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Plane Formation by Synchronous Mobile Robots without Chirality*†

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Abstract

We consider a distributed system consisting of autonomous mobile computing entities called *robots* moving in the three-dimensional space (3D-space). The robots are *anonymous*, *oblivious*, *fully-synchronous* and have neither any access to the global coordinate system nor any explicit communication medium. Each robot cooperates with other robots by observing the positions of other robots in its *local coordinate system*. One of the most fundamental agreement problems in 3D-space is the *plane formation problem* that requires the robots to land on a common plane, that is not predefined. This problem is not always solvable because of the impossibility of symmetry breaking. While existing results assume that the robots agree on the handedness of their local coordinate systems, we remove the assumption and consider the robots *without chirality*. The robots without chirality can never break the symmetry consisting of *rotation symmetry* and *reflection symmetry*. Such symmetry in 3D-space is fully described by 17 symmetry types each of which forms a group. We extend the notion of *symmetricity* [Suzuki and Yamashita, SIAM J. Compt. 1999] [Yamauchi et al., PODC 2016] to cover these 17 symmetry groups. Then we give a characterization of initial configurations from which the fully-synchronous robots without chirality can form a plane in terms of symmetricity.

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1 Introduction

Self-organization of autonomous mobile computing entities, called agents, robots, or particles, has gained much attention as real hardware robots, such as wheeled robots and drones, become cheaper and widely available. Applications in robotics cover various cooperative behavior, such as surveillance, exploration, and transportation. A swarm of autonomous

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mobile computing entities has also gained much attention in design and analysis of chemical reactions of molecules, cells, and DNA strands. Cooperative behavior in biological systems can be considered as self-organization of mobile computing entities, such as foraging ants, schooling of fishes, and migration of birds. In theoretical distributed computing, a number of models for these swarm systems have been proposed. For example, the *autonomous mobile robot model* considers an autonomous mobile robot with very weak capabilities, i.e., without any global positioning system nor any communication medium [15]. The *amoebot model* is inspired by amoeba [7], and the *population protocol model* considers delay-tolerant networks [1]. The *sticky particle model* considers particles (or square tiles) moving to the same direction according to an external force, like magnetic force [3]. Self-organization has been formalized as the *formation problem* that requires the entities to form a given shape (robot systems [15, 17, 12, 19], amoebot systems [8], and sticky particles [14]). In this paper, we consider the formation problem by autonomous mobile robots in the three-dimensional Euclidean space (3D-space). We focus on the robots with weakest abilities and investigate an essential element that determines their self-organization ability.

We adopt the conventional autonomous mobile robot model [10]. A *robot system* consists of a set of autonomous mobile robots, each of which is an *anonymous* (indistinguishable) point in a specified space. Each robot has neither any access to the global coordinate system nor any explicit communication medium. The robots cooperate with each other by observing the positions of other robots, but their local observations may be inconsistent because each robot uses its own local coordinate system. Formally, each robot repeats a *Look-Compute-Move cycle*, where it takes a snapshot (i.e., local observation) of other robots in the Look phase, computes its next position with a common algorithm in the Compute phase, and moves to the next position in the Move phase. Each local coordinate system is an orthogonal coordinate system in the specified space. It has an arbitrary unit length and the directions of coordinate axes are arbitrary. Further more, they might not agree on handedness, i.e., they can be right-handed or left-handed, thus they lack *chirality*. A robot is *oblivious* if in a Compute phase, it does not remember the past observations and the past computations and can use the observation obtained in the Look phase of the current cycle. Otherwise, a robot is *non-oblivious*, which means it is equipped with persistent local memory. A movement is *rigid* if each robot reaches its next position in each move phase, otherwise *non-rigid*. A *configuration* of a robot system is the set of positions of the robots observed in the global coordinate system, in other words, a set of points. There are three timing models for the execution of cycles: In the *fully-synchronous (FSYNC) model*, the robots execute the i-th Look-Compute-Move cycle at the same time. In the *semi-synchronous (SSYNC) model*, executions of cycles are synchronized but not all robots execute the i-th cycle at the same time. In the *asynchronous (ASYNC) model*, no assumption is made except that the length of each cycle is finite.

The *pattern formation problem* requires the robots to form a given target pattern. A sequence of results by Suzuki and Yamashita [15, 17] and Fujinaga et al. [12] showed that the set of formable pattern by anonymous robots is determined by initial *symmetry* among the robots regardless of asynchrony and obliviousness. Consider an initial configuration of four oblivious FSYNC robots with rigid movement, where they form a square and their local coordinate systems are symmetric regarding the center (Figure 1a). The four robots cannot break their symmetry forever from this initial configuration because they have the same local observation and they execute a common algorithm synchronously. Thus the robots cannot form, for example, a line. This worst case occurs in the ASYNC model (thus in the SSYNC model) with non-rigid movement because they allow stronger synchronization and rigid movement.

(a) Rotation regarding the center. **(b)** Reflection regarding a mirror plane.

Figure 1 Symmetric initial positions and local coordinate systems in 2D-space. A solid arrow shows an x-axis and a broken arrow shows an y-axis.

On the other hand, there are several crucial elements that reduce the set of formable patterns, thus the self-organization power of a robot system. Dieudonné et al. showed that anonymity prevents the robots from forming an arbitrary target pattern [9]. They showed that unique leader enables the oblivious ASYNC (thus SSYNC and FSYNC) robots to form an arbitrary target pattern. Yamauchi and Yamashita showed that when the *visibility* (observation range) is limited, the oblivious FSYNC (thus SSYNC and ASYNC) robots may increase their symmetry that cannot be resolved later [20]. Thus, limited visibility substantially reduces the formable patterns. Cicerone et al. showed that an important building block of the pattern formation algorithm for oblivious ASYNC robots in [12] does not work correctly without chirality [4]. They pointed out that robots without chirality may forever move symmetrically regarding the *center of rotation* or an *axis of reflection* (Figure 1b). Interestingly, the common cause of these three crucial elements is also *symmetry* among the robots.

As shown in Figure 1, the symmetry that the robots can never break is caused by symmetric local coordinate systems. When the robots have chirality, a local coordinate system is obtained by a uniform scaling, a translation, a rotation, or a combination of them on the global coordinate system. Thus we can obtain symmetric local coordinate systems of the robots with chirality by symmetric rotation operations. When the robots lack chirality, reflection by mirror planes are added to symmetric rotation operations because reflection changes handedness of local coordinate systems. For the robots with chirality in 2D-space, Yamashita et al. introduced the notion of *symmetricity* of a set of points. We consider a decomposition of a set P of points into regular m-gons centered at one point. We consider that one point is a regular 1-gon with an arbitrary center and two points form a regular 2-gon with the center being the midpoint. The maximum value of such m is the symmetricity $\rho(P)$ of P in 2D-space. When $\rho(P)$ is greater than one, the common center is the center of the smallest enclosing circle of P, denoted by $c(P)$, and $\rho(P)$ is generally the order of the cyclic group that acts on P. This definition is based on the fact that for each of these regular m-gons there exist symmetric local coordinate systems regarding $c(P)$, and the robots with such initial local coordinate systems cannot break the regular m-gon forever. However, when $c(P) \in P$, we have $\rho(P) = 1$, meaning that the symmetry of P can be broken. This is achieved by the robot on $c(P)$ leaving its current position.

