FO Model Checking of Geometric Graphs[*]

Petr Hliněný [†1], **Filip Pokrývka**[‡2], **and Bodhayan Roy**[3]

1   **Faculty of Informatics, Masaryk University Brno, Czech Republic**
    `hlineny@fi.muni.cz`
2   **Faculty of Informatics, Masaryk University Brno, Czech Republic**
    `xpokryvk@fi.muni.cz`
3   **Faculty of Informatics, Masaryk University Brno, Czech Republic**
    `b.roy@fi.muni.cz`

— Abstract —

Over the past two decades the main focus of research into first-order (FO) model checking algorithms has been on sparse relational structures – culminating in the FPT algorithm by Grohe, Kreutzer and Siebertz for FO model checking of nowhere dense classes of graphs. On contrary to that, except the case of locally bounded clique-width only little is currently known about FO model checking of dense classes of graphs or other structures. We study the FO model checking problem for dense graph classes definable by geometric means (intersection and visibility graphs). We obtain new nontrivial FPT results, e.g., for restricted subclasses of *circular-arc, circle, box, disk, and polygon-visibility graphs*. These results use the FPT algorithm by Gajarský et al. for FO model checking of posets of bounded width. We also complement the tractability results by related hardness reductions.

1   Introduction

Algorithmic meta-theorems are results stating that all problems expressible in a certain language are efficiently solvable on certain classes of structures, e.g. of finite graphs. Note that the model checking problem for *first-order logic* – given a graph G and an FO formula ϕ, we want to decide whether G satisfies ϕ (written as $G \models \phi$) – is trivially solvable in time $|V(G)|^{\mathcal{O}(|\phi|)}$. "Efficient solvability" hence in this context often means *fixed-parameter tractability* (FPT); that is, solvability in time $f(|\phi|) \cdot |V(G)|^{\mathcal{O}(1)}$ for some computable function f.

    In the past two decades algorithmic meta-theorems for FO logic on sparse graph classes received considerable attention. While the algorithm of [5] for MSO on graphs of bounded clique-width implies fixed-parameter tractability of FO model checking on graphs of locally bounded clique-width via Gaifman's locality, one could go far beyond that. After the result of Seese [26] proving fixed-parameter tractability of FO model checking on graphs of bounded degree there followed a series of results [6, 10, 12] establishing the same conclusion

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for increasingly rich sparse graph classes. This line of research culminated in the result of Grohe, Kreutzer and Siebertz [20], who proved that FO model checking is FPT on *nowhere dense* graph classes.

While the result of [20] is the best possible in the following sense – if a graph class $\mathcal{D}$ is *monotone* (closed on taking subgraphs) and not nowhere dense, then the FO model checking problem on $\mathcal{D}$ is as hard as that on all graphs; this does not exclude interesting FPT meta-theorems on *somewhere dense* non-monotone graph classes. Probably the first extensive work of the latter dense kind, beyond locally bounded clique-width, was that of Ganian et al. [16] studying subclasses of interval graphs for which FO model checking is FPT (when only bounded number of interval lengths is used). Another approach has been taken in the works of Bova, Ganian and Szeider [3] and Gajarský et al. [13], which studied FO model checking on posets – posets can be seen as typically quite dense special digraphs. Altogether, however, only very little is known about FO model checking of somewhere dense graph classes (except perhaps specialised [15]).

The result of Gajarský et al. [13] claims that FO model checking is FPT on posets of bounded width (size of a maximum antichain), and it happens to imply [16] in a stronger setting (see below). One remarkable message of [13] is the following (citation): *The result may also be used directly towards establishing fixed-parameter tractability for FO model checking of other graph classes. Given the ease with which it ([13]) implies the otherwise non-trivial result on interval graphs [16], it is natural to ask what other (dense) graph classes can be interpreted in posets of bounded width.* Inspired by the geometric case of interval graphs, we propose to study dense graph classes defined in geometric terms, such as intersection and visibility graphs, with respect to tractability of their FO model checking problem.

The motivation for such study is a two-fold. First, intersection and visibility graphs present natural examples of non-monotone somewhere dense graph classes to which the great "sparse" FO tractability result of [20] cannot be (at least not easily) applied. Second, their supplementary geometric structure allows to better understand (as we have seen already in [16]) the boundaries of tractability of FO model checking on them, which is, to current knowledge, terra incognita for hereditary graph classes in general.

Our results mainly concern graph classes which are related to interval graphs. Namely, we prove (Theorem 3.1) that FO model checking is FPT on *circular-arc graphs* (these are interval graphs on a circle) if there is no long chain of arcs nested by inclusion. This directly extends the result of [16] and its aforementioned strengthening in [13] (with bounding chains of nested intervals instead of their lengths). We similarly show tractability of FO model checking of interval-overlap graphs, also known as *circle graphs*, of bounded independent set size (Theorem 4.1), and of restricted subclasses of *box and disk graphs* which naturally generalize interval graphs to two dimensions (Theorem 5.1).

