Popular Matchings with Lower Quotas*

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Abstract

We consider the well-studied Hospital Residents (HR) problem in the presence of lower quotas (LQ). The input instance consists of a bipartite graph $G = (\mathcal{R} \cup \mathcal{H}, E)$ where $\mathcal{R}$ and $\mathcal{H}$ denote sets of residents and hospitals, respectively. Every vertex has a preference list that imposes a strict ordering on its neighbors. In addition, each hospital h has an associated upper-quota $q^+(h)$ and a lower-quota $q^-(h)$. A matching M in G is an assignment of residents to hospitals, and M is said to be *feasible* if every resident is assigned to at most one hospital and a hospital h is assigned at least $q^-(h)$ and at most $q^+(h)$ residents.

Stability is a de-facto notion of optimality in a model where both sets of vertices have preferences. A matching is *stable* if no unassigned pair has an incentive to deviate from it. It is well-known that an instance of the HRLQ problem need not admit a feasible stable matching. In this paper, we consider the notion of popularity for the HRLQ problem. A matching M is *popular* if no other matching M' gets more votes than M when vertices vote between M and M'. When there are no lower quotas, there always exists a stable matching and it is known that every stable matching is popular.

We show that in an HRLQ instance, although a feasible stable matching need not exist, there is always a matching that is popular in the set of feasible matchings. We give an efficient algorithm to compute a maximum cardinality matching that is popular amongst all the feasible matchings in an HRLQ instance.

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1 Introduction

In this paper we consider the Hospital Residents problem in the presence of Lower Quotas (HRLQ). The input to our problem is a bipartite graph $G = (\mathcal{R} \cup \mathcal{H}, E)$ where $\mathcal{R}$ denotes the set of residents, and $\mathcal{H}$ denotes the set of hospitals. Every resident as well as hospital has a non-empty preference ordering over a subset of elements of the other set. Every hospital $h \in \mathcal{H}$ has a non-zero upper-quota $q^+(h)$ denoting the maximum number of residents that can be assigned to h. In addition, every hospital h also has a non-negative lower-quota $q^-(h)$ denoting the minimum number of residents that have to be assigned to h. The goal is to assign residents to hospitals such that the upper and lower quotas of all the hospitals are respected (that is, it is feasible) as well as the assignment is *optimal* with respect to the preferences of the participants.

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▶ **Definition 1.** A feasible matching M in $G = (\mathcal{R} \cup \mathcal{H}, E)$ is a subset of E such that $|M(r)| \leq 1$ for each $r \in \mathcal{R}$ and $q^-(h) \leq |M(h)| \leq q^+(h)$ for each $h \in \mathcal{H}$, where $M(v)$ is the set of neighbors of v in M.

Stability is a de-facto notion of optimality in settings where both sides have preferences. A matching M (not necessarily feasible) is said to be stable if there is no *blocking pair* with respect to M. A resident-hospital pair (r, h) blocks M if r is unmatched in M or prefers h over $M(r)$, and either $|M(h)| < q^+(h)$ or h prefers r over at least one resident in $M(h)$.

There are simple instances of the HRLQ problem where there is no feasible matching that is stable. We give an example here: Let $\mathcal{R} = \{r\}, \mathcal{H} = \{h_1, h_2\}$, $q^+(h_1) = q^+(h_2) = 1$, $q^-(h_1) = 0$, and $q^-(h_2) = 1$. Let preference list of r be $\langle h_1, h_2 \rangle$. That is, r prefers h_1 over h_2. The only stable matching here is $M_1 = \{(r, h_1)\}$ which is not feasible as $|M_1(h_2)| < q^-(h_2)$. On the other hand, the only feasible matching $M_2 = \{(r, h_2)\}$ is not stable as (r, h_1) is a blocking pair with respect to M_2. This raises the question: given an HRLQ instance G, does G admit a feasible stable matching? This can be answered by constructing an HR instance G^+ by disregarding the lower quotas of all hospitals in G. It is well-known that the Gale-Shapley algorithm [5] computes a stable matching M in G^+. Furthermore, from the "Rural Hospitals Theorem" it is known that, in every stable matching of G^+, each hospital is matched to the same capacity [6, 17]. Thus G admits a stable feasible matching if and only if M is feasible for G.

The HRLQ problem is motivated by practical scenarios like assigning medical interns (residents) to hospitals. While matching residents to hospitals, rural hospitals often face the problem of being understaffed with residents, for example the National Resident Matching Program in the US [3, 16, 17]. In such real-world applications declaring that there is no feasible stable matching is simply not a solution. On the other hand, any feasible matching that disregards the preference lists completely is socially unacceptable. We address this issue by relaxing the requirement of stability by an alternative notion of optimality namely *popularity*. Our output matching M has two desirable criteria – firstly, it is a feasible matching in the instance, assuming one such exists, and hence no hospital remains understaffed. Secondly, the matching respects preferences of the participants, in particular, no majority of participants wishes to deviate to another feasible matching in the instance.

Our contribution: We consider the notion of *popularity* for the HRLQ problem. Popularity is a relaxation of stability and can be interpreted as *overall stability*. We define it formally in Section 2. In this work, we present an efficient algorithm for the following two problems in an HRLQ instance.

1. Computing a maximum cardinality matching popular in the set of feasible matchings. We give an $O(|\mathcal{R}| \cdot (|\mathcal{R}| + |\mathcal{H}| + |E|))$ time algorithm for this problem.
2. Computing a popular matching amongst maximum cardinality feasible matchings. We give an $O(|\mathcal{R}|^2 \cdot (|\mathcal{R}| + |\mathcal{H}| + |E|))$ time algorithm for this problem.

Our algorithms are based on ideas introduced in earlier works on stable marriage (SM) and HR problems[11, 3, 15]. In SM and HR problem, a popular matching is guaranteed to exist because a stable matching always exists and it is also popular. On the other hand, in the HRLQ setting even a stable matching may not exist. Yet, we prove that a feasible matching that is popular amongst all feasible matchings always exists and is efficiently computable. We believe that this is not only surprising but also a useful result in practical scenarios. Moreover, our notion of popularity subsumes the notions proposed in [3] and [15] and is more general than both. In [3], popularity is proved using linear programming, but our proofs for popularity are combinatorial.

Overview of the algorithm: Our algorithms are reductions, that is, given an HRLQ instance G, both our algorithms construct instances G' and G'' of the HR problem such that there is a natural way to map a stable matching in G' (respectively, G'') to a feasible matching in G. Moreover, any stable matching in G' (G'') gets mapped to a maximum cardinality matching that is popular amongst all the feasible matchings in G (respectively, a matching that is popular amongst all maximum cardinality matchings in G).

Organization of the paper: We define the notion of popularity in Section 2. The reduction for computing a maximum cardinality popular matching amongst feasible matchings is given in Section 3 and its correctness is proved in Section 4. Finally, Section 5 gives an overview of the algorithm for computing a feasible matching that is popular amongst maximum cardinality feasible matchings.

