Parameterized (Approximate) Defective Coloring

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Abstract

In Defective Coloring we are given a graph $G = (V, E)$ and two integers χ_d, Δ^* and are asked if we can partition V into χ_d color classes, so that each class induces a graph of maximum degree Δ^*. We investigate the complexity of this generalization of Coloring with respect to several well-studied graph parameters, and show that the problem is W-hard parameterized by treewidth, pathwidth, tree-depth, or feedback vertex set, if $\chi_d = 2$. As expected, this hardness can be extended to larger values of χ_d for most of these parameters, with one surprising exception: we show that the problem is FPT parameterized by feedback vertex set for any $\chi_d \neq 2$, and hence 2-coloring is the only hard case for this parameter. In addition to the above, we give an ETH-based lower bound for treewidth and pathwidth, showing that no algorithm can solve the problem in $n^{o(\mathrm{pw})}$, essentially matching the complexity of an algorithm obtained with standard techniques.

We complement these results by considering the problem's approximability and show that, with respect to Δ^*, the problem admits an algorithm which for any $\epsilon > 0$ runs in time $(\mathrm{tw}/\epsilon)^{O(\mathrm{tw})}$ and returns a solution with exactly the desired number of colors that approximates the optimal Δ^* within $(1 + \epsilon)$. We also give a $(\mathrm{tw})^{O(\mathrm{tw})}$ algorithm which achieves the desired Δ^* exactly while 2-approximating the minimum value of χ_d. We show that this is close to optimal, by establishing that no FPT algorithm can (under standard assumptions) achieve a better than 3/2-approximation to χ_d, even when an extra constant additive error is also allowed.

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1 Introduction

Defective Coloring is the following problem: we are given a graph $G = (V, E)$, and two integer parameters $\chi_\mathrm{d}, \Delta^*$, and are asked whether there exists a partition of V into at most χ_d sets (color classes), such that each set induces a graph with maximum degree at most Δ^*. Defective Coloring, which is also sometimes referred to in the literature as Improper Coloring, is a natural generalization of the classical Coloring problem, which corresponds to the case $\Delta^* = 0$. The problem was introduced more than thirty years ago [2, 17], and since then has attracted a great deal of attention [1, 4, 6, 13, 14, 16, 23, 25, 28, 32, 33, 34].

From the point of view of applications, Defective Coloring is particularly interesting in the context of wireless communication networks, where the assignment of colors to vertices often represents the assignment of frequencies to communication nodes. In many practical settings, the requirement of traditional coloring that all neighboring nodes receive distinct colors is too rigid, as a small amount of interference is often tolerable, and may lead to solutions that need drastically fewer frequencies. Defective Coloring allows one to model this tolerance through the parameter Δ^*. As a result the problem's complexity has been well-investigated in graph topologies motivated by such applications, such as unit-disk graphs and various classes of grids [5, 7, 8, 10, 26, 27]. For more background we refer to [22, 31].

In this paper we study Defective Coloring from the point of view of parameterized complexity [18, 19, 21, 39]. The problem is of course NP-hard, even for small values of $\chi_\mathrm{d}, \Delta^*$, as it generalizes Coloring. We are therefore strongly motivated to bring to bear the powerful toolbox of structural graph parameters, such as treewidth, which have proved extremely successful in tackling other intractable hard problems. Indeed, Coloring is one of the success stories of this domain, since the complexity of this flagship problem with respect to treewidth (and related parameters pathwidth, feedback vertex set, vertex cover) is by now extremely well-understood [37, 30]. We pose the natural question of whether similar success can be achieved for Defective Coloring, or whether the addition of Δ^* significantly alters the complexity behavior of the problem. Such results are not yet known for Defective Coloring, except for the fact that it was observed in [9] that the problem admits (by standard techniques) a roughly $(\chi_\mathrm{d}\Delta^*)^{\mathrm{tw}}$-time algorithm, where tw is the graph's treewidth. In parameterized complexity terms, this shows that the problem is FPT parameterized by $\mathrm{tw} + \Delta^*$. One of our main motivating questions is whether this running time can be improved qualitatively (is the problem FPT parameterized only by tw?) or quantitavely.

Our first result is to establish that the problem is W-hard not just for treewidth, but also for several much more restricted structural graph parameters, such as pathwidth, tree-depth, and feedback vertex set. We recall that for Coloring, the standard ${\chi_\mathrm{d}}^{\mathrm{tw}}$ algorithm is FPT by tw, as graphs of bounded treewidth also have bounded chromatic number (Lemma 1). Our result shows that the complexity of the problem changes drastically with the addition of the new parameter Δ^*, and it appears likely that tw must appear in the exponent of Δ^* in the running time, even when Δ^* is large. More strongly, we establish this hardness even for the case $\chi_\mathrm{d} = 2$, which corresponds to the problem of partitioning a graph into two parts so as to minimize their maximum degree. This identifies Defective Coloring as another member of a family of generalizations of Coloring (such as Equitable Coloring or List Coloring) which are hard for treewidth [20].

As one might expect, the W-hardness results on Defective Coloring parameterized by treewidth (or pathwidth, or tree-depth) easily carry over for values of χ_d larger than 2. Surprisingly, we show that this is *not* the case for the parameter feedback vertex set, for which the only W-hard case is 2-coloring: we establish with a simple win/win argument that

Table 1 Summary of results. Hardness results for tree-depth imply the same bounds for treewidth and pathwidth. Conversely, algorithms which apply to treewidth apply also to all other parameters.

