INTERTWINING SEMISIMPLE CHARACTERS FOR p-ADIC CLASSICAL GROUPS

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Abstract. Let G be an orthogonal, symplectic or unitary group over a nonarchimedean local field of odd residual characteristic. This paper concerns the study of the "wild part" of an irreducible smooth representation of G, encoded in its "semisimple character". We prove two fundamental results concerning them, which are crucial steps towards a complete classification of the cuspidal representations of G. First we introduce a geometric combinatorial condition under which we prove an "intertwining implies conjugacy" theorem for semisimple characters, both in G and in the ambient general linear group. Second, we prove a Skolem–Noether theorem for the action of G on its Lie algebra; more precisely, two semisimple elements of the Lie algebra of G which have the same characteristic polynomial must be conjugate under an element of G if there are corresponding semisimple strata which are intertwined by an element of G.

1. Introduction

A major motivation for the study of the representation theory of p-adic groups is, via the local Langlands correspondence, to understand Galois representations. The arithmetic core of these representations, which is rather mysterious on the Galois side, is encoded in restriction to wild inertia. On the automorphic side, this restriction corresponds to looking at certain representations of pro-p-subgroups.

For p-adic general linear groups, Bushnell and Kutzko [BK93] constructed, and classified, all cuspidal irreducible representations. At the heart of this classification sit the so-called "simple characters"; these are very particular arithmetically-defined characters of pro-p-subgroups, which exhibit remarkable rigidity properties (see below for details). These properties were exploited, and extended, by Bushnell and Henniart [BH96], who defined the notion of an "endo-class" and hence proved a Ramification Theorem [BH03] for the local Langlands correspondence for general linear groups: there is a bijection between the set of endo-classes and the set of orbits (under the Weil group) of irreducible representations of the wild inertia group. More recently, they have extended this, using the fundamental structural properties of simple characters to prove a Higher Ramification Theorem [BH17].

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For p-adic classical groups – that is, symplectic, special orthogonal and unitary groups – in odd residual characteristic, analogous characters were constructed by the second author [Ste05] as a fundamental step in the construction of all cuspidal irreducible representations [Ste08]. This required first extending the theory of simple characters to the case of "semisimple characters" (see also the work of Dat [Dat09]). However, the rigidity results which allowed Bushnell and Kutzko to obtain a *classification* were missing – partly because some of them are false.

In this paper, we prove many of these rigidity results for semisimple characters, which are new even in the case of general linear groups – in particular, we prove "intertwining implies conjugacy" and Skolem–Noether results (see below for details). In a sequel [KSS16], jointly with Kurinczuk, we are then able to put this together with other work of Kurinczuk and the second author [KS15], to turn the construction of cuspidal representations into a classification, for both complex and ℓ-modular representations, with $\ell \neq p$ prime. More precisely, we establish the following conjugacy result for cuspidal types in p-adic classical groups: if (J, λ) and (J', λ') are two types from the construction in [Ste08] which induce to give equivalent irreducible cuspidal representations, then they are conjugate.

We anticipate further work to come from these rigidity results. Semisimple characters (or, more precisely, their endo-classes) will give a decomposition of the category of smooth ℓ-modular representations of classical groups, and each subcategory should be equivalent to the subcategory of *depth zero* representations of some other (endoscopic) group, for which other techniques are available. Current work of the first author (see [Sko17] for the start of this) aims at generalizing the results proved here to proper inner forms of classical groups, where additional problems arise, analogous to those in the case of inner forms of general linear groups [BSS12]. One would then expect that a Jacquet–Langlands correspondence between inner forms would respect the decompositions of the categories by endo-class, as for general linear groups [SS16], and that this would be a major step in making such a correspondence explicit. Finally, it would be interesting to explore whether our results on semisimple characters for general linear groups can be extended to semisimple types: suppose (J, λ) and (J', λ') are Bushnell–Kutzko semisimple types for the same Bernstein component of a p-adic general linear group, so that they intertwine; what extra condition on the associated lattice sequences is required to be able to conclude that the types are conjugate? The same question can also be asked in classical groups.

Now we state our results more precisely. Let F be a nonarchimedean local field of odd residual characteristic. Let G be the isometry group of an ϵ-hermitian space with respect to some automorphism of F of order at most two, so that G is the group of fixed points under an involution on the full automorphism group $\tilde{G}$ of the underlying F-vector space V. We similarly regard the Lie algebra of G as the fixed points of an involution on $A = \mathrm{End}_F(V)$. Note that, when $\epsilon = 1$ and the involution on F is trivial, we are working with the *full* orthogonal group; however, the set of semisimple characters for the full orthogonal group and for the special orthogonal group coincide.

The starting point in the construction of semisimple characters is an algebraic combinatorial object, a so-called *semisimple stratum* $[\Lambda, q, r, \beta]$. The principal data here are: an element $\beta \in A$ which generates a sum of field extensions $E = F[\beta] = \bigoplus_{i\in I} E_i$; and a rational point Λ in the (enlarged) Bruhat–Tits building of the centralizer of β in G, which we think of as a lattice sequence in V (see [BL02]). Associated to Λ, we have a filtration $(\mathfrak{a}_n)_{n\in\mathbb{Z}}$ of A (which is the Moy–Prasad filtration) and the integer q is defined by $\beta \in \mathfrak{a}_{-q}\backslash\mathfrak{a}_{1-q}$; this is required to be positive. Finally, r is an integer between 0 and q which is small enough in the following approximate sense: the stratum $[\Lambda, q, r, \beta]$ corresponds to the coset $\beta + \mathfrak{a}_{-r}$ and r must be small enough so that the *formal intertwining* of the coset has a nice formula involving the centralizer of β. (See Section 6 for more details, and a precise definition.) A semisimple stratum $[\Lambda, q, r, \beta]$ as above splits according to the primitive idempotents 1^i of E, giving *simple* strata $[\Lambda^i, q_i, r, \beta_i]$ in $V^i = 1^i V$, which are studied in [BK93]. In particular, a semisimple stratum is simple if and only if its indexing set I has cardinality one.

Associated to any semisimple stratum $[\Lambda, q, r, \beta]$, and for any integer $m \geqslant 0$, we have a family $\mathcal{C}(\Lambda, m, \beta)$ of *semisimple characters*. We do not recall the definition here (see Section 9) but note only that, by applying the idempotents, we obtain from a semisimple character θ a collection of *simple* characters θ_i, for $i \in I$. For simple characters, the fundamental rigidity property proved in [BK93] for lattice chains (i.e. sequences without repetition), is the following:

> Suppose $\theta \in \mathcal{C}(\Lambda, m, \beta)$ and $\theta' \in \mathcal{C}(\Lambda, m, \beta')$ are *simple* characters which intertwine in $\tilde{G}$. Then they are conjugate in the parahoric subgroup $\tilde{\mathrm{U}}(\Lambda)$.

In the case of semisimple characters, this result is false as soon as $\#I > 1$: the essential reason is that one can have two lattice sequences (or even chains) Λ, Λ' which are conjugate in $\tilde{G}$ but such that the separate pieces Λ^i, Λ'^i are not (all) conjugate in $\mathrm{Aut}_F(V^i)$. Equivalently, there are points in the building of a proper Levi subgroup of $\tilde{G}$ which are not conjugate under the Levi but are conjugate under $\tilde{G}$. For similar reasons, the result would remain false if one weakened the conclusion to only conjugacy under $\tilde{G}$. Thus one needs an extra condition to ensure that intertwining implies conjugacy. In order to describe this condition, we need a "matching theorem" for semisimple characters which intertwine:

Theorem (see Theorem 10.1)**.** *Let $\theta \in \mathcal{C}(\Lambda, m, \beta)$ and $\theta' \in \mathcal{C}(\Lambda', m, \beta')$ be semisimple characters which intertwine in $\tilde{G}$ and suppose that Λ and Λ' have the same period. Then there is a unique bijection ζ between the index sets I and I' such that the simple characters θ_i and $\theta'_{\zeta(i)}$ are intertwined by an isomorphism in $\mathrm{Hom}_F(V^i, V'^{\zeta(i)})$.*

This matching theorem allows us to describe a condition which is certainly necessary for conjugacy: if θ, θ' as in the theorem are conjugate by an element of the parahoric subgroup $\tilde{\mathrm{U}}(\Lambda)$ then, with $\zeta : I \to I'$ the matching given by the previous theorem, we have

$$\dim_{\kappa_F} \Lambda^i_l/\Lambda^i_{l+1} = \dim_{\kappa_F} \Lambda'^{\zeta(i)}_l/\Lambda'^{\zeta(i)}_{l+1}, \qquad \text{for all } i \in I \text{ and } l \in \mathbb{Z}. \tag{1.1}$$

Equivalently, the isomorphism in the theorem which intertwines the characters maps the point in the building corresponding to Λ^i to a point conjugate to the point corresponding to $\Lambda'^{\zeta(i)}$. It turns out that this condition is also sufficient to obtain an "intertwining implies conjugacy" result:

Theorem (see Theorem 10.2)**.** *Let $\theta \in \mathcal{C}(\Lambda, m, \beta)$ and $\theta' \in \mathcal{C}(\Lambda, m, \beta')$ be semisimple characters which intertwine in $\tilde{G}$, let $\zeta : I \to I'$ be the matching given by Theorem 10.1, and suppose that the condition* (1.1) *holds. Then θ is conjugate to θ' by an element of* $\tilde{\mathrm{U}}(\Lambda)$.

Now we turn to our results for classical groups. Suppose that our underlying strata $[\Lambda, q, r, \beta]$ are *skew* – that is, β is in the Lie algebra of G, the associated decomposition of V is orthogonal with respect to the hermitian form, and Λ is in the building of the centralizer in G of β (see [BS09]). Our first main result here is a Skolem–Noether theorem for semisimple strata, which is crucial in the sequel [KSS16].

Theorem (see Theorem 7.12)**.** *Let $[\Lambda, q, r, \beta]$ and $[\Lambda', q, r, \beta']$ be two skew-semisimple strata which intertwine in G, and suppose that β and β' have the same characteristic polynomial. Then, there is an element $g \in G$ such that $g\beta g^{-1} = \beta'$.*

Note that, for β as in the theorem, the number of G-orbits in the Lie algebra of G with the same characteristic polynomial as β is $2^{\#I}$, $2^{\#I-1}$ or $2^{\#I-2}$, depending on G and β; thus some additional condition is certainly necessary to conclude that β, β' are conjugate.

Given a skew-semisimple stratum $[\Lambda, q, r, \beta]$, the set $\mathcal{C}_-(\Lambda, m, \beta)$ of semisimple characters for G is obtained by restricting the semisimple characters in $\mathcal{C}(\Lambda, m, \beta)$. Equivalently, one may just restrict those semisimple characters which are invariant under the involution defining G. Our final result is an "intertwining implies conjugacy" theorem for semisimple characters for G.

Theorem (see Theorem 10.3)**.** *Let $\theta_- \in \mathcal{C}_-(\Lambda, m, \beta)$ and $\theta'_- \in \mathcal{C}_-(\Lambda, m, \beta')$ be two semisimple characters of G, which intertwine over G, and assume that their matching satisfies* (1.1)*. Then, θ_- and θ'_- are conjugate under $\mathrm{U}(\Lambda) = \tilde{\mathrm{U}}(\Lambda) \cap G$.*

This is the first step in an "intertwining implies conjugacy" result for cuspidal types proved in the sequel [KSS16], which then completes the classification of cuspidal representations of G.

Let us say a few words about the proofs of these results, beginning with those for general linear groups. Since a semisimple character is defined in terms of a semisimple stratum underlying it, we must first prove similar results for strata. One major complication here is that, although a semisimple stratum $[\Lambda, q, r, \beta]$ determines the associated splitting $V = \bigoplus_{i \in I} V^i$, since it comes from the idempotents of $E = F[\beta]$, one may have *equivalent* strata with different splittings.

Thus we prove that, given two semisimple strata $[\Lambda, q, r, \beta]$ and $[\Lambda', q, r, \beta']$ which intertwine and such that Λ, Λ' have the same period, there is a canonical matching between the index sets I, I' of their splittings (see Proposition 7.1). The proof of this is by induction: when the strata are *minimal* (that is, $r = q - 1$), we match the primary factors of the characteristic polynomials of the strata (see Definition 6.6), which are equal by intertwining. The inductive step requires a careful analysis of the derived strata of a semisimple stratum. As a consequence of this, one see that if the initial strata are in fact *equivalent*, then there is an element of $\tilde{G}$ which normalizes the (equivalence class of the) strata and conjugates the two splittings (see Lemma 7.18).

As is the case for simple characters, the fact that a semisimple character does *not* determine the underlying stratum (even up to equivalence) presents additional difficulties. First, when we have a semisimple character θ which can be defined relative to two different strata, we need a matching between their associated splittings, which is given by conjugation by an element of the normalizer of θ (see Proposition 9.9). The key result, which allows one to perform induction along defining sequences for semisimple characters, is an analogue of Bushnell–Kutzko's "Translation Principle" for simple characters (see Theorem 9.16). A crucial step in this is to characterize when a stratum of the form $[\Lambda, q, q-1, \beta]$ is equivalent to a semisimple one (see Proposition 6.11). With these tools all to hand, we are able to prove the main matching and "intertwining implies conjugacy" theorems for semisimple characters.

Now we pass our attention to the skew-semisimple case. We begin with an analysis of the Witt groups $W_*(E)$ of finite field extensions E of F. Given an equivariant form $\lambda : E \to F$, we get a trace map from $W_*(E)$ and $W_*(F)$ and it is the understanding of this map that allows us to make progress. In particular, the map takes hermitian E-spaces of maximal anisotropic dimension to hermitian F-spaces of maximal anisotropic dimension (see Theorem 4.4); moreover, outside the symplectic case the map is injective on spaces of a given dimension. One deduces from this that, again outside the symplectic case, when there is a self-dual embedding of a field extension E into a hermitian F-space, it is unique up to conjugation. In the symplectic case, this is not true but we prove a Skolem–Noether for *simple* strata which intertwine (see Theorem 5.2); this is proved by using the strata to twist the symplectic form into orthogonal forms and then using a result on lifting approximate isometries.

With this to hand, the scheme of proof of "intertwining implies conjugacy" for skew-semisimple characters is formally very similar to the case of $\tilde{G}$ described above, beginning with the strata and then proceeding to characters, but we must prove that the matchings obtained along the way give isometries between the spaces V^i (which are all hermitian spaces). In general the major difficulty occurs at the base step of an induction; for example, the base case of Proposition 7.10 – that the matching for skew-semisimple strata which intertwine gives isometries – is proved using an idempotent lifting result.

We finish with a brief description of the organization of the paper. After setting up notation, we begin with some basic results on classical groups: in Section 3 we prove results on the lifting of approximate isometries in a hermitian space; in Section 4 we analyze the Witt groups $W_*(E)$ of finite field extensions E of F and trace-like maps from $W_*(E)$ to $W_*(F)$; and in Section 5 we prove the first Skolem–Noether result, for embeddings of a field (the simple case). Next we look at semisimple strata: in Section 6 we recall the definitions and some fundamental results; in Section 7 we prove that intertwining semisimple strata have a matching, and prove the Skolem–Noether theorem above; and in Section 8 we prove an intertwining implies conjugacy result for semisimple strata. Finally, we turn to semisimple characters: in Section 9 we recall the definitions and recall or prove many basic results, in particular the translation principles; and in Section 10 we prove the remaining main results.

2. Notation

Let F be a nonarchimedean local field of odd residual characteristic with valuation ν_F and equipped with an involution ρ (which may be trivial) with fixed field F_0. We write $o_F, \mathfrak{p}_F$ and κ_F for the valuation ring, its maximal ideal and the residue field of F respectively, and we assume that the image of the additive valuation $\nu := \nu_F$ is $\mathbb{Z} \cup \{\infty\}$. We also denote by $x \mapsto \bar{x}$ the reduction map $o_F \twoheadrightarrow \kappa_F = o_F/\mathfrak{p}_F$. We fix a symmetric or skew-symmetric uniformizer $\varpi \in \mathfrak{p}_F \backslash \mathfrak{p}_F^2$. We use similar notation for other nonarchimedean local fields. If $E|F$ is an algebraic field extension then we write E^{ur} for the maximal unramified subextension of $E|F$.

Let h be an ϵ-hermitian form (with $\epsilon = \pm 1$) on an F-vector space V of finite dimension, i.e. for all $v_1, v_2 \in V$ and $x, y \in F$ the bi-additive form h satisfies

$$h(v_1 x, v_2 y) = \rho(x)\epsilon\rho(h(v_2, v_1))y.$$

We denote the ring of F-endomorphisms of V by A and its group of units $A^\times$ by $\tilde{G}$. Let G be the group of all elements g of $\tilde{G}$ such that $h(gv_1, gv_2)$ is equal to $h(v_1, v_2)$, for all vectors v_1, v_2; this is the group of points of a reductive group over F_0, which is connected unless $F = F_0$ and $\epsilon = +1$, in which case it is the full orthogonal group. Let $\sigma = \sigma_h$ be the adjoint anti-involution of h on A. For a σ-stable subset M of A, we write M_+ for the set of symmetric elements and M_- for the set of skew-symmetric elements.

An o_F-lattice in V is a free o_F-module M of dimension m. The dual $M^\#$ of M with respect to h is the set of all vectors v of V such that $h(v, M)$ is a subset of $\mathfrak{p}_F$. A *lattice sequence* in V is a map Λ from $\mathbb{Z}$ to the set of o_F-lattices of V satisfying

(i) $\Lambda_s \subseteq \Lambda_t$, for all integers $s > t$, and
(ii) $\Lambda_s \varpi = \Lambda_{s+e}$ for some (unique) integer e and all integers s.

We call $e =: e(\Lambda|o_F)$ the o_F-period of Λ. An injective lattice sequence is called a *lattice chain*. For each integer s, we denote by $x \mapsto \bar{x}$ the reduction map $\Lambda_s \twoheadrightarrow \Lambda_s/\Lambda_{s+1}$. A lattice sequence Λ is called *self-dual* if there is an integer u such that $(\Lambda_s)^\# = \Lambda_{u-s}$.

As usual, a lattice sequence Λ determines the following filtrations of A and A_- (if Λ is self-dual): $\mathfrak{a}_i(\Lambda)$ is the set of all elements of A which map Λ_s into Λ_{s+i} for all integers s and $\mathfrak{a}_{-,i}(\Lambda)$ is the intersection of $\mathfrak{a}_i(\Lambda)$ with A_-. We skip the argument Λ if there is no cause of confusion and we write $\mathfrak{a}'_i$ if there is a second lattice sequence Λ' given.

The sequence Λ also induces filtrations on $\tilde{\mathrm{U}}(\Lambda) := \mathfrak{a}_0^\times$ by $\tilde{\mathrm{U}}^i(\Lambda) = 1 + \mathfrak{a}_i$ and, when Λ is self-dual, on $\mathrm{U}(\Lambda) := \tilde{\mathrm{U}}(\Lambda) \cap G$ by $\mathrm{U}^i(\Lambda) = \tilde{\mathrm{U}}^i(\Lambda) \cap G$ for $i \in \mathbb{N}$. The filtration on A defines a "valuation map" ν_Λ as follows: for $\beta \in A$, we put $\nu_\Lambda(\beta) = \sup\{i \mid \beta \in \mathfrak{a}_i\}$, an integer or ∞. The normalizer $\mathfrak{n}(\Lambda)$ of Λ is the set of elements $g \in A^\times$ such that $\nu_\Lambda(g^{-1}) = -\nu_\Lambda(g)$.