Yamauchi et al. extended symmetricity to 3D-space [19]. Symmetric rotation operations in 3D-space form $SO(3)$ of infinite order, and its subgroups with finite order are the *cyclic groups*, the *dihedral groups*, the *tetrahedral group*, the *octahedral group*, and the *icosahedral group*. Each rotation group is recognized as the set of symmetric rotation operations on a prism, a pyramid, a regular tetrahedron, a regular octahedron, and a regular icosahedron, respectively. The *symmetricity* $\varrho(P)$ of a set P of points is the set of rotation groups G such that the group action of G on P divides P into $|G|$-sets where $|G|$ is the order of G. They also showed that the robots on rotation axes can eliminate the rotation axes by leaving their current position in the same way as 2D-space. This symmetry breaking results in, for example, the fact that the robots can form a regular octahedron from a cube.

However, existing definitions of symmetricity assume chirality. In this paper, we consider symmetry among robots without chirality in 3D-space, thus composite symmetry of rotation and reflection. The combination results in $O(3)$, and its subgroups with finite order consist of seventeen *symmetry groups*, that are fully studied in group theory [2, 6]. We extend the notion of symmetricity in [19] to these seventeen symmetry groups. Then we consider the *plane formation problem* that requires the robots in 3D-space to land on a common plane without making multiplicity. As already shown, the plane formation problem is not always solvable because of the impossibility of symmetry breaking [18]. We validate our definition of symmetricity by presenting a characterization of initial configurations from which the robots can solve the plane formation problem. We first show a necessary condition based on the fact that the robots cannot break their symmetricity forever. Then we show a matching sufficient condition with a plane formation algorithm because existing plane formation algorithms [18, 19] do not work without chirality. The highlight of our results is the fact that when an initial configuration contains an "empty" mirror plane that does not contain any robot, the robots cannot break their symmetry regarding this mirror plane, and they cannot form a plane. This results in a decrease of solvable instances (initial configurations) compared to the robots with chirality, i.e., when the robots with chirality initially occupy the vertices of a cube, they can form a plane, while the robots without chirality cannot. On the other hand, we show that when a mirror plane contains some robots, these robots can eliminate the mirror plane by leaving their current positions. Thus our results partly extend symmetry breaking phenomena of rotation axes to mirror planes.

By considering the robots with weakest observation abilities, we give the complete description of symmetry of the robots in 3D-space. On the other hand, we assume strong synchronization model and movement model, i.e., the FSYNC robots with rigid movement, while existing literature considers the plane formation problem for FSYNC robots with rigid movement [18] and for SSYNC robots with non-rigid movement [16]. The reason is twofold. First, in terms of symmetry, the worst case is caused by FSYNC rigid movement. Since such a worst case occurs in the SSYNC (thus ASYNC) model with non-rigid movement, the impossibility is essentially determined by FSYNC model with rigid movement. Second, although SSYNC (thus ASYNC) robots with non-rigid movement require carefully designed algorithms, we can extend our plane formation algorithm to the SSYNC robots with non-rigid movement in the same way as Uehara et al.'s techniques [16].

Related work

In terms of the pattern formation ability of a robot system, symmetry dominates other elements such as anonymity, obliviousness, visibility, and chirality. The set of formable patterns are characterized by the initial symmetricity among the robots. Regarding 2D-space, it has been shown that irrespective of obliviousness and asynchrony, the robots with chirality can form a target pattern F from an initial configuration P if and only if $\rho(P)$ divides $\rho(F)$ except the case where F is a point with multiplicity two [12, 15, 17]. The exception is called the *rendezvous problem*, which is trivially solvable by FSYNC robots while not solvable by SSYNC (thus ASYNC) robots. Yamauchi et al. partially generalized the results to 3D-space by showing that irrespective of obliviousness, the FSYNC robots with chirality can form a target pattern F from an initial configuration P if and only if $\varrho(P)$ is a subset of $\varrho(F)$ [19].

In terms of symmetricity in 2D-space, the point formation problem and the circle formation problem are expected to be solvable from any initial configuration. While the point formation of two SSYNC (thus ASYNC) robots is unsolvable, more than two ASYNC robots can always form a point [5]. Additionally, any number of oblivious ASYNC robots can form a circle without chirality [11, 13].

Fujinaga et al. investigated the *embedded pattern formation problem*, where the target pattern is given as a set of landmarks on the plane [12]. They showed that the oblivious ASYNC robots can form any embedded target pattern by showing an algorithm that is based on the "clockwise" minimum-weight perfect matching between the positions of the robots and the landmarks. Based on this clockwise matching algorithm, Fujinaga et al. presented a pattern formation algorithm for oblivious ASYNC robots with chirality [12]. Later Cicerone et al. pointed out that the clockwise matching algorithm does not work when the robots lack chirality, and showed a new embedded target pattern formation algorithm [4].

The plane formation problem was first introduced by Yamauchi et al. for FSYNC robots with chirality and rigid movement [18]. Part of the motivation was to use these existing distributed coordination techniques for 2D-space. However, they pointed that even when the robots with chirality finish plane formation, they may not agree on the clockwise direction (thus handedness) on the plane. Consider a configuration of two robots with right-handed x-y-z local coordinate systems on a plane, where the z-axis of one robot points to the upward and that of the other robot points to the downward. Then there is no way to make the two robots agree on the clockwise direction. This fact highlights the importance of distributed algorithms that do not require chirality.

2 Preliminary

2.1 Robot Model

Let $R = \{r_1, r_2, \ldots, r_n\}$ be a set of n anonymous robots, each of which is a point in 3D-space. We use r_i just for description. We consider discrete time $t = 0, 1, 2, \ldots$ and let $p_i(t) = (x_i(t), y_i(t), z_i(t)) \in \mathbb{R}^3$ be the position of r_i at time t in the global x-y-z coordinate system Z_0, where $\mathbb{R}$ is the set of real numbers. The *configuration* of R at time t is $P(t) = \{p_1(t), p_2(t), \ldots, p_n(t)\}$. We denote the set of all possible configurations of R by $\mathcal{P}_n^3$. We assume that the initial positions of robots are distinct, i.e., $p_i(0) \neq p_j(0)$ for $r_i \neq r_j$ and $|P(0)| = n$.[1] We also assume that $n \geq 4$ since any three robots are on one plane.

Each robot r_i has no access to the global coordinate system, and it uses its local x-y-z coordinate system Z_i. The origin of Z_i is the current position of r_i while the unit distance, the directions, and the orientations of the x, y, and z axes of Z_i are arbitrary and never change. It is appropriate to denote $Z_i(t)$, but we use a shorter description. Each Z_i is either right-handed or left-handed. Thus the robots do not have *chirality*. We denote the coordinates of a point p in Z_i by $Z_i(p)$.