Parameter	Result (Exact solution)	Ref.	Result (Approximation)	Ref.
Feedback Vertex Set	W[1]-hard for $\chi_\mathrm{d} = 2$ FPT for $\chi_\mathrm{d} \neq 2$	Thm 2 Thm 20	+1-approximation in time $\mathrm{fvs}^{O(\mathrm{fvs})}$	Cor 28
Tree-depth	W[1]-hard for any $\chi_\mathrm{d} \geq 2$	Thm 2	W[1]-hard to color with $(3/2-\epsilon)\chi_\mathrm{d} + O(1)$ colors	Thm 26
Treewidth, Pathwidth	No $n^{o(\mathrm{pw})}$ or $n^{o(\mathrm{tw})}$ algorithm under ETH	Thm 14	$(1+\epsilon)$-approximation for Δ^* in $(\mathrm{tw}/\epsilon)^{O(\mathrm{tw})}$ 2-approximation for χ_d in $\mathrm{tw}^{O(\mathrm{tw})}$	Thm 23 Thm 25
Vertex Cover	$\mathrm{vc}^{O(\mathrm{vc})}$ algorithm	Thm 21		

the problem is FPT for any other value of χ_d. We also show that if one considers sufficiently restricted parameters, such as vertex cover, the problem does eventually become FPT.

Our second step is to enhance the W-hardness result mentioned above with the aim of determining as precisely as possible the complexity of Defective Coloring parameterized by treewidth. Our reduction for tree-depth and feedback vertex set is quadratic in the parameter, and hence implies that no algorithm can solve the problem in time $n^{o(\sqrt{\mathrm{tw}})}$ under the Exponential Time Hypothesis (ETH) [29]. We therefore present a second reduction, which applies only to pathwidth and treewidth, but manages to show that no algorithm can solve the problem in time $n^{o(\mathrm{pw})}$ or $n^{o(\mathrm{tw})}$ under the ETH. This lower bound is tight, as it matches asymptotically the exponent given in the algorithm of [9].

To complement the above results, we also consider the problem from the point of view of (parameterized) approximation. Here things become significantly better: we give an algorithm using a technique of [36] which for any χ_d and error $\epsilon > 0$ runs in time $(tw/\epsilon)^{O(\mathrm{tw})}n^{O(1)}$ and approximates the optimal value of Δ^* within a factor of $(1+\epsilon)$. Hence, despite the problem's W-hardness, we produce a solution arbitrarily close to optimal in FPT time.

Motivated by this algorithm we also consider the complementary approximation problem: given Δ^* find a solution that comes as close to the minimum number of colors needed as possible. By building on the approximation algorithm for Δ^*, we are able to present a $(\mathrm{tw})^{O(\mathrm{tw})}n^{O(1)}$ algorithm that achieves a 2-approximation for this problem. One can observe that this is not far from optimal, since an FPT algorithm with approximation ratio better than 3/2 would contradict the problem's W-hardness for $\chi_\mathrm{d} = 2$. However, this simple argument is unsatisfying, because it does not rule out algorithms with a ratio significantly better than 3/2, if one also allows a small additive error; indeed, we observe that when parameterized by feedback vertex set the problem admits an FPT algorithm that approximates the optimal χ_d within an additive error of just 1. To resolve this problem we present a gap-introducing version of our reduction which, for any i produces an instance for which the optimal value of χ_d is either $2i$, or at least $3i$. In this way we show that, when parameterized by tree-depth, pathwidth, or treewidth, approximating the optimal value of χ_d better than 3/2 is "truly" hard, and this is not an artifact of the problem's hardness for 2-coloring.

2 Definitions and Preliminaries

For a graph $G = (V, E)$ and two integers $\chi_\mathrm{d} \geq 1$, $\Delta^* \geq 0$, we say that G admits a $(\chi_\mathrm{d}, \Delta^*)$-coloring if one can partition V into χ_d sets such that the graph induced by each set has maximum degree at most Δ^*. Defective Coloring is the problem of deciding, given $G, \chi_{\rm d}, \Delta^*$, whether G admits a $(\chi_{\rm d}, \Delta^*)$-coloring. For $\Delta^* = 0$ this corresponds to Coloring.

We assume the reader is familiar with basic notions in parameterized complexity, such as the classes FPT and W[1]. For the relevant definitions we refer to the standard textbooks [18, 19, 21, 39]. We rely on a number of well-known graph measures: treewidth [12], pathwidth, tree-depth [38], feedback vertex set, and vertex cover, denoted respectively as $\mathrm{tw}(G), \mathrm{pw}(G), \mathrm{td}(G), \mathrm{fvs}(G), \mathrm{vc}(G)$, where we drop G if it is clear from the context.

▶ **Lemma 1.** *For any graph G we have $\mathrm{tw}(G) - 1 \le \mathrm{fvs}(G) \le \mathrm{vc}(G)$ and $\mathrm{tw}(G) \le \mathrm{pw}(G) \le \mathrm{td}(G) - 1 \le \mathrm{vc}(G)$. Furthermore, any graph G admits a $(\mathrm{tw}(G) + 1, 0)$-coloring, a $(\mathrm{pw}(G) + 1, 0)$-coloring, a $(\mathrm{td}(G), 0)$-coloring, and a $(\mathrm{fvs}(G) + 2, 0)$-coloring.*

The Exponential Time Hypothesis (ETH) states that 3-SAT on instances with n variables and m clauses cannot be solved in time $2^{o(n+m)}$ [29]. We define the k-Multi-Colored Clique problem as follows: we are given a graph $G = (V, E)$, a partition of V into k independent sets $V_1, \ldots, V_k$, such that for all $i \in \{1, \ldots, k\}$ we have $|V_i| = n$, and we are asked if G contains a k-clique. It is well-known that this problem is W[1]-hard parameterized by k, and that it does not admit any $n^{o(k)}$ algorithm, unless the ETH is false [18].

3 W-hardness for Feedback Vertex Set and Tree-depth

The main result of this section states that deciding if a graph admits a $(2, \Delta^*)$-coloring, where Δ^* is part of the input, is W[1]-hard parameterized by either fvs or td. Because of standard relations between graph parameters (Lemma 1), this implies also the same problem's W-hardness for parameters pw and tw. As might be expected, it is not hard to extend our proof to give hardness for deciding if a $(\chi_{\rm d}, \Delta^*)$-coloring exists, for any constant $\chi_{\rm d}$, parameterized by tree-depth (and hence, also treewidth and pathwidth). What is perhaps more surprising is that this cannot be done in the case of feedback vertex set. Superficially, the reason we cannot extend the reduction in this case is that one of the gadgets we use in many copies in our construction has large fvs if $\chi_{\rm d} > 2$. However, we give a much more convincing reason in Theorem 20 of Section 5 where we show that Defective Coloring is FPT parameterized by fvs for $\chi_{\rm d} \ge 3$, and therefore, if we could extend our reduction in this case it would prove that FPT=W[1].