Related work: The notion of popularity was first proposed by Gärdenfors [7] in the stable-marriage (SM) setting, where each vertex has capacity 1. The notion of popularity has been well-studied since then [2, 10, 11, 9, 4, 12]. A linear-time algorithm to compute a maximum cardinality popular matching in an HR instance is given in [3] and [15] with different notions of popularity. Furthermore, for the SM and HR problem, it is known that a matching that is *popular amongst the maximum cardinality matchings* exists and can be computed in $O(|E|(|\mathcal{R}| + |\mathcal{H}|))$ time [11, 15]. The reductions in our paper are inspired by the work of [3, 4, 11, 15]. In all these earlier works, the main idea is to execute Gale-Shapley algorithm on the HR instance and then allow unmatched residents to propose with *increased priority* [11] certain number of times. As mentioned in [11], this idea was first proposed by Király [13] in the context of approximation algorithms for the SM problem with ties. The HRLQ problem has been recently considered in [1] and [8] in different settings. Very recently, Yokoi [18] considered the notion of *envy free* matchings for the HRLQ problem. Similar to popularity, envy-freeness is also a relaxation of stability. However, unlike popular matchings, every instance of the HRLQ problem need not admit an envy free matching.

2 Notion of popularity

The notion of popularity uses votes from vertices to compare two matchings. For $r \in \mathcal{R}$, and any matching M in G, if r is unmatched in M then, $M(r) = \bot$. A vertex prefers any of its neighbours over $\bot$. For a vertex $u \in \mathcal{R} \cup \mathcal{H}$, let $x, y \in N(u) \cup \{\bot\}$, where $N(u)$ denotes the neighbours of u in G. We define $vote_u(x, y) = 1$ if u prefers x over y, -1 if u prefers y over x and 0 if $x = y$. Given two matchings M_1 and M_2 in the instance, for a resident $r \in \mathcal{R}$, we define $vote_r(M_1, M_2) = vote_r(M_1(r), M_2(r))$.

Voting for a hospital: A hospital h is assigned $q^+(h)$-many votes to compare M_1 and M_2; this is one vote per position of the hospital. If a position is not filled in a matching, we put a $\bot$ there, so that $|M_1(h)| = |M_2(h)| = q^+(h)$.

In our voting scheme, hospital h is indifferent between M_1 and M_2 as far as its $|M_1(h) \cap M_2(h)|$ positions are concerned. To compare between the two sets $M_1(h) \setminus M_2(h)$ and $M_2(h) \setminus M_1(h)$, a hospital can decide any pairing of the elements of these two sets. We denote this correspondence by $\mathbf{corr}_h$ and call it the *correspondence function of* h. Note that $\mathbf{corr}_h$ is dependent on M_1 and M_2. Under this correspondence, for a resident $r \in M_1(h) \setminus M_2(h)$,

$\mathbf{corr}_h(r, M_1, M_2)$ is the resident in $M_2(h) \setminus M_1(h)$ corresponding to r. We define

$$vote_h(M_1, M_2, \mathbf{corr}_h) \quad = \sum_{r \in M_1(h) \setminus M_2(h)} vote_h(r, \mathbf{corr}_h(r, M_1, M_2))$$

A hospital h prefers M_1 over M_2 under $\mathbf{corr}_h$ if $vote_h(M_1, M_2, \mathbf{corr}_h) > 0$. There are several ways for a hospital to define the $\mathbf{corr}_h$ function. For example, a hospital h may decide to order and compare the two sets in the decreasing order of preferences (as in [15]) or in the *most adversarial order* (as in [3]). That is, the order due to which h gives the least votes to M_1 when comparing it with M_2. We believe that our definition offers flexibility to hospitals to compare residents in $M_1(h) \setminus M_2(h)$ and $M_2(h) \setminus M_1(h)$ according to their custom designed criteria. To compare M_1 and M_2, each hospital h fixes $\mathbf{corr}_h$. The disjoint union of these functions, $\mathbf{corr} = \biguplus_h \mathbf{corr}_h$, called the *correspondence function from* M_1 *to* M_2 is then used to define the collective votes of M_1 compared to M_2

$$\Delta(M_1, M_2, \mathbf{corr}) = \sum_{r \in \mathcal{R}} vote_r(M_1, M_2) + \sum_{h \in \mathcal{H}} vote_h(M_1, M_2, \mathbf{corr}_h)$$

We can now define popularity.

▶ **Definition 2.** A matching M_1 is more popular than M_2 (denoted as $M_1 \succ_{\mathbf{corr}} M_2$) under $\mathbf{corr}$ if $\Delta(M_1, M_2, \mathbf{corr}) > 0$. A matching M_1 is *popular* if there is no matching M_2 such that $M_2 \succ_{\mathbf{corr}} M_1$ for any choice of $\mathbf{corr}$ from M_2 to M_1.

It is important to note that, between two matchings M_1 and M_2, matching M_1 may get more votes than M_2 under one correspondence function, but not under another correspondence function. For M_1 to be popular, we require that any other matching M_2 does not get more votes than M_1 under *any* choice of correspondence function. Surprisingly, such a matching indeed exists, as shown in Sections 3 and 4. We also note that both our algorithms (in fact reductions), do not need as an input the correspondence function $\mathbf{corr}$.

Decomposing $M_1 \oplus M_2$: In the one-to-one setting, $M_1 \oplus M_2$ for any two matchings M_1 and M_2 is a collection of vertex-disjoint paths and cycles. Our setting is many-to-one and hence $M_1 \oplus M_2$ has a more complex structure. Here, we recall a simple algorithm from [15] which, given two matchings M_1 and M_2 and a correspondence function $\mathbf{corr}$ from M_1 to M_2, decomposes the edges of $M_1 \oplus M_2$ into (possibly non-simple) alternating paths and cycles. Consider the graph $\tilde{G} = (\mathcal{R} \cup \mathcal{H}, M_1 \oplus M_2)$, for any two feasible matchings of the HRLQ instance. We note that the degree of every resident in $\tilde{G}$ is at most 2 and the degree of every hospital in $\tilde{G}$ is at most $2 \cdot q^+(h)$. Consider any connected component $\mathcal{C}$ of $\tilde{G}$ and let $e \in M_1$ be any edge in $\mathcal{C}$. We show how to construct a unique maximal M_1-alternating path or cycle ρ containing e: Start with $\rho = \langle e \rangle$. Use the following inductive procedure.

1. Let $r \in \mathcal{R}$ be one end-point of ρ, and let $(r, M_1(r)) \in \rho$. We grow ρ by adding the edge $(r, M_2(r))$. Similarly if $(r, M_2(r)) \in \rho$, add $(r, M_1(r))$ to ρ.
2. Let $h \in \mathcal{H}$ be an end-point of ρ, and let the last edge (r, h) on ρ be in $M_1 \setminus M_2$. We extend ρ by adding $\mathbf{corr}_h(r, M_1, M_2)$ if it is not equal to $\perp$. A similar step is performed if the last edge on ρ is $(r, h) \in M_2 \setminus M_1$.
3. We stop the procedure when we complete a cycle (ensuring that the two adjacent residents of a hospital h are $\mathbf{corr}_h$ for each other according to h), or the path can no longer be extended. Otherwise we go to Step 1 or Step 2 as applicable and repeat.

r_1	:	h_1, h_3, h_4, h_5	$(0,1)$	h_1	:	r_1, r_2, r_3
r_2	:	h_2, h_1, h_3	$(0,1)$	h_2	:	r_2, r_3, r_4
r_3	:	h_2, h_1	$(0,1)$	h_3	:	r_1, r_2
r_4	:	h_2	$(0,1)$	h_4	:	r_1
			$(1,1)$	h_5	:	r_1

Figure 1 Resident and hospital preferences in G. The $(0, 1)$ beside h_1 denote the lower and upper quotas of h_1 respectively. Preferences can be read as: r_1 prefers h_1 followed by h_3 and so on.