On the other hand, for all of the studied cases we also show that whenever we relax our additional restrictions (parameters), the FO model checking problem becomes as hard on our intersection classes as on all graphs (Corollary 6.2). Some of our hardness claims hold also for the weaker ∃FO model checking problem (Proposition 6.3).

Another well studied dense graph class in computational geometry are *visibility graphs* of polygons, which have been largely explored in the context of recognition, partition, guarding and other optimization problems [17,25]. We consider some established special cases, involving *weak visibility, terrain* and *fan* polygons. We prove that FO model checking is FPT for the visibility graphs of a weak visibility polygon of a convex edge, with bounded number of reflex (non-convex) vertices (Theorem 7.2). On the other hand, without bounding reflex vertices, FO model checking remains hard even for the much more special case of polygons that are terrain and convex fans at the same time (Theorem 7.1).

As noted above, our fixed-parameter tractability proofs use the strong result [13] on FO model checking of posets of bounded width. We refer to Section 2 for a detailed explanation of the technical terms used here. Briefly, for a given graph G from the respective class and a formula ϕ, we show how to efficiently construct a poset $\mathcal{P}_G$ of bounded width and a related FO formula ϕ^I such that $G \models \phi$ iff $\mathcal{P}_G \models \phi^I$, and then solve the latter problem.

With respect to the previously known results, we remark that our graph classes are not sparse, as they all contain large complete or complete bipartite subgraphs. For some of them, namely unit circular-arc graphs, circle graphs of bounded independence number, and box graphs (with parameter $k = 2$ as in Theorem 5.1), it can be shown that they are of locally unbounded clique-width by an adaptation of the corresponding argument from [16].

Lastly, we particularly emphasize the seemingly simple tractable case (Corollary 4.2) of permutation graphs of bounded clique size: in relation to so-called stability notion (cf. [1]), already the hereditary class of triangle-free permutation graphs has the n-order property (i.e., is *not* stable), and yet FO model checking of this class is FPT. This example presents a natural hereditary and non-stable graph class with FPT FO model checking other than, say, graphs of bounded clique-width. We suggest that if we could fully understand the precise breaking point(s) of FP tractability of FO model checking on simply described intersection classes like the permutation graphs, then we would get much better insight into FP tractability of FO model checking of general hereditary graph classes.

Due to space restrictions, most of the proofs and some illustrating pictures have had to be removed from this short paper. The statements with removed proofs are marked by ***** and they can be found, for example, in the arXiv version.

2   Preliminaries

Graphs and intersection graphs.   We work with *finite simple undirected graphs* and use standard graph theoretic notation. We refer to the vertex set of a graph G as to $V(G)$ and to its edge set as to $E(G)$, and we write shortly uv for an edge $\{u, v\}$. As it is common in the context of FO logic on graphs, vertices of our graphs can carry arbitrary labels.

Considering a family of sets $\mathcal{S}$ (in our case, of geometric objects in the plane), the *intersection graph of* $\mathcal{S}$ is the simple graph G defined by $V(G) := \mathcal{S}$ and $E(G) := \{AB : A, B \in \mathcal{S}, A \cap B \neq \emptyset\}$. In respect of algorithmic questions, it is important to distinguish whether an intersection graph G is given on the input as an abstract graph G, or alongside with its intersection representation $\mathcal{S}$.

One folklore example of a widely studied intersection graph class are *interval graphs* – the intersection graphs of intervals on the real line. Interval graphs enjoy many nice algorithmic properties, e.g., their representation can be constructed quickly, and generally hard problems like clique, independent set and chromatic number are solvable in polynomial time for them.

For a general overview and extensive reference guide of intersection graph classes we suggest to consult the online system ISGCI [7].

FO logic.   The *first-order logic of graphs* (abbreviated as FO) applies the standard language of first-order logic to a graph G viewed as a relational structure with the domain $V(G)$ and the single binary (symmetric) relation $E(G)$. That is, in graph FO we have got the standard predicate $x = y$, a binary predicate $edge(x, y)$ with the usual meaning $xy \in E(G)$, an arbitrary number of unary predicates $L(x)$ with the meaning that x holds the label L, usual logical connectives $\wedge, \vee, \rightarrow$, and quantifiers $\forall x, \exists x$ over the vertex set $V(G)$.

