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Averaged null energy condition in a classical curved background

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Abstract

The Averaged Null Energy Condition (ANEC) states that the integral along a complete null geodesic of the projection of the stress-energy tensor onto the tangent vector to the geodesic cannot be negative. Exotic spacetimes, such as those allow wormholes or the construction of time machines are possible in general relativity only if ANEC is violated along achronal geodesics. Starting from a conjecture that flat-space quantum inequalities apply with small corrections in spacetimes with small curvature, we prove that ANEC is obeyed by a minimally-coupled, free quantum scalar field on any achronal null geodesic surrounded by a tubular neighborhood whose curvature is produced by a classical source.

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I. INTRODUCTION

It is always possible to invent a spacetime with exotic features, such as wormholes, superluminal travel, or the construction of time machines, and then determine what stress-energy tensor is necessary to support the given spacetime. To rule out such exotic spacetimes we would like to prove energy conditions that restrict the stress-energy tensor that might arise from quantum fields and show that the stress-energy necessary to support an exotic spacetime is impossible. We need a condition which is strong enough to rule out exotic cases while simultaneously weak enough to be proven correct, or at least to be free of known counterexamples.

The best possibility for such a condition seems to be the achronal averaged null energy condition [1], which requires the following. Let M be a manifold with Lorentzian metric g and T be the stress-energy tensor of some fields on M. Let γ be a complete null geodesic with tangent vector $\boldsymbol{\ell}$. Suppose that γ is achronal, i.e., no two points of γ can be connected by a timelike curve. Then

$$\int_\gamma T_{ab}\ell^a\ell^b \geq 0\,. \tag{1}$$

That is to say, we require that the projection of the stress-energy tensor along a geodesic integrate to a non-negative value, but only for geodesics that are achronal. As far as we know there are no known violations of achronal ANEC using minimally-coupled scalar fields.[1] Achronal ANEC is sufficient to rule out many exotic spacetimes [1].

Reference [2] proved that the averaged null energy condition (ANEC) holds for geodesics traveling through empty, flat space, even if elsewhere in the spacetime there are boundaries or spacetime curvature, providing that these stay some minimum distance from the geodesic and do not affect the causal structure of the spacetime near the geodesic. Here we will extend this work to geodesics traveling in curved spacetime, with the restriction that the spacetime near the geodesic must obey the null convergence condition,

$$R_{ab}V^aV^b \geq 0 \tag{2}$$

for any null vector V^a. Equation (2) holds whenever the curvature is generated by a "classical background" whose stress tensor obeys the null energy condition (NEC),

$$T_{ab}V^aV^b \geq 0\,. \tag{3}$$

'We stress that Eqs. (2) and (3) need not hold in general, but only in a neighborhood of the null geodesic on which we seek to prove ANEC. Thus, for example, the results of this paper apply to any geodesic which does not encounter any material source, even if such sources exist elsewhere in the spacetime.

Reference [2] used a null-contracted timelike-averaged quantum inequality proved for flat space in Ref. [3]. Here we will conjecture that this quantum inequality holds with a small modification in spacetimes with small curvature. We will then be able to rule out ANEC violation, subject to several conditions.

In the next section we give the conditions on which our theorem depends. In Sec. III we state our theorem. In Sec. IV we discuss what it means to have small curvature and

[1] Non-minimally coupled scalar fields have some unique properties, which we discuss briefly in Sec. VI.

state our conjecture. In Sec. V we prove the theorem, and in Sec. VI we conclude with a discussion of remaining possibilities for the generation of exotic spacetimes. We use the sign convention $(+,+,+)$ in the classification of Misner, Thorne and Wheeler [4].

II. ASSUMPTIONS

A. Congruence of geodesics

As in Ref. [2], we will not be able to rule out ANEC violation on a single geodesic. However, a single geodesic would not lead to an exotic spacetime. It would be necessary to have ANEC violation along a finite congruence of geodesics in order to have a physical effect.

So let us suppose that our spacetime contains a null geodesic γ with tangent vector $\boldsymbol{\ell}$ and that there is a "tubular neighborhood" M' of γ composed of a congruence of achronal null geodesics, defined as follows. Let p be a point of γ, and let M_p be a normal neighborhood of p. Let $\mathbf{v}$ be a null vector at p, linearly independent of $\boldsymbol{\ell}$, and let $\mathbf{x}$ and $\mathbf{y}$ be spacelike vectors perpendicular to $\mathbf{v}$ and $\boldsymbol{\ell}$. Let q be any point in M_p such that p can be connected to q by a geodesic whose tangent vector is in the span of $\{\mathbf{v},\mathbf{x},\mathbf{y}\}$. Let $\gamma(q)$ be the geodesic through q whose tangent vector is the vector $\boldsymbol{\ell}$ parallel transported from p to q. If a neighborhood M' of γ is composed of all geodesics $\gamma(q)$ for some choice of p, M_p, $\mathbf{v}$, $\mathbf{x}$ and $\mathbf{y}$, we will say that M' is a tubular neighborhood of γ.

B. Coordinate system

Given the above construction, we can define Fermi-like coordinates [5] on M' as follows. Without loss of generality we can take the vector $\mathbf{v}$ to be normalized so that $\mathbf{v}\cdot\boldsymbol{\ell} = -1$, and $\mathbf{x}$ and $\mathbf{y}$ to be unit vectors. Then we have a pseudo-orthonormal tetrad at p given by $\mathbf{E}_{(u)} = \boldsymbol{\ell}$, $\mathbf{E}_{(v)} = \mathbf{v}$, $\mathbf{E}_{(x)} = \mathbf{x}$, and $\mathbf{E}_{(y)} = \mathbf{y}$. The point $q = (u,v,x,y)$ in these coordinates is found as follows. Let $q^{(1)}$ be found by traveling unit affine parameter from p along the geodesic generated by $v\mathbf{E}_{(v)} + x\mathbf{E}_{(x)} + y\mathbf{E}_{(y)}$. Then q is found by traveling unit affine parameter from $q^{(1)}$ along the geodesic $u\mathbf{E}_{(u)}$. During this process the tetrad is parallel transported. All vectors and tensors will be described using this transported tetrad unless otherwise specified. We will use Latin letters from the beginning of the alphabet to denote arbitrary components in the tetrad basis.

The points with u varying but other coordinates fixed form one of the null geodesics of the previous section.

C. Curvature

We suppose that the curvature inside M' obeys the null convergence condition, Eq. (2). We will refer to this as a "classical background", but the only way it need be classical is Eq. (2).

We would not expect any energy conditions to hold when the curvature is arbitrarily large, because then we would be in the regime of quantum gravity, so we will require that the curvature be bounded. In the coordinate system of Sec. II B we require

$$|R_{abcd}| < R_{\max}\,, \tag{4}$$

everywhere in M'.

