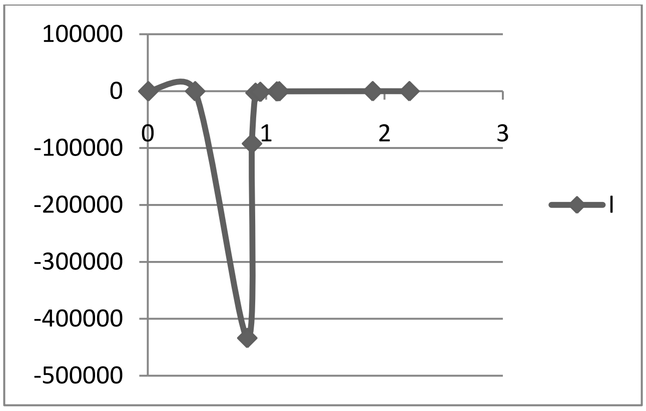

Fig.4. Angular displacement of first segment at different damping values (k_1)

Equation of motion and simulation of quintuple pendulum model was carried out analytically. The triple pendulum was extended to the quintuple pendulum. Damping was put into consideration. The bars length, masses and the moment of inertia have significant effect on the dynamics of the system. The damping effect can be seen in the figures 1,2,3 and 4. The damping effect reduced the effect of the other parameters on the displacement of the pendulum system. The mass, the length of the bars, the moment of inertia and damping all affect the dynamics of the quintuple pendulum system.

IV. CONCLUSION

Using Lagrangian – analytical methods, mathematical model featuring a set of coupled ordinary differential equations of motion for the dynamic compound quintuple pendulum system was created. These equations of motion were simulated analytically. The behaviour of the model shows that the inclusion of damping force significantly affects the dynamics of the system after the first few seconds.

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