For example, $\phi(x,y) \equiv \exists z\big(edge(x,z) \wedge edge(y,z) \wedge red(z)\big)$ states that the vertices x, y have a common neighbour in G which has got label 'red'. One can straightforwardly express in FO properties such as k-clique $\exists x_1, \ldots, x_k\big(\bigwedge_{i<j=1}^{k}(edge(x_i, x_j) \wedge x_i \neq x_j)\big)$ and k-dominating set $\exists x_1, \ldots, x_k \forall y\big(\bigvee_{i=1}^{k}(edge(x_i, y) \vee y = x_i)\big)$. Specially, an FO formula ϕ is *existential* (abbreviated as $\exists$FO) if it can be written as $\phi \equiv \exists x_1, \ldots, x_k \psi$ where ψ is quantifier-free. For example, k-clique is $\exists$FO while k-dominating set is not.

Likewise, FO logic of posets treats a poset $\mathcal{P} = (P, \sqsubseteq)$ as a finite relational structure with the domain P and the (antisymmetric) binary predicate $x \sqsubseteq y$ (instead of the predicate *edge*) with the usual meaning. Again, posets can be arbitrarily labelled by unary predicates.

Parameterized model checking.   Instances of a parameterized problem can be considered as pairs $\langle I, k \rangle$ where I is the main part of the instance and k is the *parameter* of the instance; the latter is usually a non-negative integer. A parameterized problem is *fixed-parameter tractable (FPT)* if instances $\langle I, k \rangle$ of size n can be solved in time $O(f(k) \cdot n^c)$ where f is a computable function and c is a constant independent of k. In *parameterized model checking*, instances are considered in the form $\langle (G, \phi), |\phi| \rangle$ where G is a structure, ϕ a formula, the question is whether $G \models \phi$ and the parameter is the size of ϕ.

When speaking about the FO model checking problem in this paper, we always implicitly consider the formula ϕ (precisely its size) as a parameter. We shall use the following result:

▶ **Theorem 2.1** ([13]). *The FO model checking problem of (arbitrarily labelled) posets, i.e., deciding whether $\mathcal{P} \models \phi$ for a labelled poset $\mathcal{P}$ and FO ϕ, is fixed-parameter tractable with respect to $|\phi|$ and the width of $\mathcal{P}$ (this is the size of the largest antichain in $\mathcal{P}$).*

We also present, for further illustration, a result on FO model checking of interval graphs with bounded nesting. A set $\mathcal{A}$ of intervals (interval representation) is called *proper* if there is no pair of intervals in $\mathcal{A}$ such that one is contained in the other. We call $\mathcal{A}$ a *k-fold proper* set of intervals if there exists a partition $\mathcal{A} = \mathcal{A}_1 \cup \cdots \cup \mathcal{A}_k$ such that each $\mathcal{A}_j$ is a proper interval set for $j = 1, \ldots, k$. Clearly, $\mathcal{A}$ is k-fold proper if and only if there is no chain of $k+1$ inclusion-nested intervals in $\mathcal{A}$. From Theorem 2.1 one can, with help of relatively easy arguments (Lemma 3.2), derive the following:

▶ **Theorem 2.2** ([13], cf. Proposition 2.4 and Lemma 3.2). *Let G be an interval graph given alongside with its k-fold proper interval representation $\mathcal{A}$. Then FO model checking of G is FPT with respect to the parameters k and the formula size.*

Parameterized hardness.   For some parameterized problems, like the k-clique on all graphs, we do not have nor expect any FPT algorithm. To this end, the theory of parameterized complexity of Downey and Fellows [8] defines complexity classes $W[t]$, $t \geq 1$, such that the k-clique problem is complete for $W[1]$ (the least class). Furthermore, theory also defines a larger complexity class $AW[*]$ containing all of $W[t]$. Problems that are $W[1]$-hard do not admit an FPT algorithm unless the established Exponential Time Hypothesis fails.

▶ **Theorem 2.3** ([9]). *The FO model checking problem (where the formula size is the parameter) of all simple graphs is $AW[*]$-complete.*

Dealing with parameterized hardness of FO model checking, one should also mention the related *induced subgraph isomorphism problem*: for a given input graph G, and a graph H as the parameter, decide whether G has an induced subgraph isomorphic to H. Note that this includes the clique and independent set problems. Induced subgraph isomorphism

(parameterized by the subgraph size) is clearly a weaker problem than parameterized FO model checking, since one may "guess" the subgraph with $|V(H)|$ existential quantifiers and then verify it edge by edge. Consequently, every parameterized hardness result for induced subgraph isomorphism readily implies same hardness results for $\exists$FO and FO model checking.

FO interpretations. Interpretations are a standard tool of logic and finite model theory. To keep our paper short, we present here only a simplified description of them, tailored specifically to our need of interpreting geometric graphs in posets.