Labels on edges: While comparing M_1 with M_2 using **corr**, the voting scheme induces a label on edges of M_2 with respect to M_1. Let $(r, h) \in M_2$. The label of (r, h) is (a, b) where $a = vote_r(M_1(r), M_2(r))$ and $b = vote_h(\mathbf{corr}_h(r, M_2, M_1), r)$. Thus $a, b \in \{-1, 1\}$. Here it is important to note that $\mathbf{corr}_h$ is a bijection between $M_1(h) \setminus M_2(h)$ and $M_2(h) \setminus M_1(h)$.

3 Maximum cardinality popular matching

We first give some intuition and an example which illustrates the overall idea of our algorithm. We then present a reduction from HRLQ to HR which simulates the algorithm.

At the high-level, our algorithm has three phases. In Phase-0 we simply execute the hospital-proposing Gale-Shapley algorithm on G by disregarding lower quotas of all hospitals. Let M_0 be a matching obtained. Phase-1 is the "second chance phase". In this phase, all hospitals that are *under-subscribed* in M_0 (a hospital h is under-subscribed in M if $|M(h)| < q^+(h)$), propose to residents in their preference list with increased priority and a new matching M_1 is obtained. The priority is simulated by assigning *levels* to hospitals. Phase-0 is executed with all the hospitals at level 0. The priority of a hospital is increased by increasing its level. A resident prefers a higher level hospital to any lower level hospital *irrespective* of the relative positions of the hospitals in the his preference list.

Finally, Phase-2 is the "feasibility phase". If there are *deficient hospitals* in M_1 (h is deficient in M if $|M(h)| < q^-(h)$), they again apply with an even higher priority. This process is repeated until there is no deficient hospital (we prove that $|\mathcal{R}|$ repetitions are sufficient) and every under-subscribed hospital has got a second chance.

Example: Let $G = (\mathcal{R} \cup \mathcal{H}, E)$ where $\mathcal{R} = \{r_1, \ldots, r_4\}$ and $\mathcal{H} = \{h_1, \ldots, h_5\}$. Figure 1 shows the quotas and the preferences of vertices in G.

The first part of the algorithm is an execution of hospital-proposing Gale-Shapley algorithm, with all hospitals at *level*-0. This results in a stable but infeasible matching $M_0 = \{(r_1, h_1^0), (r_2, h_2^0)\}$. As h_3, h_4, h_5 are under-subscribed in M_0, their level is increased, and h_3^1, h_4^1, h_5^1 propose from the beginning of the respective preference lists. The hospital h_3^1 proposes to r_1 which is accepted. This leaves h_1 under-subscribed. However, note that h_1^0 has not yet exhausted its preference list, so h_1^0 proposes to r_3 and this proposal is accepted. Also note that h_4^1 and h_5^1 do not get matched, as the only resident in their preference list, r_1, prefers h_3^1 over them. This results in a larger but infeasible matching $M_1 = \{(r_1, h_3^1), (r_2, h_2^0), (r_3, h_1^0)\}$.

We further increase the level of the deficient hospital h_5, thus h_5^2 proposes to r_1. This triggers a series of proposals (see Figure 2(b) and Figure 2(c) in Appendix A for details) and we finally obtain the feasible matching $M_2 = \{(r_1, h_5^2), (r_2, h_1^1), (r_3, h_2^0)\}$. As all the under-subscribed hospitals have exhausted their preference lists at *level*-0 and *level*-1 and there are no deficient hospitals, the algorithm terminates.

Note that all the hospitals are allowed to propose at *level*-0, any under-subscribed hospital is allowed to propose at *level*-1 (all hospitals except h_2, in the example), whereas only deficient hospitals are allowed to propose at *level*-2 and higher (h_5 in the example).

3.1 The reduced graph G'

To simulate the above algorithm on an HRLQ instance G, we convert G to an HR instance $G' = (\mathcal{R}' \cup \mathcal{H}', E')$. The main idea is to have in G' multiple copies of every hospital in G – the first two copies have capacity equal to upper-quota and the rest of the copies have capacity equal to lower-quota. Furthermore, we need a suitably large set of dummy residents to ensure that a matching in G' matches at most upper-quota many non-dummy residents across all copies of a hospital. We describe our reduction – we begin with vertices in G'.

The set $\mathcal{H}'$: For each hospital $h \in \mathcal{H}$ we have ℓ copies $h^0, \ldots, h^{\ell-1}$ of h in $\mathcal{H}'$. Here $\ell = 2 + \sum_{h \in \mathcal{H}} q^-(h)$. We need to define the capacities of all hospitals $h \in \mathcal{H}'$ (recall G' is an HR instance, so we do not have lower quotas for $h \in \mathcal{H}'$). For the upper quota of a hospital, we use the term "capacity" in an HR instance whereas "upper quota" in an HRLQ instance. The hospitals in $\mathcal{H}'$ and their capacities are as described below:

$$\begin{aligned} \mathcal{H}' &= \{h^0, \ldots, h^{\ell-1} \mid h \in \mathcal{H}\} \\ \text{Capacities of } h \in \mathcal{H}'\text{: } q^+(h^s) &= q^+(h), \quad s \in \{0, 1\} \\ q^+(h^s) &= q^-(h), \quad s \in \{2, \ldots, \ell-1\} \end{aligned}$$

We call hospital $h^s \in \mathcal{H}'$ a *level-s* copy of h. Note that, if $h \in \mathcal{H}$ has zero lower-quota, then $h^2, \ldots, h^{\ell-1}$ have zero capacity in $\mathcal{H}'$. As will be seen later, a resident prefers a hospital at a higher level to *any* hospital at a lower level. The following observation is immediate.

▶ **Observation 1.** *For a hospital $h \in \mathcal{H}$, the sum of capacities of all the copies of h in G' is $q_h = 2 \cdot q^+(h) + (\ell - 2) \cdot q^-(h)$.*

The set $\mathcal{R}'$: The set of residents $\mathcal{R}'$ consists of the set $\mathcal{R}$ along with a set of dummy residents $\mathcal{D}_h$ corresponding to every hospital $h \in \mathcal{H}$. The sets $\mathcal{R}'$ and $\mathcal{D}_h$ are as defined below:

$$\begin{aligned} \mathcal{R}' &= \mathcal{R} \cup \left(\bigcup_{h \in \mathcal{H}} \mathcal{D}_h \right) \text{ where } \qquad \mathcal{D}_h = \bigcup_{s \in \{0, \ldots, \ell-2\}} \mathcal{D}_h^s \qquad \forall h \in \mathcal{H} \\ \text{Here } \mathcal{D}_h^s &= \{d_{h,1}^s, \ldots, d_{h,q^+(h)}^s\}, \quad s \in \{0, 1\} \\ \text{and } \mathcal{D}_h^s &= \{d_{h,1}^s, \ldots, d_{h,q^-(h)}^s\}, \quad s \in \{2, \ldots, \ell-2\} \end{aligned}$$

We refer to $\mathcal{D}_h$ as *dummy residents corresponding to h* and $\mathcal{D}_h^s$ as *s-th set of dummy residents corresponding to h*. Note that, for a hospital h, for any level s except the last level, we have dummy residents in $\mathcal{D}_h^s$ exactly equal to the capacity of h^s in G'. For $h \in \mathcal{H}$, if $q^-(h) = 0$, then $\mathcal{D}_h^s = \emptyset$ for each $s \in \{2, \ldots, \ell - 2\}$.