The main theorem of this section is stated below. We then present the reduction in Sections 3.1, 3.2, and give the Lemmata that imply Theorem 2 in Section 3.3.

▶ **Theorem 2.** *Deciding if a graph G admits a $(2, \Delta^*)$-coloring, where Δ^* is part of the input, is W[1]-hard parameterized by $\mathrm{fvs}(G)$. Deciding if a graph G admits a $(\chi_{\rm d}, \Delta^*)$-coloring, where $\chi_{\rm d} \ge 2$ is any fixed constant and Δ^* is part of the input is W[1]-hard parameterized by $\mathrm{td}(G)$.*

3.1 Basic Gadgets

Before we proceed, we present some basic gadgets that will be useful in all the reductions of this paper (Theorems 2, 14, 26). We first define a building block $\mathcal{T}(i, j)$ which is a graph that can be properly colored with i colors, but admits no $(i - 1, j)$-coloring (similar constructions appears in [28]). We then use this graph to build two gadgets: the Equality Gadget and the Palette Gadget (Definitions 5 and 8). Informally, for given $\chi_{\rm d}, \Delta^*$, the equality gadget allows us to express the constraint that two vertices v_1, v_2 of a graph must receive the same color in any valid $(\chi_{\rm d}, \Delta^*)$-coloring. The palette gadget will be used to express the constraint that, among three vertices v_1, v_2, v_3, there must exist two with the same color. For both gadgets we first prove formally that they express these constraints (Lemmata 6 and 9). We then show that, under certain conditions, these gadgets can be added to any graph without significantly increasing its tree-depth or feedback vertex set (Lemmata 7 and 10).

▶ **Definition 3.** Given two integers $i > 0, j \geq 0$, we define the graph $\mathcal{T}(i,j)$ recursively as follows: $T(1,j) = K_1$ for all j; for $i > 1$, $T(i,j)$ is the graph obtained by taking $(j+1)$ disjoint copies of $T(i-1,j)$ and adding to the graph a new universal vertex.

▶ **Lemma 4.** *For all $i > 0, j \geq 0$ we have: $\mathcal{T}(i,j)$ admits an $(i,0)$-coloring; $\mathcal{T}(i,j)$ does not admit an $(i-1,j)$-coloring; $\mathrm{td}(\mathcal{T}(i,j)) = \mathrm{pw}(\mathcal{T}(i,j)) + 1 = \mathrm{tw}(\mathcal{T}(i,j)) + 1 = i$.*

▶ **Definition 5.** (Equality Gadget) For $i \geq 2, j \geq 0$, we define the graph $Q(u_1, u_2, i, j)$ as follows: Q contains $ij + 1$ disjoint copies of $\mathcal{T}(i-1,j)$ as well as two vertices u_1, u_2 which are connected to all vertices except each other.

▶ **Lemma 6.** *Let $G = (V, E)$ be a graph with $v_1, v_2 \in V$ and let G' be the graph obtained from G by adding to it a copy of $Q(u_1, u_2, \chi_{\mathrm{d}}, \Delta^*)$ and identifying u_1 with v_1 and u_2 with v_2. Then, any $(\chi_{\mathrm{d}}, \Delta^*)$-coloring of G' must give the same color to v_1, v_2. Furthermore, if there exists a $(\chi_{\mathrm{d}}, \Delta^*)$-coloring of G that gives the same color to v_1, v_2, this coloring can be extended to a $(\chi_{\mathrm{d}}, \Delta^*)$-coloring of G'.*

▶ **Lemma 7.** *Let $G = (V, E)$ be a graph, $S \subseteq V$, and G' be a graph obtained from G by repeated applications of the following operation: we select two vertices $v_1, v_2 \in V$ such that $v_1 \in S$, add a new copy of $Q(u_1, u_2, \chi_{\mathrm{d}}, \Delta^*)$ and identify u_i with v_i, for $i \in \{1, 2\}$. Then $\mathrm{td}(G') \leq \mathrm{td}(G \setminus S) + |S| + \chi_{\mathrm{d}} - 1$. Furthermore, if $\chi_{\mathrm{d}} = 2$ we have $\mathrm{fvs}(G') \leq \mathrm{fvs}(G \setminus S) + |S|$.*

▶ **Definition 8.** (Palette Gadget) For $i \geq 3, j \geq 0$ we define the graph $P(u_1, u_2, u_3, i, j)$ as follows: P contains $\binom{i}{2}j + 1$ copies of $\mathcal{T}(i-2,j)$, as well as three vertices u_1, u_2, u_3 which are connected to every vertex of P except each other.

▶ **Lemma 9.** *Let $G = (V, E)$ be a graph with $v_1, v_2, v_3 \in V$ and let G' be the graph obtained from G by adding to it a copy of $P(u_1, u_2, u_3, \chi_{\mathrm{d}}, \Delta^*)$ and identifying u_i with v_i for $i \in \{1, 2, 3\}$. Then, in any $(\chi_{\mathrm{d}}, \Delta^*)$-coloring of G' at least two of the vertices of $\{v_1, v_2, v_3\}$ must share a color. Furthermore, if there exists a $(\chi_{\mathrm{d}}, \Delta^*)$-coloring of G that gives the same color to two of the vertices of $\{v_1, v_2, v_3\}$, this coloring can be extended to a $(\chi_{\mathrm{d}}, \Delta^*)$-coloring of G'.*