An *FO interpretation* is a pair $I = (\nu, \psi)$ of poset FO formulas $\nu(x)$ and $\psi(x, y)$ (of one and two free variables, respectively). For a poset $\mathcal{P}$, this defines a graph $G := I(\mathcal{P})$ such that $V(G) = \{v : \mathcal{P} \models \nu(v)\}$ and $E(G) = \{uv : u, v \in V(G), \mathcal{P} \models \psi(u, v) \vee \psi(v, u)\}$. Possible labels of the elements are naturally inherited from $\mathcal{P}$ to G. Moreover, for a graph FO formula ϕ the interpretation I defines a poset FO formula ϕ^I recursively as follows: every occurrence of $edge(x, y)$ is replaced by $\psi(x, y) \vee \psi(y, x)$, every $\exists x\, \sigma$ is replaced by $\exists x\, (\nu(x) \wedge \sigma)$ and $\forall x\, \sigma$ by $\forall x\, (\nu(x) \rightarrow \sigma)$. Then, obviously, $\mathcal{P} \models \phi^I \iff G \models \phi$.

Usefulness of the concept is illustrated by the following trivial claim:

▶ **Proposition 2.4.** * *Let $\mathcal{P}$ be a class of posets such that the FO model checking problem of $\mathcal{P}$ is FPT, and let $\mathcal{G}$ be a class of graphs. Assume there is a computable FO interpretation I, and for every graph $G \in \mathcal{G}$ we can in polynomial time compute a poset $\mathcal{P} \in \mathcal{P}$ such that $G = I(\mathcal{P})$. Then the FO model checking problem of $\mathcal{G}$ is in FPT.*

3 Circular-arc Graphs

Circular-arc graphs are intersection graphs of arcs (curved intervals) on a circle. They clearly form a superclass of interval graphs, and they enjoy similar nice algorithmic properties as interval graphs, such as efficient construction of the representation [24], and easy computation of, say, maximum independent set or clique.

Since the FO model checking problem is $AW[*]$-complete on interval graphs [16], the same holds for circular-arc graphs in general. Furthermore, by [21, 23] already $\exists$FO model checking is $W[1]$-hard for interval and circular-arc graphs. A common feature of these hardness reductions (see more discussion in Section 6) is their use of unlimited chains of nested intervals/arcs. Analogously to Theorem 2.2, we prove that considering only *k-fold proper circular-arc* representations (the definition is the same as for *k*-fold proper interval representations) makes FO model checking of circular-arc graphs tractable.

▶ **Theorem 3.1.** *Let G be a circular-arc graph given alongside with its k-fold proper circular-arc representation $\mathcal{A}$. Then FO model checking of G is FPT with respect to the parameters k and the formula size.*

Note that we can (at least partially) avoid the assumption of having a representation $\mathcal{A}$ in the following sense. Given an input graph G, we compute a circular-arc representation $\mathcal{A}$ using [24], and then we easily determine the least k' such that $\mathcal{A}$ is k'-fold proper. However, without further considerations, this is not guaranteed to provide the minimum k over all circular-arc representations of G, and not even k' bounded in terms of the minimum k.

Our proof will be based on the following extension of the related argument from [13]:

▶ **Lemma 3.2** (parts from [13, Section 5]). * *Let $\mathcal{B}$ be a k-fold proper set of intervals for some integer $k > 0$, such that no two intervals of $\mathcal{B}$ share an endpoint. There exist formulas ν, ψ, ϑ depending on k, and a labelled poset $\mathcal{P}$ of width $k + 1$ computable in polynomial time from $\mathcal{B}$, such that all the following hold:*
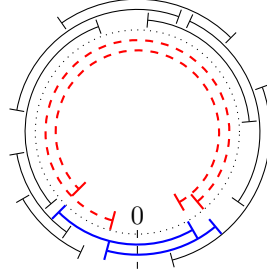
■ **Figure 1** An illustration; a proper circular-arc representation $\mathcal{A}$ (ordinary black and thick blue arcs), giving raise to a 2-fold proper interval set $\mathcal{B}$ (ordinary black and dashed red arcs), as in the proof of Theorem 3.1. The red arcs are complements of the corresponding blue arcs.

- *The domain of $\mathcal{P}$ includes (the intervals from) $\mathcal{B}$, and $\mathcal{P} \models \nu(x)$ iff $x \in \mathcal{B}$,*
- *$\mathcal{P} \models \psi(x, y)$ for intervals $x, y \in \mathcal{B}$ iff $x \cap y \neq \emptyset$ (edge relation of the interval graph of $\mathcal{B}$),*
- *$\mathcal{P} \models \vartheta(x, y)$ for intervals $x, y \in \mathcal{B}$ iff $x \subseteq y$ (containment of intervals).*

Proof of Theorem 3.1. We consider each arc of $\mathcal{A}$ in angular coordinates as $[\alpha, \beta]$ clockwise, where $\alpha, \beta \in [0, 2\pi)$. By standard arguments (a "small perturbation"), we can assume that no two arcs share the same endpoint, and no arc starts or ends in (the angle) 0. Let $\mathcal{A}_0 \subseteq \mathcal{A}$ denote the subset of arcs containing 0. Note that for every arc $[\alpha, \beta] \in \mathcal{A}_0$ we have $\alpha > \beta$, and we subsequently define $\mathcal{A}_1 := \{[\beta, \alpha] : [\alpha, \beta] \in \mathcal{A}_0\}$ as the set of their "complementary" arcs avoiding 0. For $a \in \mathcal{A}_0$ we shortly denote by $\bar{a} \in \mathcal{A}_1$ its complementary arc.