We will also need to bound the first and second derivatives of the Riemann tensor,

$$|R_{abcd,e}| < R'_{\max}, \qquad |R_{abcd,ef}| < R''_{\max} \tag{5}$$

everywhere in M'. The bounds $R_{\max}$, $R'_{\max}$ and $R''_{\max}$ are some (independent) finite numbers, but they need not be small.

We will also assume that the curvature is smooth.

D. Causal structure

We will also require that conditions outside M' do not affect the causal structure of the spacetime in M' [2][2]

$$J^+(p, M) \cap M' = J^+(p, M') \tag{6}$$

for all $p \in M'$. Otherwise the curvature outside M' may be arbitrary.

E. Quantum field theory

We consider a quantum scalar field in M. We will work entirely inside M', and there we require that the field be free and minimally coupled. It may be massive or massless. Outside M', however, we can allow different curvature coupling, interactions with other fields, and even boundary surfaces with specified boundary conditions.

Because M may not be globally hyperbolic, it is not completely straightforward to specify what we mean by a quantum field theory on M. We will use the same strategy as Ref. [2]. Our results will hold for any quantum field theory on M that reduces to the usual quantum field theory on each globally hyperbolic subspacetime of M. The states of interest will be those that reduce to Hadamard states on each globally hyperbolic subspacetime, and we will refer to any such state as "Hadamard". See Sec. II B of Ref. [2] for further details.

III. THE THEOREM

We can now state our theorem.

Theorem 1. Let (M, g) be a (time-oriented) spacetime and let γ be a null geodesic on (M, g), and suppose that γ is surrounded by a tubular neighborhood M' in the sense of Sec. II A, obeying the null convergence condition, Eq. (2), and that we have constructed coordinates by the procedure of Sec. II B. Suppose that the curvature in this coordinate system is smooth and obeys the bounds of Sec. II C, that the curvature in the system is localized, i.e., in the distant past and future the spacetime is flat, and that the causal structure of M' is not affected by conditions elsewhere in M, Eq. (6).

[2] This condition is equivalent to $J^-(p, M) \cap M' = J^-(p, M')$ for all $p \in M'$.

Let ω be a state of the free minimally coupled quantum scalar field on M' obeying the conditions of Sec. II E, and let T be the renormalized expectation value of the stress-energy tensor in state ω.

Under these conditions, it is impossible for the ANEC integral,

$$A = \int_{-\infty}^{\infty} d\lambda\, T_{ab}\ell^a \ell^b(\Gamma(\lambda)), \tag{7}$$

to converge uniformly to negative values on all geodesics $\Gamma(\lambda)$ *in* M'.

In the next section, we will conjecture that a known flat-space quantum inequality can be extended to spacetimes with small curvature in a particular way. From this conjecture we will be able to prove Theorem 1.

IV. QUANTUM INEQUALITY

The proof will proceed very much along the lines of Ref. [2]. That paper used the following quantum inequality for the null-projected but timelike-averaged stress-energy tensor, derived by Fewster and Roman [3, 6]. Let $w(\tau)$ be a timelike geodesic segment parameterized by proper time $\tau \in (-\tau_0, \tau_0)$. Let $g(\tau)$ be a smooth real function with compact support contained in $(-\tau_0, \tau_0)$. Let $\mathbf{k}$ be the tangent vector to $w(\tau)$ and let $\boldsymbol{\ell}$ be a constant null vector. Let T be the renormalized stress-energy tensor of a massless or massive[3] minimally-coupled quantum scalar field in a Hadamard state. Then the projection of T on the null vector $\boldsymbol{\ell}$ obeys a quantum inequality when integrated along the timelike geodesic w,

$$\int_{-\tau_0}^{\tau_0} d\tau\, T_{ab}(w(\tau))\ell^a \ell^b g(\tau)^2 \geq -\frac{(k_a \ell^a)^2}{12\pi^2} \int_{-\tau_0}^{\tau_0} d\tau g''(\tau)^2 \,. \tag{8}$$

Equation (8) is a consequence of the result of Ref. [6], which applies to general worldlines in curved spacetime. This more general result is in the form of a "difference inequality" that restricts the amount by which the left-hand side of Eq. (8) can be more negative than the same quantity evaluated in a reference state. We need an absolute bound, such as Eq. (8), but applicable to curved spacetime. While such a bound has not been proven, we conjecture that Eq. (8) can be extended to spacetimes of small curvature.

The basic idea was given by Ford and Roman [7]. Suppose that we want to test Eq. (8) in a laboratory on the surface of the earth. We are not in flat space, but rather in space with curvature of order $GM_\oplus/R_\oplus^3$. Furthermore the apparatus for measuring T might not be in free fall but rather accelerating with the acceleration due to gravity at the earth's surface, $a = GM_\oplus/R_\oplus^2$. But in a laboratory-scale experiment, these differences should not matter. We expect Eq. (8) to hold with a small correction for almost geodesic $w(\tau)$ in spacetimes with small curvature.

What does it mean for the curvature to be small? First of all, since the curvature has dimensions (length)$^{-2}$, we have to multiply by the square of some length to get a number that we can require to be much less than 1. The obvious length in the present example is τ_0.

[3] The derivation of [3] was for the massless case, but the same argument holds in the massive case as well [2].

We also face a problem that curvature is a tensor, and we would like to make coordinate-invariant statements. In a Riemannian space, we could require, for example, that the sectional curvature of each plane in the tangent space at each point be small. But in a Lorentzian spacetime this does not work: the sectional curvature is never bounded unless it is constant [8, 9]. A simple example of the problem is that the spacetime could contain a plane gravitational wave. The amplitude of such a wave is entirely dependent on the reference frame; it can be made arbitrarily small or arbitrarily large by the choice of coordinates. Thus one cannot say that all components of the Riemann tensor are small without regard to coordinate system.

Fortunately, in our case, we have a privileged observer whose stress-energy tensor we want to integrate. Thus the worldline of that observer can be used to generate a preferred coordinate system.[4] This works straightforwardly on that worldline, but to apply this idea to other places in the spacetime we will have to parallel transport the observer's 4-velocity. Fortunately, in the case where the curvature is in fact small, the precise details of this transport will not matter.