We consider the *fully-synchronous (FSYNC) model*, where the robots start the t-th *Look-Compute-Move cycle* at the beginning of time $(t-1)$ and finishes it before time t $(t = 1, 2, \ldots)$. Each of the Look phase, the Compute phase, and the Move phase of a cycle is completely synchronized in each time step. At time t, each robot r_i obtains a set

[1] This assumption is necessary because when more than one robots are initially at one point, it is impossible to separate them by a deterministic algorithm. While the impossibility results in this paper considers executions possibly with multiplicities, the proposed algorithm does not make any multiplicity in any execution.

$Z_i(P(t)) = \{Z_i(p_1(t)), Z_i(p_2(t)), \ldots, Z_i(p_n(t))\}$ in the Look phase. We call $Z_i(P(t))$ the local observation of r_i at time t. Then r_i computes its next position by a common algorithm ψ in the Compute phase. A robot is *oblivious* if it does not remember the past observations and the past computations, thus the input to ψ is $Z_i(P(t))$. Otherwise, it is *non-oblivious* and the input to ψ contains the past observations and the past computations. Finally, r_i moves to the next point in the Move phase. We assume that each robot always reaches its next position in a move phase and we do not care for the route to reach there. Thus we consider *rigid movement*.

An execution of an algorithm ψ from an initial configuration $P(0)$ is a sequence of configurations $P(0), P(1), P(2), \ldots$. Given an algorithm ψ, there are multiple FSYNC executions starting from $P(0)$, however when the initial local coordinate systems of $P(0)$ and initial local memory content (if any) are fixed, the FSYNC execution is uniquely determined.

The *plane formation problem* requires that the robots land on a plane, which is not predefined, without making any multiplicity. Hence point formation is not a solution for the plane formation problem. We say that an algorithm ψ *forms a plane* from an initial configuration $P(0)$, if, regardless of the choice of initial local coordinate systems Z_i for each $r_i \in R$, in any execution $P(0), P(1), P(2) \ldots$ there exists a finite $t \geq 0$ such that (i) $P(t)$ is contained in a plane, (ii) $|P(t)| = n$, i.e., all robots occupy distinct positions, and (iii) once the system reaches $P(t)$, the robots do not move anymore.

For a set P of points, we denote the smallest enclosing ball (SEB) of P by $B(P)$ and its center by $b(P)$. A point on the sphere of a ball is said to be *on* the ball, and we assume that the *interior* and the *exterior* of a ball do not include its sphere. The *innermost empty ball* $I(P)$ is the ball whose center is $b(P)$, that contains no point of P in its interior and contains at least one point of P on its sphere. When all points of P are on $B(P)$, we say P is *spherical*. When the robots occupy the vertices of a polyhedron, we say that the robots *form* the polyhedron.

2.2 Symmetry by Rotations and Reflections

A symmetric rotation operation is an operation that rotates an object without changing its appearance. If we consider a unit ball, there are infinitely many symmetric rotation operations, and these operations form $SO(3)$. The subgroups of $SO(3)$ with finite order are the *cyclic groups* C_k $(k = 1, 2, \ldots)$[2], the *dihedral groups* D_ℓ $(\ell = 2, 3, \ldots)$, the *tetrahedral group* T, the *octahedral group* O, and the *icosahedral group* I. Each of them can be recognized as a set of rotation operations on a pyramid with a regular k-gon base, a prism with a regular ℓ-gon bases, a regular tetrahedron, a regular octahedron, and a regular icosahedron, respectively. In other words, they are determined by a set of rotation axes and their arrangements. A k-fold axis admits rotations by $2\pi i/k$ $(i = 1, 2, \ldots, k)$. These k operations form the *cyclic group* C_k of order k.

The *dihedral group* D_ℓ consists of a single ℓ-fold axis called the *principal axis* and ℓ 2-fold axes perpendicular to the principal axis, and its order is 2ℓ. We abuse the term "principal axis" for the single rotation axis of a cyclic group.

The *tetrahedral group* T consists of three 2-fold axes and four 3-fold axes, and its order is 12. The *octahedral group* O consists of six 2-fold axes, four 3-fold axes, and three 4-fold axes, and its order is 24. The *icosahedral group* I consists of fifteen 2-fold axes, ten 3-fold axes, and six 5-fold axes, and its order is 60. Clearly, multiple rotation axes of a rotation group

[2] C_1 consists of an identity element.

intersect at one point. We call the cyclic groups and the dihedral groups *2D rotation groups*, and we call the remaining three rotation groups T, O, and I *3D rotation groups* because a 3D rotation group does not act on a point on a plane.

Reflection by a mirror plane is an indirect symmetric operation. We consider composite symmetry of rotation axes and mirror planes, i.e., subgroups of $O(3)$ with finite order. Each symmetry type also forms a group. The *bilateral symmetry* C_s consists of one mirror plane and its order is 2. When there are more than one mirror planes, an intersection of mirror planes introduces a rotation axis. Clearly, the rotation axes and mirror planes of the symmetry type intersect at one point. The composition of C_k ($k > 1$) and a horizontal mirror plane regarding the principal axis is denoted by C_{kh}. The order of C_{kh} is $2k$. The composition of C_k ($k > 1$) and k vertical mirror planes each of which contains the principal axis is denoted by C_{kv}. The order of C_{kv} is $2k$.

The composition of D_ℓ ($\ell \geq 2$) and a horizontal mirror plane regarding the principal axis is denoted by $D_{\ell h}$. However, this horizontal mirror plane together with rotation axes forces vertical mirror planes and the order of $D_{\ell h}$ is 4ℓ. The composition of D_ℓ ($\ell \geq 2$) and ℓ vertical mirror planes is denoted by $D_{\ell v}$. The order of $D_{\ell v}$ is 4ℓ.

The composition of T and three mutually perpendicular mirror planes, each of which contains two 2-fold axes is denoted by T_h. The order of T_h is 24. The composition of T and six mirror planes, each of which contains two 3-fold axes is denoted by T_d. The order of T_d is 24. The composition of O and nine mirror planes is denoted by O_h. The order of O_h is 48. The composition of I and fifteen mirror planes, each of which contains two 5-fold axes is denoted by I_h. The order of I_h is 120.

Another type of composite symmetry is *rotation reflection* S_m, where a rotation regarding a single m-fold axis and reflection by a horizontal mirror plane are alternated. S_2 corresponds to the central inversion, which is denoted by C_i.

Let $\mathbb{S} = \{C_1, C_i, C_s, C_k, C_{kh}, C_{kv}, D_\ell, D_{\ell h}, D_{\ell v}, S_m, T, T_d, T_h, O, O_h, I, I_h \mid k = 2, 3, \ldots, \ell = 2, 3, \ldots, m = 3, 4, \ldots\}$. We call each element of $\mathbb{S}$ *symmetry group*. These seventeen types of symmetry groups describe all symmetry in 3D-space [6].

We denote the order of $G \in \mathbb{S}$ with $|G|$. When G' is a subgroup of G ($G, G' \in \mathbb{S}$), we denote it by $G' \preceq G$. If G' is a proper subgroup of G (i.e., $G \neq G'$), we denote it by $G' \prec G$. For example, we have $D_2 \prec T$, $T \prec O, I$, but $O \not\preceq I$. If $G \in \mathbb{S}$ has a k-fold axis, $C_{k'} \preceq G$ if k' divides k. For symmetry groups containing mirror planes, $T \prec T_h \prec O_h$ but $T_h \not\prec O$. For S_m, we have $C_{m/2} \prec S_m \prec C_{mh}$.