Now, the set $\mathcal{B} := (\mathcal{A} \setminus \mathcal{A}_0) \cup \mathcal{A}_1$ is an ordinary interval representation contained in the open line segment $(0, 2\pi)$. See Figure 1. Since each of $\mathcal{A} \setminus \mathcal{A}_0$ and $\mathcal{A}_1$ is k-fold proper by the assumption on $\mathcal{A}$, the representation $\mathcal{B}$ is $2k$-fold proper. Note the following facts; every two intervals in $\mathcal{A}_0$ intersect, and an interval $a \in \mathcal{A}_0$ intersects $b \in \mathcal{A} \setminus \mathcal{A}_0$ iff $b \not\subseteq \bar{a}$.

We now apply Lemma 3.2 to the set $\mathcal{B}$, constructing a (labelled) poset $\mathcal{P}$ of width at most $2k+1$. We also add a new label *red* to the elements of $\mathcal{P}$ which represent the arcs in $\mathcal{A}_1$. The final step will give a definition of an FO interpretation $I = (\nu, \psi_1)$ such that $I(\mathcal{P})$ will be isomorphic to the intersection graph G of $\mathcal{A}$. Using the formulas ψ, ϑ from Lemma 3.2, the latter is also quite easy. As mentioned above, intersecting pairs of intervals from $\mathcal{A}$ can be described using intersection and containment of the corresponding intervals of $\mathcal{B}$:

$$\psi_1(x, y) \equiv \big(red(x) \wedge red(y)\big) \vee \big(\neg red(x) \wedge \neg red(y) \wedge \psi(x, y)\big) \vee \big(red(x) \wedge \neg red(y) \wedge \neg \vartheta(y, x)\big)$$

It is routine to verify that, indeed, $G \simeq I(\mathcal{P})$ (using the obvious bijection of $\mathcal{A}_0$ to $\mathcal{A}_1$).

We then finish simply by Theorem 2.1 and Proposition 2.4. ◀

4 Circle graphs

Another graph class closely related to interval graphs are *circle graphs*, also known as *interval overlap graphs*. These are intersection graphs of chords of a circle, and they can equivalently be characterised as having an *overlap* interval representation $\mathcal{C}$ such that $a, b \in \mathcal{C}$ form an edge, if and only if $a \cap b \neq \emptyset$ but neither $a \subseteq b$ nor $b \subseteq a$ hold (see Figure 2). A circle representation of a circle graph can be efficiently constructed [2].

Related *permutation graphs* are defined as intersection graphs of line segments with the ends on two parallel lines, and they form a complementation-closed subclass of circle graphs.
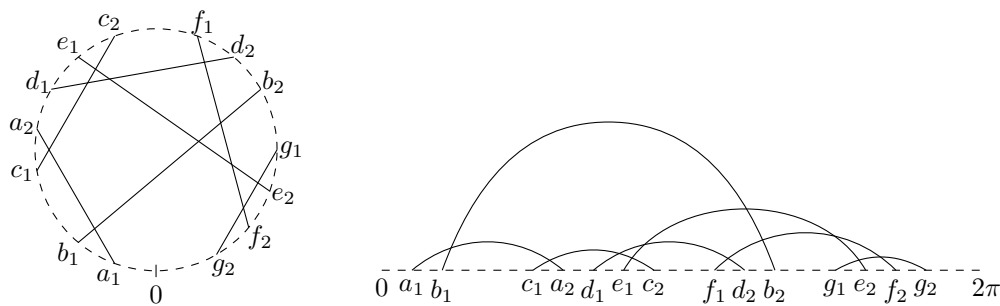
Figure 2 "Opening" a circle representation (left; an intersecting system of chords of a circle) into an overlap representation (right; the depicted arcs to be flattened into intervals on the line).

Note another easy characterization: let G be a graph and G_1 be obtained by adding one vertex adjacent to all vertices of G; then G is a permutation graph if and only if G_1 is a circle graph. We will see in Section 6 that the $\exists$FO model checking problem is $W[1]$-hard for circle graphs, and the FO model checking problem is $AW[*]$-complete already for permutation graphs. However, there is also a positive result using a natural additional parameterization. The proof of it uses arguments similar to those of Theorem 3.1.