▶ **Lemma 10.** *Let $G = (V, E)$ be a graph, $S \subseteq V$, and G' be a graph obtained from G by repeated applications of the following operation: we select three vertices $v_1, v_2, v_3 \in V$ such that $v_1, v_2 \in S$, add a new copy of $P(u_1, u_2, u_3, \chi_{\mathrm{d}}, \Delta^*)$ and identify u_i with v_i, for $i \in \{1, 2, 3\}$. Then $\mathrm{td}(G') \leq \mathrm{td}(G \setminus S) + |S| + \chi_{\mathrm{d}} - 2$.*

3.2 Construction

We are now ready to present a reduction from k-Multi-Colored Clique. In this section we describe a construction which, given an instance of this problem (G, k) as well as an integer $\chi_{\mathrm{d}} \geq 2$ produces an instance of Defective Coloring. Recall that we assume that in the initial instance $G = (V, E)$ is given to us partitioned into k independent sets $V_1, \ldots, V_k$, all of which have size n. We will produce a graph $H(G, k, \chi_{\mathrm{d}})$ and an integer Δ^* with the property that H admits a $(\chi_{\mathrm{d}}, \Delta^*)$-coloring if and only if G has a k-clique. In the next section we prove the correctness of the construction and give bounds on the values of $\mathrm{td}(H)$ and $\mathrm{fvs}(H)$ to establish Theorem 2.

In our new instance we set $\Delta^* = |E| - \binom{k}{2}$. Let us now describe the graph H. Since we will repeatedly use the gadgets from Definitions 5 and 8, we will use the following convention: whenever v_1, v_2 are two vertices we have already introduced to H, when we say that we add an equality gadget $Q(v_1, v_2)$, this means that we add to H a copy of $Q(u_1, u_2, \chi_{\mathrm{d}}, \Delta^*)$ and then identify u_1, u_2 with v_1, v_2 respectively (similarly for palette gadgets). To ease presentation we will gradually build the graph by describing its different conceptual parts.

Palette Part. Informally, the goal of this part is to obtain two vertices (p_A, p_B) which are guaranteed to have different colors. This part contains the following:

1. Two vertices called p_A, p_B which we will call the main palette vertices.
2. For all $i \in \{1, \ldots, \Delta^*\}$, $j \in \{A, B\}$ a vertex p_j^i.
3. For all $i \in \{1, \ldots, \Delta^*\}$, $j \in \{A, B\}$ we add an equality gadget $Q(p_j, p_j^i)$.
4. An edge between p_A, p_B.
5. For all $i \in \{1, \ldots, \Delta^*\}$, $j \in \{A, B\}$ an edge from p_j to p_j^i.

Choice Part. Informally, the goal of this part is to encode a choice of a vertex in each V_i. To this end we make $2n$ choice vertices for each color class of the original instance. The selection will be encoded by counting how many of the first n of these vertices have the same color as p_A. Formally, this part contains the following:

6. For all $i \in \{1, \ldots, k\}$, $j \in \{1, \ldots, 2n\}$ the vertex c_j^i. We call these the choice vertices.
7. For all $i \in \{1, \ldots, k\}$, $j \in \{A, B\}$ the vertex g_j^i. We call these the guard vertices.
8. For all $i \in \{1, \ldots, k\}$, $j \in \{1, \ldots, 2n\}$ edges between c_j^i and the vertices g_A^i and g_B^i.
9. For all $i \in \{1, \ldots, k\}$, $j \in \{A, B\}$ we add an equality gadget $Q(p_j, g_j^i)$.
10. If $\chi_{\mathrm{d}} \geq 3$, for all $i \in \{1, \ldots, k\}$, $j \in \{1, \ldots, 2n\}$ we add a palette gadget $P(p_A, p_B, c_j^i)$.

Transfer Part. Informally, the goal of this part is to transfer the choices of the previous part to the rest of the graph. For each color class of the original instance we make $(k-1)$ "low" transfer vertices, whose deficiency will equal the choice made in the previous part, and $(k-1)$ "high" transfer vertices, whose deficiency will equal the complement of the same value. Formally, this part of H contains the following:

11. For $i, j \in \{1, \ldots, k\}$, $i \neq j$ the vertex $h_{i,j}$ and the vertex $l_{i,j}$. We call these the high and low transfer vertices.
12. For $i, j \in \{1, \ldots, k\}$, $i \neq j$ and for all $l \in \{1, \ldots, n\}$ an edge from $l_{i,j}$ to c_l^i.
13. For $i, j \in \{1, \ldots, k\}$, $i \neq j$ and for all $l \in \{n+1, \ldots, 2n\}$ an edge from $h_{i,j}$ to c_l^i.
14. For all $i, j \in \{1, \ldots, k\}$, $i \neq j$ we add an equality gadget $Q(p_A, l_{i,j})$ and an equality gadget $Q(p_A, h_{i,j})$.

Edge representation. Informally, this part contains a gadget representing each edge of G. Each gadget will contain a special vertex which will be able to receive the color of p_B if and only if the corresponding edge is part of the clique. Formally, we assume that all the vertices of each V_i are numbered $\{1, \ldots, n\}$. For each edge e of G, if e connects the vertex with index i_1 from V_{j_1} with the vertex with index i_2 from V_{j_2} (assuming without loss of generality $j_1 < j_2$) we add the following vertices and edges to H:

15. Four independent sets $L_e^1, H_e^1, L_e^2, H_e^2$ with respective sizes $n - i_1$, i_1, $n - i_2$, i_2.
16. Edges connecting the vertex l_{j_1,j_2} (respectively, $h_{j_1,j_2}, l_{j_2,j_1}, h_{j_2,j_1}$) with all vertices of the set L_e^1 (respectively the sets H_e^1, L_e^2, H_e^2).
17. A vertex c_e, connected to all vertices in $L_e^1 \cup H_e^1 \cup L_e^2 \cup H_e^2$.
18. If $\chi_{\mathrm{d}} \geq 3$, for each $v \in L_e^1 \cup H_e^1 \cup L_e^2 \cup H_e^2 \cup \{c_e\}$ we add a palette gadget $P(p_A, p_B, v)$.