▶ **Observation 2.** *For a hospital $h \in \mathcal{H}$, the total number of dummy residents corresponding h in $\mathcal{R}'$ is $|\mathcal{D}_h| = 2 \cdot q^+(h) + (\ell - 3) \cdot q^-(h)$.*

Preference lists: We denote by $\langle list_r \rangle$ and $\langle list_h \rangle$ the preference lists of r and h in G, respectively. Furthermore, $\langle \mathcal{D}_h^s \rangle$ denotes the strict list consisting of elements of $\mathcal{D}_h^s$ in increasing order of indices. Finally, $\circ$ denotes the concatenation of two lists. We now describe the preferences of hospitals and residents in G'.
Hospitals' preference lists: For a hospital h in $\mathcal{H}$, the preference lists of its copies $h^s \in \mathcal{H}'$ for $s \in \{0, 1, \dots, \ell - 1\}$ are given by:

$$\begin{aligned}
s = 0 &\quad : \quad \langle list_h \rangle \circ \langle \mathcal{D}_h^0 \rangle \\
s = 1 &\quad : \quad \langle \mathcal{D}_h^0 \rangle \circ \langle list_h \rangle \circ \langle \mathcal{D}_h^1 \rangle \\
s = 2 &\quad : \quad \langle d^1_{h,k(h)}, \dots, d^1_{h,q^+(h)} \rangle \circ \langle list_h \rangle \circ \langle \mathcal{D}_h^2 \rangle, \qquad k(h) = q^+(h) - q^-(h) + 1 \\
s \in \{3, 4, \dots, \ell - 2\} &\quad : \quad \langle \mathcal{D}_h^{s-1} \rangle \circ \langle list_h \rangle \circ \langle \mathcal{D}_h^s \rangle \\
s = \ell - 1 &\quad : \quad \langle \mathcal{D}_h^{(\ell-2)} \rangle \circ \langle list_h \rangle
\end{aligned}$$

The preference list of the *level*-2 copy of h is slightly different. In $\mathcal{D}_h^1$ we have $q^+(h)$ dummy residents but we want only $q^-(h)$ many out of them to be in the preference list of h^2.
Residents' preference lists: For any $s \in \{0, \dots, \ell - 1\}$, and any $r \in \mathcal{R}$, let $\langle list_r \rangle^s$ denote the list obtained by replacing every hospital h in $\langle list_r \rangle$ by its level-s copy h^s. Then the preference list of r in G' is given by:

$$\text{For } r \in \mathcal{R} \quad : \quad \langle list_r \rangle^{\ell-1} \circ \langle list_r \rangle^{\ell-2} \circ \dots \circ \langle list_r \rangle^0$$

Thus r prefers any level-s hospital to any level-$(s-1)$ hospital. For two hospitals at the same level s, r prefers h^s over h'^s in G' iff r prefers h over h' in G. We now give the preference list of every dummy resident in $\mathcal{D}_h$. Recall that for any $h \in \mathcal{H}$, $k(h) = q^+(h) - q^-(h)$.

$$\begin{aligned}
\text{For } h \in \mathcal{H}, \quad d^s_{h,i} \in \mathcal{D}_h &\quad : \\
s = 0 &\quad : \quad h^0, h^1 \\
s = 1, \quad i \in \{1, \dots, k(h)\} &\quad : \quad h^1 \\
s = 1, \quad i \in \{k(h) + 1, \dots, q^+(h)\} &\quad : \quad h^1, h^2 \\
s \in \{2, \dots, \ell - 2\} &\quad : \quad h^s, h^{s+1}
\end{aligned}$$

3.2 Properties of the stable matching M' in G'

Having described the reduction from an HRLQ instance G to an HR instance G', we now discuss some useful properties of a stable matching M' in G'. With respect to a stable matching M' in G' we introduce the following definitions.

▶ **Definition 3. Level-s resident:** A non-dummy resident $r \in \mathcal{R}'$ is said to be at *level-s* in M' if r is matched to a *level-s* hospital in M'. Let $\mathcal{R}'_s$ denote the set of *level-s* residents.

▶ **Definition 4. Active hospital:** A hospital h^s is said to be *active* in M' if $M'(h^s)$ contains at least one non-dummy resident. Otherwise, (when all positions of h^s are matched to dummy residents), h^s is said to be *inactive*.

In Lemma 5, we prove some invariants for any stable matching M' in G'. These invariants allow us to define a natural map from M' to a matching M in G, and to show that M is feasible as well as popular among feasible matchings.

▶ **Lemma 5.** *The following hold for any stable matching M' in G':*
1. *For any $h \in \mathcal{H}$, M' matches at most $q^+(h)$ non-dummy residents across all its level copies in G'.*

2. *The matching M' in G' leaves only the level-$(\ell-1)$ copy of any hospital (if it exists) under-subscribed.*
3. *Let $h^s \in \mathcal{H}'$ be active in M'. Then,*
   (a) *$M'(h^{s-1})$ contains at least one resident from the $(s-1)$-th set of dummy residents.*
   (b) *For $0 \le j \le s-2$, h^j is inactive in M' and all positions of h^j are matched to the j-th set of dummy residents.*
   (c) *For $s+2 \le j \le \ell-1$, h^j is inactive in M' and all positions of h^j are matched to the $(j-1)$-th set of dummy residents.*
4. *For any $h \in \mathcal{H}$, at most two consecutive level copies h^s and h^{s+1} are active in M'.*
5. *A level-s resident r in M' does not have any hospital h in its preference list which is active at level-$(s+2)$ or more in M'.*

Proof.