▶ **Theorem 4.1.** ∗ *The FO model checking problem of circle graphs is FPT with respect to the formula and the maximum independent set size.*

An interesting question is whether 'independent set size' in Theorem 4.1 can also be replaced with 'clique size'. We think the right answer is 'yes', but we have not yet found the algorithm. At least, the answer is positive for the subclass of permutation graphs:

▶ **Corollary 4.2.** ∗ *The FO model checking problem of permutation graphs is FPT with respect to the formula size, and either the maximum clique or the maximum independent set size.*

▶ **Corollary 4.3.** ∗ *The subgraph isomorphism (not induced) problem of permutation graphs is FPT with respect to the subgraph size.*

5 Box and Disk graphs

Box (intersection) graphs are graphs having an intersection representation by rectangles in the plane, such that each rectangle (box) has its sides parallel to the x- and y-axes. The recognition problem of box graphs is NP-hard [28], and so it is essential that the input of our algorithm would consist of a box representation. *Unit-box graphs* are those having a representation by unit boxes.

The $\exists$FO model checking problem is $W[1]$-hard already for unit-box graphs [22], and we will furthermore show that it stays hard if we restrict the representation to a small area in Proposition 6.3. Here we give the following slight extension of Theorem 2.2:

▶ **Theorem 5.1.** ∗ *Let G be a box intersection graph given alongside with its box representation $\mathcal{B}$ such that the following holds: the projection of $\mathcal{B}$ to the x-axis is a k-fold proper set of intervals, and the projection of $\mathcal{B}$ to the y-axis consists of at most k distinct intervals. Then FO model checking of G is FPT with respect to the parameters k and the formula size.*
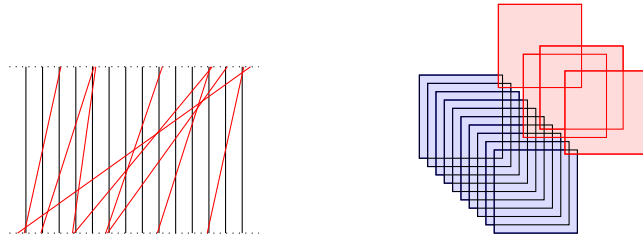
Figure 3 Constructing witnesses of the consecutive neighbourhood representation property – as permutation graphs (left) and as unit-box graphs (right); cf. Corollary 6.2.

Furthermore, *disk graphs* are those having an intersection representation by disks in the plane. Their recognition problem is NP-hard already with unit disks [4], and the ∃FO model checking problem is $W[1]$-hard again for unit-disk graphs by [22]. Similarly to Theorem 5.1, we have identified a tractable case of FO model checking of unit-disk graph, based on restricting the y-coordinates of the disks. Due to space restrictions, we leave this case only for the full paper.

6 Hardness for intersection classes

Our aim is to provide a generic reduction for proving hardness of FO model checking (even without labels on vertices) using only a simple property which is easy to establish for many geometric intersection graph classes. We will then use it to derive hardness of FO for quite restricted forms of intersection representations studied in our paper (Corollary 6.2).

We say that a graph G *represents consecutive neighbourhoods* of order ℓ, if there exists a sequence $S = (v_1, v_2, \ldots, v_\ell) \subseteq V(G)$ of distinct vertices of G and a set $R \subseteq V(G)$, $R \cap S = \emptyset$, such that for each pair i, j, $1 \leq i < j \leq \ell$, there is a vertex $w \in R$ whose neighbours in S are precisely the vertices $v_i, v_{i+1} \ldots, v_j$. (Possible edges other than those between R and S do not matter.) A graph class $\mathcal{G}$ has the *consecutive neighbourhood representation* property if, for every integer $\ell > 0$, there exists an efficiently computable graph $G \in \mathcal{G}$ such that G or its complement $\overline{G}$ represents consecutive neighbourhoods of order ℓ.

Note that our notion of 'representing consecutive neighbourhoods' is related to the concepts of "n-order property" and "stability" from model theory (mentioned in Section 1). This is not a random coincidence, as it is known [1] that on monotone graph classes stability coincides with nowhere dense (which is the most general characterization allowing for FPT FO model checking on monotone classes). In our approach, we stress easy applicability of this notion to a wide range of geometric intersection graphs and, to certain extent, to ∃FO model checking.

The main result is as follows. A *duplication* of a vertex v in G is the operation of adding a *true twin* v' to v, i.e., new v' adjacent to v and precisely to the neighbours of v in G.