With these considerations in mind we proceed as follows. Let (N, g) be a globally hyperbolic spacetime and let $w(\tau)$ be a timelike path in N, parameterized by proper time $\tau \in (-\tau_0, \tau_0)$, with tangent vector $\mathbf{k}$. In general we will only need to consider the "double cone" $N = J_-(w(\tau_0)) \cap J_+(w(-\tau_0))$. Let $\epsilon \ll 1$. We will say that (N, g) has small curvature ϵ relative to w if N is a normal neighborhood of the point $p = w(0)$ and there exists a set of three unit spacelike vectors $\mathbf{E}_{(i)}$, $i = 1, 2, 3$ at p, orthogonal to each other and to $\mathbf{E}_{(0)} = k$, such that at each point q, every component of the Riemann tensor in the tetrad basis formed by parallel transporting the tetrad $\{\mathbf{E}_{(a)}\}$ along the geodesic connecting p and q obeys

$$|R_{abcd}|\tau_0^2 < \epsilon\,. \tag{9}$$

Suppose (N, g) has small curvature ϵ by the above definition, and we consider the curvature components in a different tetrad basis resulting from a choice of $\mathbf{E}_{(i)}$ other than the one which satisfies Eq. (9). Changing to such a basis will given curvature components that are linear combinations of the ones we had before, and so may be larger than the bound of Eq. (9), but only by factors of order 1.

We could also choose a different starting point p on w. Since the curvature is small, the different parallel transport would change the Riemann tensor components only by factors of $1 + O(\epsilon)$, so the condition would be the same at first order.

We will also require that the proper acceleration of the path on which we want the quantum inequality to hold should be small. Since acceleration has the units of inverse time, we will multiply by the time τ_0 to get a dimensionless measure limiting the total acceleration along the path of interest.

Once we are in curved spacetime, we must address ambiguities in the definition of the stress-energy tensor T. We will adopt the axiomatic definition given by Wald [10], but there remains the ambiguity of adding local curvature terms with arbitrary coefficients. These terms are the metric, the Einstein tensor, and two terms that are second order in the curvature or involve second derivatives of the curvature [11],

$$
\begin{aligned}
{}^{(1)}H_{ab} &= 2R_{;ab} - 2g_{ab}\Box R + g_{ab}R^2/2 - 2RR_{ab} &\quad (10\text{a})\\
{}^{(2)}H_{ab} &= R_{;ab} - \Box R_{ab} - g_{ab}\Box R/2 + g_{ab}R^{cd}R_{cd}/2 - 2R^{cd}R_{acbd}\,. &\quad (10\text{b})
\end{aligned}
$$

[4] A similar technique was used in Ref. [7].

A multiple of the the metric will not concern us here, because it vanishes when contracted with the null vector $\boldsymbol{\ell}$. A term proportional to the Einstein tensor can be absorbed into renormalization of Newton's constant, and we assume that that has been done.

As it turns out, the remaining ambiguity will not affect our proof below. However, it must be taken into account in the present conjecture. Following an idea in Ref. [12], we will allow any definition of T_{ab} and absorb the ambiguity into a local curvature term in our bound.

We now can now conjecture that Eq. (8) holds with a modification of order ϵ and a local curvature term.

Conjecture 1. Let (N, g) be a globally hyperbolic spacetime and let $w(\tau)$ be a timelike path in N, parameterized by proper time $\tau \in (-\tau_0, \tau_0)$. Let $\mathbf{k}$ be the tangent vector to w and let $\boldsymbol{\ell}$ be a null vector field obeying $k^a \nabla_a \ell^b = 0$. Let $g(\tau)$ be a smooth real function with compact support contained in $(-\tau_0, \tau_0)$. Let T be any definition (obeying Wald's axioms [10]) of the renormalized stress-energy tensor of a massless or massive minimally-coupled quantum scalar field in a Hadamard state. If (N, g) has small curvature ϵ relative to w and $|D^2 w^a/d\tau^2|\tau_0 < \epsilon$ everywhere on w, then

$$\int_{-\tau_0}^{\tau_0} d\tau\, T_{ab}(w(\tau))\ell^a \ell^b g(\tau)^2 \geq -\frac{(k_a \ell^a)^2}{12\pi^2} \int_{-\tau_0}^{\tau_0} d\tau\, g''(\tau)^2 [1 + c(\epsilon)] + \int_{-\tau_0}^{\tau_0} d\tau\, g(\tau)^2 C_{ab} \ell^a \ell^b\,, \quad (11)$$

where $c(\epsilon)$ is a function that goes to zero as $\epsilon \to 0$, and C_{ab} is a linear combination of Eqs. (10). The form of $c(\epsilon)$ and the coefficients of ${}^{(1)}H$ and ${}^{(2)}H$ in C_{ab} do not depend on the spacetime or the quantum state. Note that terms in Eqs. (10) whose tensor structure is that of the metric do not contribute in Eq. (11) because $\boldsymbol{\ell}$ is null.

We intend to prove Conjecture 1 in future work.

V. PROOF OF THE THEOREM

A. Outline of the proof

Following Ref. [2], we will prove Theorem 1 by contradiction using integrals over a parallelogram shown below in Fig. 2. By considering this parallelogram as made up of segments of the null geodesics of M', and assuming Theorem 1 is violated, we set a negative upper bound on the integral of the null-contracted stress-energy tensor over the parallelogram. Then we consider the same set of points as being made up of timelike paths, and demonstrate that these paths obey the conditions of Conjecture 1. Thus using Eq. (11), we can set a lower bound on the same integral over the parallelogram. In the limit where the parallelogram becomes long and narrow, these bounds conflict, proving the theorem.

B. The parallelogram

We will use the (u, v, x, y) coordinates of Sec. II B. Let r be a positive number small enough such that whenever $|v|, |x|, |y| < r$, the point $(0, v, x, y)$ is inside the normal neighborhood N_p defined in Sec. II A. Then the point $(u, v, x, y) \in M'$ for any u.

Now consider the points

$$\Phi(u, v) = (u, v, 0, 0)\,. \quad (12)$$

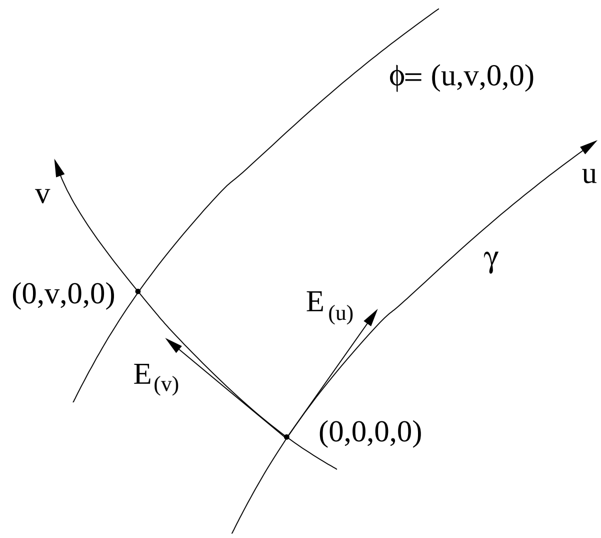

FIG. 1: Construction of the family of null geodesics Φ using Fermi normal coordinates

With v fixed and u varying, these are null geodesics in M'. (See Fig. 1.) Write the ANEC integral

$$A(v) = \int_{-\infty}^{\infty} du\, T_{uu}(\Phi(u,v))\,. \tag{13}$$

Suppose that, contrary to Theorem 1, Eq. (13) converges uniformly to negative values for all $|v| < r$. We will prove that this leads to a contradiction.