The *translation* of Λ by $s \in \mathbb{Z}$ is the lattice sequence $(\Lambda + s)_i := \Lambda_{i-s}$, and we define the direct sum $\Lambda \oplus \Lambda'$ of two lattice sequences Λ and Λ' of the same period as $(\Lambda_j \oplus \Lambda'_j)_{j \in \mathbb{Z}}$. The lattice sequence

$$\Lambda \oplus (\Lambda + 1) \oplus \cdots \oplus (\Lambda + e(\Lambda|o_F) - 1)$$

is always a lattice chain. By this construction, many theorems in [BK93] proven for lattice chains are valid for lattice sequences (cf. [Ste05], and also [KS15], where this is called a †-construction). If this is the case, or the proof of a result for lattice chains is valid for lattice sequences without change, then, in the following, we just refer to the statement for lattice chains.

Finally, for x a real number, we denote by $\lfloor x \rfloor$ the greatest integer not greater than x.

3. Lifting isometries

The isomorphism type of the hermitian space (V, h) is encoded in any self-dual lattice sequence of V, as explained in this section. The main results are Proposition 3.1 and Corollary 3.2, which explain how an approximate isometry (for example, one which induces an isometry at the level of residue fields) can be lifted to a genuine isometry. Let us state the main proposition:

Proposition 3.1. *Let $F|F'$ be a finite field extension. Suppose we are given two finite-dimensional ϵ-hermitian spaces (V, h) and (V', h') with respect to (F, ρ), an F'-linear isomorphism $f : V \to V'$ and two self-dual o_F-lattice sequences Λ and Λ' of (V, h) and (V', h'), respectively, such that, for all $i \in \mathbb{Z}$,*

- $f(\Lambda_i) = \Lambda'_i$,
- $\overline{f((\Lambda_i)^\#)} = \overline{f(\Lambda_i)^\#}$,
- $\overline{h'(f(v), f(w))} = \overline{h(v, w)} \in \kappa_F$, *for all* $v \in \Lambda_i$, $w \in (\Lambda_{i+1})^\#$ *and*
- $\overline{f(vx)} = \overline{f(v)x} \in \Lambda'_{i+e(\Lambda'|o_F)\nu_k(x)}/\Lambda'_{1+i+e(\Lambda'|o_F)\nu_k(x)}$, *for all* $v \in \Lambda_i$, $x \in F^\times$.

Then there is an F-linear isometry g from (V, h) to (V', h') mapping Λ to Λ' such that $(f - g)(\Lambda_i) \subseteq \Lambda'_{i+1}$ for all integers i.

Later it will be useful to have a stronger approximation statement. For that we introduce a generalization of the adjoint anti-involution. For two finite-dimensional ϵ-hermitian spaces (V,h) and (V',h') with respect to (F,ρ) there is a map $\sigma_{h,h'}$ from $\mathrm{Hom}_F(V,V')$ to $\mathrm{Hom}_F(V',V)$ defined, for $f\in\mathrm{Hom}_F(V,V')$, by the equation

$$h'(f(v),w)=h(v,\sigma_{h,h'}(f)(w)),\qquad \text{for } v\in V,\ w\in V'.$$

Corollary 3.2. *Let (V^1,h_1) and (V^2,h_2) be two ϵ-hermitian spaces over F (for the same ϵ), let Λ^1 be a self-dual lattice sequence and let $f:V^1\to V^2$ be an F-linear isomorphism such that $\Lambda^2:=f(\Lambda^1)$ is self-dual. Suppose U_i is a closed subgroup of $\tilde{\mathrm{U}}^1(\Lambda^1)$ which is invariant under σ_{h_i} for $i=1,2$, such that $\sigma_{h_1,h_2}(f)\in U_1f^{-1}U_2$. Then there is an isometry from (V^1,h_1) to (V^2,h_2) contained in U_2fU_1.*

Proof. The ϵ-hermitian spaces (V_1,h_1) and (V_2,h_2) are isometric by a map which sends Λ^1 to Λ^2, by Proposition 3.1. Thus we can restrict to the case where $(V_1,h_1)=(V_2,h_2)=:(V,h)$ and $f(\Lambda_1)=\Lambda_2=\Lambda$. By assumption, the double coset U_2fU_1 is invariant under the automorphism $g\mapsto\sigma_h(g^{-1})$, and this double coset thus has a fixed point, by [KS15, 2.7(ii)(a)] and [Ste01a, 2.2]. □

We need a sequence of lemmas to prove Proposition 3.1.

Lemma 3.3. *Suppose that Λ is a self-dual lattice chain of period 1 such that $\Lambda_0^{\#}=\Lambda_1$. Consider the form*

$$\bar h:\Lambda_0/\Lambda_1\times\Lambda_0/\Lambda_1\to\kappa_F$$

defined by $\bar h(\bar v,\bar w)=\overline{h(v,w)}$. Then every Witt basis of $(\Lambda_0/\Lambda_1,\bar h)$ lifts to a Witt basis of (V,h) contained in Λ_0, under the projection $\Lambda_0\twoheadrightarrow\Lambda_0/\Lambda_1$.

Proof. Let $\mathcal{B}$ be a Witt basis of $\bar h$. We have

$$\mathcal{B}=\mathcal{B}_0\,\dot\cup\,\mathcal{B}_{1,-1}\,\dot\cup\,\mathcal{B}_{2,-2}\,\dot\cup\ldots\dot\cup\,\mathcal{B}_{r,-r},$$

where $\mathcal{B}_{i,-i}$ spans a hyperbolic space, $\mathcal{B}_0$ spans an anisotropic space, and all these spaces are pairwise orthogonal to each other in Λ_0/Λ_1. Further we have a decomposition

$$\mathcal{B}_0=\mathcal{B}_{0,1}\,\dot\cup\,\mathcal{B}_{0,2}\,\dot\cup\ldots\dot\cup\,\mathcal{B}_{0,t}$$

into pairwise orthogonal sets of cardinality one. Take an arbitrary lift $\mathcal{B}'^{(0)}$ of $\mathcal{B}$ to Λ_0; for an element $\bar v\in\mathcal{B}$, we write $v\in\mathcal{B}'^{(0)}$ for its lift.

Step 1: Consider $\mathcal{B}_{0,1}=\{\bar v_0\}$; put $W:=v_0^{\perp}$ and define

$$\mathcal{B}'^{(1)}:=\{\mathrm{proj}_W(v)\mid v\in\mathcal{B}'^{(0)}\backslash\{v_0\}\}\cup\{v_0\},$$

where proj_W denotes the orthogonal projection onto W. We recall the formula

$$\mathrm{proj}_W(v)=v-v_0\frac{h(v_0,v)}{h(v_0,v_0)}$$

and conclude that $\overline{h(\mathrm{proj}_W(v),\mathrm{proj}_W(v'))}$ is equal to

$$\overline{h(v,v')}-\overline{\frac{h(v_0,v')h(v,v_0)}{h(v_0,v_0)}}-\overline{\frac{\rho(h(v_0,v))h(v_0,v')}{\rho(h(v_0,v_0))}}+\overline{\frac{\rho(h(v_0,v))h(v_0,v')}{\rho(h(v_0,v_0))}}$$

and therefore equal to $\overline{h(v,v')}$ for all $v, v' \in B'^{(0)}$. Thus, replacing (V,h) by $(W, h_{|W})$ and Λ by its intersection with W and then repeating, we can assume that $\mathcal{B}_0$ is empty.

Step 2: Consider $\mathcal{B}_{1,-1} = \{\bar{v}_1, \bar{v}_{-1}\}$ and define now $W := \{v_1, v_{-1}\}^{\perp}$. Then, as in Step 1, elements v and v' of

$$\mathcal{B}'^{(1)} := \{\mathrm{proj}_W(v) \mid v \in \mathcal{B}'^{(0)} \backslash \{v_1, v_{-1}\}\} \cup \{v_1, v_{-1}\}$$

satisfy $\overline{h(\mathrm{proj}_w(v), \mathrm{proj}_W(v'))} = \overline{h(v,v')}$, because if $v \in \mathcal{B}'^{(0)} \backslash \{v_1, v_{-1}\}$ then

$$v \equiv \mathrm{proj}_W(v) + v_{-1} h(v_1, v) + v_1 \epsilon h(v_{-1}, v) \pmod{\Lambda_1}.$$

Thus we have reduced to the hyperbolic case that $\mathcal{B}$ is equal to $\mathcal{B}_{1,-1}$.

Step 3: We have $\mathcal{B} = \mathcal{B}_{1,-1} = \{\bar{v}_1, \bar{v}_{-1}\}$. The sequence $(w_i)_{i \geqslant 1}$,defined by $w_1 := v_1$ and

$$w_{i+1} := w_i - v_{-1} \frac{h(w_i, w_i)}{2}, \qquad \text{for } i \geqslant 1,$$

has a limit v_1' which satisfies $h(v_1', v_1') = 0$ and $\bar{v}_1' = \bar{v}_1$, and analogously we find v_{-1}' with similar properties. Then

$$\mathcal{B}'^{(1)} := \left\{ \frac{1}{v_1' \rho(h(v_1', v_{-1}'))}, v_{-1}' \right\}$$

is a Witt basis of V which lifts $\mathcal{B}$. □

Lemma 3.4. *Suppose that Λ is a self-dual lattice chain of period 1 such that $\Lambda_0^{\#} = \Lambda_0$. Consider the form*

$$\bar{h} : \Lambda_0/\Lambda_1 \times \Lambda_0/\Lambda_1 \to \kappa$$

defined by $\bar{h}(\bar{v}, \bar{w}) = \overline{h(v,w)\varpi^{-1}}$. For every Witt basis $\mathcal{B} = \mathcal{B}_0 \cup \mathcal{B}^- \cup \mathcal{B}^+$ of $(\Lambda_0/\Lambda_1, \bar{h})$, with isotropic parts $\mathcal{B}^-$ and $\mathcal{B}^+$ and anisotropic part $\mathcal{B}_0$, there is a Witt basis $\mathcal{B}' = \mathcal{B}_0' \cup \mathcal{B}'^+ \cup \mathcal{B}'^-$ contained in Λ_{-1} of (V,h) such that $\mathcal{B}_0'$, $\mathcal{B}'^+$ and $\mathcal{B}'^- \varpi$ are lifts of $\mathcal{B}_0$, $\mathcal{B}^+$ and $\mathcal{B}^-$ under the projection $\Lambda_0 \twoheadrightarrow \Lambda_0/\Lambda_1$, respectively.

Here we explicitly make use of the fact that $\rho(\varpi) \in \{\varpi, -\varpi\}$.

Proof. This follows directly from Lemma 3.3 if we substitute h by $h\varpi^{-1}$. □

We need a third base case for period 2.

Lemma 3.5. *Suppose that Λ is a self-dual lattice chain of period 2 such that $\Lambda_0^{\#} = \Lambda_0$. Then h has anisotropic dimension zero and for any basis $\mathcal{B}_0$ of Λ_0/Λ_1 there is a Witt basis for h,*

$$\mathcal{B}' = \mathcal{B}_{-1}' \cup \mathcal{B}_0',$$

such that $\mathcal{B}_i'$ is a subset of $\Lambda_i \backslash \Lambda_{i+1}$ for all i and such that $\mathcal{B}_0'$ is a lift of $\mathcal{B}_0$ under the projection $\Lambda_0 \twoheadrightarrow \Lambda_0/\Lambda_1$. Further, h vanishes on $\mathcal{B}_0' \times \mathcal{B}_0'$.

Proof. First we prove that h is hyperbolic. Suppose for contradiction that it has positive anisotropic dimension, i.e. let v be an anisotropic vector and part of a Witt basis for h which splits Λ. We can multiply v by a scalar such that $h(v,v)$ is a unit or a uniformizer of F. We treat only the second case, because the first one is similar. There is an index i such that $\Lambda_i \cap vF$ is equal to vo_F, and then this is equal to $\Lambda_i \cap vF$ because $h(v,v)$ is uniformizer. Since, for all lattices in the image of Λ the homothety class is invariant under dualization, we obtain that the index has to be zero. Thus, $\Lambda_{-1} \cap vF = \mathfrak{p}_F^{-1}$ is equal to $\varpi^{-2}(\Lambda_1 \cap vF)$, which is a contradiction.

Now let us construct the lift. We start with a Witt basis $\mathcal{B}''$ for h which splits Λ. Let $\mathcal{B}_0''$ be the set of elements v of $\mathcal{B}''$ such that

$$vF \cap \Lambda_0 \neq vF \cap \Lambda_1,$$

and let W_0 be the span of $\mathcal{B}_0''$. We prove that the restriction of h to W_0 is zero. We define, for $v \in \mathcal{B}''$, the element v^* to be the element of $\mathcal{B}''$ such that $h(v,v^*)$ is non-zero, i.e. equal to 1 or -1. If there is an element $v \in W_0 \cap \mathcal{B}''$ such that $v^* \in W_0$ then $\Lambda_{-1} \cap (vF + v^*F) = \Lambda_0 \cap (vF + v^*F)$ and thus this coincides with $(\Lambda_{-1})^\# \cap (vF + v^*F)$. This is a contradiction because $(\Lambda_{-1})^\#$ is equal to Λ_1. This shows that h is zero on W_0. Thus, multiplying elements of $\mathcal{B}_0''$ by scalars if necessary, we can assume that $\mathcal{B}_0''$ is a subset of $\Lambda_0 \backslash \Lambda_1$. By the definition of W_0 we have that, for all $v \in \mathcal{B}_0''$, the intersection of vF with Λ_{-1} is vo_F for all $v \in \mathcal{B}_0''$ and thus taking duals we get that the intersection of v^*F with Λ_1 is $v^*\mathfrak{p}_F$, and thus $\mathcal{B}''$ is a subset of $\Lambda_{-1} \backslash \Lambda_0$. Thus, we have now found a basis $\mathcal{B}''$ satisfying all the conditions except that $\mathcal{B}_0''$ need not be a lift of $\mathcal{B}_0$. Now a base change from $\mathcal{B}_0''$ to a lift of $\mathcal{B}_0$ in W_0, together with the adjoint base change on the span of $\mathcal{B}'' \backslash \mathcal{B}_0''$, finishes the proof. □

Corollary 3.6. *Under the assumptions of Lemma 3.5 there is a unique κ-basis $\mathcal{B}_{-1}$ of Λ_{-1}/Λ_0 such that, for all elements x of $\mathcal{B}_0$, there is exactly one element y of $\mathcal{B}_{-1}$ such that*

$$\bar{h}(y,z) \begin{cases} 1 & , \text{ if } z = x \\ 0 & , \text{ if } z \in \mathcal{B}_0 \backslash \{x\} \end{cases}$$

where $\bar{h} : \Lambda_{-1}/\Lambda_0 \times \Lambda_0/\Lambda_1 \to \kappa$ is the form induced from h. Further, there is a Witt basis for h which lifts $\mathcal{B}_0 \cup \mathcal{B}_{-1}$.

Proof. By Lemma 3.5 the form $\bar{h}$ is non-degenerate and thus identifies the dual of Λ_0/Λ_1 with Λ_{-1}/Λ_0 with σ-twisted κ-action. We take for $\mathcal{B}_{-1}$ the basis dual to $\mathcal{B}_0$. The remaining part follows from Lemma 3.5. □

We put together the two previous results to treat the general case.

Lemma 3.7. *Let Λ be a self-dual lattice chain of period e and let $\mathcal{B}$ be a subset of V satisfying the following conditions:*

(i) $(\Lambda_0)^\# \in \{\Lambda_0, \Lambda_1\}$;

(ii) $\mathcal{B} = \bigcup_{i=\lfloor \frac{1-e}{2} \rfloor}^{\lfloor \frac{e-1}{2} \rfloor} \mathcal{B}_i$, *with* $\mathcal{B}_i \subseteq \Lambda_i \backslash \Lambda_{i+1}$;

(iii) *$\bar{\mathcal{B}}_i$, the image of $\mathcal{B}_i$ in Λ_i/Λ_{i+1}, is a basis of Λ_i/Λ_{i+1};*
(iv) *for all $i \in \{0, 1, \ldots, \lfloor \frac{e-1}{2} \rfloor\}$ with $(\Lambda_i)^\# \notin \{\Lambda_{i+1}, \Lambda_{i+1-e}\}$ and all $v \in \mathcal{B}_i$ there exists a unique $v' \in \mathcal{B} \cap (\Lambda_{i+1})^\# \backslash (\Lambda_i)^\#$ such that $\overline{h(v, v')} = \bar{1}$;*
(v) *if $(\Lambda_0)^\# = \Lambda_1$ then $\bar{\mathcal{B}}_0$ is a Witt basis for $(\Lambda_0/\Lambda_1, \bar{h})$;*
(vi) *if $(\Lambda_{\lfloor \frac{e-1}{2} \rfloor})^\# = \Lambda_{\lfloor \frac{1-e}{2} \rfloor}$ then $\bar{B}_{\lfloor \frac{e-1}{2} \rfloor}$ is a Witt basis of $(\Lambda_{\lfloor \frac{e-1}{2} \rfloor}/\Lambda_{\frac{e+1}{2}}, \overline{h\varpi^{-1}})$.*

Then there is a basis $\mathcal{B}'$ of (V, h) such that

(a) *$\mathcal{B}' = \bigcup_{i=\lfloor \frac{1-e}{2} \rfloor}^{\lfloor \frac{e-1}{2} \rfloor} \mathcal{B}'_i$, where $\mathcal{B}'_i := \mathcal{B}' \cap (\Lambda_i \backslash \Lambda_{i+1})$, for all i,*
(b) *$\bar{\mathcal{B}}'_i = \bar{\mathcal{B}}_i$, for all i, and*
(c) *$\mathcal{B}'$ is a Witt basis of (V, h) up to multiplication of some isotropic elements of $\mathcal{B}'_{\lfloor \frac{e-1}{2} \rfloor}$ by ϖ^{-1}.*

Proof. The lattice chain Λ is split by a Witt decomposition; that is, there are pairwise orthogonal ϵ-hermitian spaces V^i, $i \in \{0, \ldots, \lfloor \frac{e-1}{2} \rfloor\}$ whose sum is V such that

$$(V^i \cap \Lambda_i) + \Lambda_{i+1} = \Lambda_i \text{ and } V^i \cap (\Lambda_{i+1})^\# + (\Lambda_i)^\# = (\Lambda_{i+1})^\#.$$

Counting dimensions we deduce that $V^i \cap \Lambda_j$ is a subset of Λ_{j+1} for all j with $(\Lambda_j)^\# \notin \{\Lambda_i a, (\Lambda_{i+1})^\# a \mid a \in F^\times\}$. Now consider, for $1 \leqslant i \leqslant \lfloor \frac{e-1}{2} \rfloor$,

$$\tilde{\mathcal{B}}_i := \left\{ \operatorname{proj}_{V^i}(v) \mid v \in \mathcal{B}_j \text{ and } \Lambda_j \in \{\Lambda_i, (\Lambda_{i+1})^\#\} \right\}.$$

For each i, the lattice sequence $\Lambda \cap V^i$ in V^i is a multiple of a lattice chain of period 1. Thus, after scaling, we can apply Lemma 3.3 or 3.4 or Corollary 3.6 on $(V^i, \Lambda \cap V^i, \bar{\tilde{\mathcal{B}}}_i)$ to obtain $\mathcal{B}'_i$. □

Proof of Proposition 3.1. We only have to prove that we can replace f by an F-linear isomorphism, i.e. that we can reduce the argument to $F = F'$. The rest follows directly from Lemma 3.7.