Finally, once we have added a gadget (as described above) for each $e \in E$, we add the following structure to H in order to ensure that we have a sufficient number of edges included in our clique:

19. A vertex c_U (universal checker) connected to all c_e for $e \in E$.
20. An equality gadget $Q(p_A, c_U)$.

Budget-Setting. Our construction is now almost done, except for the fact that some crucial vertices have degree significantly lower than Δ^* (and hence are always trivially colorable). To fix this, we will effectively lower their deficiency budget by giving them some extra neighbors. Formally, we add the following:

21. For each guard vertex g_j^i, with $j \in \{A, B\}$, we construct an independent set G_j^i of size $\Delta^* - n$ and connect it to g_j^i. For each $v \in G_j^i$ we add an equality gadget $Q(p_j, v)$.
22. For each transfer vertex $l_{i,j}$ (respectively $h_{i,j}$), we construct an independent set of size $\Delta^* - n$ and connect all its vertices to $l_{i,j}$ (or respectively to $h_{i,j}$). For each vertex v of this independent set we add an equality gadget $Q(p_A, v)$.
23. For each vertex c_e we add an independent set of size Δ^* and connect all its vertices to c_e. For each vertex v of this independent set we add an equality gadget $Q(p_B, v)$.

This completes the construction of the graph H.

3.3 Correctness

To establish Theorem 2 we need to establish three properties of the graph $H(G, k, \chi_\mathrm{d})$ described in the preceding section: that the existence of a k-clique in G implies that H admits a $(\chi_\mathrm{d}, \Delta^*)$-coloring; that a $(\chi_\mathrm{d}, \Delta^*)$-coloring of H implies the existence of a k-clique in G; and that the tree-depth and feedback vertex set of G are bounded by some function of k. These are established in the Lemmata below.

▶ **Lemma 11.** *For any $\chi_\mathrm{d} \geq 2$, if G contains a k-clique, then the graph $H(G, k, \chi_\mathrm{d})$ described in the previous section admits a $(\chi_\mathrm{d}, \Delta^*)$-coloring.*

Proof. Consider a clique of size k in G that includes exactly one vertex from each V_i. We will denote this clique by a function $f : \{1, \ldots, k\} \to \{1, \ldots, n\}$, that is, we assume that the clique contains the vertex with index $f(i)$ from V_i. We produce a $(\chi_\mathrm{d}, \Delta^*)$-coloring of H as follows: vertex p_A receives color 1, while vertex p_B receives color 2. All vertices for which we have added an equality gadget with one endpoint identified with p_A (respectively p_B) take color 1 (respectively 2). We use Lemma 6 to properly color the internal vertices of the equality gadgets.

We have still left uncolored the choice vertices c_j^i as well as the internal vertices $L_e^1, H_e^1, L_e^2, H_e^2, c_e$ of the edge gadgets. We proceed as follows: for all $i \in \{1, \ldots, k\}$ we use color 1 on the vertices c_l^i such that $l \in \{1, \ldots, f(i)\} \cup \{n+1, \ldots, 2n - f(i)\}$; we use color 2 on all remaining choice vertices. For every $e \in E$ that is contained in the clique we color all vertices of the sets $L_e^1, H_e^1, L_e^2, H_e^2$ with color 1, and c_e with color 2. For all other edges we use the opposite coloring: we color all vertices of the sets $L_e^1, H_e^1, L_e^2, H_e^2$ with color 2, and c_e with color 1. We use Lemma 9 to properly color the internal vertices of palette gadgets, since all palette gadgets that we add use either color 1 or color 2 twice in their endpoints. This completes the coloring.

To see that the coloring we described is a $(\chi_\mathrm{d}, \Delta^*)$-coloring, first we note that by Lemmata 6,9 internal vertices of equality and palette gadgets are properly colored. Vertices p_A, p_B have exactly Δ^* neighbors with the same color; guard vertices g_j^i have exactly n neighbors with the same color among the choice vertices, hence exactly Δ^* neighbors with the same color overall; choice vertices have at most k neighbors of the same color, and we can assume that $k < |E| - \binom{k}{2}$; the vertex c_U has exactly $\Delta^* = |E| - \binom{k}{2}$ neighbors with color 1, since the clique contains exactly $\binom{k}{2}$ edges; all internal vertices of edge gadgets have at most one neighbor of the same color. Finally, for the transfer vertices $l_{i,j}$ and $h_{i,j}$, we note that $l_{i,j}$ (respectively $h_{i,j}$) has exactly $f(i)$ (respectively $n - f(i)$) neighbors with color 1 among the choice vertices. Furthermore, when $i < j$, $l_{i,j}$ (respectively $h_{i,j}$) has $|L_e^1|$ (respectively $|H_e^1|$) neighbors with color 1 in the edge gadgets, those corresponding to the edge e that belongs in the clique between V_i and V_j. But by construction $|L_e^1| = n - f(i)$ and $|H_e^1| = f(i)$, and with similar observations for the case $j < i$ we conclude that all vertices have deficiency at most Δ^*. ◀

▶ **Lemma 12.** *For any $\chi_{\mathrm{d}} \geq 2$, if the graph $H(G, k, \chi_{\mathrm{d}})$ described in the previous section admits a $(\chi_{\mathrm{d}}, \Delta^*)$-coloring, then G contains a k-clique.*

▶ **Lemma 13.** *For any $\chi_{\mathrm{d}} \geq 2$, the graph $H(G, k, \chi_{\mathrm{d}})$ described in the previous section has $\mathrm{td}(H) = O(k^2 + \chi_{\mathrm{d}})$. Furthermore, if $\chi_{\mathrm{d}} = 2$, then $\mathrm{fvs}(H) = O(k^2)$.*

Theorem 2 now follows directly from the reduction we have described and Lemmata 11,12,13.