▶ **Theorem 6.1.** * *Let $\mathcal{G}$ be a class of unlabelled graphs having the consecutive neighbourhood representation property, and $\mathcal{G}$ be closed on induced subgraphs and duplication of vertices. Then the FO model checking of $\mathcal{G}$ is $AW[*]$-complete with respect to the formula size.*

Graphs witnessing the consecutive neighbourhood representation property can be easily constructed within our intersection classes, even with strong further restrictions. See some illustrating examples in Figure 3. So, we obtain the following hardness results:

▶ **Corollary 6.2. *** *The FO model checking problem is AW[∗]-complete with respect to the formula size, for each of the following geometric graph classes (all unlabelled):*

(a) *circular-arc graphs with a representation consisting or arcs of lengths from $[\pi - \varepsilon, \pi + \varepsilon]$ on the circle of diameter 1, for any fixed $\varepsilon > 0$,*

(b) *connected permutation graphs,*

(c) *unit-box graphs with a representation contained within a square of side length $2 + \varepsilon$, for any fixed $\varepsilon > 0$,*

(d) *unit-disk graphs (that is of diameter 1) with a representation contained within a rectangle of sides $1 + \varepsilon$ and 2, for any fixed $\varepsilon > 0$.*

It is worthwhile to notice that for each of the classes listed in Corollary 6.2, the k-clique and k-independent set problems are all easily FPT, and yet FO model checking is not.

Finally, we return to the weaker ∃FO model checking problem. In fact, this problem can be treated "the same" as the aforementioned parameterized induced subgraph isomorphism problem: precisely, one of them admits an FPT algorithm on any given (unlabelled) graph class if and only if the other does so.

The hardness construction in the proof of Theorem 6.1 can be turned into ∃FO, but only if vertex labels are allowed. Though, we can modify some of the constructions from Corollary 6.2 to capture also ∃FO without labels.

▶ **Proposition 6.3. *** *The ∃FO model checking problem is $W[1]$-hard with respect to the formula size, for both the following unlabelled geometric graph classes:*

(a) *circle graphs,*

**(b)** *unit-box graphs with a representation contained within a square of side length 3.*

One complexity question that remains open after Proposition 6.3 is about ∃FO on unlabelled permutation graphs (for labelled ones, this is $W[1]$-hard by the remark after Corollary 6.2). While induced subgraph isomorphism is generally NP-hard on permutation graphs by [21], we are not aware of results on the parameterized version, and we currently have no plausible conjecture about its parameterized complexity.

7   Polygonal visibility graphs

Given a polygon W in the plane, two vertices p_i and p_j of W are said to be *mutually visible* if the line segment $p_i p_j$ does not intersect the exterior of W. The *visibility graph G* of W is defined to have vertices v_i corresponding to each vertex p_i of W, and edge (v_i, v_j) if and only if p_i and p_j are mutually visible. Our aim is to study the visibility graphs of some special established classes of polygons with respect to FO model checking.

If there is an edge e of the polygon W, such that for any point p of W, there is a point on e that sees p, then W is called a *weak visibility polygon*, and e is called a *weak visibility edge* of W (Figure 4a) [17, 18]. A vertex v_i of W is called a *reflex vertex* if the interior angle of W formed at v_i by the two edges of W incident to v_i is more than π. Otherwise, v_i is called a *convex vertex*. If both of the end vertices of an edge of W are convex vertices, then the edge is called a *convex edge*.

If the boundary of W consists only of an x-monotone polygonal arc touching the x-axis at its two extreme points, and an edge contained in the x-axis joining the two points, then it is called a *terrain* (Figure 4b) [11, 17]. All terrains are weak visibility polygons with respect to their edge that lies on the x-axis. If all points of a W are visible from a single vertex v of the polygon, then W is called a *fan* (Figure 4c) [17, 19]. If W is a fan with respect to a
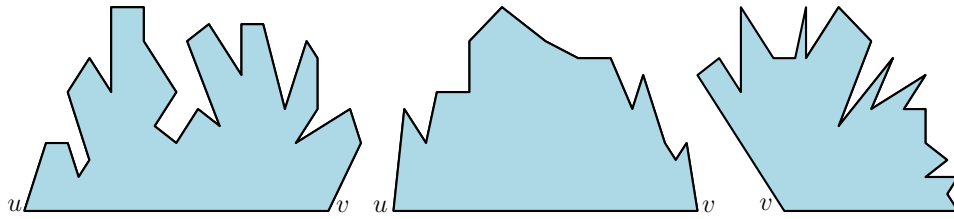
Figure 4 From left to right: (a) a weak visibility polygon with respect to edge uv; (b) a terrain; (c) a convex fan visible from the vertex v.

convex vertex v, then W is called a *convex fan* [25]. If W is a convex fan with respect to a vertex v, then both of the edges of W incident to v are convex edges, and W is also a weak visibility polygon with respect to any of them.