- *Proof of 1*: We first note that the total capacity of all the copies of h in G' is $q_h = 2 \cdot q^+(h) + (\ell-2) \cdot q^-(h)$. Furthermore, the total number of dummy residents for h is given by $|\mathcal{D}_h| = 2 \cdot q^+(h) + (\ell-3) \cdot q^-(h)$. We now show that at most $q^+(h) - q^-(h)$ dummy residents out of $\mathcal{D}_h$ can remain unmatched in a stable matching M'. Assuming this, it is immediate that the total number of non-dummy residents that can be matched across all copies of h is at most $q_h - \{|\mathcal{D}_h| - (q^+(h) - q^-(h))\} = q^+(h)$.
  We now argue that at most $q^+(h) - q^-(h)$ dummy residents of $\mathcal{D}_h$ can remain unmatched in M'. Consider the set of dummy residents corresponding to a hospital $h \in \mathcal{H}$ i.e. $\bigcup_{s=0}^{\ell-2} \mathcal{D}_h^s$. With the exception of h^2, for any h^s, $\mathcal{D}_h^{(s-1)}$ are the most preferred $q^+(h^s)$ dummy residents of h^s. A dummy resident $d_{h,j}^{s-1}$ which is a top choice for h^s cannot remain unmatched in M', else $(d_{h,j}^{s-1}, h^s)$ blocks M'. Thus, these dummy residents can never remain unmatched in M'. The only dummy residents that are not the first choice of any hospital and hence can remain unmatched are the subset of $\mathcal{D}_h^1$ consisting of the first $q^+(h) - q^-(h)$ dummy residents from $\mathcal{D}_h^1$. This is because, by construction of G', only the last $q^-(h)$ dummy residents from $\mathcal{D}_h^1$ are present in the preference list of h^2 as its top $q^+(h^2)$ top-choices. This establishes that the number of dummy residents of $\mathcal{D}_h$ that can remain unmatched in M' is at most $q^+(h) - q^-(h)$.
- *Proof of 2:* Consider a hospital $h \in \mathcal{H}$. For each copy h^s of h in $\mathcal{H}'$, where $s < \ell - 1$, the s-th set of dummy residents have h^s as their first choice. Further, their number is same as $q^+(h^s)$. Thus h^s can not remain under-subscribed in any stable matching M' of G', otherwise these dummy residents will form a blocking pair with h^s.
- *Proof of 3a:* For the sake of contradiction, assume that h^{s-1} is not matched to any resident from the $(s-1)$-th set of dummy residents and still h^s is matched to a non-dummy resident. As there are exactly $q^+(h^s)$ many dummy residents in the preference list of h^s from the $(s-1)$-th set and each dummy resident from the $(s-1)$-th set has only h^{s-1} and h^s in its preference list, this means that there is a dummy resident d from its $(s-1)$-th set of dummy residents unmatched in M'. But h^s prefers any dummy resident in its $(s-1)$-th set of dummy residents over any non-dummy resident. Thus (d, h^s) forms a blocking pair with respect to M', contradicting the stability of M'.
  Proof of 3b: If h^s is active and h^j is matched to a non-dummy resident r for some $0 \le j \le s-2$, then $(r, h^{(s-1)})$ is a blocking pair with respect to M'. This is because, as proved above, $h^{(s-1)}$ must be matched to at least one resident in $\mathcal{D}_h^{(s-1)}$, and $h^{(s-1)}$ prefers any non-dummy resident over any dummy resident in $\mathcal{D}_h^{(s-1)}$.
  Proof of 3c: If h^s is active then h^j can not be active for $s+2 \le j \le \ell-1$ else $h^{(j-1)}$ must be matched to a resident from $\mathcal{D}_h^{(j-1)}$ as proved above, and then each non-dummy resident r in $M'(h^s)$ forms a blocking pair with h^j contradicting the stability of M'. But

if h^s is active, then h^j can not be matched to a dummy resident from $\mathcal{D}_h^j$ either, otherwise a resident in $M'(h^s)$ forms a blocking pair with h^{j-1}. The latter is true because any resident in $\langle list_h \rangle$ prefers h^j over h^s for $j > s$ and h^j prefers any resident in $list_h$ to any dummy resident in $\mathcal{D}_h^j$. Hence h^j must be matched to only dummy residents in $\mathcal{D}_h^{(j-1)}$.

- *Proof of 4:* Assume the contrary. Thus let h be a hospital such that there are two levels i and j, $j < i-1$, where h^i and h^j are active in M'. Further, assume that h^i is matched to r_i and h^j be matched to r_j. Then, by part 3a above, h^{i-1} must be matched to at least one resident from the $(i-1)$-th set of dummy resident. But, by the structure of preference lists, h^{i-1} prefers a non-dummy resident, and hence r_j, over any resident in the $(i-1)$-th set of dummy residents. Also, r^j prefers h^{i-1} over h^j since $j < i-1$. Thus (r_j, h^{i-1}) forms a blocking pair in G' w.r.t. M', contradicting the stability of M'.
- *Proof of 5:* Let there be an edge (r, h^t) in G' such that r is a level-s resident and $t \geq s+2$ and h^t is active in M'. Then, by part 3a above, h^{t-1} has at least one resident from its $(t-1)$-th set of dummy residents in $M'(h^{s+1})$. As r has edge to h^t, r also has an edge to h^{t-1} by construction of G'. Also, again by construction of G', r prefers any level-$(t-1)$ hospital over any level-s hospital and h^{t-1} prefers any non-dummy resident in its preference list over any dummy resident in its $(t-1)$-th set of dummy residents. Thus (r, h^{t-1}) forms a blocking pair with respect to M' in G', contradicting its stability.

This completes the proof of all invariants. ◀

4 Proof of popularity

In this section, we show how to use the reduction in the previous section to compute a maximum cardinality matching that is popular amongst all feasible matchings. Thus, amongst all feasible matchings, our algorithm outputs the largest popular matching. We call such a matching a *maximum cardinality popular matching.*

Our algorithm reduces the HRLQ instance G to an HR instance G' as described in Section 3. We then compute a stable matching M' in G'. Finally, to obtain a matching M in G we describe a simple map function. For every $h \in \mathcal{H}$, let $M(h) = \mathcal{R} \cap \left(\bigcup_{s=0}^{\ell-1} M'(h^s)\right)$. Note that $M(h)$ denotes the set of non-dummy residents matched to any copy h^s of h in M'. Thus, a resident r is matched to a hospital h in M if and only if r is matched to a level-s copy of h in M' for some $s \in \{0, \ldots, \ell-1\}$. We say that $M = map(M')$.

Division of $\mathcal{R}$ and $\mathcal{H}$ into subsets: We divide the residents and hospitals in G into subsets depending upon a matching M' in G'. Let R_i be the set of non-dummy residents matched to a *level-i* hospital h^i in M'. We define the same set R_i in G as well. Further, define H_j to be the set of hospitals $h \in H$ such that $\mathcal{R} \cap M'(h^j) \neq \emptyset$, that is, *level-$j$* copy h^j of h is active in M'. Define the unmatched residents to be in R_0. Also, a non-lower-quota hospital h such that $M(h) = \emptyset$ is defined to be in H_1, and a lower-quota hospital h with $M(h) = \emptyset$ is defined to be in $H_{\ell-1}$. The following lemma summarizes the properties of the sets R_i and H_j. See full-version [14] for proof.

▶ **Lemma 6.** *For a stable matching M' in G', let $M = map(M')$. The following hold:*
1. *Each hospital is present in at most two sets H_j, H_{j+1} for some j. We say that $h \in H_j \cap H_{j+1}$.*
2. *If $h \in H_j \cap H_{j+1}$, then there is no edge from h to any $r \in R_i$ where $i \leq j-1$.*
3. *All the non-lower-quota hospitals that are under-subscribed in M are in H_1.*
4. *All the deficient lower-quota hospitals from M are in $H_{\ell-1}$.*

5. *If a non-lower-quota hospital is under-subscribed, it has no edge to any resident in R_0. If a lower-quota hospital is deficient, it does not have an edge to any resident in R_i for $i < \ell - 1$. Similarly an unmatched resident does not have an edge to any hospital in $H_1 \cup \ldots \cup H_{\ell-1}$.*
6. *Let $h \in \mathcal{H}$ be such that $|M(h)| > q^-(h)$. Then $h \notin H_2 \cup \ldots \cup H_{\ell-1}$.*

Throughout the following discussion, assume that M is a matching which is a map of a stable matching M' in G' and N is any feasible matching in G. Additionally, whenever we consider the decomposition of $M \oplus N$ into paths and cycles, we use an arbitrary correspondence function **corr**. Using Theorem 7 we prove that M is feasible in G.