Since the statement depends only on $\operatorname{im}(\Lambda)$, without loss of generality assume that Λ, and therefore Λ' also, is a chain. Take a κ_F-basis $(\bar{v}_{ij})_j$ of Λ_i/Λ_{i+1} and lift it to $(v_{ij})_j$, for $i = 0, \ldots, e(\Lambda|o_F)$. Then $(v_{ij})_{ij}$ is an F-splitting basis of Λ. Similarly we choose a lift $(w_{ij})_{ij}$ for $(\overline{f(v_{ij})})_{ij}$. The F-linear map $\tilde{f}$ which maps v_{ij} to w_{ij} satisfies the assumptions of the Proposition and $(f - \tilde{f})(\Lambda_i) \subseteq \Lambda_{i+1}$, for all $i \in \mathbb{Z}$. Thus we can replace f by $\tilde{f}$. □

4. Witt groups

In this section we fix a finite field extension $E|F$ and an involution ρ' extending ρ and we denote E_0 the set of ρ'-fixed points in E. We fix a non-zero ρ'-ρ-equivariant F-linear map

$$\lambda : E \to F.$$

We heavily use in this section that the residue characteristic of F is odd. We will see that the map λ induces in a natural way a map from the Witt group $W_{\rho',\epsilon}(E)$ of (ρ',ϵ)-hermitian forms over E to the Witt group $W_{\rho,\epsilon}(F)$.

We recall that the Witt group $W_{\rho,\epsilon}(F)$ is the set of equivalence classes of (ρ,ϵ)-hermitian forms over F, where we say two such forms are equivalent if their maximal anisotropic direct summands are isometric. We write $\langle h\rangle$ for the class in $W_{\rho,\epsilon}(F)$ of signed forms equivalent to h; similarly, for a (skew-) symmetric matrix M, we write $\langle M\rangle$ for the class of signed hermitian forms equivalent to the form with Gram-matrix M under the standard basis.

The group structure on $W_{\rho,\epsilon}(F)$ is induced by the orthogonal sum. Let us recall its structure:

Theorem 4.1. *The Witt group $W_{\rho,\epsilon}(F)$ is isomorphic to*

(i) *the trivial group if ρ is trivial and $\epsilon=-1$;*
(ii) *$C_2\times C_2$ if $-1\in(F^\times)^2$ and ρ is non-trivial;*
(iii) *C_4 if ρ non-trivial and $-1\notin(F^\times)^2$;*
(iv) *$C_2\times C_2\times C_2\times C_2$ if $-1\in(F^\times)^2$, $\epsilon=1$ and ρ is trivial;*
(v) *$C_4\times C_4$ if $-1\notin(F^\times)^2$, $\epsilon=1$ and ρ is trivial.*

Proof. The proof is an easy conclusion of the classification of the hermitian forms using Witt bases, given for example in [BT87, 1.14], and is left to the reader. □

When it is non-trivial, the group $W_{\rho,\epsilon}(F)$ is generated by the classes of one-dimensional anisotropic spaces. For example, if $\epsilon=1$ then: in the case $F\neq F_0$, the one-dimensional anisotropic spaces are $\langle(1)\rangle$ and $\langle(\delta)\rangle$, with $\delta\in F_0^\times\backslash N_{F/F_0}(F^\times)$; in the case $F=F_0$, the one-dimensional anisotropic spaces are $\langle(1)\rangle$, $\langle(\varpi)\rangle$, $\langle(\delta)\rangle$ and $\langle(\delta\varpi)\rangle$, with δ a non-square unit in o_F.

Definition 4.2. We define $\mathrm{Tr}_{\lambda,\epsilon}$ from $W_{\rho',\epsilon}(E)$ to $W_{\rho,\epsilon}(F)$ by

$$\langle\tilde{h}\rangle\mapsto\langle\lambda\circ\tilde{h}\rangle=:\mathrm{Tr}_{\lambda,\epsilon}(\langle\tilde{h}\rangle).$$

If $E|F$ is tamely ramified and $\lambda=\mathrm{tr}_{E|F}$ then we write $\mathrm{Tr}_{E|F,\rho',\epsilon}$ for $\mathrm{Tr}_{\lambda,\epsilon}$.

For the remainder of the section we often skip the subscripts in Tr.

Example 4.3. In general, the map $\mathrm{Tr}_{E|F,\rho',\epsilon}$ is not injective, even if $\epsilon=1$. For example consider $E=\mathbb{Q}_3(\sqrt{3},\sqrt{5})$, $F=\mathbb{Q}_3(\sqrt{5})$ and $\rho'(\sqrt{5})=-\sqrt{5}$. Then

$$\mathrm{Tr}_{E|F,\rho',\epsilon}(\langle(\sqrt{3})\rangle)=\left\langle\begin{pmatrix}0&6\\6&0\end{pmatrix}\right\rangle=0,$$

so that $\mathrm{Tr}_{E|F,\rho',\epsilon}$ is not injective. On the other hand, we have that

$$\mathrm{Tr}_{E|F,\rho',\epsilon}(\langle(1)\rangle)=\left\langle\begin{pmatrix}2&0\\0&6\end{pmatrix}\right\rangle\neq 0,$$

In particular, $\mathrm{Tr}_{E|F,\rho',\epsilon}$ maps the class $\langle(\sqrt{3})\oplus(1)\rangle$ of maximal anisotropic dimension to the class $W_{\rho,\epsilon}(F)$ of maximal anisotropic dimension. We will see that this is always the case.

There is a unique element X in $W_{\rho,\epsilon}(F)$ with maximal anisotropic dimension, which we denote by $X_{\rho,\epsilon,F}$. The main result of this section is the following theorem:

Theorem 4.4. $\mathrm{Tr}_\lambda(X_{\rho',\epsilon,E}) = X_{\rho,\epsilon,F}$.

The following definition will be useful both in the proof of Theorem 4.4 and in several other proofs later.

Definition 4.5. Let γ be a (skew-)symmetric element of $\mathrm{Aut}_F(V)$. We define the signed hermitian form

$$h^\gamma : V \times V \to F$$

via

$$h^\gamma(v,w) := h(v,\gamma w),\ v,w \in V.$$

We call h^γ the *(skew-)symmetric twist of h by γ.*

Note that, if h is an ϵ-hermitian form, then h^γ is ϵ-hermitian when γ is symmetric, and $(-\epsilon)$-hermitian when γ is skew-symmetric. Twisting by a symmetric element γ induces a permutation of $W_{\rho,\epsilon}(F)$ and we observe that, by an easy check, the only classes in $W_{\rho,\epsilon}(F)$ which are preserved by every symmetric twist are the trivial class and the class $X_{\rho,\epsilon,F}$ of maximal anisotropic dimension. Indeed, twisting by all symmetric elements gives a transitive action on the classes of spaces of fixed odd (anisotropic) dimension.

Proposition 4.6. *If $E|F$ has odd degree, then,* Tr_λ *is injective.*

Proof. There is nothing to say in the symplectic case, so we assume $\epsilon = 1$ or $F \neq F_0$. Moreover, we can assume that $\epsilon = 1$ because, if $F \neq F_0$ then a twist by a skew-symmetric element of $F^\times$ induces bijections $W_{\rho',1}(E) \to W_{\rho',-1}(E)$ and $W_{\rho,1}(F) \to W_{\rho,-1}(F)$, commuting with Tr_λ. Now $\mathrm{Tr}_\lambda(\langle\langle(1)\rangle\rangle)$ is a class of odd anisotropic dimension, all classes of this anisotropic dimension are symmetric twists of $\mathrm{Tr}_\lambda(\langle\langle(1)\rangle\rangle)$ and they generate $W_{\rho,1}(F)$. Thus Tr_λ is surjective and, moreover, bijective, since $W_{\rho,1}(F)$ is isomorphic to $W_{\rho',1}(E)$ as groups. □

Lemma 4.7. *Suppose $E|F$ is of degree 2 and $F = F_0$. Then* $\mathrm{im}(\mathrm{Tr}_{E|F,\rho',1})$ *has at least four elements. Further:*

- (i) *If $E \neq E_0$ then* $\mathrm{Tr}_{E|F,\rho',1}$ *is injective.*
- (ii) *If $E = E_0$ then the kernel of* $\mathrm{Tr}_{E|F,\rho',1}$ *has exactly 4 elements and they have anisotropic dimension at most 2.*

Proof. Take an element $\delta \in E_0^\times$ and a uniformizer α of E which is skew-symmetric with respect to the generator τ of $\mathrm{Gal}(E|F)$. Then $\mathrm{Tr}_{E|F,\rho',1}(\langle\delta\rangle)$ has Gram matrix

$$\begin{pmatrix} \delta+\tau(\delta) & \alpha(\delta-\tau(\delta)) \\ \pm\alpha(\delta-\tau(\delta)) & \pm\alpha^2(\delta+\tau(\delta)) \end{pmatrix}$$

with respect to the F-basis $\{1, \alpha\}$, where we have $+$ if ρ' is trivial and $-$ if not. Its determinant is $d := \pm 4\alpha^2 N_{E|F}(\delta)$ and we only have to choose δ such that $-d$ is not a square in F to get that $\mathrm{Tr}_{E|F,\rho',1}(\langle(\delta)\rangle)$ is non-zero.

If $-1 \in (E^\times)^2$ then, since p is odd, also $-1 \in N_{E|F}(E^\times)$ and thus we can find $\delta \in E_0^\times$ such that $-d = 4\alpha^2$; this is not a square in $F^\times$ because $\alpha \notin F$. If $-1 \notin (E^\times)^2$ then $E|F$ is ramified and $\nu_F(\alpha^2) = 1$ so we can take $\delta = 1$ to get $-d \notin (F^\times)^2$.

In either case, we have that $\mathrm{Tr}_{E|F,\rho',1}(\langle(\delta)\rangle)$ is non-zero for a suitable δ, and thus of anisotropic dimension 2. Taking symmetric twists of $\mathrm{Tr}_{E|F,\rho',1}(\langle(\delta)\rangle)$ by elements of F (which commute with $\mathrm{Tr}_{E|F,\rho',1}$), we see that the image of $\mathrm{Tr}_{E|F,\rho',1}$ has at least two non-trivial elements and thus, as a subgroup of a 2-group, at least four elements in total. This also shows (i).

We consider now the case $E = E_0$. Take $y \in o_F$ to be a non-square unit if $E|F$ is ramified and a uniformizer of F if $E|F$ is unramified. Then (α) and $(y\alpha)$ are not isomorphic and both are in the kernel of $\mathrm{Tr}_{E|F,\mathrm{id},1}$. Since the kernel consists of at most four elements, it is the subgroup generated by $\langle(\alpha)\rangle$ and $\langle(y\alpha)\rangle$, which is of order four and consists of classes of spaces of anisotropic dimensions $0, 1, 1, 2$. □

Proof of Theorem 4.4. As in the proof of Proposition 4.6, we may assume that $\epsilon = 1$. We only need to prove the statement for one λ, because given two such maps λ_1, λ_2 there is a symmetric element z of E such that $\lambda_1(zx) = \lambda_2(x)$ for all $x \in E$. (We thank R. Kurinczuk for pointing this out.) Moreover, we only have to prove that $\mathrm{Tr}_\lambda(X_{\rho',E,1})$ is non-zero for a suitable λ, since its image is invariant under any symmetric twist with an element of $F_0^\times$, so must be trivial or $X_{\rho,1,F}$.

If E/F is of odd degree then the result follows immediately from Proposition 4.6. Since the result is transitive in towers of extensions, this means we can reduce to the case that E/F is quadratic; in particular, $E|F$ is at worst tamely ramified and we can take $\lambda = \mathrm{tr}_{E|F}$. Moreover, we may replace $E|F$ by $E|F_0$ since, if $\mathrm{Tr}_{E|F_0,\rho',1}(X_{\rho',1,E})$ is non-zero then $\mathrm{Tr}_{E|F,\rho',1}(X_{\rho',1,E})$ is non-zero also. But then, by transitivity again and considering the extensions $E|E_0$ and $E_0|F_0$, we reduce to the case $E|F$ quadratic with $F = F_0$. Now Lemma 4.7 implies that $X_{\rho',1,E}$ is not in the kernel of $\mathrm{Tr}_{E|F,\rho',1}$, as required. □

5. Skolem–Noether

In this section we consider Skolem–Noether-like theorems for classical groups. We take the notation E, ρ', λ from Section 4. We fix two ρ'-σ-equivariant F-algebra embeddings

$$\phi_i : (E, \rho') \to (A, \sigma),\ i = 1, 2.$$

We attach to each ϕ_i an ϵ-hermitian form

$$h_{\phi_i} : V \times V \to E$$

with respect to ρ' such that

$$h = \lambda \circ h_{\phi_i}.$$

For the proof that such a form exists and is unique, see [BS09]. Note that the ϵ-hermitian forms h_{ϕ_i} may differ because the maps ϕ_i may induce different E-actions on V. In particular, two such embeddings ϕ_1, ϕ_2 are conjugate by an element of G if and only if (V, h_{ϕ_1}) is isomorphic to (V, h_{ϕ_2}) as an hermitian E-space.

We then get the following corollary of Theorem 4.4.

Corollary 5.1. *Suppose that ρ' is non-trivial and that either $\epsilon = 1$ or $F \neq F_0$. Then ϕ_1, ϕ_2 are conjugate by an element of $g \in G$, that is*

$$g\phi_1(x)g^{-1} = \phi_2(x), \text{for all } x \in E.$$

Proof. We write $W_{\rho',\epsilon}(E)^0$ for the set of classes of $W_{\rho',\epsilon}(E)$ with even-dimensional anisotropic part. Then $W_{\rho',\epsilon}(E)^0$ only consists of the trivial element and $X_{\rho',\epsilon,E}$ so, by Theorem 4.4, there is a map λ such that Tr_λ is injective on $W_{\rho',\epsilon}(E)^0$. Since $\mathrm{Tr}_\lambda(\langle h_{\phi_i}\rangle) = \langle h\rangle$, we deduce that (V, h_{ϕ_1}) and (V, h_{ϕ_2}) are isomorphic as hermitian E-spaces and the result follows. □

In the symplectic case, the analogous result is false without further hypotheses. The following theorem gives a sufficient additional condition which will be useful.

Theorem 5.2. *For $i = 1, 2$, let Λ^i be a self-dual lattice sequence in V normalized by $\phi_i(E)^\times$. Let β be a non-zero skew-symmetric element generating E over F and write $r_i := 1 + \nu_{\Lambda^i}(\phi_i(\beta))$. Suppose that there is an element g of G such that*

$$g^{-1}\left(\phi_1(\beta) + \mathfrak{a}_{r_1,-}(\Lambda^1)\right) g \cap \left(\phi_2(\beta) + \mathfrak{a}_{r_2,-}(\Lambda^2)\right) \neq \varnothing.$$

Then ϕ_1, ϕ_2 are conjugate by an element of G.

In the language of strata below (see Section 6), the hypotheses here say that the pure skew strata $[\Lambda^i, r_i + 1, r_i, \phi_i(\beta)]$ intertwine. We will need the following lemma, where we recall that h^γ denotes the twist of h by a (skew-)symmetric element γ (see Definition 4.5)

Lemma 5.3. *Let Λ be a self-dual lattice sequence and let a_1, a_2 be two non-zero symmetric or skew-symmetric elements of the normalizer of Λ such that $a_1a_2^{-1} \in \tilde{\mathrm{U}}^s(\Lambda)$, for some $s > 0$. Then there is an F-linear isometry from (V, h^{a_1}) to (V, h^{a_2}) in $\tilde{\mathrm{U}}^s(\Lambda)$.*

Proof. We apply Proposition 3.1 for $f = \mathrm{id}_V$ to see that the spaces (V, h^{a_1}) and (V, h^{a_2}) are isometric. Now we apply Corollary 3.2, with $f = \mathrm{id}_V$ again, to finish the proof. □

Proof of Theorem 5.2. By Corollary 5.1 we only need to treat the case that $F = F_0$ and $\epsilon = -1$. By hypothesis, there are elements $g \in G$ and $c_i \in \phi_i(\beta) + \mathfrak{a}_{r_i,-}(\Lambda^i)$ such that $gc_1g^{-1} = c_2$. Thus, by Lemma 5.3, we have isometries

$$h^{\phi_1(\beta)} \cong h^{c_1} \cong h^{c_2} \cong h^{\phi_2(\beta)},$$

where the middle isomorphism is given by g. Let f be an isomorphism from $h^{\phi_1(\beta)}$ to $h^{\phi_2(\beta)}$. Since $h^{\phi_i(\beta)}$ are orthogonal forms, Corollary 5.1 applied to the embeddings $x \mapsto f\phi_1(x)f^{-1}$

and ϕ_2 implies that there is an isomorphism from $h^{\phi_1(\beta)}$ to $h^{\phi_2(\beta)}$ which conjugates ϕ_1 to ϕ_2. But any such isomorphism is an isometry of (V, h), as required. □

We will also need the following integral version of the Skolem–Noether theorem:

Proposition 5.4 ([Sko14, Theorem 1.2])**.** *Let $\phi_i : (E, \rho') \to (A, \sigma)$ be a ρ'-σ-equivariant F-algebra embedding, for $i = 1, 2$. Suppose further that (V, h_{ϕ_1}) is isomorphic to (V, h_{ϕ_2}) as hermitian E-spaces and that there is a self-dual lattice chain Λ normalized by $\phi_i(E)^\times$, $i = 1, 2$. Then ϕ_1, ϕ_2 are conjugate by an element of* $\mathrm{U}(\Lambda)$.

6. Semisimple strata

We now turn to the notion of semisimple stratum for G. The background can be found in [BK93, Ste02, Ste05], whose notation we adopt. However, many of the results in the literature are only available for lattice chains, while other results on semisimple strata were omitted in [Ste05] (jumping directly to semisimple *characters*). Thus we gather together here various results which we will need in our work.

A *stratum* is a quadruple $[\Lambda, q, r, \beta]$ consisting of an o_F-lattice sequence Λ, non-negative integers $q \geqslant r$ and an element $b \in \mathfrak{a}_{-q}(\Lambda)$. This stratum is called *strict* if Λ is a lattice chain. The stratum is *skew* if $\beta \in A_-$ and Λ is self-dual, and it is called *null* if $\beta = 0$ and $q = r$.

Two strata $[\Lambda, q, r, \beta]$ and $[\Lambda', q', r', \beta']$ are *equivalent* if $\beta + \mathfrak{a}_{-r-j} = \beta' + \mathfrak{a}'_{-r'-j}$, for all non-negative integers j. This is equivalent to saying that Λ is a translate of Λ', $r = r'$ and the cosets $\beta + \mathfrak{a}_{-r} = \beta' + \mathfrak{a}'_{-r'}$ coincide. They *intertwine* under a subgroup H of $\tilde{G}$ if there is an element g of H such that $g(\beta + \mathfrak{a}_{-r})g^{-1}$ intersects $\beta' + \mathfrak{a}'_{-r'}$. We denote the set of such elements by $I_H([\Lambda, q, r, \beta], [\Lambda', q', r', \beta'])$. If both strata are equal we skip the second argument and if H is $\tilde{G}$ we skip H in the notation. The two strata are *conjugate* under H if there is a $g \in H$ such that $[g\Lambda, q, r, g\beta g^{-1}]$ is equal to $[\Lambda', q', r', \beta']$. Two equivalence classes of strata are called *conjugate* under H if there are representatives of either classes which are conjugate under H.

Definition 6.1 (Simple stratum)**.** A stratum $[\Lambda, q, r, \beta]$ is called

(i) *pure* if $F[\beta]$ is a field such that $F[\beta]^\times \subseteq \mathfrak{n}(\Lambda)$ and $\nu_\Lambda(\beta) = -q < -r$;
(ii) *simple* if either it is null, or it is pure and the degree $[F[\beta] : F]$ is minimal among all equivalent pure strata.

This is equivalent to [Ste05, Definition 1.5], or [BK93] in the case of lattice chains (see Proposition 6.4 below).