See Appendix A for the detailed description of symmetry groups.

2.3 Rotation Group, Symmetry Group, and Symmetricity

We start with the symmetry of positions of robots, called *symmetry group* of a configuration, that the robots can agree irrespective of their local coordinate systems. Then we introduce symmetry of local coordinate systems, called *symmetricity*, that the robots can never break. Note that the robots do not know local coordinate systems of other robots.

The *rotation group* $\gamma(P)$ of a set $P \in \mathcal{P}_n^3$ of points is the rotation group that acts on P and none of its proper supergroup in $\{C_k, D_\ell, T, O, I \mid k = 1, 2, \ldots, \ell = 2, 3, \ldots\}$ acts on P. The *symmetry group* $\theta(P)$ of P is the symmetry group that acts on P and none of its proper supergroup in $\mathbb{S}$ acts on P. Clearly, $\gamma(P)$ and $\theta(P)$ are uniquely determined,[3] and

[3] See for example [6], that shows an algorithm to uniquely determine the symmetry group of a polyhedra. The algorithm checks rotation axes, mirror planes, and a point of inversion. Since we consider a set of

Table 1 Rotation group, symmetry group, and symmetricity of regular polyhedra.

Polyhedron	Rotation group	Symmetry group	Symmetricity
Regular tetrahedron	T	T_d	$\{D_2, S_4\}$
Regular octahedron	O	O_h	$\{D_3, S_6\}$
Cube	O	O_h	$\{D_4, D_{2h}, D_{2v}, C_{4h}, S_4\}$
Regular dodecahedron	I	I_h	$\{D_5, D_2, S_{10}\}$
Regular icosahedron	I	I_h	$\{T, D_3, S_6\}$

$\gamma(P)$ is a subgroup of $\theta(P)$ ($\gamma(P) \preceq \theta(P)$). By the definition, when $\theta(P)$ is either C_1, C_i, C_s, $\gamma(P) = C_1$. Table 1 shows the rotation group of a set of vertices of each regular polyhedron.

The group action of $\theta(P)$ decomposes P into disjoint subsets. Let $Orb(p) = \{g * p \mid g \in \theta(P)\}$ be the orbit of $p \in P$ where $*$ denotes the action of g on s, and the orbit space $\{Orb(p) \mid p \in P\} = \{P_1, P_2, \dots, P_m\}$ is called the $\theta(P)$-*decomposition* of P. Each element P_i is *transitive* because it is one orbit regarding $\theta(P)$.

Yamauchi et al. [18] showed that in configuration P without any multiplicity, the robots with chirality can agree on the $\gamma(P)$-decomposition $\{P_1, P_2, \dots, P_m\}$ of P and a total ordering among the elements so that (i) P_1 is on $I(P)$, (ii) P_m is on $B(P)$, and (iii) P_{i+1} is not in the interior of the ball centered at $b(P)$ and containing P_i on its sphere. Though their techniques rely on chirality, we can extend it to robots without chirality. In [18], each robot translates its local observations to a "celestial map" by considering $I(P)$ as the earth. It considers that its current position is on the half line from $b(P)$ containing the north pole. Then, the robot selects an appropriate robot to define the prime meridian and translates the position of each robot to a triple consisting of its altitude, latitude, and longitude. The ordered sequence of these triples is the *local view* of the robots. However, the lack of chirality does not allow the robots to agree on the direction of longitude. Then we make a robot consider both directions and select the direction that produces the smallest sequence. In the same way as [18], we have the following property. We omit the proof because of the page limitation.

▶ **Lemma 1.** *Let $P \in \mathcal{P}_n^3$ and $\{P_1, P_2, \dots, P_m\}$ be a configuration of n robots represented as a set of points and its $\theta(P)$-decomposition, respectively. Then we have the following two properties:*

1. *For each P_i ($i = 1, 2, \dots, m$), all robots in P_i have the same local view.*
2. *Any two robots, one in P_i and the other in P_j, have different local views, for all $i \neq j$.*

By Lemma 1, the robots can agree on a total ordering of the elements $P_1, P_2, \dots, P_m$ by using the lexicographical ordering of the local views of each element. In the following, we assume that $P_1, P_2, \dots, P_m$ is sorted by this ordering.

We denote the set of local coordinate systems for configuration P with a set Q of quadruples such that $Q = \{(o_i, x_i, y_i, z_i) \mid p_i \in P\}$ where o_i is the position of $p_i \in P$ (i.e., the origin of Z_i) and x_i, y_i, and z_i are the coordinates of $(1,0,0)$, $(0,1,0)$, and $(0,0,1)$ of Z_i observed in the global coordinate system Z_0. We use (P, Q) to explicitly show the set of local coordinate systems for P though Q contains P as $\{o_1, o_2, \dots, o_n\}$. We define the *symmetry group* $\sigma(P, Q)$ of (P, Q) as the symmetry group that acts on (P, Q) and none of its proper supergroup in $\mathbb{S}$ acts on it. Clearly, we have $\sigma(P, Q) \preceq \theta(P)$. We define the $\sigma(P, Q)$-decomposition of (P, Q) in the same way as the $\theta(P)$-decomposition of P. We note that the robots of P cannot obtain Q nor $\sigma(P, Q)$ because they can observe only the positions of themselves.

points and their convex-hulls, we can use the same algorithm.

Given a set P of points, $\theta(P)$ determines the arrangement of its rotation axes and mirror planes in P. We thus use $\theta(P)$ and the arrangement of its rotation axes and mirror planes in P interchangeably. For two groups $G, H \in \mathbb{S}$, an *embedding* of G to H is an embedding of each rotation axis and each mirror plane of G to one of the rotation axes and one of the mirror planes of H with keeping their arrangement in G. Any k-fold axis of G is embedded so that it overlaps a k'-fold axis of H, where k divides k'. Observe that we can embed G to H if and only if $G \preceq H$.

We also consider a G-decomposition of a set P of points for some $G \prec \theta(P)$ ($G \in \mathbb{S}$) for an embedding of G in $\theta(P)$. Note that the robots cannot agree the ordering among the elements of such a G-decomposition of P.

▶ **Definition 2.** Let $P \in \mathcal{P}_n^3$ be a set of points. The symmetricity $\varrho(P)$ of P is the set of symmetry groups $G \in \mathbb{S}$ that acts on P (thus $G \preceq \theta(P)$) and there exists an embedding of G to $\theta(P)$ such that each element of the G-decomposition of P is a $|G|$-set.

We define $\varrho(P)$ as a set because the "maximal" symmetry group that satisfies the definition is not uniquely determined. Maximality means that there is no proper supergroup in $\mathbb{S}$ that satisfies the condition of Definition 2. When it is clear from the context, we denote $\varrho(P)$ by the set of such maximal elements. See Table 1 as an example.

We can rephrase the definition of symmetricity of P as a set of symmetry groups formed by rotation axes and mirror planes of $\theta(P)$ that do not contain any point of P. This is because a point on a rotation axis (a mirror plane, respectively) does not allow any decomposition into $|G|$-sets for any G that contains the rotation axis. [4]

We conclude this section with the following two lemmas, that validate the definition of symmetricity.