▶ **Theorem 7.** *If G admits a feasible matching, then $M = map(M')$ is feasible for G.*

Proof. Suppose M is not feasible. Thus, there is a deficient lower-quota hospital h in M. Let N be a feasible matching in G. Consider decomposition of $M \oplus N$ into (possibly non-simple) paths and cycles using an arbitrary correspondence function **corr** (recall decomposition from Section 2). As h is deficient in M and not deficient in N, there must be a path ρ in $M \oplus N$ ending in h. Moreover, if the other end of ρ is a hospital h' then $|M(h')| - |N(h')| > 0$. Note that in this case, ρ has even-length and hence ends with a M-edge. The other case is where ρ ends in a resident r and hence ends with a N-edge. We consider the two cases below:

- **ρ ends in a hospital h':** As h is deficient in M, $h \in H_{\ell-1}$ by part 4 of Lemma 6. Also, since $|M(h')| > |N(h')| \geq q^-(h')$, by part 6 of Lemma 6, $h' \in H_0 \cup H_1$. Thus ρ starts at $H_{\ell-1}$ and ends in H_0 or H_1. Let $\rho = \langle h, r_1, h_1, r_2, h_2, \ldots, r_t, h_t, r', h' \rangle$, where $(r_i, h_i) \in M$ and $(r', h') \in M$. We show below that such a path ρ can not exist and hence M must be feasible.

  By part 5 of Lemma 6, h has edges only to residents in $R_{\ell-1}$. Hence $r_1 \in R_{\ell-1}$ and hence $h_1 \in H_{\ell-1}$. By part 2 of Lemma 6, h_1 has no edges to residents in $R_0 \cup \ldots \cup R_{\ell-3}$. Therefore $r_2 \in R_{\ell-1} \cup R_{\ell-2}$ and $h_2 \in H_{\ell-1} \cup H_{\ell-2}$. Thus each $h_i \in \rho$ can not be in H_j, for any $j < \ell - i$. But $h' \in H_0 \cup H_1$ and hence $r' \in R_0 \cup R_1$. Therefore $h_t \notin H_3 \cup \ldots \cup H_{\ell-1}$ by part 2 of Lemma 6, otherwise (h_t, r') edge can not exist in G. In other words, ρ has to contain at least one hospital from each level i, $1 \leq i \leq \ell - 1$. Thus $t \geq \ell - 2$. Moreover, all the hospitals in ρ which are in $H_{\ell-1} \cup \ldots \cup H_2$ are lower-quota hospitals. Thus ρ has at least $t + 1 = \ell - 1$ lower-quota hospitals. Note that this count includes repetitions, as a hospital can appear multiple times in ρ. However, any hospital in $H_2 \cup \ldots \cup H_{\ell-1}$ can not be matched to more than $q^-(h)$ residents in M by part 6 and hence can appear at most $q^-(h)$ times on ρ. But then the sum of lower quotas of all the hospitals is $\ell - 2$, contradicting that ρ has a total of $\ell - 1$ occurrences of lower-quota hospitals. Thus such a path ρ can not exist and M must be feasible.
- **ρ ends in a resident r:** Now consider the case where ρ ends at a resident r. Then the last edge on ρ must be a N-edge and hence r is unmatched in M. Therefore $r \in R_0$. Let $\rho = \langle h, r_1, h_1, r_2, h_2, \ldots, r_t, h_t, r \rangle$ where $(r_i, h_i) \in M$ for $1 \leq i \leq t$ and the remaining edges are in N. Consider the first hospital, say h_j on ρ such that $h_j \in H_2$ and for each $h_i, i < j$, $h_i \in H_3 \cup \ldots \cup H_{\ell-1}$. Such an h_j has to exist by the argument given for the previous case. Moreover, $j \geq \ell - 2$ as ρ has to contain at least one hospital from each level as described in the previous case. Thus the number of occurrences of lower-quota hospitals on ρ exceeds the sum of lower quotas and hence such a ρ can not exist.

This completes the proof of the lemma. ◀

In Lemma 8 and Theorem 9 below, we give crucial properties of the division of $\mathcal{R}$ and $\mathcal{H}$ that will be helpful in proving popularity of the matching M which is a map of a stable matching

M' in G'. For both of them, assume that **corr** is an arbitrary correspondence function from N to M. The proofs appear in the full-version [14].

▶ **Lemma 8.** *Let N be any feasible matching. Let $(r, h) \in M$ and $(r', h) \in N$ such that $r' = \textbf{corr}_h(r, M, N)$. Further let $h \in H_j \cap H_{j+1}$ and $r \in R_{j+1}$. Further, let $r' \in R_j$. Then the label on (r', h) edge is $(-1, -1)$.*

Let $\rho = \langle h_0, r_1, h_1, r_2, h_2, \ldots, h_t, r_{t+1}\rangle$ be a path in $M \oplus N$ where $M = map(M')$ and N is any feasible matching in G. Also, label the edges of $N \setminus M$. See Section 2 for the decomposition and labeling of edges.

▶ **Theorem 9.** *Let $\rho = \langle h_0, r_1, h_1, r_2, h_2, \ldots, h_t, r_{t+1}\rangle$ be a path in $M \oplus N$ as described above. Here $(r_k, h_k) \in M$ for all k and $(h_k, r_{k+1}) \in N$ with $r_{k+1} = \textbf{corr}_{h_k}(r_k, M, N)$. Moreover, let $h_0 \in H_p \cap H_{p+1}$ and $r_{t+1} \in R_q$. Then the number of $(1, 1)$ edges in ρ is at most the number of $(-1, -1)$ edges plus $q - p$.*

Proof. We prove this by induction on the number of $(-1, -1)$ edges. Note that, except h_0, all the h_is are matched in M', and hence we can consider them at the same level as their matched residents.

Base case: Let ρ have no $(-1, -1)$ edges. As ρ starts at $h \in H_p \cap H_{p+1}$, r_1 has to be in *level*-$(p+1)$ or above. This is because there is no edge from h to a resident in $R_0 \cup \ldots \cup R_{p-1}$, and if $r_1 \in R_p$ then by Lemma 8 the label on (h_0, r_1) must be $(-1, -1)$. By assumption, there is no $(-1, -1)$ edge in ρ. So $r_1 \in R_j$ for some j, $p + 1 \leq j \leq \ell$. Therefore $h_1 \in H_j$.

Thus the path can only use edges from a hospital at a lower level to a resident at the same or higher level. Further, there is no $(1, 1)$ edge in $H_k \times R_k$ for any k; otherwise the same edge blocks M' in G' contradicting the stability of M'. So $(1, 1)$ edges can appear in ρ only when it goes from a hospital in a lower level to a resident in a higher level. So there can be at most $q - p$ many $(1, 1)$ edges on ρ.