Since the convergence is uniform, $A(v)$ is continuous. Then since $A(v) < 0$ for all $|v| < r$, we can choose a positive number $v_0 < r$ and a negative number $-A$ larger than all $A(v)$ with $v \in (-v_0, v_0)$. Then it is possible to find some number u_1 large enough that

$$\int_{u_-(v)}^{u_+(v)} du\, T_{uu}(\Phi(u,v)) < -A/2 \tag{14}$$

for any $v \in (-v_0, v_0)$ as long as

$$u_+(v) \;>\; u_1 \tag{15a}$$
$$u_-(v) \;<\; -u_1\,. \tag{15b}$$

As in Ref. [2], we will define a series of parallelograms in the (u, v) plane, and derive a contradiction by integrating over each parallelogram in null and timelike directions. Each parallelogram will have the form

$$v \;\in\; (-v_0, v_0) \tag{16a}$$
$$u \;\in\; (u_-(v), u_+(v))\,, \tag{16b}$$

where $u_-(v), u_+(v)$ are linear functions of v obeying Eqs. (15). On each parallelogram we will construct a weighted integral of Eq. (14) as follows. Let $f(a)$ be a smooth function supported only within the interval $(-1, 1)$ and normalized

$$\int_{-1}^{1} da f(a)^2 = 1\,. \tag{17}$$

Then we can write

$$\int_{-v_0}^{v_0} dv\, f(v/v_0)^2 \int_{u_-(v)}^{u_+(v)} du\, T_{uu}(\Phi(u,v)) < -v_0 A/2\,. \tag{18}$$

We can construct this same parallelogram as follows. First choose a velocity V. Eventually we will take the limit $V \to 1$. Define the Doppler shift parameter

$$\delta = \sqrt{\frac{1+V}{1-V}}\,. \tag{19}$$

Let α be some fixed number with $0 < \alpha < 1/3$ and then let

$$\tau_0 = \delta^{-\alpha} r\,. \tag{20}$$

As $V \to 1$, $\delta \to \infty$ and $\tau_0 \to 0$.

Now define the set of points

$$\Phi_V(\eta,\tau) = \Phi(\eta + \frac{\delta\tau}{\sqrt{2}}, \frac{\tau}{\sqrt{2}\delta})\,. \tag{21}$$

We will be interested in the paths given by $\Phi_V(\eta,\tau)$ with η fixed and τ ranging from $-\tau_0$ to τ_0. In flat space, such paths would be timelike geodesic segments, parameterized by τ and moving at velocity V with respect to the original coordinate frame. In our curved spacetime, this is nearly the case, as we will show below. Define

$$\begin{aligned} \eta_0 &= u_1 + \tau_0\delta/\sqrt{2} && \text{(22a)}\\ v_0 &= \tau_0/(\sqrt{2}\delta) && \text{(22b)}\\ u_\pm(v) &= \pm\eta_0 + \delta^2 v && \text{(22c)} \end{aligned}$$

so that $u_\pm$ satisfies Eqs. (15). Then the range of points given by Eq. (12) with coordinate ranges specified by Eqs. (16) is the same as that given by Eq. (21) with coordinate ranges

$$\begin{aligned} -\tau_0 &< \tau < \tau_0 && \text{(23a)}\\ -\eta_0 &< \eta < \eta_0 && \text{(23b)} \end{aligned}$$

The parallelogram is shown in Fig. 2.

The Jacobian

$$\left|\frac{\partial(u,v)}{\partial(\eta,\tau)}\right| = \frac{1}{\sqrt{2}\delta} \tag{24}$$

so Eq. (18) becomes

$$\int_{-\eta_0}^{\eta_0} d\eta \int_{-\tau_0}^{\tau_0} d\tau\, T_{uu}(\Phi_V(\eta,\tau)) f(\tau/\tau_0)^2 < -A\tau_0/2\,. \tag{25}$$

We will show that this is impossible by applying the quantum inequality of Sec. IV.

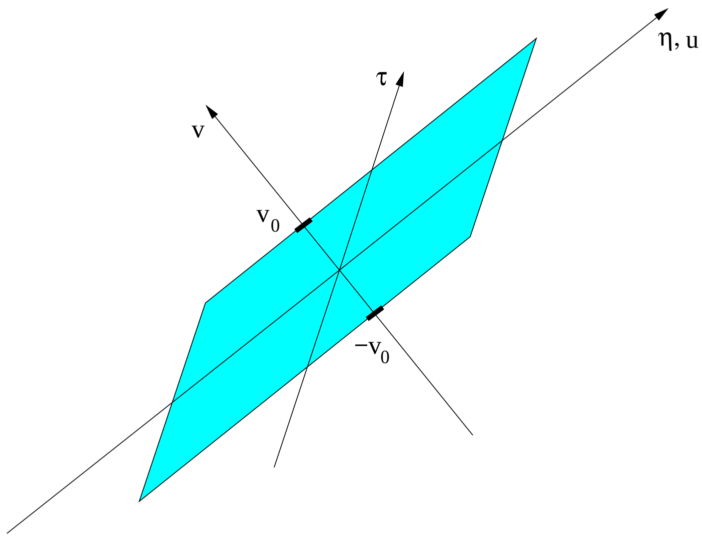

FIG. 2: The parallelogram $\Phi(u,v)$, $v \in (-v_0, v_0)$, $u \in (u_-(v), u_+(v))$, or equivalently $\Phi_V(\eta,\tau)$, $\tau \in (-\tau_0, \tau_0)$, $\eta \in (-\eta_0, \eta_0)$

C. Transformation of the Riemann tensor

We would like to work in coordinates which bring to rest, as much as possible, the path $\Phi_V(\eta,\tau)$ with η fixed. So let us construct new Fermi coordinates by a Lorentz transformation. We define

$$x^{\alpha'} = \Lambda^{\alpha'}_{\alpha} x^{\alpha}\,, \tag{26}$$

where Λ is diagonal with

$$\Lambda^{u'}_{u} = \delta^{-1} \tag{27a}$$
$$\Lambda^{v'}_{v} = \delta \tag{27b}$$
$$\Lambda^{x'}_{x} = \Lambda^{y'}_{y} = 1\,. \tag{27c}$$