▶ **Lemma 3.** *For an arbitrary initial configuration $P \in \mathcal{P}_n^3$ and any $G \in \varrho(P)$, there exists a set of local coordinate systems Q such that $\sigma(P, Q) = G$.*

Proof. Let $\{P_1, P_2, \ldots, P_m\}$ be the G-decomposition of P for some embedding of G to $\theta(P)$. We show a construction of Q for P and G. Clearly, such embedding exists since $G \preceq \theta(P)$. From the definition, $|P_j| = |G|$ for $j = 1, 2, \ldots, m$. For each P_j, we arbitrary fix a local coordinate system of one robot $p_i \in P_j$. Then for each $p_k \in P_j$, there exists a unique element of G such that $p_k = g_k * p_i$ and $g_k \neq g_\ell$ if $p_k \neq p_\ell$ for any $p_\ell \in P_j$. Then we fix the local coordinate system of p_k by applying g_k to the local coordinate system of p_i. The local coordinate systems Q obtained by this procedure satisfies the property. ◀

Lemma 3 shows that there exists an arrangement of local coordinate systems Q for any initial configuration P and $G \in \varrho(P)$ such that $\sigma(P, Q) = G$. Then, Lemma 4 shows that the robots may forever caught in this initial symmetry.

▶ **Lemma 4.** *Irrespective of obliviousness, for an arbitrary initial configuration $P \in \mathcal{P}_n^3$, any $G \in \varrho(P)$, and any algorithm ψ, there exists an execution $P(0)(= P), P(1), P(2), \ldots$ such that $\theta(P(t)) \succeq G$.*

Proof. Let Q be initial local coordinate systems for P such that $\sigma(Q, P) = G$ for arbitrary $G \in \varrho(P)$. By Lemma 3, such Q always exists. Let $\{P_1, P_2, \ldots, P_m\}$ be the G-decomposition of P. From this arrangement of initial local coordinate systems, for each P_j ($j = 1, 2, \ldots, m$), the robots forming P_j keep their symmetry group G forever for any algorithm ψ. We first show

[4] We assume that a set P of points does not contain any multiplicity. In other words, we consider an initial configuration P.

an induction for the oblivious FSYNC robots. For any $p_i, p_k \in P_j$, when $\psi(Z_i[P(0)]) = x$ holds, we have $\psi(Z_k[P(0)]) = x$ and $Z_0[Z_k(\psi(Z_k[P(0)]))] = g_k * Z_0[Z_i(\psi(Z_i[P(0)]))]$. Let $P_j(1) \subseteq P(1)$ be the positions of robots of P_j in $P(1)$. Thus $\theta(P_j(1)) = G$ and $\theta(P(1)) \succeq G$. By an easy induction for $t = 1, 2, \ldots$, we have the property for any $P(t)$.

Non-obliviousness does not improve the situation. When the initial memory contents are identical (for example, empty), the above discussion holds for the transition from $P(0)$ to $P(1)$. During this transition, the robots in the same element P_j obtain the same local observation, performs the same computation, and exhibits the same movement. Thus, their local memory content are still identical in $P(1)$ and they continue symmetric movement during the transition from $P(1)$ to $P(2)$. ◀

3 Impossibility of Plane Formation

The following theorem shows a necessary condition for the FSYNC robots without chirality to form a plane, that will be shown to be a sufficient condition in Section 4.

▶ **Theorem 5.** *Irrespective of obliviousness, the FSYNC robots without chirality can form a plane from an initial configuration P only if $\varrho(P)$ contains neither any 3D rotation group nor any 2D rotation group with horizontal mirror plane regarding the principal axis (except C_{2h} and S_m).*

Proof. Let ψ be an arbitrary plane formation algorithm for an initial configuration P such that $\varrho(P)$ contains a 3D rotation group, C_{kh} $(k \geq 3)$, or $D_{\ell h}$ $(\ell \geq 2)$. We have the following three cases.

Case A: $\varrho(P)$ contains C_{kh} for some $k \geq 3$.

Let Q be a set of initial local coordinate systems for P such that $\sigma(P, Q) = C_{kh} \in \varrho(P)$ $(k \geq 3)$. From Lemma 4, irrespective of obliviousness, for any algorithm ψ, there exists an execution $P = P(0), P(1), P(2), \ldots$ such that $\theta(P(t)) \succeq C_{kh}$ for any $t \geq 0$. Assume that $P(t')$ be a terminal configuration. Then $\theta(P(t'))$ is a supergroup of C_{kh}, and $\theta(P(t'))$ has the mirror plane of $\theta(P)$. The robots are on this mirror plane, otherwise the robots are not on one plane because of their symmetry.

Let $\{P_1, P_2, \ldots, P_m\}$ be the $\sigma(P, Q)$-decomposition of $P(= P(0))$. For each P_i $(1 \leq i \leq m)$ and $p \in P_i$, there exists $q \in P_i$ such that p and q are at symmetric positions regarding the mirror plane of C_{kh}. By Lemma 4, the robots of P_i move with keeping the rotation axis and the mirror plane of the embedding of C_{kh} in P. Thus p and q occupy the same point on the mirror plane of C_{kh} in $P(t')$. Hence the robots cannot avoid multiplicity and $P(t')$ is not a terminal configuration of the plane formation problem.

Case B: $\varrho(P)$ contains $D_{\ell h}$ for some $\ell \geq 2$.

Let Q be a set of initial local coordinate systems for P such that $\sigma(P, Q) = D_{\ell h} \in \varrho(P)$ $(\ell \geq 2)$. By Lemma 4, irrespective of obliviousness, for any algorithm ψ, there exists an execution $P = P(0), P(1), P(2), \ldots$ such that $\theta(P(t)) \succeq D_{\ell h}$ for any $t \geq 0$. We have the same discussion as Case A. If there exists a terminal configuration, the robots are on the initial horizontal mirror plane of $D_{\ell h}$. Hence, the robots cannot avoid multiplicity and $P(t')$ is not a terminal configuration of the plane formation problem.

Case C: $\varrho(P)$ contains a 3D-rotation group.

The impossibility for this case has been shown for robots with chirality in [18] and the result holds for our robots because our model allows the robots with chirality. We note that when $\varrho(P)$ contains T_d, T_h, O_h or I_h, then it contains the corresponding rotation group because it is a subgroup without any mirror plane.

Consequently, when $\varrho(P)$ contains the above three types of symmetry groups, they cannot form a plane. ◀

4 Plane Formation Algorithm

In this section, we show a plane formation algorithm for oblivious FSYNC robots without chirality and prove our main theorem. Theorem 6 gives a necessary and sufficient condition for the FSYNC robots without chirality to solve the plane formation problem. In other words, it gives a characterization of the initial configurations from which the plane formation problem is solvable.