We now want to consider strata where $F[\beta]$ is a direct sum of (not necessarily separable) field extensions. Given a decomposition $V = \bigoplus_i V^i$ we write $A^{i,j}$ for $\mathrm{Hom}_F(V^j, V^i)$ and 1^i for the projection onto V^i with kernel $\bigoplus_{j \neq i} V^j$. A stratum $[\Lambda, q, r, \beta]$ is *split* by the decomposition if $1^i \beta 1^j = 0$ for $i \neq j$ and if the decomposition splits Λ, i.e. Λ is the direct sum of the lattice sequences $\Lambda^i := \Lambda \cap V^i$. We write $\beta_i := 1^i\beta 1^i$ and $q_i := -\min\{\nu_\Lambda(\beta_i), -r\}$. We are now in a position to define a semisimple stratum.

Definition 6.2 ([Ste05, Definition 3.2])**.** A stratum $[\Lambda, q, r, \beta]$ is called *semisimple* if either it is null or $\nu_\Lambda(\beta) = -q < -r$ and there is a splitting $V = \bigoplus_i V^i$ such that

(i) for every i the stratum $[\Lambda^i, q_i, r, \beta_i]$ in $A^{i,i}$ is simple,
(ii) for all $i \neq j$ the stratum $[\Lambda^i \oplus \Lambda^j, \max\{q_i, q_j\}, r, \beta_i + \beta_j]$ is not equivalent to a simple stratum.

A semisimple stratum is called *skew-semisimple* if the decomposition of V is orthogonal and all strata occurring in (i) are skew.

For later, to describe the intertwining of $[\Lambda, q, r, \beta]$, we need an integer $k_0(\beta, \Lambda)$ which characterizes the semisimplicity of a stratum. Denote by $a_\beta : A \to A$ the map $a_\beta(x) = x\beta - \beta x$ and put $\mathfrak{n}_l = a_\beta^{-1}(\mathfrak{a}_l) \cap \mathfrak{a}_0$.

If $F[\beta]$ is a field we define, as in [Ste05, Definition 1.4]:

$$k_0(\beta, \Lambda) := \max\{-q, \max\{l \in \mathbb{Z} \mid \mathfrak{n}_l \nsubseteq \mathfrak{b}_0 + \mathfrak{a}_1\}\},\ k_0(0, \Lambda) = -\infty.$$

and one writes $k_F(\beta)$ for $k_0(\beta, \mathfrak{p}_E^{\mathbb{Z}})$, where $\mathfrak{p}_E^{\mathbb{Z}}$ denotes the lattice sequence $i \mapsto \mathfrak{p}_E^i$, the unique o_F-lattice chain in the F-vector space E whose normalizer contains $E^\times$. We have that

$$k_0(\beta, \Lambda) = e(\Lambda|o_E)k_F(\beta), \tag{6.3}$$

by the remark after [Ste01a, Lemma 5.6]. We now prove that Definition 6.1 is equivalent to that in [Ste05, 1.5]

Proposition 6.4. *Given a non-negative integer s, a pure stratum $[\Lambda, q, s, \beta]$ is simple if and only if $-s < k_0(\beta, \Lambda)$. Further, writing $\tilde{\Lambda} = \bigoplus_{l=0}^{e-1}(\Lambda - l)$, with $e = e(\Lambda|o_F)$, we have $k_0(\beta^{\oplus e}, \tilde{\Lambda}) = k_0(\beta, \Lambda)$.*

Note that the lattice sequence $\tilde{\Lambda}$ in the statement is in fact a lattice chain, with the same period as Λ.

Proof. The second assertion follows directly from (6.3), and we thus only concentrate on the first, which is true if Λ is a lattice chain by [BK93, Theorem 2.4.1]. We compare the two notions of simple: a stratum which is simple in the sense of Definition 6.1 is called *degree-simple*, and a stratum which is either null or pure satisfying $-s < k_0(\beta, \Lambda)$ is called *k_0-simple.*

If $[\Lambda, q, s, \beta]$ is k_0-simple then so is $[\tilde{\Lambda}, q, s, \beta^{\oplus e}]$, by the second assertion, and thus it is degree-simple, because $\tilde{\Lambda}$ is a lattice chain. Thus $[\Lambda, q, s, \beta]$ is degree-simple.

If $[\Lambda, q, s, \beta]$ is degree- but not k_0-simple, then $[\mathfrak{p}_E^{\mathbb{Z}}, -\nu_E(\beta), \left\lfloor \frac{s}{e(\Lambda|o_E)} \right\rfloor, \beta]$ is not k_0-simple. But then, the latter is not degree-simple, because $\mathfrak{p}_E^{\mathbb{Z}}$ is a lattice chain, and thus $[\Lambda, q, s, \beta]$ is not degree-simple, using a (W, E)-decomposition as in [BK99, 5.3]. □

Corollary 6.5. *$[\bigoplus_{l=0}^{e-1}(\Lambda - l), q, s, \beta^{\oplus e}]$ is simple if and only if $[\Lambda, q, s, \beta]$ is simple.*

If $F[\beta]$ is not a field we define for a semisimple stratum, as in [Ste05, (3.6)],

$$k_0(\beta, \Lambda) := -\min\{s \in \mathbb{Z} \mid [\Lambda, q, s, \beta] \text{ is not semisimple}\}.$$

This integer is negative because $[\Lambda, q, r, \beta]$ is semisimple and $r \geqslant 0$.

Minimal strata. We begin now with an analysis of semisimple strata of the form $[\Lambda, q, q-1, \beta]$. For the simple case, we recall that an element β of an extension $E|F$ is called *minimal* if it satisfies the following two conditions:

(i) $\gcd(\nu_E(\beta), e(E|F)) = 1$;
(ii) $\beta^{e(E|F)}\varpi^{\nu_E(\beta)} + \mathfrak{p}_E$ generates the extension $\kappa_E|\kappa_F$.

Then, by [BK93, 1.4.13(ii),1.4.15], a pure stratum $[\Lambda, q, q-1, \beta]$ is simple if and only if β is minimal. By a slight abuse, we call a semisimple stratum of the form $[\Lambda, q, q-1, \beta]$ a *minimal semisimple stratum*.

For minimal semisimple strata, the *characteristic polynomial* is very important for distinguishing the summands. For b an element of a finite dimensional semisimple algebra B over some field K, we denote the reduced characteristic polynomial of b in $B|K$, defined in [Rei03, (9.20)], by $\chi_{b,B|K}$, and the minimal polynomial by $\mu_{b,B|K}$.

Definition 6.6. Let $[\Lambda, q, q-1, \beta]$ be a stratum with $\nu_\Lambda(\beta) = -q$ and set $y_\beta := \beta^{\frac{e}{g}}\varpi^{\frac{q}{g}}$, where $g = \gcd(e, q)$, with characteristic polynomial $\Phi(X) = \chi_{y_\beta, A|F} \in o_F[X]$. We define the *characteristic polynomial* of the stratum $[\Lambda, q, q-1, \beta]$ to be the reduction $\phi_\beta := \bar{\Phi} \in \kappa_F[X]$. It depends only on the equivalence class of the stratum.

For a null stratum we define $y_0 := 0$ and $\phi_0(X) := X^N$, where $N = \dim_F(V)$.

Remark 6.7. If $[\Lambda, q, q-1, \beta]$ and $[\Lambda, q, q-1, \gamma]$ intertwine then $\phi_\beta = \phi_\gamma$.

Proposition 6.8. *If $[\Lambda, q, q-1, \beta]$ is semisimple with associated splitting $V = \bigoplus_{i \in I} V^i$, then we have the following:*

(i) *ϕ_β is the product of the polynomials ϕ_{β_i}, which are pairwise coprime polynomials;*
(ii) *each polynomial ϕ_{β_i} is a power of an irreducible polynomial;*
(iii) *the F-algebra homomorphism induced by $\beta \mapsto \sum_{i \in I} \beta_i$ is a bijection from $F[\beta]$ to the product of the $E_i := F[\beta_i]$;*
(iv) *$\kappa_F[\bar{y}_\beta]$ is canonically isomorphic to $\prod_{i \in I} \kappa_F[\bar{y}_{\beta_i}]$.*

Proof. For all indices i, we have $e = e(\Lambda|o_F) = e(\Lambda^i|o_F)$ and $q = q_i$ for all indices i with $\beta_i \neq 0$. Since also $\beta = \sum_i \beta_i$ with $\beta_i \in A^{i,i}$, we get

$$y_\beta = \sum_i \beta_i^{\frac{e}{g}} \varpi^{\frac{q}{g}} = \sum_i y_{\beta_i},$$

and ϕ_β is equal to the product of the ϕ_{β_i}. That ϕ_{β_i} is primary now follows from the fact that $[\Lambda^i, q_i, r, \beta_i]$ is a simple stratum and the remaining assertions are a consequence of [Ste05, Remark 3.3]. □

It will also be useful to have another criterion by which to recognize a minimal semisimple stratum. Recall that a stratum $[\Lambda, q, q-1, \beta]$ is called *fundamental* if the coset $\beta + \mathfrak{a}_{1-q}$ contains no nilpotent elements; in this case, the rational number $\frac{q}{e}$ is called the *level* of the stratum, where $e = e(\Lambda|o_F)$. We also define the level of the null stratum $[\Lambda, q, q, 0]$ to be $\frac{q}{e}$.

Proposition 6.9. *A stratum $[\Lambda, q, q-1, \beta]$ is fundamental if and only if its characteristic polynomial is not a power of X. Two fundamental strata which intertwine have the same level. If a null stratum intertwines a fundamental stratum, then they have different levels.*

Proof. Suppose $[\Lambda, q, q-1, \beta]$ has characteristic polynomial X^m and put $e = e(\Lambda|o_F)$; then the element β satisfies

$$\beta^{em} \in \varpi^{-qm}\mathfrak{a}_1 = \mathfrak{a}_{1-qme}.$$

Then, by [Bus87, Lemma 2.1], there is a nilpotent element in $\beta + \mathfrak{a}_{1-q}$, so the stratum is not fundamental. (The proof of that Lemma is valid for lattice sequences if one allows block matrices with block sizes $0 \times l$ or $l \times 0$.) Conversely, if $[\Lambda, q, q-1, \beta]$ is not fundamental, then y_β is congruent to a nilpotent element modulo $\mathfrak{a}_1$, and thus the characteristic polynomial of the stratum is a power of X. The remaining assertions now follow easily, because if one of them were false, then there would be a fundamental stratum whose characteristic polynomial is a power of X. □

We now give criteria for a fundamental stratum to be simple or semisimple. We recall that a fundamental stratum is called *non-split* if the characteristic polynomial of the stratum is a power of an irreducible polynomial. Given a fundamental stratum $[\Lambda, q, q-1, b]$ we define the following κ_F-algebra

$$\mathcal{R}([\Lambda, q, q-1, b]) := \{\bar{x} \in \mathfrak{a}_0/\mathfrak{a}_1 \mid xb \equiv bx \pmod{\mathfrak{a}_{1-q}}\}.$$

The following result is stated in [BK93, 2.4.13] for strict strata but, because the quotient $\mathfrak{a}_0/\mathfrak{a}_1$ depends only on the image of Λ, is also valid for arbitrary lattice sequences.

Proposition 6.10 ([BK93, 2.4.13])**.** *A non-split fundamental stratum $[\Lambda, q, q-1, b]$ is equivalent to a simple stratum if and only if $\mathcal{R}([\Lambda, q, q-1, b])$ is semisimple.*

To get a similar result for semisimple strata we need, for an element $b \in \mathfrak{a}_{-q}(\Lambda)$ and an integer n, the map

$$m_{n,q,b} : \mathfrak{a}_{-nq}/\mathfrak{a}_{1-nq} \to \mathfrak{a}_{-(n+1)q}/\mathfrak{a}_{1-(n+1)q}$$

induced by multiplication by b.

Proposition 6.11. *A fundamental stratum $[\Lambda, q, q-1, b]$ is equivalent to a semisimple stratum if and only if $\mathcal{R}([\Lambda, q, q-1, b])$ is semisimple and, for all non-negative integers n, the kernel of $m_{n+1,q,b}$ and the image of $m_{n,q,b}$ intersect trivially.*

Proof. Since the algebra $\mathcal{R}([\Lambda, q, q-1, b])$ and the maps $m_{n,q,b}$ depend only on the equivalence class of the stratum, we are free to move to an equivalent stratum at any point.

Suppose first that $\mathcal{R}([\Lambda, q, q-1, b])$ is semisimple and, for all non-negative integers n, the kernel of $m_{n+1,q,b}$ and the image of $m_{n,q,b}$ intersect trivially. We inductively find a splitting. For this, assume that ϕ_b is a product of two coprime monic factors $\bar{f}_0$ and $\bar{f}_1$. Let Φ be the characteristic polynomial of $y_b = \varpi^{\frac{q}{g}} b^{\frac{e}{g}}$, where g is the greatest common divisor of $e = e(\Lambda|o_F)$ and q. Hensel's Lemma implies that we can factorize Φ as $f_0 f_1$ where f_i is a monic lift of $\bar{f}_i$. By Bézout's Lemma, there are polynomials $a_0, a_1 \in o_F[X]$ such that $a_0 f_0 + a_1 f_1 = 1$. The map $1_i = a_i(y_b) f_i(y_b)$ is the projection onto the kernel of f_{1-i}, and the sum $\ker(f_0) \oplus \ker(f_1) = V$ splits the stratum $[\Lambda, q, q-1, b]$. Moreover, we have $\mathcal{R}([\Lambda, q, q-1, b]) \simeq \mathcal{R}([\Lambda^0, q_0, q-1, b_0]) \oplus \mathcal{R}([\Lambda^1, q_1, q-1, b_1])$, by the coprimality of f_0, f_1, so that both $\mathcal{R}([\Lambda^i, q_i, q-1, b_i])$ are semisimple.

Thus, by Proposition 6.10 and 6.8, we only have to show that strata equivalent to null strata are the only non-fundamental strata for which the kernel of $m_{n+1,q,b}$ and the image of $m_{n,q,b}$ intersect trivially. Now let us assume that $[\Lambda, q, q-1, b]$ is non-fundamental. Then without loss of generality we can assume that b is nilpotent. The conditions on the maps imply that $m_{n,q,b} \circ m_{(n-1),q,b} \circ \cdots \circ m_{0,q,b}$ is injective on the image of $m_{0,q,b}$. If n is big enough, the first product is the zero map, so the image of $m_{0,q,b}$ is zero, i.e. $[\Lambda, q, q-1, b]$ is equivalent to a null stratum.

For the converse, suppose that $[\Lambda, q, q-1, b]$ is a semisimple stratum with associated splitting $V = \bigoplus_{i \in I} V^i$. Since the characteristic polynomials ϕ_{b_i} are pairwise coprime, we have $\mathcal{R}([\Lambda, q, q-1, b]) \simeq \bigoplus_{i \in I} \mathcal{R}([\Lambda^i, q_i, q-1, b_i])$ and, since each stratum $[\Lambda^i, q_i, q-1, b_i]$ is simple, this algebra is semisimple by Proposition 6.10. (Note that the algebra is clearly semisimple for the null stratum.)

The maps $m_{n,q,b}$ preserve the decomposition $A = \bigoplus_{i,j} A^{i,j}$ so we may work blockwise. On the diagonal blocks $A^{i,i}$, the map $m_{n,q,b}$ is either zero (in the case $b_i = 0$) or bijective. On the non-diagonal blocks $A^{i,j}$, with $i \neq j$, the map is bijective or zero by [BK99, 3.7 Lemma 4]. ☐

Semisimple strata. Now we turn to the case of general semisimple strata $[\Lambda, q, r, \beta]$. A very important tool to prove properties of semisimple strata by an inductive procedure is the tame corestriction map, which was introduced in [BK93, 1.3.3] in the simple case.

Definition 6.12. Let $E|F$ be a field extension and B be the centralizer of E in A. A non-zero B-B-bimodule map $s : A \to B$ is called a *tame corestriction (relative to $E|F$)* if, for all $\mathfrak{o}_F$-lattice sequences Λ normalized by $E^\times$, we have

$$s(\mathfrak{a}_j(\Lambda)) = \mathfrak{a}_j(\Lambda) \cap B,$$

for all integers j.

If $E = F[\gamma]$ we often write s_γ for a (choice of) tame corestriction relative to $E|F$.

Remark 6.13. (i) By [BK93, 1.3.4], tame corestrictions exist: if ψ_F and ψ_E are additive characters of F and E respectively then there is a unique map $s : A \to B$ such that

$$\psi_F \circ \operatorname{tr}_{A|F}(ab) = \psi_E \circ \operatorname{tr}_{B|E}(s(a)b), \qquad a \in A,\ b \in B.$$

This map is a tame corestriction and every tame corestriction arises in this way. Moreover, tame corestrictions are unique up to multiplication by an element of $o_E^\times$.

(ii) If γ generates the extension $E|F$ then, by [BK93, 1.3.2 (i)], the kernel of s_γ is equal to the image of the adjoint map $a_\gamma : A \to A$.

(iii) If E is σ-invariant, we can arrange the additive characters ψ_F and ψ_E in (i) to be σ-invariant also, and then the tame corestriction s is σ-equivariant.

Given a simple stratum $[\Lambda, q, r+1, \gamma]$ in A and an element $c \in \mathfrak{a}_{-r}$, the tame corestriction map allows us to define a *derived stratum* $[\Lambda, r+1, r, s_\gamma(c)]$ in B_γ, the centralizer in A of γ, and we can ask whether this derived stratum is (equivalent to) a fundamental or simple stratum. The following theorem is particularly useful.

Theorem 6.14 ([BK93, Theorems 2.2.8, 2.4.1])**.** *Let $[\Lambda, q, r+1, \beta]$ be a stratum equivalent to a simple stratum $[\Lambda, q, r+1, \gamma]$. Then $[\Lambda, q, r, \beta]$ is equivalent to a simple stratum if and only if the derived stratum $[\Lambda, r+1, r, s_\gamma(\gamma - \beta)]$ is equivalent to a simple stratum.*

As an immediate corollary, we get the following result on semisimple strata.

Corollary 6.15. *Let $[\Lambda, q, r+1, \beta]$ be a stratum equivalent to a simple stratum $[\Lambda, q, r+1, \gamma]$. Assume that we have a decomposition $V = \bigoplus_i V^i$ into β- and γ-invariant F-subspaces. Then $[\Lambda, r+1, r, s_\gamma(\gamma - \beta)]$ is equivalent to a semisimple stratum with associated splitting $V = \bigoplus_i V^i$ if and only if $[\Lambda, q, r, \beta]$ is equivalent to a semisimple stratum with associated splitting $V = \bigoplus_i V^i$.*

Suppose now that $[\Lambda, q, 0, \beta]$ is semisimple so that, for any $0 \leqslant r < q$, the stratum $[\Lambda, q, r+1, \beta]$ is equivalent to a semisimple stratum $[\Lambda, q, r+1, \gamma]$. Then we can realize the assumption on γ in the previous corollary (that is, we can find γ such that the splitting associated to $[\Lambda, q, 0, \beta]$ is preserved by γ) by the following theorem.