In the primed coordinates, we have

$$\Phi_V(\eta,\tau) = (\eta/\delta + \tau/\sqrt{2}, \tau/\sqrt{2}, 0, 0)\,. \tag{28}$$

Equation (4) gives a bound on the components of the Riemann tensor, measured in the original tetrad. The covariant components R_{abcd} transform oppositely to the coordinate components, so

$$R_{a'b'c'd'} = \Lambda^{a}_{a'}\Lambda^{b}_{b'}\Lambda^{c}_{c'}\Lambda^{d}_{d'} R_{abcd}\,, \tag{29}$$

where

$$\Lambda^{u}_{u'} = \delta \tag{30a}$$
$$\Lambda^{v}_{v'} = \delta^{-1} \tag{30b}$$
$$\Lambda^{x}_{x'} = \Lambda^{y}_{y'} = 1 \tag{30c}$$

Since we are taking $\delta \to \infty$, components of R with more u's than v's diverge after the transformation. Components of R with fewer u's than v's go to zero and components with equal numbers of u's and v's remain the same. We want the curvature to be bounded by $R_{\max}$ in the primed coordinate system, which will be true if all components of the Riemann tensor with more u's than v's are zero. We will now show that this is the case in our system.

All points of interests are on achronal null geodesics, which thus must be free of conjugate points. Using Eq. (2) and proposition 4.4.5 of Ref. [13], each geodesic must violate the "generic condition". That is to say, we must have

$$\ell^c \ell^d \ell_{[a} R_{b]cd[e} \ell_{f]} = 0 \tag{31}$$

everywhere in M'.

The only nonvanishing components of the metric in the tetrad basis are $g_{uv} = g_{vu} = -1$ and $g_{xx} = g_{yy} = 1$. The tangent vector $\boldsymbol{\ell}$ has only one nonvanishing component $\ell^u = 1$, while the covector has only one nonvanishing component $\ell_v = -1$. Thus Eq. (31) becomes

$$\ell_{[a} R_{b]uu[e} \ell_{f]} = 0\,. \tag{32}$$

Let j, k, l, m and n denote indices chosen only from $\{x, y\}$. Choosing $a = m$, $e = n$, and $a = f = v$ we find

$$R_{muun} = 0 \tag{33}$$

for all m and n. Thus

$$R_{uu} = 0\,. \tag{34}$$

Equation (34) also follows immediately from the fact that since R_{uu} cannot be negative, any positive R_{uu} would lead to conjugate points.

If we apply the null convergence condition, Eq. (2), to $\mathbf{V} = \mathbf{E}_{(u)} + \epsilon \mathbf{E}_{(m)} + (\epsilon^2/2)\mathbf{E}_{(v)}$, where $\epsilon \ll 1$, we get

$$R_{uu} + 2R_{mu}\epsilon + O(\epsilon^2) \geq 0\,. \tag{35}$$

Since $R_{uu} = 0$ from Eq. (34), in order to have Eq. (35) hold for both signs of ϵ, we must have

$$R_{mu} = 0\,. \tag{36}$$

Since $R_{mu} = -R_{umvu} + g^{jk} R_{jmku}$,

$$R_{umvu} = g^{jk} R_{jmku}\,. \tag{37}$$

Now we use the Bianchi identity,

$$R_{luum;n} + R_{lunu;m} + R_{lumn;u} = 0\,. \tag{38}$$

From Eq. (33), $R_{luum,n} = 0$. The correction to make the derivatives covariant involves terms of the forms $R_{auum} \nabla_n \mathbf{E}_l^{(a)}$ and $R_{laum} \nabla_n \mathbf{E}_u^{(a)}$. Because of Eq. (33), the only contribution to the first of these comes from $a = v$, which we can transform using Eq. (37). For the second, we observe that $0 = \nabla_n (\mathbf{E}^{(v)} \cdot \mathbf{E}^{(v)}) = 2\nabla_n \mathbf{E}^{(v)} \cdot \mathbf{E}^{(v)} = 2\nabla_n \mathbf{E}_u^{(v)}$, so $a = v$ does not contribute. Furthermore $R_{lumn;u} = R_{lumn,u}$, because the u direction is the single final direction in the coordinate construction of Sec. II B, and so in this direction the tetrad vectors are just parallel transported. Thus we find

$$\begin{aligned} \frac{dR_{lumn}}{du} \;=\; & g^{jk} [R_{jmku} \nabla_n \mathbf{E}_l^{(v)} + R_{jlku} \nabla_n \mathbf{E}_m^{(v)} - R_{jnku} \nabla_m \mathbf{E}_l^{(v)} - R_{jlku} \nabla_m \mathbf{E}_n^{(v)}] \\ & + (R_{lkum} + R_{lukm}) \nabla_n \mathbf{E}_u^{(k)} + (R_{lknu} + R_{lunk}) \nabla_m \mathbf{E}_u^{(k)}\,. \end{aligned} \tag{39}$$

Eq. (39) is a first-order differential equation in the pair of independent Riemann tensor components R_{xuxy} and R_{yuxy}. By assumption, the curvature and its derivative vanish in the distant past, and therefore the correct solution to these equations is

$$R_{lumn} = 0\,. \tag{40}$$

Eqs. (37) and (40) then give

$$R_{umvu} = 0\,. \tag{41}$$

Combining Eqs. (33), (40), and (41) and their transformations under the usual Riemann tensor symmetries, we conclude that all components of the Riemann tensor with more u's than v's vanish as desired. It follows that

$$|R_{a'b'c'd'}| < R_{\max}\,. \tag{42}$$

everywhere in M'.

A similar argument using the Bianchi identity twice more would show that $R_{abcd} = 0$ unless 2 of a, b, c, and d are v, but we will not need that result here.