Induction step: Let the theorem hold for at most $i-1$ many $(-1, -1)$ edges. Let (h_k, r_{k+1}) be one such edge. Further, let $h_k \in H_a$ and $r_{k+1} \in R_b$. Consider the two subpaths $\rho_1 = \langle h_0, \ldots, r_k\rangle$ and $\rho_2 = \langle h_{k+1} \ldots, r_{t+1}\rangle$. As the number of $(-1, -1)$ edges in each of ρ_1 and ρ_2 is less than i, the induction hypothesis holds. Therefore, the number of $(1, 1)$ edges in ρ_1 is at most $a - p$ plus the number of $(-1, -1)$ edges in ρ_1. Similarly, the number of $(1, 1)$ edges in ρ_2 is $q - b$ plus the number of $(-1, -1)$ edges in ρ_2. The number of $(-1, -1)$ edges in ρ is one more than the total number of $(-1, -1)$ edges in ρ_1 and ρ_2. Hence the number of $(1, 1)$ edges in ρ is at most the number of $(-1, -1)$ edges in ρ plus $a - p + q - b - 1$. As there is an edge between h_k and r_{k+1}, $b \geq a - 1$ by Lemma 6 part 2. Thus $a - p + q - b - 1 \leq q - p$, which completes the proof. ◀

Theorem 10 shows that M is a popular matching amongst all the feasible matchings in G.

▶ **Theorem 10.** *Let N be any feasible matching in G and **corr** be an arbitrary correspondence function from N to M.*

1. *If ρ is an alternating cycle in the decomposition of $M \oplus N$, then $\Delta(M \oplus \rho, M, \textbf{corr})$*[1] ≤ 0.
2. *If ρ is an alternating path in the decomposition of $M \oplus N$ with exactly one end-point matched in M, then $\Delta(M \oplus \rho, M, \textbf{corr}) \leq 0$.*

[1] Note that when comparing $M \oplus \rho$ with M, we use the restriction of the correspondence function **corr** used to compare N with M. With the abuse of notation we refer to the restriction also as **corr**.

3. *If ρ is an alternating path in the decomposition of $M \oplus N$ with both the end-points matched in M then $\Delta(M \oplus \rho, M, \boldsymbol{corr}) \leq 0$.*

Proof. We prove the three cases below.

1. Let ρ be an alternating cycle in $M \oplus N$. Further, let $(r, h) \in M$. Consider $\rho' = \rho \setminus \{(r, h)\}$ which is an alternating path from h to r. The path ρ' starts and ends at the same level. Hence the number of $(1, 1)$ edges on ρ' is at most the number of $(-1, -1)$ edges on ρ'. The same holds for ρ.
2. Let ρ be an alternating path in $M \oplus N$ with exactly one end-point matched in M. Thus ρ has even length, and both its end-points are either hospitals or both are residents.

   Consider the first case. So let $\rho = \langle h_0, r_1, h_1, \ldots, r_t, h_t \rangle$ where $(r_i, h_i) \in M$ for all i. Thus $|M(h_0)| < |N(h_0)| \leq q^+(h_0)$, and hence h_0 is under-subscribed. Then by part 3 of Lemma 6 and feasibility of M, $h_0 \notin H_0$. By feasibility of N, $h_t \in H_0 \cup H_1$. As $(r_t, h_t) \in M$, $r \in R_0 \cup R_1$ by the definition of levels. Consider the subpath $\rho' = \rho \setminus \{(r_t, h_t)\}$ i.e. the path obtained by removing the edge (r_t, h_t) from ρ. Applying Theorem 9 to ρ' with $p \geq 1$ and $q = 0$ or $q = 1$, we get the number of $(1, 1)$ edges on ρ' to be at most the number of $(-1, -1)$ edges on ρ'.

   Consider the case when both the end-points of ρ are residents. Thus $\rho = \langle r_0, h_1, r_1, \ldots, h_t, r_t \rangle$ where $(h_i, r_i) \in M$ for all i. Again consider $\rho' = \rho \setminus \{(h_t, r_t)\}$. As r_0 is unmatched in M, $r_0 \in R_0$ by the definition of levels. Applying Theorem 9 to ρ' with $q = 0$, we get that the number of $(1, 1)$ edges on ρ' is at most the number of $(-1, -1)$ edges on ρ'.
3. Consider the case when both the end-points of the alternating path ρ are matched in M. Thus one end-point of ρ is a hospital whereas the other end-point is a resident. Let $\rho = \langle r_0, h_0, \ldots, r_t, h_t \rangle$ where $(r_i, h_i) \in M$ for all i. Hence $|M(h_t)| > |N(h_t)| \geq q^-(h_t)$ by feasibility of N. Therefore $h_t \in H_0 \cup H_1$ by part 6 of Lemma 6 which implies that $r_t \in R_0 \cup R_1$. Consider the subpath $\rho' = \rho \setminus \{(r_0, h_0), (r_t, h_t)\}$. Thus ρ' begins at h_0 and ends at r_t. Applying Theorem 9 with $q = 1$ and $0 \leq p \leq \ell$ gives that the number of $(1, 1)$ edges on ρ', and hence on ρ, is at most one more than the number of $(-1, -1)$ edges on ρ. These votes in favor of N are compensated by the end-points r_0 and h_t as r_0 is unmatched in N and $|M(h_t)| > |N(h_t)|$.

This completes the proof of the theorem. ◀

The following lemma proves that M is a maximum cardinality popular matching in G. The proof appears in the full-version [14].

▶ **Lemma 11.** *Let N be any feasible matching in G such that $|N| > |M|$. For any arbitrary correspondence function $\boldsymbol{corr}$ from N to M, $\Delta(N, M, \boldsymbol{corr}) < 0$.*

Size of G': Note that $|\mathcal{H}'| = \ell \cdot |\mathcal{H}|$. Furthermore, $|\mathcal{R}'| = |\mathcal{R}| + (\ell - 1) \cdot |\mathcal{H}|$. The second term in $|\mathcal{R}'|$ accounts for the number of dummy residents in G'. Finally, $|E'| \leq \ell \cdot |E| + 2(\ell - 1) \cdot |\mathcal{H}|$. The first term in $|E'|$ is because the preference list of every hospital in G appears ℓ times in G'. The second term is because every dummy resident in G' appears on the preference list of at most two hospitals in G'. Since, ℓ is upper bounded by $|\mathcal{R}|$, the size of our HR instance G' is $O(|\mathcal{R}| \cdot (|\mathcal{R}| + |\mathcal{H}| + |E|))$. This is the same as the running time of our algorithm to compute a maximum cardinality popular matching in G.

5 Popular matching amongst maximum cardinality feasible matchings

In this section our goal is to compute a maximum cardinality feasible matching that is popular amongst the set of maximum cardinality feasible matchings. Our algorithm is similar to the one described in Section 3. We give an overview of our algorithm and illustrate the execution of the same on the example instance in Figure 1.