4 ETH-based Lower Bounds for Treewidth and Pathwidth

In this section we present a reduction which strengthens the results of Section 3 for the parameters treewidth and pathwidth. In particular, the reduction we present here establishes that, under the ETH, the known algorithm for Defective Coloring for these parameters is essentially best possible.

We use a similar presentation order as in the previous section, first giving the construction and then the Lemmata that imply the result. Where possible, we re-use the gadgets we have already presented. The main theorem of this section states the following:

▶ **Theorem 14.** *For any fixed $\chi_{\mathrm{d}} \geq 2$, if there exists an algorithm which, given a graph $G = (V, E)$ and parameters $\chi_{\mathrm{d}}, \Delta^*$ decides if G admits a $(\chi_{\mathrm{d}}, \Delta^*)$-coloring in time $n^{o(\mathrm{pw})}$, then the ETH is false.*

4.1 Basic Gadgets

We use again the equality and palette gadgets of Section 3 (Definitions 5,8). Before proceeding, let us show that adding these gadgets to the graph does not increase the pathwidth too much. For the two types of gadget Q, P, we will call the vertices $u_1, u_2 (, u_3)$ the endpoints of the gadget.

▶ **Lemma 15.** *Let $G = (V, E)$ be a graph and let G' be the graph obtained from G by repeating the following operation: find a copy of $Q(u_1, u_2, \chi_{\mathrm{d}}, \Delta^*)$, or $P(u_1, u_2, u_3, \chi_{\mathrm{d}}, \Delta^*)$; remove all its internal vertices from the graph; and add all edges between its endpoints which are not already connected. Then $\mathrm{tw}(G) \leq \max\{\mathrm{tw}(G'), \chi_{\mathrm{d}}\}$ and $\mathrm{pw}(G) \leq \mathrm{pw}(G') + \chi_{\mathrm{d}}$.*

4.2 Construction

We now describe a construction which, given an instance $G = (V, E)$, k, of k-Multi-Colored Clique and a constant χ_{d} returns a graph $H(G, k, \chi_{\mathrm{d}})$ and an integer Δ^* such that H admits a $(\chi_\mathrm{d}, \Delta^*)$-coloring if and only if G has a k-clique, and the pathwidth of H is $O(k + \chi_\mathrm{d})$. We use m to denote $|E|$, and we set $\Delta^* = m - \binom{k}{2}$. As in Section 3 we present the construction in steps to ease presentation, and we use the same conventions regarding adding Q and P gadgets to the graph.

Palette Part. This part repeats steps 1-5 of the construction of Section 3. We recall that this creates two main palette vertices p_A, p_B (which are eventually guaranteed to have different colors).

Choice Part. In this part we construct a sequence of independent sets, arranged in what can be thought of as a $k \times 2m$ grid. The idea is that the choice we make in coloring the first independent set of every row will be propagated throughout the row. We can therefore encode k choices of a number between 1 and n, which will encode the clique.

6. For each $i \in \{1, \dots, k\}$, for each $j \in \{1, \dots, 2m\}$ we construct an independent set $C_{i,j}$ of size n.
7. (Backbone vertices) For each $i \in \{1, \dots, k\}$, for each $j \in \{1, \dots, 2m-1\}$, for each $l \in \{A.B\}$ we construct a vertex $b^l_{i,j}$. We connect $b^l_{i,j}$ to all vertices of $C_{i,j}$ and all vertices of $C_{i,j+1}$.
8. For each backbone vertex $b^l_{i,j}$ added in the previous step, for $l \in \{A, B\}$, we add an equality gadget $Q(p_l, b^l_{i,j})$.

Edge Representation. In the $k \times 2m$ grid of independent sets we have constructed we devote two columns to represent each edge of G. In the remainder we assume some numbering of the edges of E with the numbers $\{1, \dots, m\}$, as well as a numbering of each V_i with the numbers $\{1, \dots, n\}$. Suppose that the j-th edge of E, where $j \in \{1, \dots, m\}$ connects the j_1-th vertex of V_{i_1} to the j_2-th vertex of V_{i_2}, where $j_1, j_2 \in \{1, \dots, n\}$ and $i_1, i_2 \in \{1, \dots, k\}$. We perform the following steps for each such edge.

9. We construct four independent sets $H^1_j, L^1_j, H^2_j, L^2_j$ with respective sizes $n - j_1, j_1, n - j_2, j_2$.
10. We construct four vertices $h^1_j, l^1_j, h^2_j, l^2_j$. We connect h^1_j (respectively l^1_j, h^2_j, l^2_j) with all vertices of H^1_j (respectively L^1_j, H^2_j, L^2_j).
11. We connect h^1_j to all vertices of $C_{i_1,2j-1}$, l^1_j to all vertices of $C_{i_1,2j}$, h^2_j to all vertices of $C_{i_2,2j-1}$, l^2_j to all vertices of $C_{i_2,2j}$.
12. We add equality gadgets $Q(p_A, h^1_j), Q(p_A, l^1_j), Q(p_A, h^2_j), Q(p_A, l^2_j)$.
13. We add a checker vertex c_j and connect it to all vertices of $H^1_j \cup L^1_j \cup H^2_j \cup L^2_j$.

Validation and Budget-Setting. Finally, we add a vertex that counts how many edges we have included in our clique, as well as appropriate vertices to diminish the deficiency budget of various parts of our construction.

14. We add a universal checker vertex c_U and connect it to all vertices c_j added in step 13. We add an equality gadget $Q(p_A, c_U)$.
15. For every vertex c_j added in step 13 we construct an independent set of size Δ^* and connect all its vertices to c_j. For each vertex v in this set we add an equality gadget $Q(p_B, v)$.
16. For each vertex constructed in step 10 $(h^1_j, l^1_j, h^2_j, l^2_j)$, we construct an independent set of size $\Delta^* - n$ and connect it to the vertex. For each vertex v of this independent set we add an equality gadget $Q(p_A, v)$.