D. Timelike paths

We would like to apply Eq. (11) to the paths in Eq. (28). First we show that they are timelike. Differentiating Eq. (28), we find the components of the tangent vector $\mathbf{k} = d\Phi_V/d\tau$ in the primed Fermi coordinate basis (not the tetrad basis),

$$k^{u'} = k^{v'} = \frac{1}{\sqrt{2}}\,. \tag{43}$$

The squared length of $\mathbf{k}$ in terms of these components is $g_{\alpha'\beta'}k^{\alpha'}k^{\beta'}$. We showed in Ref. [5] that $g_{\alpha'\beta} = \eta_{\alpha'\beta'} + h_{\alpha'\beta'}$, where $h_{\alpha'\beta}$ at some point $\mathbf{X}$ is a sum of a small number of terms (6 in the present case of 2-step Fermi coordinates) each of which is a coefficient no greater than 1 times an average of

$$R_{\alpha'\gamma'\delta'\beta'}X^{\delta'}X^{\gamma'} \tag{44}$$

over one of the geodesics used in the construction of the Fermi coordinate system. The summations over δ' and γ' in Eq. (44) are only over restricted sets of indices depending on the specific term under consideration. From Eqs. (28) and (22a) the points under consideration satisfy

$$|u'| \;<\; u_1/\delta + \sqrt{2}\tau_0 \tag{45a}$$
$$|v'| \;<\; \tau_0/\sqrt{2} \tag{45b}$$
$$x' \;=\; y' = 0\,. \tag{45c}$$

From Eq. (20), the first term in Eq. (45a) decreases faster than the second, so we find that all components of $\mathbf{X}$ are $O(\tau_0)$. Using Eq. (42) we find

$$h_{\alpha'\beta'} = O(R_{\max}\tau_0^2) \tag{46}$$

so

$$g_{\alpha'\beta'}k^{\alpha'}k^{\beta'} = -1 + O(R_{\max}\tau_0^2)\,. \tag{47}$$

Thus for sufficiently large δ, and thus small τ_0, $\mathbf{k}$ is timelike.

No we consider the acceleration of our paths. Reference [5] gives the affine connection $\nabla_{\beta'} E^{\gamma'}_{(\alpha')}$ as a sum of 2 averages of terms of the form

$$R^{\gamma'}{}_{\alpha'\delta'\beta'} X^{\delta'} = O(R_{\max}\tau_0) \tag{48}$$

just as above. Thus the acceleration is given by

$$|a^{\beta'}| = \frac{Dk^{\beta'}}{d\tau} = |k^{\alpha'} \nabla_{\alpha'} k^{\beta'}| = |k^{\alpha'} k^{\gamma'} \nabla_{\alpha'} E^{\beta'}_{(\gamma')}| = O(R_{\max}\tau_0)\,. \tag{49}$$

We want to show that the components of the acceleration are small, so we will calculate the dimensionless quantity

$$|a^{\beta'}|\tau_0 = O(R_{\max}\tau_0^2)\,. \tag{50}$$

E. Causal diamond

For each η, we would like to apply Eq. (11). But what is the spacetime N in which we are to work? It must include the timelike path from $p = \Phi_V(\eta, -\tau_0)$ to $q = \Phi_V(\eta, \tau_0)$, and to be globally hyperbolic it must include all points in both the future of p in the past of q, so we can let N be the "double cone" or "causal diamond",

$$N = J^+(p) \cap J^-(q)\,. \tag{51}$$

We have shown that the curvature is small everywhere in the tube M', so we must show that $N \subset M'$.

From the previous section, we have that the metric in primed coordinates can be written as

$$g_{\alpha'\beta'} = \eta_{\alpha'\beta'} + h_{\alpha'\beta'}\,, \tag{52}$$

where $h_{\alpha'\beta'}$ consists of terms of the form $R_{\alpha'\gamma'\delta'\beta'} X^{\delta'} X^{\gamma'}$. The double cone in flat space obeys

$$|x'|, |y'|, |v'| < \tau_0\,, \tag{53}$$

so the same is true at zeroth order in the Riemann tensor R. Thus at zeroth order,

$$h_{\alpha'\beta'} = O(R_{\max}\tau_0^2)\,, \tag{54}$$

and so at first order in R,

$$|x'|, |y'|, |v'| < \tau_0(1 + O(R_{\max}\tau_0^2))\,. \tag{55}$$

Since $\tau_0 \ll r$ for large δ, we have

$$|x'|, |y'|, |v'| < r\,. \tag{56}$$

Now we can replace the primed coordinates,

$$x' = x \tag{57a}$$
$$y' = y \tag{57b}$$
$$v' = v\delta\,, \tag{57c}$$

so

$$|x|, |y|, |v| < r\,, \tag{58}$$

and $N \subset M'$ as desired.

F. Quantum Inequality

We would now like to apply Eq. (11) to give a lower bound on the integral of T_{uu} on the paths $\Phi_V(\eta,\tau)$. Because of the ambiguity involving local curvature terms in Conjecture 1, we will first bound

$$T'_{uu} = T_{uu} - C_{uu}\,, \tag{59}$$

where C_{ab} the is the particular local curvature term for which Conjecture 1 holds. We will then show that C_{uu} does not contribute.

Equation (42) shows that the curvature is small in the tetrad basis transported according to the construction of Sec. II B. These are not precisely the coordinates used in the conditions of Conjecture 1, but the difference is of no consequence, precisely because the curvature is small. Equations (47) and (50) show that, for sufficiently large δ, $\Phi_V(\eta,\tau)$ is a timelike path with small acceleration. The parameter τ is not exactly the proper time, but we show in Appendix A that this contributes only a correction of order $R_{\max}\tau_0^2$. Thus Eq. (11) gives

$$\int_{-\tau_0}^{\tau_0} d\tau\, T'_{uu}(\Phi_V(\eta,\tau)) f(\tau/\tau_0)^2 \geq \frac{(\ell_a k^a)^2}{12\pi^2\tau_0^4}\int_{-\tau_0}^{\tau_0} d\tau f''(\tau/\tau_0)^2[1+c(R_{\max}\tau_0^2)]\,, \tag{60}$$

where $c(R_{\max}\tau_0^2)$ vanishes as $\tau_0 \to 0$. In the unprimed coordinates, the only nonvanishing covariant component of $\boldsymbol{\ell}$ is $\ell_v = -1$, so $\ell_a k^a = -k^v = -1/(\sqrt{2}\delta)$ so

$$(\ell_a k^a)^2 = \frac{1}{2\delta^2}\,. \tag{61}$$

Let

$$F = \int f''(\alpha)^2 d\alpha = \frac{1}{\tau_0}\int f''(\tau/\tau_0)^2 d\tau\,. \tag{62}$$

Then Eq. (60) becomes

$$\int_{-\tau_0}^{\tau_0} d\tau\, T'_{uu}(\Phi_V(\eta,\tau)) f(\tau/\tau_0)^2 \geq -\frac{F}{24\pi^2\delta^2\tau_0^3}[1+c(R_{\max}\tau_0^2)]\,. \tag{63}$$

Integrating in η gives

$$\int_{-\eta_0}^{\eta_0} d\eta \int_{-\tau_0}^{\tau_0} d\tau\, T'_{uu}(\Phi_V(\eta,\tau)) f(\tau/\tau_0)^2 \geq -\frac{F\eta_0}{12\pi^2\delta^2\tau_0^3}[1+c(R_{\max}\tau_0^2)]\,. \tag{64}$$

Now consider

$$\int_{-\eta_0}^{\eta_0} d\eta \int_{-\tau_0}^{\tau_0} d\tau\, C_{uu}(\Phi_V(\eta,\tau))\,. \tag{65}$$

Terms from Eqs. (10) proportional to g_{ab} do not contribute, because $g_{uu}=0$. Similarly $R_{uu}=0$ from Eq. (34). The term $R^{cd}R_{ucud}$ vanishes because $R_{ucud}=0$ unless $c=d=v$, from Eqs. (33) and (41), while $R^{vv} = g^{uv}g^{uv}R_{uu} = 0$.