▶ **Theorem 6.** *Irrespective of obliviousness, the FSYNC robots without chirality can form a plane from an initial configuration P if and only if $\varrho(P)$ contains neither any 3D rotation group nor any 2D rotation group with horizontal mirror plane regarding the principal axis (except C_{2h} and S_m).*

The necessity is clear from Theorem 5. We prove the sufficiency by presenting a plane formation algorithm for solvable instances (i.e., initial configurations). Due to the page limitation, we show a sketch of the proposed algorithm.

By the condition of the theorem, solvable instances are classified into the following three types.

Type 1: Initial configuration P with $\theta(P) \in \{T, T_d, T_h, O, O_h, I, I_h\}$ is in this type. By the condition of Theorem 6, any initial configuration P of this type contains one of the following polyhedra as an element of its $\theta(P)$-decomposition, because some robots are on some rotation axes: a regular tetrahedron, a regular octahedron, a regular dodecahedron, and an icosidodecahedron.[5]

Type 2: Initial configuration P with $\theta(P) \in \{C_k, C_{kh}, C_{kv}, D_\ell, D_{\ell h}, D_{\ell v}, S_m | k = 2, 3, \ldots, \ell = 2, 3, \ldots, m = 3, 4, \ldots\}$ is in this type. By Theorem 6, $\theta(P)$ does not have the horizontal mirror plane or there are some robots on the horizontal mirror plane.

Type 3: Initial configuration P with $\theta(P) \in \{C_1, C_i, C_s\}$ is in this type.

The proposed algorithm handles these three types separately. The robots can agree on the type of the current configuration and they execute the corresponding algorithm. In a Type 1 initial configuration, the robots first break their symmetry and translates an initial Type 1 configuration to another Type 2 or Type 3 configuration. In a Type 2 initial configuration, the robots agree on a plane perpendicular to the principal axis and containing the center of their smallest enclosing ball, and land on the plane. In a Type 3 initial configuration, if it is an asymmetric configuration, the robots agree on a plane by using the total ordering among themselves (Lemma 1). Otherwise, the robots agree on a plane other than the mirror plane by using two elements of their $\theta(P)$-decomposition and land on it.

In the following, we use a point and a robot at the point interchangeably.

We put several preparation steps before the execution of the main algorithm. These steps are realized very easily in the FSYNC model without changing the initial symmetry of the robots.

Removing the center. When $b(P) \in P$, the robot on $b(P)$ leaves its current position so that a resulting configuration will be asymmetric.

For symmetry breaking of Type 1. If initial configuration P is Type 1, we send P_i of the $\theta(P)$-decomposition of P forming one of the specified polyhedra to the interior of $I(P)$.

[5] Points on rotation axes of a 3D rotation group also form a cube, a cuboctahedron, and a regular icosahedron. However, a cube allows D_{2h} to join its symmetricity, and the remaining two polyhedra allow $T(\prec O, I)$ to join their symmetricity.

We select P_i with the minimum index among the elements satisfying the condition to guarantee that no robot is on the track.

For symmetry breaking of Type 2. If initial configuration P is Type 2, we move P_j of the $\theta(P)$-decomposition of P on the horizontal mirror plane of $\theta(P)$ to the interior of $I(P)$. We select P_j with the minimum index among the elements satisfying the condition.

In the following, since the proposed algorithm is designed for the oblivious FSYNC robots, it is always described for a current configuration.

4.1 Symmetry Breaking

We consider a Type 1 configuration P. Let $\{P_1, P_2, \dots, P_m\}$ be the the $\theta(P)$-decomposition of P. The preparation step guarantees that P_1 forms one of the following four polyhedra; a regular tetrahedron, a regular octahedron, a regular dodecahedron, and an icosidodecahedron. Then the proposed plane formation algorithm first makes the robots execute the *go-to-center algorithm* (Algorithm 1) proposed in [18]. Each robot of P_1 selects an adjacent face of the polyhedron and moves to the center of the selected face. But it stops ϵ before the center to avoid collisions. Clearly, Algorithm 1 does not depend on chirality. We have the following lemma for any execution of Algorithm 1.

▶ **Lemma 7.** *Let P and $\{P_1, P_2, \dots, P_m\}$ be a Type 1 initial configuration and its $\theta(P)$-decomposition. Then there exists one element of P_i ($1 \leq i \leq m$) that forms one of the following polyhedra; a regular tetrahedron, a regular octahedron, a regular dodecahedron, or an icosidodecahedron. Then one step execution of Algorithm 1 translates P into another configuration P' that satisfies (i) $\gamma(P')$ is a 2D rotation group, and (ii) if $\gamma(P') \neq C_1, C_2$, $\theta(P')$ does not have any horizontal mirror plane.*

We present a sketch of the proof because of the page limitation. In [18], it is proved that Algorithm 1 breaks the rotation symmetry of the specified polyhedra and that the rotation group of any resulting configuration is a 2D rotation group. Since our robots lack chirality, we consider the combination of 2D rotation groups and mirror planes for a resulting configuration. We can show that the movement of the robots does not create any new mirror plane, and any resulting configuration does not have any horizontal mirror plane (except C_{2h}).

Since this symmetry breaking occurs in P_1, the (rotation) symmetry of the whole robots decreases to a 2D rotation group.

4.2 Landing Algorithm

In this section, we show a plane formation algorithm for Type 2 and Type 3 initial configurations. When $\gamma(P)$ of a current configuration P is a cyclic group or a dihedral group, our basic strategy is to make the robots agree on the plane perpendicular to the principal axis and containing $b(P)$. Each robot moves along a perpendicular to the agreed plane (Figure 2a). However, this simple strategy results in collisions on the agreed plane because the plane can be a mirror plane of $\theta(P)$. To avoid multiplicities, the proposed algorithm consumes five phases. The first three phases break the mirror plane of $\theta(P)$ and resolves the collisions on the target plane. The fourth phase makes the robots agree on the target plane and in the fifth phase each robot computes the destinations of all robots to avoid any collision.

In any configuration P, the robots execute the algorithm with the smallest phase number. The robots can easily agree on which phase to execute because the condition of the phases divide the set of all configurations with 2D rotation groups into disjoint subsets. Depending on the execution, some phases may be skipped.

Algorithm 1 Go-to-center algorithm for robot r_i [18].

Notation

P: Current configuration observed in Z_i.
$p_i \in P$: The position of r_i (i.e., the origin of Z_i).
$\{P_1, P_2, \dots, P_m\}$: $\theta(P)$-decomposition of P, where P_1 forms one of the four polyhedra.
$\epsilon = \ell/100$, where ℓ is the length of an edge of the polyhedron that P_1 forms.