17. For each backbone vertex $b^l_{i,j}$, with $l \in \{A, B\}$, we construct an independent set of size $\Delta^* - n$ and connect it to $b^l_{i,j}$. For each vertex v of this independent set we add an equality gadget $Q(p_l, v)$.
18. If $\chi_{\mathrm{d}} \geq 3$, for each vertex v added in steps 6-17 we add a palette gadget $P(p_A, p_B, v)$.

4.3 Correctness

▶ **Lemma 16.** *For any $\chi_{\mathrm{d}} \geq 2$, if G contains a k-clique then the graph $H(G, k, \chi_{\mathrm{d}})$ described in the previous section admits a $(\chi_{\mathrm{d}}, \Delta^*)$-coloring.*

▶ **Lemma 17.** *For any $\chi_{\mathrm{d}} \geq 2$, if the graph $H(G, k, \chi_{\mathrm{d}})$ described in the previous section admits a $(\chi_{\mathrm{d}}, \Delta^*)$-coloring, then G contains a k-clique.*

▶ **Lemma 18.** *For the graph $H(G, k, \chi_{\mathrm{d}})$ described in the previous section* $\mathrm{pw}(H) = O(k+\chi_{\mathrm{d}})$.

The proof of Theorem 14 now follows directly from Lemmata 16,17,18.

5 Exact Algorithms for Treewidth and Other Parameters

In this section we present several exact algorithms for Defective Coloring. Theorem 19 gives a treewidth-based algorithm which can be obtained using standard techniques. Essentially the same algorithm was already sketched in [9], but we give another version here for the sake of completeness and because it is a building block for the approximation algorithm of Theorem 23. Theorem 20 uses a win/win argument to show that the problem is FPT parameterized by fvs when $\chi_{\mathrm{d}} \neq 2$ and therefore explains why the reduction presented in Section 3 only works for 2 colors. Theorem 21 uses a similar argument to show that the problem is FPT parameterized by vc (for any χ_{d}).

▶ **Theorem 19.** *There is an algorithm which, given a graph $G = (V, E)$, parameters $\chi_{\mathrm{d}}, \Delta^*$, and a tree decomposition of G of width* tw*, decides if G admits a $(\chi_{\mathrm{d}}, \Delta^*)$-coloring in time $(\chi_{\mathrm{d}}\Delta^*)^{O(\mathrm{tw})}n^{O(1)}$.*

▶ **Theorem 20.** Defective Coloring *is FPT parameterized by* fvs *for $\chi_{\mathrm{d}} \neq 2$. More precisely, there exists an algorithm which given a graph $G = (V, E)$, parameters $\chi_{\mathrm{d}}, \Delta^*$, with $\chi_{\mathrm{d}} \neq 2$, and a feedback vertex set of G of size* fvs*, decides if G admits a $(\chi_{\mathrm{d}}, \Delta^*)$-coloring in time $\mathrm{fvs}^{O(\mathrm{fvs})}n^{O(1)}$.*

▶ **Theorem 21.** Defective Coloring *is FPT parameterized by* vc*. More precisely, there exists an algorithm which, given a graph $G = (V, E)$, parameters $\chi_{\mathrm{d}}, \Delta^*$, and a vertex cover of G of size* vc*, decides if G admits a $(\chi_{\mathrm{d}}, \Delta^*)$-coloring in time $\mathrm{vc}^{O(\mathrm{vc})}n^{O(1)}$.*

6 Approximation Algorithms and Lower Bounds

In this section we present two approximation algorithms which run in FPT time parameterized by treewidth. The first algorithm (Theorem 23) is an *FPT approximation scheme* which, given a desired number of colors χ_{d}, is able to approximate the minimum feasible value of Δ^* for this value of χ_{d} arbitrarily well (that is, within a factor $(1 + \epsilon)$). The second algorithm, which also runs in FPT time parameterized by treewidth, given a desired value for Δ^*, produces a solution that approximates the minimum number of colors χ_{d} within a factor of 2.

These results raise the question of whether it is possible to approximate χ_{d} as well as we can approximate Δ^*, that is, whether there exists an algorithm which comes within a factor $(1+\epsilon)$ (rather than 2) of the optimal number of colors. As a first response, one could observe that such an algorithm probably cannot exist, because the problem is already hard when $\chi_\mathrm{d}=2$, and therefore an FPT algorithm with multiplicative error less than $3/2$ would imply that FPT=W[1]. However, this does not satisfactorily settle the problem as it does not rule out an algorithm that achieves a much better approximation ratio, if we allow it to also have a small additive error in the number of colors. Indeed, as we observe in Corollary 28, it is possible to obtain an algorithm which runs in FPT time parameterized by feedback vertex set and has an additive error of only 1, as a consequence of the fact that the problem is FPT for $\chi_\mathrm{d} \geq 3$. This poses the question of whether we can design an FPT algorithm parameterized by treewidth which, given a $(\chi_\mathrm{d}, \Delta^*)$-colorable graph, produces a coloring with $\rho\chi_\mathrm{d} + O(1)$ colors, for $\rho < 3/2$.

In the second part of this section we settle this question negatively by showing, using a recursive construction that builds on Theorem 2, that such an algorithm cannot exist. More precisely, we present a gap-introducing version of our reduction: the ratio between the number of colors needed to color Yes and No instances remains $3/2$, even as the given χ_d increases. This shows that the "correct" multiplicative approximation ratio for this problem really lies somewhere between $3/2$ and 2, or in other words, that there are significant barriers impeding the design of a better than $3/2$ FPT approximation for χ_d, beyond the simple fact that 2-coloring is hard.