As in Section 3, the algorithm here is a three phase algorithm. Phase-0 and Phase-2 are exactly as in Section 3. The modification is in Phase-1. In Section 3, we gave every hospital a second chance to propose if it was under-subscribed in the output of Phase-0. Here, we give $(|\mathcal{R}| - 1)$ chances for every hospital to propose if it is under-subscribed at the end of Phase-0. To see this, recall the example instance in Figure 1. The output of hospital-proposing Gale-Shapley algorithm with all hospitals at *level*-0 is the matching $M_0 = \{(r_1, h_1^0), (r_2, h_2^0)\}$. When all the under-subscribed hospitals are allowed to propose at *level*-1, the matching obtained is $M_1 = \{(r_3, h_1^0), (r_2, h_2^0), (r_1, h_3^1)\}$. Note that h_4^1 and h_5^1 are still under-subscribed. The execution continues with under-subscribed hospitals being allowed to propose at level-2 and then at level-3 (see Figure 2 (d), Appendix A for a detailed proposal sequence). The matching obtained is $M_3 = \{(r_1, h_4^3), (r_2, h_3^2), (r_3, h_1^1), (r_4, h_2^0)\}$. Note that this is a maximum cardinality matching but it is infeasible as h_5 is deficient.

Subsequently, the only deficient hospital h_5 is allowed to propose at *level*-4. This results in h_5^4 applying to r_1, and thus M_3 gets changed to a feasible matching with maximum cardinality, which is $M_4 = \{(r_1, h_5^4), (r_2, h_3^2), (r_3, h_1^1), (r_4, h_2^0)\}$. A possible proposal sequence can be found in table in Figure 2 (b) followed by Figure 2 (d) in Appendix A.

To simulate this algorithm, we again reduce the HRLQ instance G to an HR instance G'' where every hospital has $\ell = |\mathcal{R}| + \sum_{h \in \mathcal{H}} q^-(h)$ many copies. In the interest of space, we give the details of the reduction and proof of correctness in the full-version [14]. A similar calculation for size of G'' as in Section 4 and the fact that $\ell \leq |\mathcal{R}|^2$ gives us the size of G'' as $O(|\mathcal{R}|^2 \cdot (|\mathcal{R}| + |\mathcal{H}| + |E|))$. This matches the running time of our algorithm in this section.

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A Execution Sequence of Algorithm in Section 3

$$\begin{array}{lll} r_1 & : & h_1, h_3, h_4, h_5 \\ r_2 & : & h_2, h_1, h_3 \\ r_3 & : & h_2, h_1 \\ r_4 & : & h_2 \end{array}$$

$$\begin{array}{llll} (0,1) & h_1 & : & r_1, r_2, r_3 \\ (0,1) & h_2 & : & r_2, r_3, r_4 \\ (0,1) & h_3 & : & r_1, r_2 \\ (0,1) & h_4 & : & r_1 \\ (1,1) & h_5 & : & r_1 \end{array}$$

(a)

Proposal	A/R	mMatching
$h_1^0 \to r_1$	✓	m$\{(r_1, h_1^0)\}$
$h_2^0 \to r_2$	✓	m$\{(r_1, h_1^0), (r_2, h_2^0)\}$
$h_3^0 \to r_1$	×	m$\{(r_1, h_1^0), (r_2, h_2^0)\}$
$h_4^0 \to r_1$	×	m$\{(r_1, h_1^0), (r_2, h_2^0)\}$
$h_5^0 \to r_1$	×	m$\{(r_1, h_1^0), (r_2, h_2^0)\}$ $= M_0$
$h_3^1 \to r_1$	✓	m$\{(r_1, h_3^1), (r_2, h_2^0)\}$
$h_4^1 \to r_1$	×	m$\{(r_1, h_3^1), (r_2, h_2^0)\}$
$h_5^1 \to r_1$	×	m$\{(r_1, h_3^1), (r_2, h_2^0)\}$
$h_1^0 \to r_2$	×	m$\{(r_1, h_3^1), (r_2, h_2^0)\}$
$h_1^0 \to r_3$	✓	m$\{(r_1, h_3^1), (r_2, h_2^0), (r_3, h_1^0)\}$ $= \mathbf{M_1}$

(b)

Proposal	A/R	mMatching
$h_5^2 \to r_1$	✓	m$\{(r_1, h_5^2), (r_2, h_2^0), (r_3, h_1^0)\}$
$h_3^1 \to r_2$	✓	m$\{(r_1, h_5^2), (r_2, h_3^1), (r_3, h_1^0)\}$
$h_2^0 \to r_3$	✓	m$\{(r_1, h_5^2), (r_2, h_3^1), (r_3, h_2^0)\}$
$h_1^1 \to r_1$	×	m$\{(r_1, h_5^2), (r_2, h_3^1), (r_3, h_2^0)\}$
$h_1^1 \to r_2$	✓	m$\{(r_1, h_5^2), (r_2, h_1^1), (r_3, h_2^0)\}$ $= \mathbf{M_2}$

(c)

Proposal	A/R	mMatching
$h_4^2 \to r_1$	✓	m$\{(r_1, h_4^2), (r_2, h_2^0), (r_3, h_1^0)\}$
$h_5^2 \to r_1$	×	m$\{(r_1, h_4^2), (r_2, h_2^0), (r_3, h_1^0)\}$
$h_3^1 \to r_2$	✓	m$\{(r_1, h_4^2), (r_2, h_3^1), (r_3, h_1^0)\}$
$h_2^0 \to r_3$	✓	m$\{(r_1, h_4^2), (r_2, h_3^1), (r_3, h_2^0)\}$
$h_1^1 \to r_1$	×	m$\{(r_1, h_4^2), (r_2, h_3^1), (r_3, h_2^0)\}$
$h_1^1 \to r_2$	✓	m$\{(r_1, h_4^2), (r_2, h_1^1), (r_3, h_2^0)\}$
$h_3^2 \to r_1$	✓	m$\{(r_1, h_3^2), (r_2, h_1^1), (r_3, h_2^0)\}$
$h_4^3 \to r_1$	✓	m$\{(r_1, h_4^3), (r_2, h_1^1), (r_3, h_2^0)\}$
$h_3^2 \to r_2$	✓	m$\{(r_1, h_4^3), (r_2, h_3^2), (r_3, h_2^0)\}$
$h_1^1 \to r_3$	✓	m$\{(r_1, h_4^3), (r_2, h_3^2), (r_3, h_1^1)\}$
$h_2^0 \to r_4$	✓	m$\{(r_1, h_4^3), (r_2, h_3^2), (r_3, h_1^1), (r_4, h_2^0)\}$ $= \mathbf{M_3}$
$h_5^3 \to r_1$	✓	m$\{(r_1, h_5^3), (r_2, h_3^2), (r_3, h_1^1), (r_4, h_2^0)\}$ $= \mathbf{M_4}$

(d)

Figure 2 (a) Preference lists of residents and hospitals are recalled from Figure 1. Tables (b) (c) and (d) above show the proposal sequence for the instance in Figure 1. In each of the tables, the ✓ denotes that the proposal is accepted and the × denotes that the proposal is rejected. A possible proposal sequence for algorithm in Section 3 can be obtained by using table (b) followed by the sequence in table (c). A possible proposal sequence for algorithm in Section 5 can be obtained by using Table (b) followed by the sequence in Table (d).