Algorithm

If $p_i \in P_1$ **then**
  If P_1 is an icosidodecahedron **then**
    Select an adjacent regular pentagon face of P_1.
    Destination d is the point ϵ before the center of the face on the line from p_i to the center.
  Else
  // P_1 is a regular tetrahedron, a regular octahedron, or a regular dodecahedron.
    Select an adjacent face of P_1.
    Destination d is the point ϵ before the center of the face on the line from p_i to the center.
  Endif
  Move to d.
Endif

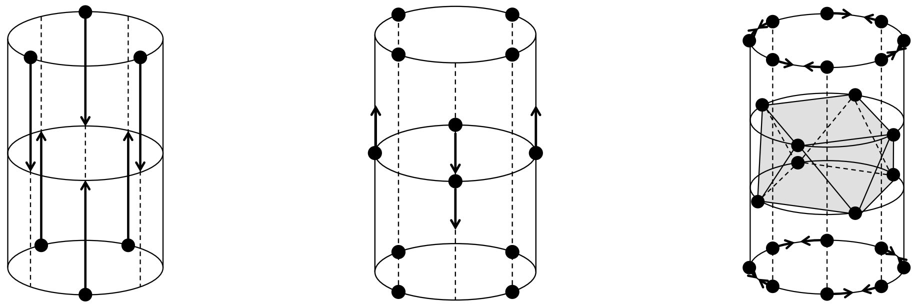
4.2.1 First phase: Removing the mirror plane

When $\theta(P)$ has a horizontal mirror plane, our basic strategy allows multiplicities. By the condition of Theorem 6, there is at least one element of the $\theta(P)$-decomposition of P on the horizontal mirror plane. To remove this mirror plane, we first make the robots of such an element leave their current positions (Fig. 2b). Intuitively, the first phase makes these robots select the upward direction or the downward direction regarding the horizontal mirror plane and they move to the selected directions. Any resulting configuration does not have the horizontal mirror plane any more because for each new positions of the robots, there is no corresponding point regarding the horizontal mirror plane.

In the following, we assume that when the current configuration P has a rotation axis, it does not have any horizontal mirror plane for the principal axis or $\theta(P) = C_{2h}$.

4.2.2 Second phase: Collision avoidance for dihedral groups

When $\gamma(P)$ is a dihedral group, say D_ℓ or $D_{\ell v}$, and the $\theta(P)$-decomposition of P contains a prism, our basic strategy makes multiplicities (Fig. 2c). By Theorem 6, $\theta(P)$ does not have a horizontal mirror plane, and there exists at least one element P_j that does not form such a prism. The robots of P_i circulates toward the nearest point of P_j and twist their prism. This movement resolves the collisions among the perpendiculars from the robots of P_i to the target plane.



(a) Antiprism **(b)** Horizontal mirror plane **(c)** Prism

Figure 2 Basic idea of the landing algorithm and collision avoidance. (a) Basic idea. (b) Removing the horizontal mirror plane. (c) Avoiding collisions by using an anti-prism.

4.2.3 Third phase: Collision avoidance on the principal axis

When $\gamma(P)$ is a dihedral group and some robots are on the principal axis, we need another trick to resolve the collisions of these robots. Clearly, these robots form element(s) of the $\theta(P)$-decomposition of P and the size of each of such elements P_k is two.

We also use an element of P_j that forms a "twisted" prism in the same way as the previous case. Each point of $p \in P_k$ selects the nearest point of P_j, however in this case, the robots of P_k do not move. Other robots consider the vertices of the twisted prism as possible destinations of this fictitious move.

4.2.4 Fourth phase: Agreement of the target plane

The robots agree on the target plane. Depending on $\theta(P)$ of the current configuration P, we have the following five cases:

(i) When $\gamma(P)$ is a cyclic group or a dihedral group, the robots agree on the plane perpendicular to the principal axis and containing $b(P)$. The exceptional case is when $\theta(P) = C_{2h}$. In this case, since the robots are not on one plane, there exists at least one element P_i of the $\theta(P)$-decomposition of P such that $|P_i| = 4$. Let i be the minimum index among such elements. Then, the robots agree on the plane formed by P_i. P_i forms a rectangle perpendicular to the horizontal mirror plane and the agreed plane is not a mirror plane for P.

(ii) When $\theta(P)$ is a rotation-reflection, the robots agree on the mirror plane of $\theta(P)$.

(iii) When $\theta(P)$ is a bilateral symmetry, the size of each element of the $\theta(P)$-decomposition of P is one or two. Since the robots are not on one plane, there exists at least one element P_i such that $|P_i| = 2$ and forms a perpendicular line regarding the mirror plane. If there is just one such element P_i, the robots agree on the plane defined by P_i (a line) and P_1 (a point). If $i = 1$, then the algorithm uses P_2 instead of P_1. Otherwise, let P_i and P_j be the elements with the minimum and second-minimum index of such elements. Then the robots agree on the plane defined by these two elements. The selected plane is not a mirror plane for P.

(iv) When $\theta(P)$ is a central inversion, the size of each element of the $\theta(P)$-decomposition of P is one or two. Since the robots are not on one plane, there are more than one elements of size two. Each of these elements form a line and they all intersect at the center of inversion. Let P_i and P_j be such elements with the minimum and second-minimum index. Then the robots agree on the plane defined by these two elements.

(v) When $\theta(P)$ is C_1, by Lemma 1, the robots can agree on the total ordering of themselves and agree on the plane defined by P_1, P_2, and P_3.

4.2.5 Fifth phase: Computation of final positions

Let P and $\{P_1, P_2, \ldots, P_m\}$ be the current configuration and its $\theta(P)$-decomposition. Through the previous four phases, for each element P_i $(i = 1, 2, \ldots, m)$, the foots of perpendiculars from the points of P_i to the target plane do not overlap. However, the foot of perpendiculars of different elements of the $\theta(P)$-decomposition of P may overlap.

Then all robots locally compute the landing positions of all robots in the order of $P_1, P_2, \ldots, P_m$. An element with a smaller index has a higher priority, i.e., if the landing positions of P_i are computed, any P_j $(j > i)$ avoids them with a common rule. For example, a point p is a destination of some robot in P_i, a small circle containing no other destination is drawn, and a destination of $q \in P_j$ is selected from this circle. Then this circle is considered as possible landing positions for P_j so that any collision will be avoided in the succeeding computation. Irrespective of handedness, all robots agree on all possible landing positions, and each robot computes its collision-free landing point. Finally, the robots move to their destinations in the same cycle.

As the non-oblivious robots can execute the proposed algorithm, we have the following theorem, that together with Theorem 5, proves our main theorem.

▶ **Theorem 8.** *Irrespective of obliviousness, the FSYNC robots without chirality can form a plane from an initial configuration P if $\varrho(P)$ contains neither any 3D rotation group nor any 2D rotation group with horizontal mirror plane regarding the principal axis (except C_{2h} and S_m).*

5 Conclusion

We considered the plane formation problem by FSYNC robots without chirality. We extended the notion of symmetricity in [19] to the composition of rotation symmetry and reflection symmetry. By using the symmetricity, we gave a characterization of initial configurations from which the FSYNC robots without chirality can form a plane. We also showed a plane formation algorithm for oblivious FSYNC robots without chirality.

One of the most important future directions is to investigate the effect of other system elements in 3D-space. The ASYNC model and the SSYNC model are not thoroughly discussed in 3D-space except the plane formation for SSYNC robots [16]. The effect of local visibility also remains to be investigated.