Theorem 6.16 ([Ste05, 3.4], [Ste01b, 1.10])**.** *Let $[\Lambda, q, r, \beta]$ be a (skew)-stratum split by $V = \bigoplus_i V^i$ ($V = \ominus_i V^i$) such that every stratum $[\Lambda, q_i, r, \beta_i]$ is equivalent to a simple stratum, and such that $[\Lambda, q, r, \beta]$ is equivalent to a simple stratum. Then $[\Lambda, q, r, \beta]$ is equivalent to a (skew)-simple stratum $[\Lambda, q, r, \gamma]$ split by the same direct sum.*

Proof. We observe only that, although this is not quite the statement in [Ste05, 3.4], this is what the proof there actually demonstrates. The skew case then follows immediately by applying [Ste01b, 1.10]. ☐

In particular, if $[\Lambda, q, r, \beta]$ is a semisimple stratum with splitting $V = \bigoplus_i V^i$ and $[\Lambda, q, r+1, \beta]$ is equivalent to a simple stratum $[\Lambda, q, r+1, \gamma]$ such that $\gamma V^i \subseteq V^i$ for each i, then Corollary 6.15 implies that the derived stratum $[\Lambda, r+1, r, s_\gamma(\gamma-\beta)]$ is equivalent to a semisimple stratum with the same splitting $V = \bigoplus_i V^i$.

Notation 6.17. For the rest of the article we use the following notation: $[\Lambda, q, r, \beta]$ always denotes a stratum, and B the centralizer of β in A. If $[\Lambda, q, r, \beta]$ is semisimple then $V = \bigoplus_{i\in I} V^i$ is the associated splitting and we have $A = \bigoplus_{i,j} A^{i,j}$ and $B = \bigoplus_{i\in I} B^{i,i}$, where $B^{i,i}$ is the centralizer of $E_i = F[\beta_i]$ in $A^{i,i}$. Further, we write $\mathfrak{b}_l$ for the intersection of $\mathfrak{a}_l$ with B. We use analogous notations for a second stratum $[\Lambda', q', r', \beta']$ but all with $()'$. If we want to specify the centralizer of γ in A, for an arbitrary element γ, we write B_γ.

Let $[\Lambda, q, r, \beta]$ be a semisimple stratum. We define a *tame corestriction* $s_\beta : A \to B$ for β by $s_\beta(a) := \sum_i s_i(a_{ii})$, where s_i is a tame corestriction for β_i as in Definition 6.12. If s_i is defined relative to additive characters ψ_F, ψ_{E^i} as in Remark 6.13(i), then we put $\psi_{B^{i,i}} = \psi_{E^i} \circ \mathrm{tr}_{B^{i,i}|E^i}$ and define an additive character of B by

$$\psi_B(b) = \prod_{i\in I} \psi_{B^{i,i}}(b_i), \qquad b = \sum_{i\in I} b_i, \quad b_i \in B^{i,i}.$$

Writing $\psi_A = \psi_F \circ \mathrm{tr}_{A|F}$, the map s_β is then a non-zero (B,B)-bimodule homomorphism satisfying

$$\psi_A(ab) = \psi_B(s_\beta(a)b), \qquad a \in A,\ b \in B,$$

and

$$s_\beta(\mathfrak{a}'_l) = \mathfrak{b}'_l$$

for all lattice sequences Λ' which are split by $V = \bigoplus_i V^i$ into a direct sum of o_{E^i}-lattice sequences.

Lemma 6.18. *The sequence $A \xrightarrow{a_\beta} A \xrightarrow{s_\beta} B$ is exact and the kernel of s_β is split by the decomposition $A = \bigoplus A^{i,j}$.*

Proof. By definition, the kernel of s_β is the direct sum of the $A^{i,j}$, for $i \neq j$, and of the kernels of s_i, for $i \in I$. The sequence is exact on the (i,i) components, by [BK93, 1.3.2], and it is therefore enough to prove that for $j \neq i$ the restriction of a_β on $A^{i,j}$ is bijective onto $A^{i,j}$. It has the form $a_\beta(a_{ij}) = \beta_i a_{ij} - a_{ij}\beta_j$, which is injective because β_i and β_j have no common eigenvalue, because their minimal polynomials are coprime since $[\Lambda^i \oplus \Lambda^j, \max\{q_i, q_j\}, r, \beta_i + \beta_j]$ is not equivalent to a simple stratum. ☐

To describe the intertwining of a semisimple stratum $[\Lambda, q, r, \beta]$, recall that we have defined the integer $k_0 = k_0(\beta, \Lambda)$ and the lattices $\mathfrak{n}_l = a_\beta^{-1}(\mathfrak{a}_l) \cap \mathfrak{a}_0$, for l an integer. We will also need the unit subgroups $1+\mathfrak{m}_l$, where $\mathfrak{m}_l=\mathfrak{n}_{l+k_0}\cap\mathfrak{a}_l$, for integers $l\geqslant 1$. As the first of several intertwining results we have:

Theorem 6.19 (see [Ste05, 4.4], [BK93, 1.5.8] for simple strata)**.** *Let $[\Lambda,q,r,\beta]$ be a semisimple stratum.*

(i) $I([\Lambda,q,r,\beta])=(1+\mathfrak{m}_{-(k_0+r)})B^\times(1+\mathfrak{m}_{-(k_0+r)})$.
(ii) *If the stratum is skew then*

$$I_G([\Lambda,q,r,\beta])=((1+\mathfrak{m}_{-(k_0+r)})\cap G)(B^\times\cap G)((1+\mathfrak{m}_{-(k_0+r)})\cap G).$$

The crucial ingredient for the proof is:

Lemma 6.20 ([Ste05, 3.7])**.** *For all integers s we have*

(i) $\mathfrak{n}^{i,j}_{-s}\subseteq\mathfrak{a}_{-(k_0+s)}$ *for $i\neq j$.*
(ii) $\mathfrak{n}_{-s}=\mathfrak{b}_0+\mathfrak{n}_{-s}\cap\mathfrak{a}_{-(k_0+s)}$

In [Ste05], this lemma was formulated for $s\leqslant -k_0$, but the case $s\geqslant -k_0$ is trivial. From this we deduce

Lemma 6.21. *Take $i\neq j$. The restriction of a_β to $A^{i,j}$ is an F-linear homeomorphism and $a_\beta^{-1}(\mathfrak{a}_s)^{i,j}$ is equal to $\mathfrak{n}_s^{i,j}$ for all integers $s\geqslant k_0$.*

Proof. The map a_β is a linear automorphism on $A^{i,j}$ by Lemma 6.18; thus the image of an o_F-lattice contains an o_F-lattice and the restriction of a_β to $A^{i,j}$ is a homeomorphism. It follows that, for s big enough, we have that $a_\beta^{-1}(\mathfrak{a}_s)^{i,j}:=a_\beta^{-1}(\mathfrak{a}_s)\cap A^{i,j}$ is contained in $\mathfrak{a}_0^{i,j}$ and is therefore equal to $\mathfrak{n}_s^{i,j}$; in particular, it is contained in $\mathfrak{a}^{i,j}_{-k_0+s}$ by Lemma 6.20(i). By periodicity we have that $a_\beta^{-1}(\mathfrak{a}_s)^{i,j}$ is contained in $\mathfrak{a}^{i,j}_{-k_0+s}$ for all integers s and thus $\mathfrak{n}_s^{i,j}$ is equal to $a_\beta^{-1}(\mathfrak{a}_s)^{i,j}$ for all integers s with $s\geqslant k_0$. ☐

Proof of Theorem 6.19. We follow the proof of [BK93, 1.5.8]. For a null stratum there is nothing to prove, so we assume the stratum is non-null. The main ingredients which have to be verified are the exact sequences of [BK93, 1.4.10], which hold by Lemma 6.20, and the analogue of [BK93, 1.4.16], which we prove now. We write d for $-(r+k_0)$ and put

- $M=\mathfrak{n}_{t+jd+k_0}\cap\mathfrak{a}_{t+jd}\cap(\gamma_1(\mathfrak{n}_{-r}\cap\mathfrak{a}_d)+(\mathfrak{n}_{-r}\cap\mathfrak{a}_d)\gamma_2+(\mathfrak{n}_{t+jd-r}\cap\mathfrak{a}_{t+(j+1)d}))$,
- $L=\mathfrak{a}_{t+jd+k_0}\cap(\gamma_1\mathfrak{a}_{-r}+\mathfrak{a}_{-r}\gamma_2+\mathfrak{a}_{t+jd-r})$,

for integers $t\geqslant 0,\ j\geqslant 1$ and elements γ_1,γ_2 of $B^\times$. The sequence $M\stackrel{a_\beta}{\to}L\stackrel{s_\beta}{\to}B$ is exact if all its restrictions on the $A^{i,j}$ are exact. For $i=j$ the proof is done in [Ste01a, (5.2)] and for $i\neq j$ it follows from Lemma 6.20(i), 6.21 and 6.18. The same cohomology argument as in [Ste05, Corollary 4.14] proves (ii). ☐

A completely analogous proof using [BK93, 1.5.12] provides:

Theorem 6.22. *Let $[\Lambda,q,r,\beta]$ and $[\Lambda',q',r',\beta]$ be semisimple strata in A. Then*

(i) $I([\Lambda,q,r,\beta],[\Lambda',q',r',\beta])=(1+\mathfrak{m}'_{-(k'_0+r')})B^\times(1+\mathfrak{m}_{-(k_0+r)})$.

(ii) *If both strata are skew then*

$$I_G([\Lambda,q,r,\beta],[\Lambda',q',r',\beta])=((1+\mathfrak{m}'_{-(k'_0+r')})\cap G)(B^\times\cap G)((1+\mathfrak{m}_{-(k_0+r)})\cap G).$$

7. Matching for intertwining strata

In this chapter we show that, if we have semisimple strata which intertwine, then there is a canonical bijection between their associated splittings. This will then allow us to deduce a Skolem–Noether theorem for skew-semisimple strata which intertwine.

7.1. **For general linear groups.** We fix a pair of semisimple strata $[\Lambda,q,r,\beta]$, $[\Lambda',q',r',\beta']$, with splittings $\bigoplus_{i\in I}V^i$ and $\bigoplus_{j\in I'}V'^j$ respectively. The main result of this subsection is:

Proposition 7.1. *Suppose that $[\Lambda,q,r,\beta]$ intertwines $[\Lambda',q',r,\beta']$ and that Λ,Λ' have the same period. Then*

(i) *If one stratum is null and $q=q'$ then the other stratum is null.*

(ii) *If both strata are non-null then $q=q'$ and there is a unique bijection $\zeta:I\to I'$ such that $[\Lambda^i\oplus\Lambda'^{\zeta(i)},\max\{q_i,q'_{\zeta(i)}\},r,\beta_i+\beta'_{\zeta(i)}]$ is equivalent to a simple stratum, for all indices $i\in I$. Moreover, V^i and $V'^{\zeta(i)}$ have the same F-dimension.*

Note that, in case (i), both splittings are trivial so we trivially have a (unique) bijection ζ as in (ii). We call the bijection ζ a *matching* of $[\Lambda',q',r,\beta']$ and $[\Lambda,q,r,\beta]$.

Remark 7.2. If Λ,Λ' do not have the same period then we can scale them so that they do. In particular, we only require the intertwining hypothesis in Proposition 7.1 in order to get a matching ζ.

To prove Proposition 7.1 (and, later, other results on semisimple strata), we introduce the notion of a defining sequence for a semisimple stratum, which allows us to prove properties of semisimple strata by an inductive process (cf. [BK93] for the simple case). Let $\Delta=[\Lambda,q,r,\beta]$ be a (skew)-semisimple stratum with associated splitting $V=\bigoplus_{i\in I}V^i$. A *defining sequence* for Δ is a finite sequence of (skew)-semisimple strata $(\Delta^k)_{k=0,\ldots,q-r-1}$ defined as follows:

- $\Delta^0=\Delta$;
- for $0<k\leqslant q-r-1$, we have $\Delta^k=[\Lambda,q,r+k,\gamma_k]$ a (skew)-semisimple stratum equivalent to $[\Lambda,q,r+k,\beta]$ with $\gamma_k\in\prod_i A^{i,i}$ (see Theorem 6.16).

Note that there is a significant degree of choice in producing a defining sequence for a (skew)-semisimple stratum.

Suppose now we want to prove a statement $P(\Delta,\Delta')$ for all pairs of semisimple strata Δ,Δ'. The inductive procedure, which we call *strata induction*, to prove P is given by the following steps.

- The base case: Here one proves P for all minimal semisimple strata.
- The induction step:
  (i) The step $r+1$ to r: From the induction hypothesis and possibly an auxiliary statement **(S1)** we restrict to the case where the first elements $\Delta^{(1)}$ and $\Delta'^{(1)}$ of defining sequences of Δ and Δ' have the same element γ, and hence the same associated splitting.
  (ii) Taking a second auxiliary statement **(S2)**, we show that the derived strata $s_\gamma(\Delta)$ and $s_\gamma(\Delta')$ satisfy the assumptions of P. In this article, (S2) will always be the description of the intertwining of $\Delta^{(1)}$ with $\Delta'^{(1)}$ (see Proposition 7.3 below).
  (iii) The base case shows $P(s_\gamma(\Delta), s_\gamma(\Delta'))$ and, together with a third auxiliary statement **(S3)**, provides $P(\Delta, \Delta')$. For (S3) we will use Theorem 6.14.

  Strata induction can be restricted to simple strata by substituting the word semisimple by simple.

In the following, we use the notation for tame corestrictions as in the previous section.

Proposition 7.3. *Let $[\Lambda, q, r, \beta]$ and $[\Lambda', q', r', \beta]$ be semisimple strata with splitting $V = \bigoplus_{i\in I} V^i$. Suppose we are given elements $a \in \mathfrak{a}_{-r}$ and $a' \in \mathfrak{a}'_{-r'}$ such that there is an element g of $\tilde{G}$ which intertwines $[\Lambda, q, r-1, \beta + a]$ with $[\Lambda', q', r'-1, \beta + a']$. Using Theorem 6.22, write $g = (1+u')b(1+v)$, with $b \in B^\times$. Then the component $b^{i,i}$ intertwines $[\Lambda, r, r-1, s_i(a^{i,i})]$ with $[\Lambda', r', r'-1, s_i(a'^{i,i})]$, for all $i \in I$.*

Proof. Again we only have to consider a non-zero element β. This is essentially the calculation in [BK93, 2.6.1] which we want to recall, to show that its validity for different semisimple strata. Note that the hypotheses imply that g certainly intertwines $[\Lambda, q, r, \beta]$ and $[\Lambda', q', r', \beta]$ so that, by Theorem 6.22, we can write $g = (1+u')b(1+v)$, with $u' \in \mathfrak{m}'_{-(k'_0+r')}$, $b \in B^\times$ and $v \in \mathfrak{m}_{-(k_0+r)}$.

Let $(1+w')$ be the inverse of $(1+u')$. By the intertwining property of g, we have

$$g(\beta + a) \equiv (\beta + a')g \pmod{g\mathfrak{a}_{1-r} + \mathfrak{a}'_{1-r'}g}.$$

Multiplying by $(1+w')$ on the left and $(1+v)^{-1}$ on the right we obtain

$$b(1+v)(\beta+a)(1+v)^{-1} \equiv (1+w')(\beta+a')(1+w')^{-1}b \pmod{b\mathfrak{a}_{1-r} + \mathfrak{a}'_{1-r'}b}. \tag{7.4}$$

We firstly consider the right hand side.

$$\begin{aligned}(1+w')(\beta+a') &= \beta - a_\beta(w') + a' + w'a' + \beta w' \\ &= (\beta - a_\beta(w') + a')(1+w') + w'a' - a'w' + a_\beta(w')w' \\ &\equiv (\beta - a_\beta(w') + a')(1+w') \pmod{\mathfrak{a}'_{1-r'}},\end{aligned}$$

because $a' \in \mathfrak{a}'_{-r'}$, $w' \in \mathfrak{a}'_1$ and $a_\beta(w') \in \mathfrak{a}'_{-r'}$. A similar calculation for the left hand side and equation (7.4) leads to

$$b(\beta - a_\beta(v) + a) \equiv (\beta - a_\beta(w') + a')b \pmod{b\mathfrak{a}_{1-r} + \mathfrak{a}'_{1-r'}b}. \tag{7.5}$$

We apply s_β to get

$$bs_\beta(a) \equiv s_\beta(a')b \pmod{b\mathfrak{b}_{1-r} + \mathfrak{b}'_{1-r'}b}$$

and thus $b^{i,i}$ intertwines the derived strata $[\Lambda, r, r-1, s_i(a^{i,i})]$ and $[\Lambda', r', r'-1, s_i(a'^{i,i})]$, for all $i \in I$. □

Proposition 7.6 (cf. [BK93, 2.2.1])**.** *Suppose that $[\Lambda, q, r, \beta]$ and $[\Lambda', q', r', \beta]$ are simple and that there are $a \in \mathfrak{a}_{-r}$ and $a' \in \mathfrak{a}'_{-r'}$ such that $s_\beta(a) \equiv s_\beta(a') \pmod{\mathfrak{b}_{1-r} + \mathfrak{b}'_{1-r'}}$. Then, there are elements $w' \in \mathfrak{m}'_{-(k'_0+r')}$ and $v \in \mathfrak{m}_{-(k_0+r)}$ such that*

$$(1+w')(\beta+a')(1+w')^{-1} \equiv (1+v)(\beta+a)(1+v)^{-1} \pmod{\mathfrak{a}_{1-r} + \mathfrak{a}'_{1-r'}}.$$

Moreover, if the strata and the elements a and a' are skew and the strata intertwine in G, then we can choose $1+v$ and $1+w'$ in G.

Before the proof let us recall that the *Cayley transform* of an element v of $(\mathfrak{a}_1)_-$ is the element $\left(1+\frac{v}{2}\right)\left(1-\frac{v}{2}\right)^{-1}$. It is an element of $\mathrm{U}^1(\Lambda) \subset G$.

Proof. Let β be non-zero and write C for the kernel of s_β. Without loss of generality we can assume that $s_\beta(a)$ and $s_\beta(a')$ are equal, since the map $s_\beta : \mathfrak{a}_{1-r} \to \mathfrak{b}_{1-r}$ is surjective. The map

$$\mathfrak{m}_{-r-k_0} + \mathfrak{m}'_{-r'-k'_0} \xrightarrow{a_\beta} C \cap (\mathfrak{a}_{-r} + \mathfrak{a}'_{-r'})$$

is surjective because $C \cap (\mathfrak{a}_{-r} + \mathfrak{a}'_{-r'})$ is equal to $(C \cap \mathfrak{a}_{-r}) + (C \cap \mathfrak{a}'_{-r'})$ by [BK93, 1.3.17] and $a_\beta^{-1}(\mathfrak{a}_{-r})$ is a subset of $B + \mathfrak{m}_{-r-k_0}$ by Lemma 6.20(ii). Thus, we can find $w' \in \mathfrak{m}_{-r'-k'_0}$ and $v \in \mathfrak{m}_{-r-k_0}$ to satisfy (7.5) for $b = 1$. We now follow the calculation in the proof of Proposition 7.3 backwards to show the desired congruence. In the skew situation we can find skew-symmetric elements, say $\tilde{v}$ and $\tilde{w}'$, which satisfy (7.5) and we define $1+v$ and $1+w'$ to be the Cayley transforms of $\tilde{v}$ and $\tilde{w}'$ respectively. □

We need one final lemma before we can prove Proposition 7.1, which will play the role of (S1).