The remaining term is $R_{;uu}$. As explained in conjunction with Eq. (38), the covariant nature of the derivatives does not matter, and $R_{;uu}$ is a total derivative in u. In Eq. (65), it is integrated $d\eta$ which is just du. In the limit where $\eta_0 \to \infty$, the boundary term vanishes, because the curvature is localized. Thus C_{uu} does not contribute and we can use T_{uu} in place of T'_{uu} in Eq. (64).

Now we compare Eq. (64) to Eq. (25). Equation (64) says that integral over the parallelogram is no more negative that something that goes to zero in the $\delta \to \infty$ limit as

$$\frac{\eta_0}{\delta^2 \tau_0^3} \sim \delta^{2\alpha-1} . \tag{66}$$

Equation (25) says that the same integral is more negative than something that goes to zero as $\tau_0 \sim \delta^{-a}$. Since $\alpha < 1/3$, the lower bound in Eq. (64) goes to zero more quickly than the upper bound in Eq. (25). Thus for sufficiently large δ, the lower bound will be above the upper bound, so they cannot simultaneously be satisfied. This contradiction proves Theorem 1.

VI. DISCUSSION

As discussed in Ref. [1], to have an exotic spacetime there would have to be violation of ANEC on achronal geodesics, generated by a state of quantum fields in that same spacetime. We have proved, subject to Conjecture 1 and the various assumptions above, that minimally-coupled, free quantum scalar fields can only violate ANEC on geodesics traveling through parts of spacetime that violate the null convergence condition. Could it be that a single effect both violates ANEC and produces the curvature that allows ANEC to be violated? The following heuristic argument casts doubt on this possibility.

Suppose ANEC violation and NEC violation have the same source. We will say that they are produced by an exotic stress-energy tensor T_{exotic}. This T_{exotic} gives rise to an exotic Einstein curvature tensor,

$$G_{\text{exotic}} = 8\pi l_{\text{Planck}}^2 T_{\text{exotic}} , \tag{67}$$

in units where $c = \hbar = 1$. It is G_{exotic} that permits T_{exotic} to arise from the quantum field. Without G_{exotic}, the spacetime would obey the null convergence condition, and so, since T_{exotic} violates ANEC, it would have to vanish. A reasonable conjecture is that as $G_{\text{exotic}} \to 0$, $T_{\text{exotic}} \to 0$ at least linearly.[5] Then we can write schematically

$$|T_{\text{exotic}}| \lesssim l^{-2} |G_{\text{exotic}}| , \tag{68}$$

where l is a constant length obeying $l \gg l_{\text{Planck}}$. The parameter l, needed on dimensional grounds, might be the wavelength of some excited modes of the quantum field. Equation (68) is just schematic because we have not said anything about the places at which these tensors should be compared, or in what coordinate system they should be measured.

Combining Eqs. (67) and (68), we find

$$|T_{\text{exotic}}| \lesssim (l_{\text{Planck}}/l)^2 |T_{\text{exotic}}| , \tag{69}$$

which is impossible since $l \gg l_{\text{Planck}}$.

Given the assumptions of this paper, it appears that the only remaining possibility for self-consistent achronal ANEC violation using minimally coupled free fields is to have first a quantum field that violates NEC but obeys ANEC, and then a second quantum field (or a second, weaker effect produced by the same field) that violates ANEC when propagating

[5] Not, for example, changing discontinuously for infinitesimal but nonzero G_{exotic} or going as $G_{\text{exotic}}^{1/2}$.

in the spacetime generated by the first field. The stress-energy tensor of the second field would be a small correction to that of the first, but perhaps this correction might lead to ANEC violation on geodesics that were achronal (and thus obeyed ANEC only marginally) taking into account only the first field. This idea seems rather unlikely to us, and we will attempt to rule it out in future work.

If one considers quantum scalar fields with non-minimal curvature coupling, the situation is rather different. Even classical non-minimally coupled scalar fields can violate ANEC [14, 15], with large enough (Planck-scale) field values. However, as the field values increase toward such levels, the effective Newton's constant first diverges and the becomes negative. Such situations may not be physically realizable. If one excludes such field values, some restrictions are known, but there are no quantum inequalities of the usual sort [16, 17], and there are general [18] and specific [19, 20] cases where conformally coupled quantum scalar fields violate ANEC in curved space. It may be possible to control such situations by considering only cases where a spacetime is produced self-consistently by fields propagating in that spacetime, but the status of this "self-consistent achronal ANEC" for non-minimally coupled scalar fields outside the large-field region is not known.

Appendix A: Proper time

We start with $\Phi_V(\eta,\tau)$ given by Eq. (21) with tangent vector

$$k = \frac{\partial}{\partial\tau}\Phi_V(\eta,\tau) = \left(\frac{\delta}{\sqrt{2}}, \frac{1}{\sqrt{2\delta}}\right) \tag{A1}$$

in the coordinate basis. We would like to reparameterize the path $\Phi_V(\eta,\tau)$ in terms of proper time, which we will denote τ'. Then $g_{ab}k'^a k'^b = -1$ where k' is the tangent vector to the reparameterized path,

$$k' = \frac{\partial}{\partial\tau'}\Phi_V(\eta,\tau(\tau')) = \frac{k}{d\tau'/d\tau} \tag{A2}$$

so

$$\frac{d\tau'}{d\tau} = \sqrt{-g_{ab}k^a k^b} = \sqrt{h} \tag{A3}$$

with

$$h = 1 - h_{ab}k^a k^b\,. \tag{A4}$$

Now the left hand side of Eq. (60) can be written

$$\begin{aligned}\int_{-\tau_0}^{\tau_0} d\tau T'_{ab}(\Phi_V(\eta,\tau))k^a k^b f(\tau/\tau_0)^2 &= \int_{-\tau_0}^{\tau_0} d\tau' T'_{ab}(\Phi_V(\eta,\tau(\tau')))k'^a k'^b \sqrt{h} f(\tau/\tau_0)^2 \\ &= \int_{-\tau_0}^{\tau_0} d\tau' T'_{ab}(\Phi_V(\eta,\tau'))k'^a k'^b g(\tau')^2\,,\end{aligned} \tag{A5}$$

where we let

$$g(\tau') \equiv f(\tau(\tau')/\tau_0)h^{1/4}\,. \tag{A6}$$

Now we can apply the quantum inequality for the function g and proper time τ'. Since the curvature and the function f are smooth, so is g. We get

$$\int_{-\tau_0}^{\tau_0} d\tau' T'_{ab}(\Phi_V(\eta,\tau'))k'^a k'^b g(\tau')^2 \geq -\frac{(\ell_a k'^a)^2}{12\pi}\int_{-\tau_0}^{\tau_0} d\tau' g''(\tau')^2[1 + c(R_{\max}\tau_0^2)] \tag{A7}$$