References

1. Dana Angluin, James Aspnes, Zoe Diamadi, Michael J. Fischer, and René Peralta. Computation in networks of passively mobile finite-state sensors. In *Proceedings of the 23rd annual ACM symposium on Principles of distributed computing (PODC 2004)*, pages 290–299, 2004. doi:10.1145/1011767.1011810.
2. Mark A. Armstrong. *Groups and symmetry*. Springer-Verlag New York Inc., 1988.
3. Aaron Becker, Erik D. Demaine, Sándor P. Fekete, Golnaz Habibi, and James McLurkin. Reconfiguring massive particle swarms with limited, global control. In *Proceedings of the 9th International Symposium on Algorithms and Experiments for Sensor Systems, Wireless Networks and Distributed Robotics (ALGOSENSORS 2013)*, pages 51–66, 2013. doi:10.1007/978-3-642-45346-5_5.

4 Serafino Cicerone, Gabriele Di Stefano, and Alfredo Navarra. Asynchronous embedded pattern formation without orientation. In *Proceedings of the 30th International SymposiumDistributed on Computing (DISC 2016)*, pages 85–98, 2016. doi:10.1007/978-3-662-53426-7_7.

5 Mark Cieliebak, Paola Flocchini, Giuseppe Prencipe, and Nicola Santoro. Distributed computing by mobile robots: gathering. *SIAM Journal on Computing*, 41(4):829–879, 2012. doi:10.1137/100796534.

6 Peter R. Cromwell. *Polyhedra.* University Press, 1997.

7 Zahra Derakhshandeh, Shlomi Dolev, Robert Gmyr, Andréa W. Richa, Christian Scheideler, and Thim Strothmann. Brief announcement: Amoebot – a new model for programmable matter. In *Proceedings of the 26th ACM symposium on Parallelism in algorithms and architectures*, pages 220–222, 2014. doi:10.1145/2612669.2612712.

8 Zahra Derakhshandeh, Robert Gmyr, Andréa W. Richa, Christian Scheideler, and Thim Strothmann. Universal shape formation for programmable matter. In *Proceedings of the 28th ACM Symposium on Parallelism in Algorithms and Architectures (SPAA 2016)*, pages 289–299, 2016. doi:10.1145/2935764.2935784.

9 Yoann Dieudonné, Franck Petit, and Vincent Villain. Leader election problem versus pattern formation problem. In *Proceedings of the 24th International Symposium on Distributed Computing (DISC 2010)*, pages 267–281, 2010. doi:10.1007/978-3-642-15763-9_26.

10 Paola Flocchini, Giuseppe Prencipe, and Nicola Santoro. *Distributed computing by oblivious mobile robots.* Morgan & Claypool, 2012.

11 Paola Flocchini, Giuseppe Prencipe, Nicola Santoro, and Giovanni Viglietta. Distributed computing by mobile robots: Solving the uniform circle formation problem. In *Proceedings of the 18th International Conference on Principles of Distributed Systems (OPODIS 2014)*, pages 217–232, 2014. doi:10.1007/978-3-319-14472-6_15.

12 Nao Fujinaga, Yukiko Yamauchi, Hirotaka Ono, Shuji Kijima, and Masafumi Yamashita. Pattern formation by oblivious asynchronous mobile robots. *SIAM Journal on Computing*, 44(3):740–785, 2015. doi:10.1137/140958682.

13 Marcello Mamino and Giovanni Viglietta. Square formation by asynchronous oblivious robots. In *Proceedings of the 28th Canadian Conference on Computational Geometry (CCCG 2016)*, pages 1–6, 2016.

14 Sheryl Manzoor, Samuel Sheckman, Jarrett Lonsford, Hoyeon Kim, Min Jun Kim, and Aaron T. Becker. Parallel self-assembly of polyominoes under uniform control inputs. *IEEE Robotics and Automation Letters*, 2(4):2040–2047, 2017. doi:10.1109/LRA.2017.2715402.

15 Ichiro Suzuki and Masafumi Yamashita. Distributed anonymous mobile robots: Formation of geometric patterns. *SIAM Journal on Computing*, 28(4):1347–1363, 1999. doi:10.1137/S009753979628292X.

16 Taichi Uehara, Yukiko Yamauchi, Shuji Kijima, and Masafumi Yamashita. Plane formation by semi-synchronous robots in the three dimensional euclidean space. In *Proceedings of the 18th International Symposium on Stabilization, Safety, and Security of Distributed Systems (SSS 2016)*, pages 383–398, 2016. doi:10.1007/978-3-319-49259-9_30.

17 Masafumi Yamashita and Ichiro Suzuki. Characterizing geometric patterns formable by oblivious anonymous mobile robots. *Theoretical Computer Science*, 411:2433–2453, 2010. doi:10.1016/j.tcs.2010.01.037.

18 Yukiko Yamauchi, Taichi Uehara, Shuji Kijima, and Masafumi Yamashita. Plane formation by synchronous mobile robots in the three dimensional euclidean space. *Journal of the ACM*, 64(16), 2017. doi:10.1145/3060272.

19 Yukiko Yamauchi, Taichi Uehara, and Masafumi Yamashita. Brief announcement: Pattern formation problem for synchronous mobile robots in the three dimensional euclidean

space. In *Proceedings of the 2016 ACM Symposium on Principles of Distributed Computing (PODC 2016)*, pages 447–449, 2016. doi:10.1145/2933057.2933063.

20 Yukiko Yamauchi and Masafumi Yamashita. Pattern formation by mobile robots with limited visibility. In *Proceedings of the 20th International Colloquium on Structural Information and Communication Complexity (SIROCCO 2013)*, pages 201–212, 2013. doi:10.1007/978-3-319-03578-9_17.

A Seventeen Symmetry Groups in 3D-space

We summarize the seventeen symmetry groups in 3D-space. Each rotation group is determined by rotation axes, mirror planes, and their arrangement. Our results also heavily rely on the order of each symmetry group and horizontal mirror planes. The following three tables show these properties together with typical polyhedra each of which is obtained as an orbit of a seed point in each symmetry group.

Table 2 Symmetry groups without rotation axis

	Symmetry	Order
C_1	Identity element	1
C_i	Point of inversion	2
C_s	Single mirror plane	2

Table 3 Symmetry groups with 2D rotation groups

	Principal axis	Other axes	Horizontal mirror	Order	Orbits
C_k	k-fold	-	N	k	
C_{kh}	k-fold	-	Y	$2k$	
C_{kv}	k-fold	-	N	$2k$	Pyramid with regular k-gon base
D_ℓ	ℓ-fold	ℓ 2-fold axes	N	2ℓ	
$D_{\ell h}$	ℓ-fold	ℓ 2-fold axes	Y	4ℓ	Hexagonal prism for D_6
$D_{\ell v}$	ℓ-fold	ℓ 2-fold axes	N	4ℓ	Hexagonal anti-prism for D_{6v}
S_m	m-fold	-	Y	m	Hexagonal anti-prism for S_{12}

Table 4 Symmetry groups with 3D rotation groups

	Rotation axes (folding)				Mirror planes	Order	Orbits
	2	3	4	5			
T	3	4	-	-	0	12	Snub tetrahedron
T_d	3	4	-	-	6	24	Regular tetrahedron
T_h	3	4	-	-	3	24	
O	6	4	3	-	0	24	Snub cube
O_h	6	4	3	-	9	48	Cube, regular octahedron
I	15	10	-	6	0	60	Snub icosahedron
I_h	15	10	-	6	15	120	Regular dodecahedron