Lemma 7.7. *Let $[\Lambda, q, r, \beta]$ and $[\Lambda', q, r, \beta']$ be semisimple strata for which there is a (unique) bijection $\zeta : I \to I'$ such that $[\Lambda^i \oplus \Lambda'^{\zeta(i)}, \max\{q_i, q'_{\zeta(i)}\}, r, \beta_i + \beta'_{\zeta(i)}]$ is equivalent to a simple stratum, for all indices $i \in I$, and $\dim_F V^i = \dim_F V'^{\zeta(i)}$. Then there are an element g of $\tilde{G}$ and an element $\gamma \in \prod_i A^{i,i}$ such that $V^i = gV'^{\zeta(i)}$, for all indices $i \in I$, and:*

- *$[g\Lambda', q, r, g\beta' g^{-1}]$ is equivalent to $[g\Lambda', q, r, \gamma]$;*
- *$[\Lambda, q, r, \beta]$ is equivalent to $[\Lambda, q, r, \gamma]$; and*
- *$[g\Lambda', q, r, \gamma]$ and $[\Lambda, q, r, \gamma]$ are semisimple strata with the same associated splitting $V = \bigoplus_{i \in I} V^i$.*

Proof. Applying Theorem 6.16 to the strata $[\Lambda^i \oplus \Lambda'^{\zeta(i)}, \max\{q_i, q_{\zeta(i)}\}, r, \beta_i + \beta'_{\zeta(i)}]$, for each i, we find an equivalent simple stratum $[\Lambda^i \oplus \Lambda'^{\zeta(i)}, \max\{q_i, q_{\zeta(i)}\}, r, \gamma_i + \gamma'_{\zeta(i)}]$; in particular, γ_i

and $\gamma'_{\zeta(i)}$ have the same irreducible minimal polynomial, and the same characteristic polynomial since $\dim_F V^i = \dim_F V'^{\zeta(i)}$.

Further, for $i \neq j$ in I, the stratum $[\Lambda^i \oplus \Lambda^j, \max\{q_i, q_j\}, r, \gamma_i + \gamma_j]$ is equivalent to $[\Lambda^i \oplus \Lambda^j, \max\{q_i, q_j\}, r, \beta_i + \beta_j]$, which is not equivalent to a simple stratum, so that the stratum $[\Lambda, q, r, \gamma]$ is semisimple, where $\gamma = \sum_{i\in I} \gamma_i$. The same applies to $[\Lambda', q, r, \gamma']$, where $\gamma' = \sum_{i\in I} \gamma'_{\zeta i}$. Finally, since γ_i and $\gamma'_{\zeta(i)}$ have the same characteristic polynomial, we can find $g \in G$ such that $V^i = gV'^{\zeta(i)}$ and $g\gamma' g^{-1} = \gamma$, and the result follows. □

Proof of Proposition 7.1. (i) and the equality $q = q'$ in (ii) follow from the results on level in Proposition 6.9. The existence of ζ in (ii) is proved by strata induction, where we take Lemma 7.7 for (S1), Proposition 7.3 for (S2), and Theorem 6.14 for (S3). The base case follows because the characteristic polynomials are equal, so we match the primary factors using [Ste05, 3.3(ii)]. The equality of dimensions follows from the fact that the degree of the ith primary factor is the dimension of V^i.

For the inductive step, suppose that $\Delta = [\Lambda, q, r, \beta]$ and $\Delta' = [\Lambda', q, r, \beta']$ are semisimple strata as in the proposition which intertwine. Then the stratum $[\Lambda, q, r + 1, \beta]$ is equivalent to a semisimple stratum $\Delta_\gamma = [\Lambda, q, r + 1, \gamma]$ whose splitting is a coarsening of that of Δ, by Theorem 6.16; similarly we have a semisimple stratum $\Delta'_\gamma = [\Lambda', q, r + 1, \gamma']$. Since the strata Δ_γ, Δ'_γ intertwine, we may apply the inductive hypothesis to them. In particular, they satisfy the hypotheses of Lemma 7.7 and, replacing Δ' by its conjugate $g\Delta'$, we may assume $\gamma' = \gamma$.

Now we apply (S2) – Proposition 7.3 – to the strata Δ_γ and Δ'_γ, with $a = \beta-\gamma$ and $a' = \beta'-\gamma$. The conclusion is that the derived strata intertwine so that the base case gives us a bijection between the index sets $\zeta : I \to I'$ such that, for each $i \in I$, the stratum $[\Lambda^i \oplus \Lambda'^{\zeta(i)}, r + 1, r, s_{\gamma_i}(\beta_i - \gamma_i) + s_{\gamma_{\zeta(i)}}(\beta'_{\zeta(i)} - \gamma_{\zeta(i)})]$ is equivalent to a simple stratum. (Here $\gamma_i = 1^i\gamma 1^i$, where 1^i is the idempotent corresponding to V^i, and similarly $\gamma'_{\zeta(i)} = 1'^{\zeta(i)}\gamma 1'^{\zeta(i)}$, corresponding to $V'^{\zeta(i)}$; note also that γ_i and $\gamma_{\zeta(i)}$ have the same characteristic polynomial so that we can view both V^i and $V'^{\zeta(i)}$ as $F[\gamma_i]$-vector spaces.) But then $[\Lambda^i \oplus \Lambda'^{\zeta(i)}, \max\{q_i, q'_{\zeta(i)}\}, r, \beta_i + \beta'_{\zeta(i)}]$ is equivalent to a simple stratum, by (S3) – Theorem 6.14.

The existence of ζ implies, in particular, that both strata have the same number of blocks, i.e. the sets I and I' have the same cardinality. Finally, we prove the uniqueness of ζ. Assume, for contradiction, that there are two distinct indices $i, j \in I$ and an index $i' \in I'$ such that $[\Lambda^i \oplus \Lambda'^{i'}, \max\{q_i, q'_{i'}\}, r, \beta_i + \beta'_{i'}]$ and $[\Lambda^j \oplus \Lambda'^{i'}, \max\{q_j, q'_{i'}\}, r, \beta_j + \beta'_{i'}]$ are both equivalent to simple strata. From this (and the equality of periods) it follows that the integers q_i, q_j and $q'_{i'}$ are all equal; we denote this integer by q.

By the proof of the existence, the spaces V^i and $V^{i'}$ have the same dimension, and thus, by conjugating, we can assume that they are equal. By Theorem 6.16, the strata $[\Lambda^i, q, r, \beta_i]$ and $[\Lambda'^{i'}, q, r, \beta_{i'}]$ intertwine. Then the stratum $[\Lambda^i \oplus \Lambda^j, q, r, \beta_i + \beta_j]$ intertwines with $[\Lambda'^{i'} \oplus \Lambda^j, q, r, \beta_{i'} + \beta_j]$ and the latter is equivalent to a simple stratum. Thus the semisimple stratum $[\Lambda^i \oplus \Lambda^j, q, r, \beta_i + \beta_j]$, which has two blocks, is intertwined with a simple stratum, which has only one block. This is a contradiction since the existence shows that semisimple strata which intertwine have the same number of blocks. □

As a useful consequence, we see that, given two semisimple strata which intertwine, we can find equivalent semisimple strata with elements which are conjugate.

Corollary 7.8. *Suppose that the semisimple strata $[\Lambda, q, r, \beta]$ and $[\Lambda', q, r, \beta']$ intertwine and that Λ, Λ' have the same period, and let $\zeta : I \to I'$ be the matching between their index sets. Then there are semisimple strata $[\Lambda, q, r, \tilde{\beta}]$ and $[\Lambda', q, r, \tilde{\beta}']$, equivalent to, and with the same associated splitting as, $[\Lambda, q, r, \beta]$ and $[\Lambda', q, r, \beta']$ respectively, such that $\tilde{\beta}'_{\zeta(i)}$ has the same characteristic polynomial as $\tilde{\beta}_i$, for all indices $i \in I$.*

Proof. This follows immediately from Lemma 7.7 (whose hypotheses are satisfied, thanks to Proposition 7.1) by putting, in the notation of the Lemma, $\tilde{\beta} = \gamma$ and $\tilde{\beta}' = g^{-1}\gamma g$. □

If $[\Lambda, q, r, \beta]$ and $[\Lambda', q', r', \beta']$ are strata in spaces V and V' respectively, then we put

$$I([\Lambda^i, q, r, \beta], [\Lambda^{\zeta(i)}, q, r, \beta']) = \left\{ g \mid V \xrightarrow{\sim} V' \mid g(\beta + \mathfrak{a}_{-r})g^{-1} \cap (\beta' + \mathfrak{a}'_{-r'} \neq \varnothing \right\}.$$

This generalizes the notion of intertwining and we say that any element of this set intertwines $[\Lambda, q, r, \beta]$ with $[\Lambda', q', r', \beta']$.

Corollary 7.9. *Suppose that the semisimple strata $[\Lambda, q, r, \beta]$ and $[\Lambda', q, r, \beta']$ intertwine and that Λ, Λ' have the same period, and let $\zeta : I \to I'$ be the matching between their index sets. Then the intertwining set $I([\Lambda, q, r, \beta], [\Lambda', q, r, \beta'])$ is equal to*

$$(1 + \mathfrak{m}'_{-(r+k'_0)})(\prod_i I([\Lambda^i, q, r, \beta], [\Lambda^{\zeta(i)}, q, r, \beta']))(1 + \mathfrak{m}_{-(r+k_0)}).$$

Proof. By Corollary 7.8 we may replace the strata with equivalent strata such that there is $g \in \tilde{G}$ such that $\beta' = g^{-1}\beta g$. The result now follows by applying Theorem 6.22 to the strata $[\Lambda, q, r, \beta]$ and $[g\Lambda', q, r, \beta]$. □

7.2. **For classical groups.** We continue with the notation from the previous section but assume now that all our strata are skew. We will prove the following strengthening of Proposition 7.1 in this case.

Proposition 7.10. *Suppose that $[\Lambda, q, r, \beta]$ and $[\Lambda', q, r, \beta']$ are two skew-semisimple strata which intertwine in G and let $\zeta : I \to I'$ be the matching given by Proposition 7.1. Then:*

(i) $(V^i, h|_{V^i}) \cong (V'^{\zeta(i)}, h|_{V'^{\zeta(i)}})$, *for all $i \in I$;*
(ii) *the intertwining set $I_G([\Lambda, q, r, \beta], [\Lambda', q, r, \beta'])$ is equal to*

$$((1 + \mathfrak{m}'_{-(r+k'_0)}) \cap G)((\prod_i I_i) \cap G)((1 + \mathfrak{m}_{-(r+k_0)}) \cap G),$$

where $I_i = I([\Lambda^i, q, r, \beta], [\Lambda'^{\zeta(i)}, q, r, \beta'])$.

Remark 7.11. Part (ii) of Proposition 7.10 is a consequence of (i): Indeed, if (i) is true then, by conjugating, we can assume that $V^i = V'^{\zeta(i)}$, for all $i \in I$, and (ii) follows from Corollary 7.9 and a simple cohomology argument as in [Ste05, Corollary 4.14].

As an immediate consequence of Proposition 7.10 and the simple Skolem–Noether Theorem 5.2, we get a Skolem–Noether Theorem for semisimple strata.

Theorem 7.12. *Let $[\Lambda, q, r, \beta]$ and $[\Lambda', q, r, \beta']$ be two skew-semisimple strata which intertwine in G, and suppose that β and β' have the same characteristic polynomial. Then there is an element $g \in G$ such that $g\beta g^{-1}$ is equal to β'.*

For the proof of Proposition 7.10 we need the following idempotent lifting lemma.

Lemma 7.13. *Let $(\mathfrak{k}_r)_{r\geqslant 0}$ be a decreasing sequence of o_F-lattices in A such that $\mathfrak{k}_r\mathfrak{k}_s \subseteq \mathfrak{k}_{r+s}$, for all $r, s \geqslant 0$, and $\bigcap_{r\geqslant 1} \mathfrak{k}_r = \{0\}$. Suppose there is an element α of $\mathfrak{k}_0$ which satisfies $\alpha^2 - \alpha \in \mathfrak{k}_1$. Then there is an idempotent $\tilde{\alpha} \in \mathfrak{k}_0$ such that $\tilde{\alpha} - \alpha \in \mathfrak{k}_1$. Moreover, if $\sigma(\alpha) = \alpha$ then we can choose $\tilde{\alpha}$ such that $\sigma(\tilde{\alpha}) = \tilde{\alpha}$.*

Proof. We define $e_1 := \alpha$, and put $e_2 := 3e_1^2 - 2e_1^3 \in \mathfrak{k}_0$. A straightforward calculation shows that

$$e_2^2 - e_2 = 4(e_1^2 - e_1)^3 - 3(e_1^2 - e_1)^2 \in \mathfrak{k}_{2r}.$$

Continuing this process, we construct a sequence $(e_n)_{n\geqslant 1}$ in $\mathfrak{h}_0$ which satisfies

(i) $e_n \equiv e_i \pmod{\mathfrak{k}_{2^i r}}$ and
(ii) $e_n \equiv e_n^2 \pmod{\mathfrak{k}_{2^n r}}$,

for all positive integers $i < n$. This sequence has a limit $\tilde{\alpha}$ in $\mathfrak{b}_0$ which is, by construction, an idempotent congruent to α modulo $\mathfrak{k}_r$. Moreover, by construction the sequence (e_n) is symmetric if α is, in which case the limit $\tilde{\alpha}$ is also symmetric. ☐

Corollary 7.14. *Let $(\mathfrak{k}_r)_{r\geqslant 0}$ be as in Lemma 7.13. Suppose that $\alpha_1, \ldots, \alpha_l$ are elements of $\mathfrak{k}_0$ such that $\alpha_i^2 - \alpha_i \in \mathfrak{k}_1$ and $\alpha_i\alpha_j \in \mathfrak{k}_1$, for all $i \neq j$. Suppose further that $\sum_i \alpha_i \equiv 1 \pmod{\mathfrak{k}_1}$. Then there are idempotents $\tilde{\alpha}_i$ such that $\tilde{\alpha}_i - \alpha_i \in \mathfrak{k}_1$, with $\tilde{\alpha}_i\tilde{\alpha}_j = 0$, for $i \neq j$, and $\sum_i \tilde{\alpha}_i = 1$. If further $\sigma(\alpha_i) = \alpha_i$, for all i, then we can choose the $\tilde{\alpha}_i$ such that $\sigma(\tilde{\alpha}_i) = \tilde{\alpha}_i$, for all i.*

Proof. We find $\tilde{\alpha}_1$ by Lemma 7.13 and set $\tilde{\alpha}_1^\perp = 1 - \tilde{\alpha}_1$. Consider the space $V^{(1)} := \tilde{\alpha}_1^\perp V$, the lattices $\mathfrak{k}_r^{(1)} := \tilde{\alpha}_1^\perp \mathfrak{k}_r \tilde{\alpha}_1^\perp$ and the elements $\alpha_i^{(1)} := \tilde{\alpha}_1^\perp \alpha_i \tilde{\alpha}_1^\perp$ for $i \geqslant 2$. These satisfy the hypotheses of the corollary, which now follows by induction. If $\sigma(\alpha_1) = \alpha_1$ then we choose $\tilde{\alpha}_1$ such that $\sigma(\tilde{\alpha}_1) = \tilde{\alpha}_1$, and $V^{(1)}$ is then the orthogonal complement of $\tilde{\alpha}_1 V$ so that the result again follows by induction. ☐

We also need the classical group analogue of Lemma 7.7.

Lemma 7.15. *Let $[\Lambda, q, r, \beta]$ and $[\Lambda', q, r, \beta']$ be skew-semisimple strata which intertwine in G and let $\zeta : I \to I'$ be the matching given by Proposition 7.1. Suppose moreover* *that* $(V^i, h|_{V^i}) \cong (V'^{\zeta(i)}, h|_{V'^{\zeta(i)}})$*, for all* $i \in I$*. Then there are an element* $g \in G$ *and a skew element* $\gamma \in \prod_i A^{i,i}$ *such that* $V^i = gV'^{\zeta(i)}$*, for all indices* $i \in I$*, and:*

- $[g\Lambda', q, r, g\beta' g^{-1}]$ *is equivalent to* $[g\Lambda', q, r, \gamma]$*;*
- $[\Lambda, q, r, \beta]$ *is equivalent to* $[\Lambda, q, r, \gamma]$*; and*
- $[g\Lambda', q, r, \gamma]$ *and* $[\Lambda, q, r, \gamma]$ *are skew-semisimple strata with the same associated splitting* $V = \bigoplus_{i\in I} V^i$*.*

Proof. The proof is the same as that of Lemma 7.7. We only need to note that, once we have found γ_i and $\gamma'_{\zeta(i)}$ with the same irreducible minimal polynomial then there is an element $g \in G$ such that $V^i = gV'^{\zeta(i)}$, since $(V^i, h|_{V^i}) \cong (V'^{\zeta(i)}, h|_{V'^{\zeta(i)}})$, and then the elements $g\gamma'_{\zeta(i)}g^{-1}$ and γ_i are conjugate in $G_i = A^{i,i} \cap G$ by Remark 7.11 and Theorem 5.2. □

Proof of Proposition 7.10. It is sufficient to prove (i) by Remark 7.11. We prove (i) by strata induction, giving first the inductive step. Suppose that $\Delta = [\Lambda, q, r, \beta]$ and $\Delta' = [\Lambda', q, r, \beta']$ are skew-semisimple strata as in the proposition which intertwine in G. Then the stratum $[\Lambda, q, r+1, \beta]$ is equivalent to a skew-semisimple stratum $\Delta_\gamma = [\Lambda, q, r+1, \gamma]$ whose splitting is a coarsening of that of Δ, by Theorem 6.16; similarly we have a skew-semisimple stratum $\Delta'_\gamma = [\Lambda', q, r+1, \gamma']$. Since the strata Δ_γ, Δ'_γ intertwine, we may apply Lemma 7.15 and, replacing Δ' by its conjugate $g\Delta'$, we may assume $\gamma' = \gamma$.

Now, if $h \in G$ intertwines the strata Δ_γ and Δ'_γ then Theorem 6.22 allows us to write $h = xby$, with $b \in B \cap G$ (and x, y in certain compact subgroups). Then Proposition 7.3, applied as in the proof of Proposition 7.1, implies that the derived strata $[\Lambda, r+1, r, s_\gamma(\beta - \gamma)]$ and $[\Lambda', r+1, r, s_\gamma(\beta' - \gamma)]$ intertwine. On the other hand, these derived strata are equivalent to skew-semisimple strata so the base step (below) now implies that the bijection $\zeta : I \to I'$ has the property that $(V^i, h_{i,\phi_i}) \cong (V'^{\zeta(i)}, h'^{\phi'_{\zeta(i)}}_{\zeta(i)})$, where $\phi_i : F[\gamma_i] \to A^{i,i}$ is the embedding given by the splitting, and h_{i,ϕ_i} is such that $h_i = \lambda_i \circ h_{i,\phi_i}$ and $h_i = h|_{V^i}$. But then we also have $(V^i, h_i) \cong (V'^{\zeta(i)}, h'_{\zeta(i)})$, as required.

It remains to show the base case $r = q - 1$. The lattice sequences have the same level, and so the same period, by Proposition 6.9. Since they intertwine, the strata have the same characteristic polynomial. If two minimal strata with the same level intertwine and one of them is null, then the other is null, by Proposition 6.9, and both have the trivial associated splitting. Thus we need only consider non-null semisimple strata.

Replacing $[\Lambda', q, q-1, \beta']$ by a conjugate if necessary, we may assume that the strata are intertwined by 1 so that

$$(\beta + \mathfrak{a}_{1-q}) \cap (\beta' + \mathfrak{a}'_{1-q}) \neq \emptyset.$$

Thus there are elements $a \in \mathfrak{a}_1$ and $a' \in \mathfrak{a}'_1$ such that

$$z := y_\beta + a = y_{\beta'} + a'.$$

By the bijectivity of ζ we can assume that $I = I'$ and ζ is the identity. Let $i \in I$.