Now let us determine h. Since we are working only in the u-v plane, we have two-step Fermi coordinates with one index in each step. Thus we can use Eq. (27) of Ref. [5] to get

$$h_{ab}(\mathbf{X}) = 2F_{ab} = 2\int_0^1 d\lambda \alpha_{2m}(\lambda)(1-\lambda)R_{acdb}(\mathbf{X}_{(1)} + \lambda \mathbf{X}_{(2)})X^d_{(2)}X^c_{(2)} \tag{A8}$$

where $\mathbf{X}_{(1)} = \Phi(0, v)$ and $\mathbf{X}_{(2)} = \Phi(u, 0)$. Because of the symmetry of the Riemann tensor, the only nonvanishing case is

$$h_{vv} = 2\int_0^1 d\lambda(1-\lambda)R_{vuuv}(\Phi(\lambda u, v))u^2 \tag{A9}$$

and

$$h = 1 - h_{vv}k^v k^v = 1 - \frac{h_{vv}}{2\delta^2} \tag{A10}$$

The maximum magnitude of u is $u_1 + \sqrt{2}\tau_0\delta$, so in the limit $\delta \to \infty$, $h = 1 + O(R_{\max}\tau_0^2)$.

We are not interested in $O(R_{\max}\tau_0^2)$ correction terms, and we will write $\approx$ to show that such terms have been ignored. Thus we can take $h \approx 1$, except where it is differentiated, and we will not worry about the difference between k'^a and k^a, and that between $d\tau'$ and $d\tau$ on the right hand side of Eq. (A7).

We would like to write g'' in terms of f'' and a correction that vanishes in the limit $\delta \to \infty$. So we will calculate the derivatives of g,

$$\frac{dg}{d\tau'} = \frac{1}{\sqrt{h}}\frac{dg}{d\tau} = h^{-1/4}\frac{df}{d\tau} + \frac{f}{4}h^{-5/4}\frac{dh}{d\tau} \approx \frac{df}{d\tau} + \frac{f}{4}\frac{dh}{d\tau} \tag{A11}$$

$$\frac{d^2g}{d\tau'^2} \approx \frac{d^2f}{d\tau^2} - \frac{5f}{16}\left(\frac{dh}{d\tau}\right)^2 + \frac{f}{4}\frac{d^2h}{d\tau^2} \tag{A12}$$

To compute the derivatives of h, we will change variables to $q = \lambda u$ in Eq. (A9) to get

$$h = 1 - \frac{1}{\delta^2}\int_0^u dq(u-q)R_{vuuv}(\Phi(q, v)) \tag{A13}$$

Now we can calculate the first derivative,

$$\begin{aligned}\frac{dh}{d\tau} &= k^u\frac{dh}{du} + k^v\frac{dh}{dv} = \frac{\delta}{\sqrt{2}}\frac{dh}{du} + \frac{1}{\sqrt{2}\delta}\frac{dh}{dv} \\ &= -\frac{1}{\sqrt{2}\delta}\int_0^u dq\, R_{vuuv}(\Phi(q, v)) - \frac{1}{\sqrt{2}\delta^3}\int_0^u dq\,(u-q)R_{vuuv,v}(\Phi(q, v))\,.\end{aligned} \tag{A14}$$

Using the bounds from Sec. II C, we find in the $\delta \to \infty$ limit,

$$\left|\frac{dh}{d\tau}\right| \le \tau_0 R_{\max} + \frac{\tau_0^2}{\sqrt{2}\delta}R'_{\max}\,. \tag{A15}$$

For sufficiently large δ the second term is negligible compared to the first.

For the second derivative we can write

$$\begin{aligned}\frac{d^2h}{d\tau^2} &= \frac{\delta^2}{2}\frac{d^2h}{du^2} + \frac{d^2h}{dudv} + \frac{1}{2\delta^2}\frac{d^2h}{dv^2} \\ &= -\frac{1}{2}R_{vuuv} - \frac{1}{\delta^2}\int_0^u dq\, R_{vuuv,v}(\Phi(q, v)) - \frac{1}{2\delta^4}\int_0^u dq\,(u-q)R_{vuuv,vv}(\Phi(q, v)))\end{aligned} \tag{A16}$$

Again using the bounds from Sec. II C, we find

$$\left|\frac{d^2h}{d\tau^2}\right| \le \frac{1}{2}R_{\max} + \frac{\sqrt{2}\tau_0}{\delta}R'_{\max} + \frac{\tau_0^2}{2\delta^2}R''_{\max} \tag{A17}$$

As before, for sufficiently large δ, the second and third term can be neglected in comparison to the first.

Keeping only the most important corrections, we then find

$$\left|\frac{d^2g}{d\tau'^2}\right| \lesssim \frac{f''}{\tau_0^2} - \frac{5}{16}fR_{\max}^2\tau_0^2 + \frac{1}{8}fR_{\max} = \frac{1}{\tau_0^2}[f'' + O(R_{\max}\tau_0^2)]\,, \tag{A18}$$

which justifies ignoring the difference between τ and τ' in Eq. (60).

A similar argument applies to the acceleration. In Sec. V D we found that the acceleration was small,

$$\frac{Dk^{\beta'}}{d\tau} = O(R_{\max}\tau_0)\,. \tag{A19}$$

Changing to the proper time τ' means that we should consider instead

$$\frac{Dk'^{\beta'}}{d\tau'} \approx \frac{Dk'^{\beta'}}{d\tau} = \frac{D}{d\tau}\left(\frac{k^\beta}{\sqrt{h}}\right) \approx \frac{Dk^\beta}{d\tau} - \frac{1}{2}\frac{dh}{d\tau}k^{\beta'} = O(R_{\max}\tau_0) \tag{A20}$$

from Eqs. (A19), (A15), and (43). Thus Eq. (50) holds for the proper acceleration as well.

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