In this section we identify some interesting tractable and hard cases of the FO model checking problem on these visibility classes.

We first argue that the FO model checking problem of polygon visibility graphs stays hard even when the polygon is a terrain and a convex fan. Our approach is very similar to that in Theorem 6.1 above, that is, we show that a given FO model checking instance of general graphs can be interpreted in another instance of the visibility graph of a specially constructed polygon which is a terrain and a convex fan at the same time.

▶ **Theorem 7.1. *** *The FO model checking problem of unlabelled polygon visibility graphs (given alongside with the representing polygon) is $AW[*]$-complete with respect to the formula size, even when the polygon is a terrain and a convex fan at the same time.*

Second, we prove that FO model checking of the visibility graph of a given weak visibility polygon of a convex edge is FPT when additionally parameterized by the number of reflex vertices. We remark that, for example, the independent set problem is NP-hard on polygonal visibility graphs [27], but Ghosh et al. [18] showed that the maximum independent set of the visibility graph of a given weak visibility polygon of a convex edge, is computable in quadratic time. In Theorem 7.1, we have seen that the latter result does not generalise to arbitrary FO properties, since FO model checking remains hard even for a very special subcase of weak visibility polygons. So, an additional parameterization is necessary.

▶ **Theorem 7.2. *** *Let W be a given polygon weakly visible from one of its convex edges, with k reflex vertices, and let G be the visibility graph of W. Then FO model checking of G is FPT with respect to the parameters k and the formula size.*

While we cannot fit the whole algorithm in the short paper, we at least give an informal overview of how the algorithm works. As in the previous intersection graph cases, our aim is to construct, from given W, a poset $\mathcal{P}$ such that the width of $\mathcal{P}$ is bounded by a function of k and that we have an FO interpretation of the visibility graph of W in this $\mathcal{P}$.

Let W be weakly visible from its convex edge uv, and denote by C_{uv} the clockwise sequence of the vertices of W from u to v. The subsequence of C_{uv} between two reflex vertices v_a and v_b, such that all vertices in it are convex, is called an *ear* of W. The length of this sequence can be 0 as well. Additionally, the first (last) ear of W is defined as the subsequence between u and the first reflex vertex of C_{uv} (between the last reflex vertex and v, respectively). We have got $k + 1$ ears in W. With a slight abuse of terminology at u, v, we may simply say that an ear is a sequence of convex vertices between two reflex vertices.

The crucial idea of our construction of the poset $\mathcal{P}$ (which contains all vertices of W, in particular) is that the visibility edges between the internal (convex) vertices of the ears are

nicely structured: withing one ear E_a, they form a clique, and between two ears E_a, E_b, the visibility edges exhibit a "shifting pattern" not much different from the left and right ends of intervals in a proper interval representation (cf. Lemma 3.2). Consequently, we may "encode" all the edges between E_a and E_b with help of an extra subposet of $\mathcal{P}$ of fixed width, and since we have got only $k + 1$ ears, this together gives a poset of width bounded in k.

The last step concerns visibility edges incident with one of the k reflex vertices or u, v. These can be easily encoded in $\mathcal{P}$ with only $2(k+2)$ additional labels, without any assumption on the structure of $\mathcal{P}$: for each reflex vertex x of C_{uv}, or $x \in \{u, v\}$, we assign one new label L_x^0 to x itself and another new label L_x^1 to all the neighbours of x. Altogether, we can efficiently construct an FO interpretation of G in $\mathcal{P}$ such that the formulas depend only on k. Then we may finish by Theorem 2.1.

8 Conclusions

We have identified several FP tractable cases of the FO model checking problem of geometric graphs, and complemented these by hardness results showing quite strict limits of FP tractability on the studied classes. Overall, this presents a nontrivial new contribution towards understanding on which (hereditary) dense graph classes can FO model checking be FPT.

All our tractability results rely on the FO model checking algorithm of [13], which is mainly of theoretical interest. However, in some cases one can employ, in the same way, the simple and practical $\exists$FO model checking algorithm of [14]. We would also like to mention the possibility of enhancing the result of [13] via interpreting posets in posets. While this might seem impossible, we actually have one positive indication of such an enhancement. It is known that interval graphs are C_4-free complements of comparability graphs (i.e., of posets) – the width of which is the maximum clique size of the original interval graph. Then, among k-fold proper interval graphs there are ones of unbounded clique size, which have FPT FO model checking by Theorem 2.2. This opens a promising possibility of an FP tractable subcase of FO model checking of posets of unbounded width, for future research.

Finally, we list two concrete open problems related to our results. We conjecture that FO model checking is FPT for

- circle graphs additionally parameterized by the maximum clique size,
- visibility graphs of weak visibility polygons additionally parameterized by the maximum independent set size.

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