6.1 Approximation Algorithms

Our first approximation algorithm, which is an approximation scheme for the optimal value of Δ^*, relies on a method introduced in [36] (see also [3]), and a theorem of [11]. The high-level idea is the following: intuitively, the obstacle that stops us from obtaining an FPT running time with the dynamic programming algorithm of Theorem 19 is that the dynamic program is forced to store some potentially large values for each vertex. More specifically, to characterize a partial solution we need to remember not just the color of each vertex in a bag, but also how many neighbors with the same color this vertex has already seen (which is a value that can go up to Δ^*). The main trick now is to "round" these values in order to decrease the number of possible states a vertex can be found in. To do this, we select an appropriate value δ (polynomial in $\frac{\epsilon}{\log n}$), and try to replace every value that the dynamic program would calculate with the next higher integer power of $(1+\delta)$. This has the advantage of limiting the number of possible values from Δ^* to $\log_{(1+\delta)} \Delta^* \approx \frac{\log \Delta^*}{\delta}$, and this is sufficient to obtain the promised running time. The problem is now that the rounding we applied introduces an approximation error, which is initially a factor of at most $(1+\delta)$, but may increase each time we apply an arithmetic operation as part of the algorithm. To show that this error does not get out of control we show that in any bag of the tree all values stored are within a factor $(1+\delta)^h$ of the correct ones, where h is the height of the bag. We then use a theorem of Bodlaender and Hagerup [11] which states that any tree decomposition can be balanced in such a way that its height is at most $O(\log n)$, and as a result we obtain that all values are sufficiently close to being correct.

The second algorithm we present in this section (Theorem 25) uses the approximation scheme for Δ^* to obtain an FPT 2-approximation for χ_d. The idea here is that, given a $(\chi_\mathrm{d}, \Delta^*)$-colorable graph, we first produce a $(\chi_\mathrm{d}, (1+\epsilon)\Delta^*)$-coloring using the algorithm of Theorem 23, and then apply a procedure which uses 2 colors for each color class of this solution but manages to divide by two the number of neighbors with the same color of every vertex. This is achieved with a simple polynomial-time local search procedure.

▶ **Theorem 22.** *[11] There is a polynomial-time algorithm which, given a graph $G = (V, E)$ and a tree decomposition of G of width tw, produces a tree decomposition of G of width at most $3\mathrm{tw} + 2$ and height $O(\log n)$.*

▶ **Theorem 23.** *There is an algorithm which, given a graph $G = (V, E)$, parameters $\chi_\mathrm{d}, \Delta^*$, a tree decomposition of G of width tw, and an error parameter $\epsilon > 0$, either returns a $(\chi_\mathrm{d}, (1 + \epsilon)\Delta^*)$-coloring of G, or correctly concludes that G does not admit a $(\chi_\mathrm{d}, \Delta^*)$-coloring, in time $(\mathrm{tw}/\epsilon)^{O(\mathrm{tw})} n^{O(1)}$.*

▶ **Lemma 24.** *There exists a polynomial-time algorithm which, given a graph with maximum degree Δ, produces a two-coloring of that graph where all vertices have at most $\Delta/2$ neighbors of the same color.*

▶ **Theorem 25.** *There is an algorithm which, given a graph $G = (V, E)$, parameters $\chi_\mathrm{d}, \Delta^*$, and a tree decomposition of G of width tw, either returns a $(2\chi_\mathrm{d}, \Delta^*)$-coloring of G, or correctly concludes that G does not admit a $(\chi_\mathrm{d}, \Delta^*)$-coloring, in time $(\mathrm{tw})^{O(\mathrm{tw})} n^{O(1)}$.*

6.2 Hardness of Approximation

The main result of this section is that χ_d cannot be approximated with a factor better than 3/2 in FPT time (for parameters tree-depth, pathwidth, or treewidth), even if we allow the algorithm to also have a constant additive error. We remark that an FPT algorithm with additive error 1 is easy to obtain for feedback vertex set (Corollary 28).

▶ **Theorem 26.** *For any fixed $\chi_\mathrm{d} > 0$, if there exists an algorithm which, given a graph $G = (V, E)$ and a $\Delta^* \geq 0$, correctly distinguishes between the case that G admits a $(2\chi_\mathrm{d}, \Delta^*)$-coloring, and the case that G does not admit a $(3\chi_\mathrm{d} - 1, \Delta^*)$-coloring in FPT time parameterized by $\mathrm{td}(G)$, then FPT=W[1].*

▶ **Corollary 27.** *For any constants $\delta_1, \delta_2 > 0$, if there exists an algorithm which, given a graph $G = (V, E)$ that admits a $(\chi_\mathrm{d}, \Delta^*)$-coloring and parameters $\chi_\mathrm{d}, \Delta^*$, is able to produce a $((\frac{3}{2} - \delta_1)\chi_\mathrm{d} + \delta_2, \Delta^*)$-coloring of G in FPT time parameterized by $\mathrm{td}(G)$, then FPT=W[1].*

▶ **Corollary 28.** *There is an algorithm which, given a graph $G = (V, E)$, parameters $\chi_\mathrm{d}, \Delta^*$, and a feedback vertex set of G of size fvs, either returns a $(\chi_\mathrm{d} + 1, \Delta^*)$-coloring of G, or correctly concludes that G does not admit a $(\chi_\mathrm{d}, \Delta^*)$-coloring, in time $(\mathrm{fvs})^{O(\mathrm{fvs})} n^{O(1)}$.*

7 Conclusions

In this paper we classified the complexity of Defective Coloring with respect to some of the most well-studied graph parameters, given essentially tight ETH-based lower bounds for pathwidth and treewidth, and explored the parameterized approximability of the problem. Though this gives a good first overview of the problem's parameterized complexity landscape, there are several questions worth investigating next. First, is it possible to make the lower bounds of Section 4 even tighter, by precisely determining the base of the exponent in the algorithm's dependence? This would presumably rely on a stronger complexity assumption such as the SETH, as in [37]. Second, can we determine the complexity of the problem with respect to other structural parameters, such as clique-width [15], modular-width [24], or neighborhood diversity [35]? For some of these parameters the existence of FPT algorithms is already ruled out by the fact that Defective Coloring is NP-hard on cographs [9], however the complexity of the problem is unknown if we also add χ_d or Δ^* as a parameter. Finally, it would be very interesting to close the gap between 2 and 3/2 on the performance of the best treewidth-parameterized FPT approximation for χ_d.

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