We show that there is an idempotent e such that $e \equiv 1^i \pmod{\mathfrak{a}_1}$ and $e \equiv 1'^i \pmod{\mathfrak{a}'_1}$: There is a polynomial $Q \in o_F[X]$ such that $Q(y_\beta) \equiv 1^i \pmod{\mathfrak{a}_1}$. Moreover, by replacing $Q(X)$ by $\frac{1}{2}(Q(X) + \sigma(Q)(\pm X))$, we can choose Q such that, for all j, the coefficient of X^j is symmetric (resp. skew-symmetric) if and only if y_β^j is symmetric (resp. skew-symmetric). We have a canonical isomorphism from $\kappa[\bar{y}_\beta]$ to $\kappa[\bar{y}_{\beta'}]$ (mapping $\bar{y}_\beta$ to $\bar{y}_{\beta'}$) so $Q(y_{\beta'})$ is congruent to some idempotent modulo $\mathfrak{a}'_1$, and indeed $Q(y_{\beta'}) \equiv 1'^i \pmod{\mathfrak{a}'_1}$ since the matching ζ is given by matching minimal polynomials. By Proposition 7.13 applied with $\mathfrak{k}_r = \mathfrak{a}_r \cap \mathfrak{a}'_r$, there is a symmetric idempotent $e \in \mathfrak{a}_0 \cap \mathfrak{a}'_0$ congruent to $Q(z)$ modulo both radicals.

The idempotent e gives a new splitting $V = \tilde{V}^i \oplus (\tilde{V}^i)^\perp$ for *both* lattice sequences.

Finally, we show that V^i and $\tilde{V}^i$ are isomorphic signed hermitian spaces. We define the map $\psi : V^i \to \tilde{V}^i$ to be the restriction of e to V^i. We first show that the map is injective. If v is a non-zero element of its kernel, then there is an integer l such that $v \in \Lambda_l^i \setminus \Lambda_{l+1}^i$. But then

$$0 \not\equiv v \equiv 1^i v \equiv ev \equiv 0 \pmod{\Lambda_{l+1}^i},$$

where the third congruence uses that $e \equiv 1^i \pmod{\mathfrak{a}_1}$. Similarly, the restriction of 1^i to $\tilde{V}^i$ is injective and these maps induce pairwise inverse κ_F-isomorphisms between $\Lambda_l^i/\Lambda_{l+1}^i$ and $\tilde{\Lambda}_l^i/\tilde{\Lambda}_{l+1}^i$ where $\tilde{\Lambda}_l^i$ is the intersection of Λ_l with $\tilde{V}^i$. Thus $\psi(\Lambda^i)$ is equal to $\tilde{\Lambda}^i$.

We now compare the hermitian structures. For $v \in \Lambda_l^i$ and $w \in (\Lambda_{l+1}^i)^\#$ we have

$$\overline{h(v,w)} = \overline{h(v,1^i w)} = \overline{h(v,ew)} = \overline{h(v,e^2 w)} = \overline{h(ev,ew)} = \overline{h(\psi(v),\psi(w))}$$

By Proposition 3.1 there is an F-linear isometry

$$\tilde{\psi} : (V^i, h|_{V^i}) \to (\tilde{V}^i, h|_{\tilde{V}^i})$$

such that

- $\tilde{\psi}(\Lambda^i) = \tilde{\Lambda}^i$ and
- $\tilde{\psi}$ and ψ induce the same isomorphism on $\Lambda_l^i/\Lambda_{l+1}^i$, for all integers l.

Thus, V^i, $\tilde{V}^i$, and similarly V'^i, are isomorphic signed hermitian spaces. □

7.3. **Matching for equivalent strata.** We need to understand the matching between two equivalent strata, and for that reason we have the following three results.

Lemma 7.16. *Suppose that e is an idempotent in $\prod_i A^{i,i}$ such that every non-zero element x of $\prod_i A^{i,i}$ satisfies $\nu_\Lambda(ex - xe) > \nu_\Lambda(x)$. Then e is a central idempotent of $\prod_i A^{i,i}$, i.e. commutes with all elements of $\prod_i A^{i,i}$.*

Proof. Let x be an element of $\prod_i A^{i,i}$ and put $x' = a_e(x) = ex - xe$. Then $ex'e$ is zero and one checks $a_e(a_e(x')) = a_e(x')$. The condition on e now implies $a_e(x') = 0$ and thus $ex' = ea_e(x') = 0 = a_e(x')e = x'e$. This implies $ex = exe = xe$ and, since x was arbitrary, e is central. □

Lemma 7.17. *Let $[\Lambda, q, m, \beta]$ and $[\Lambda, q, m, \beta']$ be two semisimple strata, such that*

$$\tilde{\mathrm{U}}^1(\mathfrak{a})B_{\beta'}\tilde{\mathrm{U}}^1(\mathfrak{a}) = \tilde{\mathrm{U}}^1(\mathfrak{a})B_{\beta}\tilde{\mathrm{U}}^1(\mathfrak{a})$$

then there are a bijection $\tau : I \to I'$ and an element g of $\tilde{\mathrm{U}}^1(\mathfrak{a})$ such that

(i) $1^i \equiv 1^{\tau(i)} \pmod{\mathfrak{a}_1}$, *and*
(ii) $g1^ig^{-1} = 1^{\tau(i)}$

for all indices $i \in I$. Moreover, the bijection τ satisfies

$$\dim_{\kappa_F}(\Lambda^i_j/\Lambda^i_{j+1}) = \dim_{\kappa_F}(\Lambda^{\tau(i)}_j/\Lambda^{\tau(i)}_{j+1}), \quad \textit{for all } i \in I,\ j \in \mathbb{Z}.$$

Proof. By the equality of the two sets and Lemmas 7.13 and 7.16, every primitive central idempotent of B_β has to be congruent modulo $\mathfrak{a}_1(\Lambda)$ to a sum of primitive central idempotents of $B_{\beta'}$, and vice versa. The first part follows from this. For the second part, take the map g which sends $v \in V$ to $\sum_i 1^{\tau(i)}1^iv$. Finally, the map $v \mapsto 1^iv$ induces, for each $j \in \mathbb{Z}$, a linear map $\Lambda^{\tau(i)}_j/\Lambda^{\tau(i)}_{j+1} \to \Lambda^i_j/\Lambda^i_{j+1}$ whose inverse is induced by $v \mapsto 1^{\tau(i)}v$. □

Lemma 7.18. *Let $[\Lambda, q, m, \beta]$ and $[\Lambda, q, m, \beta']$ be equivalent semisimple strata. Then there is an element g of $1 + \mathfrak{m}_{-(k_0+m)}(\beta, \Lambda)$ such that $[\Lambda, q, m, \beta]$ and $[\Lambda, q, m, g\beta'g^{-1}]$ have the same associated splitting.*

Proof. Note that we may replace $[\Lambda, q, m, \beta]$ by an equivalent stratum with the same splitting (and likewise for $[\Lambda, q, m, \beta']$). Thus, by applying Corollary 7.8, we may assume that β and β' have the same characteristic polynomial and thus there is an element x of $\tilde{G}$ such that $x\beta x^{-1} = \beta'$. Note that this implies that $xV^i = V^{\zeta(i)}$.

Since the strata intertwine, Proposition 7.1 gives us a matching $\zeta : I \to I'$ such that the minimal polynomials satisfy $\mu_{\beta_i} = \mu_{\beta'_{\zeta(i)}}$ and $\dim_F V^i = \dim_F V^{\zeta(i)}$, for each $i \in I$. We can also compare the intertwining sets of the strata (which are equal) and then Lemma 7.17 gives us a map $\tau : I \to I'$ such that $1^i \equiv 1^{\tau(i)} \pmod{\mathfrak{a}_1}$, for all $i \in I$. Since the identity intertwines the two strata, Corollary 7.9 implies that we can write the identity as uyv, with $u, v \in \tilde{\mathrm{U}}^1(\Lambda)$ and $y = \prod_{i\in I} y_i$ such that $y_iV^i = V^{\zeta(i)}$. Moreover, we have $y = u^{-1}v^{-1} \in \tilde{\mathrm{U}}^1(\Lambda)$ so that $y_i\Lambda^i = \Lambda^{\zeta(i)}$. Thus

$$1^i \equiv y1^iy^{-1} = 1^{\zeta(i)} \pmod{\mathfrak{a}_1}.$$

In particular, we get $1^{\tau(i)} \equiv 1^{\zeta(i)} \pmod{\mathfrak{a}_1}$ so that $\zeta = \tau$, and then Lemma 7.17 also implies that ζ satisfies the extra condition

$$\dim_{\kappa_F}(\Lambda^i_j/\Lambda^i_{j+1}) = \dim_{\kappa_F}(\Lambda^{\zeta(i)}_j/\Lambda^{\zeta(i)}_{j+1}), \quad \text{for all } i \in I,\ j \in \mathbb{Z}.$$

Now Λ^i and $x^{-1}\Lambda^{\zeta(i)}$ are $o_{F[\beta_i]}$-lattice sequences in V^i with successive quotients of the same dimensions so there is an element z_i of $B^\times_{\beta_i}$ such that $z_i\Lambda^i = x^{-1}\Lambda^{\zeta(i)}$. In particular, writing $z = \prod_{i\in I} z_i$, the element xz conjugates β to β' and lies in $\tilde{\mathrm{U}}(\Lambda)$.

Finally, since the strata $[\Lambda, q, m, \beta]$ and $[\Lambda, q, m, \beta']$ are equivalent, the element xz also lies in $\mathfrak{n}_{-m}(\beta, \Lambda) \cap \tilde{\mathrm{U}}(\Lambda)$ which, by Lemma 6.20, is $(1 + \mathfrak{m}_{-(k_0+m)}(\beta, \Lambda))\mathfrak{b}_0^\times$. Hence we can write $xz = gb$, with $b \in \mathfrak{b}_0^\times$ and $g \in 1 + \mathfrak{m}_{-(k_0+m)}(\beta, \Lambda)$. □

We end this section with a criterion for a minimal semisimple stratum to be equivalent to a skew-semisimple stratum, in terms of its characteristic polynomial.

Lemma 7.19. *Suppose $[\Lambda, m, m-1, \beta]$ is a semisimple stratum such that Λ is self-dual and $\sigma(\beta) \equiv -\beta \pmod{\mathfrak{a}_{1-m}}$. Put $e_0 = e(F|F_0)$, $e = e(\Lambda|\mathfrak{o}_F)$ and $g = \gcd(m, e)$, and set $\eta = (-1)^{(2m+ee_0)/ge_0}$. Let ϕ be the characteristic polynomial of $[\Lambda, m, m-1, \beta]$ and suppose that its primary factors ϕ_i satisfy*

$$\sigma(\phi_i)(X) = \eta^{\deg(\phi_i)}\phi_i(\eta X).$$

Then the stratum is equivalent to a skew-semisimple stratum.

Proof. Let (1^i) be the idempotents of the associated splitting of β. By hypothesis, the stratum $[\Lambda, m, m-1, \beta]$ is equivalent to the $[\Lambda, m, m-1, -\sigma(\beta)]$, which is also semisimple, with associated idempotents $(\sigma(1^i))$. Then Lemma 7.17 implies that there is a bijection τ of I such that $\sigma(1^i)$ is congruent to $1^{\tau(i)}$ modulo $\mathfrak{a}_1$, for all indices i. Recalling that ϕ is the characteristic polynomial of (the reduction of) $y_\beta = \beta^{e/g}\varpi^{m/g}$ and noting that $\sigma(y_\beta)$ is congruent to ηy_β modulo $\mathfrak{a}_1$, it follows that $\sigma(\phi_i)(X)$ is equal to $\eta^{\deg(\phi_{\tau(i)})}(\eta X)$, whence $\phi_i = \pm\phi_{\tau(i)}$, by the hypotheses of the lemma. However, the characteristic polynomials for different simple blocks (i.e. for different i) are coprime and thus $\tau(i) = i$ for all i.

Now we apply Corollary 7.14 to the elements $(1^i + \sigma(1^i))/2$ (and $\mathfrak{k}_r = \mathfrak{a}_r$) to obtain pairwise orthogonal symmetric idempotents e_i with $\sum_i e_i = 1$. Then we conjugate the stratum by $\sum_i e_i 1^i$ to obtain a stratum equivalent to $[\Lambda, m, m-1, \beta]$ whose simple blocks are equivalent to skew-simple strata by [Ste01b, 1.10]. This finishes the proof. □

8. Intertwining and conjugacy for semisimple strata

In the case of simple strata on a fixed lattice *chain*, intertwining implies conjugacy up to equivalence (see [BK93, 2.6.1]). The same result is true for arbitrary lattice sequences and, as we prove here, for simple skew strata (that is, G-intertwining implies G-conjugacy). However, the analogous result is no longer true for semisimple strata. As well as giving some examples to illustrate this, we give a useful sufficient additional condition to guarantee that the strata are indeed conjugate.

8.1. For general linear groups.

Theorem 8.1 (cf. [BK93, 2.6.1])**.** *Suppose $[\Lambda, q, r, \beta]$ and $[\Lambda, q, r, \beta']$ are simple strata which intertwine. Then, up to equivalence they are conjugate by an element of $\tilde{\mathrm{U}}(\Lambda)$.*

Proof. By Corollary 7.8, we can assume that β and β' have the same characteristic polynomial. By [BH96, Lemma 1.6], there is an element of $\tilde{\mathrm{U}}(\Lambda)$ which conjugates β to β'. □

In contrast to simple strata we cannot achieve intertwining implies conjugacy for semisimple strata.

Example 8.2. Let V be a 4-dimensional vector space over F with basis $v_1, \ldots, v_4$ and let Λ be the lattice chain of period 2 such that

$$\Lambda_0 = v_1 o_F + v_2 o_F + v_3 o_F + v_4 o_F, \qquad \Lambda_1 = v_1 o_F + v_2 o_F + v_3 o_F + v_4 \mathfrak{p}_F.$$

Then, with respect to the basis, $\mathfrak{a}_0(\Lambda)$ is

$$\begin{pmatrix} o_F & o_F & o_F & o_F \\ o_F & o_F & o_F & o_F \\ o_F & o_F & o_F & o_F \\ \mathfrak{p}_F & \mathfrak{p}_F & \mathfrak{p}_F & o_F \end{pmatrix}$$

The two elements:

$$b := \mathrm{diag}(\varpi^{-1}, \varpi^{-1}, -\varpi^{-1}, -\varpi^{-1}), \qquad b' := \mathrm{diag}(-\varpi^{-1}, -\varpi^{-1}, \varpi^{-1}, \varpi^{-1})$$

give two semisimple strata $[\Lambda, 2, 1, b]$ and $[\Lambda, 2, 1, b']$ which intertwine but whose equivalence classes are not conjugate over $\mathrm{Aut}_F(V)$. Indeed, suppose for contradiction that the strata are conjugate under an element of $\tilde{G}$; then this element has to be an element of the normalizer of Λ and thus by Lemma 7.18 we can assume after conjugation that the associated splittings, which are the same for both strata, are conjugated to each other. Note that this splitting is given by $V^1 = v_1 F + v_2 F$ and $V^2 = v_3 F + v_4 F$. The minimal polynomials of the strata force that the matching has to be given by exchanging the two blocks V^1 and V^2. But this is not possible, because the image of $\Lambda^1 = \Lambda \cap V^1$ contains only one homothety class of lattices, while the image of $\Lambda^2 = \Lambda \cap V^2$ contains two.

Thus we impose an extra condition in the following Theorem.

Theorem 8.3. *Suppose that $[\Lambda, q, r, \beta]$ and $[\Lambda, q, r, \beta']$ are two non-null semisimple strata which intertwine and let ζ be the matching between their index sets. Suppose moreover that*

$$\dim_{\kappa_F}(\Lambda^i_j/\Lambda^i_{j+1}) = \dim_{\kappa_F}(\Lambda^{\zeta(i)}_j/\Lambda^{\zeta(i)}_{j+1}), \quad \text{for all } i \in I,\ j \in \mathbb{Z}. \tag{8.4}$$

Then the strata are conjugate by an element of $\tilde{\mathrm{U}}(\Lambda) \cap \prod_i A^{i,\zeta(i)}$.

Proof. Fix an index i of I. By Proposition 7.1 and Corollary 7.9 there is an F-linear isomorphism $g_i : V^i \to V^{\zeta(i)}$ such that $[g_i\Lambda^i, q, r, g_i\beta^i g_i^{-1}]$ intertwines $[\Lambda^{\zeta(i)}, q, r, \beta'^{\zeta(i)}]$. Now condition (8.4) implies that the lattice sequences $g_i\Lambda^i$ and $\Lambda^{\zeta(i)}$ are conjugate so, modifying g_i if necessary, we may assume $g_i\Lambda^i_j = \Lambda^{\zeta(i)}_j$, for each $j \in \mathbb{Z}$. Now we can apply

Theorem 8.1 so that, replacing g_i by a translate by an element of $\tilde{\mathrm{U}}(\Lambda^{\zeta(i)})$, we can assume that $[g_i\Lambda^i, q, r, g_i\beta^i g_i^{-1}]$ is equivalent to $[\Lambda^{\zeta(i)}, q, r, \beta'^{\zeta(i)}]$. Then

$$\left(\prod_{i\in I} g_i\right)\Lambda_j = \bigoplus_{i\in I} g_i\Lambda_j^i = \bigoplus_{i\in I}\Lambda_j^{\zeta(i)} = \Lambda_j.$$

so that $\prod_{i\in I} g_i \in \tilde{\mathrm{U}}(\Lambda)$ conjugates the first to the second stratum. □

8.2. **For classical groups.** We give here the similar "intertwining implies conjugacy" statements for skew-semisimple strata, beginning with the simple case.

Theorem 8.5. *Suppose $[\Lambda, q, r, \beta]$ and $[\Lambda, q, r, \beta']$ are two skew-simple strata which intertwine in G. Then they are conjugate over $\mathrm{U}(\Lambda)$.*

Proof. The proof is mutatis mutandis that of Theorem 8.1: we apply Corollary 7.8, then Theorem 5.2, and then Proposition 5.4. □

As in the non-skew case, this is no longer true if one replaces simple by semisimple.

Example 8.6. Consider a ramified quadratic field extension $F|F_0$ and a skew-hermitian form on $V = F^4$ whose Gram matrix (h_{ij}) with respect to the standard basis is the anti-diagonal matrix with entries

$$h_{41} = h_{32} = -1 = -h_{23} = -h_{14},$$

and write G for the isometry group of this form. Let ϖ be a skew-symmetric uniformizer of F and let z be a non-square in F_0. Let Λ be the self-dual lattice chain corresponding to the hereditary order

$$\begin{pmatrix} o_F & \mathfrak{p}_F & \mathfrak{p}_F & \mathfrak{p}_F \\ o_F & o_F & \mathfrak{p}_F & \mathfrak{p}_F \\ o_F & o_F & o_F & \mathfrak{p}_F \\ o_F & o_F & o_F & o_F \end{pmatrix}$$

We define the skew-symmetric elements:

$$b := \mathrm{diag}(\varpi^{-1}z, \varpi^{-1}, \varpi^{-1}, z\varpi^{-1}), \qquad b' := \mathrm{diag}(\varpi^{-1}, \varpi^{-1}z, \varpi^{-1}z, \varpi^{-1}).$$

The minimal skew-semisimple strata $[\Lambda, 4, 3, b]$ and $[\Lambda, 4, 3, b']$ intertwine over G because b is conjugate to b' under G, but the strata are not conjugate under G because they are not conjugate under $\tilde{\mathrm{U}}(\Lambda)$.

As an immediate consequence of Proposition 7.10 and Theorem 8.5 (as in the non-skew case above) we have:

Theorem 8.7. *Suppose that $[\Lambda, q, r, \beta]$ and $[\Lambda, q, r, \beta']$ are two non-null skew-semisimple strata which intertwine in G, with matching ζ, such that (8.4) holds. Then the strata are conjugate by an element of $\mathrm{U}(\Lambda)$.*

9. Semisimple characters

Associated to the semisimple strata studied in the previous sections, we have sets of characters of certain compact open subgroups, which are call *semisimple characters*. The purpose of this section is both briefly to recall their definitions and properties (from [BK93] and [Ste05]) and to ensure that all the results we need are available for arbitrary lattice sequences.

9.1. Semisimple characters for $\tilde{G}$.

Fix a semisimple stratum $[\Lambda, q, 0, \beta]$. Define $r := -k_0(\beta, \Lambda)$ and let $[\Lambda, q, r, \gamma]$ be a semisimple stratum equivalent to $[\Lambda, q, r, \beta]$ such that γ commutes with the projections 1^i of the associated splitting of β. If $[\Lambda, q, r, \beta]$ is minimal then we take γ to be zero.

The rings of a semisimple stratum (cf. [BK93, 3.1]). We start with the orders $\mathfrak{h}(\beta, \Lambda)$ and $\mathfrak{j}(\beta, \Lambda)$, defined inductively by

- $\mathfrak{h}(\beta, \Lambda) = \mathfrak{b}_{\beta,0} + \mathfrak{h}(\gamma, \Lambda) \cap \mathfrak{a}_{\lfloor \frac{r}{2} \rfloor + 1}$,
- $\mathfrak{j}(\beta, \Lambda) = \mathfrak{b}_{\beta,0} + \mathfrak{j}(\gamma, \Lambda) \cap \mathfrak{a}_{\lfloor \frac{r+1}{2} \rfloor}$,

with $\mathfrak{h}(0, \Lambda) = \mathfrak{j}(0, \Lambda) = \mathfrak{a}_0$. We define now the groups

$$H^{m+1}(\beta, \Lambda) := \mathfrak{h}(\beta, \Lambda) \cap \tilde{\mathrm{U}}^{m+1}(\Lambda), \qquad J^{m+1}(\beta, \Lambda) := \mathfrak{j}(\beta, \Lambda) \cap \tilde{\mathrm{U}}^{m+1}(\Lambda),$$

for $m \geqslant -1$, and write H and J instead of H^0 and J^0.

We now begin the proofs of the statements in [BK93, Section 3.1] for semisimple strata. (Note that some of these are already in [Ste05].)

Proposition 9.1 (cf. [BK93, (3.1.9)]).
(i) *For all $-1 \leqslant t \leqslant r$, the lattice $\mathfrak{h}^{\lfloor \frac{t}{2} \rfloor}(\beta, \Lambda)$ is a bimodule over the ring $\mathfrak{n}_{-t}(\beta, \Lambda)$.*
(ii) *If $r < n$, $\mathfrak{h}^k(\beta, \Lambda)$ is equal to $\mathfrak{h}^k(\gamma, \Lambda)$ for $k \geqslant \lfloor \frac{r}{2} \rfloor + 1$.*
(iii) *For $k \geqslant 0$, $\mathfrak{h}^k(\beta, \Lambda)$ is a $\mathfrak{b}_\beta$-bimodule.*
(iv) *$\mathfrak{h}(\beta, \Lambda)$ is a ring and in particular an o_F-order in A and $\mathfrak{h}^k(\beta, \Lambda)$ is a two-sided ideal of $\mathfrak{h}(\beta, \Lambda)$, for all non-negative integers k.*
(v) *Let $t \leqslant r - 1$ and let $[\Lambda, q, t, \beta']$ be a semisimple stratum equivalent to $[\Lambda, q, t, \beta]$. Then $\mathfrak{h}^k(\beta, \Lambda)$ is equal to $\mathfrak{h}^k(\beta', \Lambda)$, for all non-negative integers $k > t - \lfloor \frac{r+1}{2} \rfloor$.*

Proof. In [BK93, (3.1.9)] the statement is proven for strict simple strata. In the case of a non-strict simple stratum $[\Lambda, q, m, \beta]$, the stratum

$$[\bigoplus_{l=0}^{e(\Lambda)-1} (\Lambda - l), q, m, \beta^{\oplus e}]$$

is a strict simple stratum and, using the identity

$$1_V \mathfrak{h}^k(\beta^{\oplus e}, \oplus_l (\Lambda - l)) 1_V = \mathfrak{h}^k(\beta, \Lambda)$$

(where 1_V denotes projection onto the first copy of V in $\bigoplus_{l=0}^{e(\Lambda)-1} V$) we get the result for all simple strata. Thus we continue with the case of semisimple strata.

We begin with the proof of (v), but only for the case where the strata in (v) have the same associated splitting; we prove the general case after the next four lemmas. We use the idea of [Ste05, Lemma 3.9]. Assume that $\mathfrak{h}^k(\beta,\Lambda)$ and $\mathfrak{h}^k(\beta',\Lambda)$ are defined using the same γ and that $[\Lambda,q,t,\beta']$ has the same associated splitting as $[\Lambda,q,t,\beta]$. In particular, we immediately get that $\mathfrak{h}^k(\beta,\Lambda)\cap A^{i,j}=\mathfrak{h}^k(\beta',\Lambda)\cap A^{i,j}$, for $i\neq j$, from the definition, while $\mathfrak{h}^k(\beta,\Lambda)\cap A^{i,i}=\mathfrak{h}^k(\beta',\Lambda)\cap A^{i,i}$ follows from the simple case.

Before proving (v) in general, we show how the remaining assertions follow from it. (ii) is straightforward while induction and (v) imply that the definition of $\mathfrak{h}(\beta,\Lambda)$ does not depend on the choice of γ. Now (i) follows from [Ste05, 3.10(ii)], (iv) follows by induction from (i) and (ii), and finally (iii) follows from (iv) and (i).

To finish the proof of Proposition 9.1(v) we need the following sequence of lemmas.

Lemma 9.2. $\mathfrak{h}^{\max\{0,1+t-\lfloor\frac{r+1}{2}\rfloor\}}$ *is an* $\mathfrak{n}_{-t}(\beta,\Lambda)$*-bimodule for all* $r\geqslant t\geqslant 0$.

For this we need the analogue of [Ste05, Lemma 3.10] for $\mathfrak{h}$ instead of $\mathfrak{j}$ (see the sentence following *loc. cit.*).

Lemma 9.3 (cf. [Ste05, Lemma 3.10(i)])**.**

(i) *For all integers* $k<\frac{r}{2}$, *we have* $\mathfrak{n}_{-k}\cap\mathfrak{a}_{r-k}\subseteq\mathfrak{h}^{r-k}(\beta,\Lambda)$.
(ii) *For all integers* $k\leqslant\frac{r}{2}$, *we have* $\mathfrak{n}_{-k}\cap\mathfrak{a}_{r-k}\subseteq\mathfrak{j}^{r-k}(\beta,\Lambda)$

Proof of Lemma 9.2. The proof is by induction on $r=-k_0(\beta,\Lambda)$. We have the two important identities:

$$\mathfrak{h}^t(\beta,\Lambda)=\mathfrak{b}_{\beta,t}+\mathfrak{h}^{\max\{t,\lfloor\frac{r}{2}\rfloor+1\}}(\gamma,\Lambda)$$

and

$$\mathfrak{n}_{-t}=\mathfrak{b}_{\beta,0}+\mathfrak{n}_{-t}(\beta,\Lambda)\cap\mathfrak{a}_{r-t}$$

We write t_0 for $\max\{0,1+t-\lfloor\frac{r+1}{2}\rfloor\}$ so that

$$t_0+r-t\geqslant 1+\left\lfloor\frac{r}{2}\right\rfloor \text{ and } 2(r-t+t_0)>r. \tag{9.4}$$

We have to show that $\mathfrak{n}_{-t}\mathfrak{h}^{t_0}(\beta,\Lambda)$ is a subset of $\mathfrak{h}^{t_0}(\beta,\Lambda)$. We have

- $\mathfrak{h}^{\max\{t_0,\lfloor\frac{r}{2}\rfloor+1\}}(\gamma,\Lambda)=\mathfrak{h}^{\max\{t_0,\lfloor\frac{r}{2}\rfloor+1\}}(\beta,\Lambda)$ is a $\mathfrak{b}_{\beta,0}$-module.
- $(\mathfrak{n}_{-t}(\beta,\Lambda)\cap\mathfrak{a}_{r-t})\mathfrak{b}_{\beta,t_0}$ is contained in $\mathfrak{n}_{t_0-t}(\beta,\Lambda)\cap\mathfrak{a}_{t_0+r-t}$, which is a subset of $\mathfrak{h}^{\max\{t_0,\lfloor\frac{r}{2}\rfloor+1\}}(\gamma,\Lambda)$ by (9.4) and Lemma 9.3.

The last containment we need, that $\mathfrak{n}_{-t}(\beta,\Lambda)\cap\mathfrak{a}_{r-t}\mathfrak{h}^{\max\{t,\lfloor\frac{r}{2}\rfloor+1\}}(\gamma,\Lambda)$ is a subset of $\mathfrak{h}^{t_0}(\beta,\Lambda)$, is proved by induction. The case of $\gamma=0$ is trivial, while the induction step is a result of the equality $\mathfrak{n}_{-t}(\beta,\Lambda)\cap\mathfrak{a}_{r-t}=\mathfrak{n}_{-t}(\gamma,\Lambda)\cap\mathfrak{a}_{r-t}$ and the induction hypothesis. □

Finally, we see that the proof of the general case of Proposition 9.1(v) follows from Lemmas 9.2 and 7.18. □

Given now the preliminary results on semisimple strata that we have obtained in previous sections and Proposition 9.1, we can follow the definitions and proofs of [BK93, Section 3.1], from (3.1.3) to (3.1.21), to see that if one makes the obvious substitution

- "replace $\mathfrak{b}_{\beta,t}\mathfrak{n}_l$ by $\mathfrak{a}_t \cap \mathfrak{n}_{t+l}$",

then everything is true except possibly for the equalities in (3.1.9)(iii), in (3.1.10)(iii) and in (3.1.11). (Some of these are already described in [Ste05].) Thus, from now on, we will use these statements from [BK93] for semisimple strata by referring to [BK93] (and giving the reference to [Ste05] if there is one).

Characters (cf. [BK93, 3.2]). Here we introduce the semisimple characters and their groups exactly the same way as it was done in [BK93, Section 3.2] for simple characters. This definition is equivalent to the definition given in [Ste05, Section 3]. We fix an additive character ψ_F of F of level one (that is, trivial on $\mathfrak{p}_F$ but not on o_F). We define $\psi_A := \psi_F \circ \mathrm{tr}_{A|F}$ and a character

$$\psi_\beta : \tilde{\mathrm{U}}^{\lfloor \frac{q}{2} \rfloor + 1}(\Lambda) \to \mathbb{C}^\times, \quad \psi_\beta(1+x) := \psi_A(\beta x).$$

The kernel of ψ_β contains $\tilde{\mathrm{U}}^{q+1}(\Lambda)$ because ψ_F has level one.

Definition 9.5. Suppose $0 \leqslant m < r$. If $q = r$ then we define the set $\mathcal{C}(\Lambda, m, \beta)$ to be the set of all characters $\theta : H^{m+1}(\beta, \Lambda) \to \mathbb{C}$ such that:

(i) the restriction of θ to $H^{m+1}(\beta, \Lambda) \cap \tilde{\mathrm{U}}^{\lfloor \frac{q}{2} \rfloor + 1}(\Lambda)$ is equal to ψ_β;
(ii) the restriction of θ to $H^{m+1} \cap B_\beta^\times$ factors through the determinant map $\det_{B_\beta} : B_\beta^\times \to F[\beta]^\times$.

If $q > r$ then we define $\mathcal{C}(\Lambda, m, \beta)$ inductively to be the set of all characters $\theta : H^{m+1}(\beta, \Lambda) \to \mathbb{C}$ such that:

(i) θ is normalized by $\mathfrak{n}(\Lambda) \cap B_\beta^\times$;
(ii) the restriction of θ to $H^{m+1} \cap B_\beta^\times$ factors through the determinant map $\det_{B_\beta} : B_\beta^\times \to F[\beta]^\times$;
(iii) if $m' = \max(m, \lfloor \frac{r}{2} \rfloor)$, then $\theta|_{H^{m'+1}(\beta,\Lambda)}$ is of the form $\theta_0\psi_c$, for some $\theta_0 \in \mathcal{C}(\Lambda, m', \gamma)$ and $c = \beta - \gamma$.

We also define $\mathcal{C}(\Lambda, m, 0)$ to be the set consisting of the trivial character on $\tilde{\mathrm{U}}^{m+1}(\Lambda)$.

Remark 9.6. (i) For $m \geqslant \lfloor \frac{q}{2} \rfloor$ we have $\mathcal{C}(\Lambda, m, \beta) = \{\psi_\beta\}$
(ii) This definition of the set of semisimple characters is *a priori* different from the one introduced in [Ste05, 3.13], which we temporarily call $\mathcal{C}(\Lambda, m, \beta)_{BKS}$:

(a) If Λ is strict and the stratum is simple then $\mathcal{C}(\Lambda, m, \beta)_{BKS}$ is defined as in Definition 9.5, see [BK93, (3.2.1)(3.2.3)]. In [Ste05], the set $\mathcal{C}(\Lambda, m, 0)_{BKS}$ is defined to be the set consisting of the trivial character on $\tilde{\mathrm{U}}^{m+1}(\Lambda)$.

(b) If Λ is a lattice sequence and the stratum is simple then $\mathcal{C}(\Lambda, m, \beta)_{BKS}$ is defined in [BK94, Section 5] in the following way. We take an o_E-lattice chain Λ^0 of the same period as Λ, in some space V^0, and restrict all elements of $\mathcal{C}(\Lambda^0 \oplus \Lambda, m, \beta \oplus \beta)_{BKS}$ to $H^{m+1}(\beta, \Lambda)$. The obtained restrictions form the set $\mathcal{C}(\Lambda, m, \beta)_{BKS}$.

(c) If Λ is a lattice sequence and the stratum is semisimple and non-null with associated splitting $V = \bigoplus_i V^i$ then $\mathcal{C}(\Lambda, m, \beta)_{BKS}$ is defined to be the set of characters θ such that $\theta_i := \theta|_{H^{m+1}(\beta_i, \Lambda^i)}$ is an element of $\mathcal{C}(\Lambda^i, m, \beta_i)_{BKS}$ and, for $m' = \max(m, \lfloor \frac{-k_0(\beta,\Lambda)}{2} \rfloor)$, the restriction of θ to $H^{m'+1}(\beta, \Lambda)$ is equal to $\psi_{\beta-\gamma}\theta_0$ for some element θ_0 of $\mathcal{C}(\Lambda, m', \gamma)_{BKS}$ where $[\Lambda, q, -k_0(\beta, \Lambda), \gamma]$ is an element of a defining sequence of the stratum with β (see [Ste05, 3.13]).

We write $I_H(\theta, \theta')$ for the intertwining of two characters in a group H, i.e. $g \in H$ is an element of $I_H(\theta, \theta')$ if and only if $\theta^g : x \mapsto \theta(gxg^{-1})$ and θ' agree on the intersection of their domains. In the case $H = \tilde{G}$ we omit the subscript.

Proposition 9.7. *The sets $\mathcal{C}(\Lambda, m, \beta)_{BKS}$ and $\mathcal{C}(\Lambda, m, \beta)$ coincide. In particular the definition is independent of the choice of γ.*

Proof. Let us first remark that the definition of $\mathcal{C}(\Lambda, m, \beta)$ is independent of the choice of γ once we have established the equality for all possible strata which can occur as a first member with respect to a jump sequence of (Λ, β), by [Ste05, 3.14(ii)]. We prove the equality at first for simple strata. Note that we fix here q and that we do an induction on the critical exponent k_0. If the stratum is null, then both sets only consists of the trivial character on $\tilde{\mathrm{U}}^{m+1}(\Lambda)$.

Suppose now that $k_0 \geqslant -q$: The set $\mathcal{C}(\Lambda, m, \beta)_{BKS}$ is a subset of $\mathcal{C}(\Lambda, m, \beta)$ because of [BK93, (3.2.1)(3.2.3)]. (The normalizing property is also satisfied in the case of lattice sequences because the lattice chain Λ^0 in Remark 9.6(ii)(b) can be chosen to be principal.)

For the other containment in the case of $m \geqslant \lfloor \frac{-k_0}{2} \rfloor$ we have that $\mathcal{C}(\Lambda, m, \beta)$ is contained in $\mathcal{C}(\Lambda, m, \gamma)\psi_{\beta-\gamma}$ which is equal to

$$\mathcal{C}(\Lambda, m, \gamma)_{BKS}\psi_{\beta-\gamma} = \mathcal{C}(\Lambda, m, \beta)_{BKS}$$

by induction hypothesis. In the case $m < \lfloor \frac{-k_0}{2} \rfloor$ we follow an induction on m, where we use $\lfloor \frac{-k_0}{2} \rfloor$ as the start for the induction, which is know by the first case.

Take an element θ of $\mathcal{C}(\Lambda, m, \beta)$. Consider case (ii)(b) in Remark 9.6. Writing $\bar{\Lambda} = \Lambda^0 \oplus \Lambda$, we have to show that there is an element of $\mathcal{C}(\bar{\Lambda}, m, \beta \oplus \beta)_{BKS}$ which restricts to θ. By the induction hypothesis on m there is a character $\tilde{\theta}$ in $\mathcal{C}(\bar{\Lambda}, m+1, \beta \oplus \beta)_{BKS}$ which restricts to $\theta|_{H^{m+2}(\beta,\Lambda)}$. On the other hand, by definition $\theta|_{\tilde{\mathrm{U}}^{m+1}(\Lambda_E)}$ factorizes through the determinant, i.e. has the form $\chi \circ \det_B$, with χ a smooth character of $E^\times$. We define $\bar{\theta}$ a character on

$$H^{m+1}(\beta \oplus \beta, \bar{\Lambda}) = \tilde{\mathrm{U}}^{m+1}(\bar{\Lambda}_E) H^{m+2}(\beta \oplus \beta, \bar{\Lambda})$$

via $\bar{\theta}(bx) = \chi(\det_{\bar{B}}(b))\tilde{\theta}(x)$. If this is well-defined then it lies in $\mathcal{C}(\bar{\Lambda}, m, \beta \oplus \beta)_{BKS}$ and its restriction to $H^{m+1}(\beta, \Lambda)$ is θ.

For $\bar{\theta}$ to be well-defined we only need that $\chi \circ \det_{\bar{B}}|_{\tilde{\mathrm{U}}^{m+1}(\bar{\Lambda}_E)}$ and $\tilde{\theta}$ coincide on the intersection of their domains, which is $\tilde{\mathrm{U}}^{m+2}(\bar{\Lambda}_E)$. The image of $\det_{\bar{B}}$ on $\tilde{\mathrm{U}}^{m+2}(\bar{\Lambda}_E)$ coincides with the image of $\det_B$ on $\tilde{\mathrm{U}}^{m+2}(\Lambda_E)$ (it is equal to $\tilde{\mathrm{U}}^{\lfloor \frac{m+1}{e(\Lambda_E)} \rfloor + 1}(o_E)$) and the restriction of $\tilde{\theta}$ to $\tilde{\mathrm{U}}^{m+2}(\bar{\Lambda}_E)$ factorizes through $\det_{\bar{B}}$. Thus this restriction has to be equal to the corresponding restriction of $\chi \circ \det_{\bar{B}}$ because $\tilde{\theta}$ and θ coincide on $\tilde{\mathrm{U}}^{m+2}(\Lambda_E)$. This finishes the proof in the simple case.

We consider now the semisimple case for $k_0 \geqslant -q$. We have that $\mathcal{C}(\Lambda, m, \beta)_{BKS}$ is a subset of $\mathcal{C}(\Lambda, m, \beta)$ because the simple restrictions of an element θ of the first set satisfies the normalizing and the factorizing condition and θ is trivial on the unipotent parts of the Iwahori decomposition of $H^{m+1}(\beta, \Lambda)$ with respect to β by [Ste05, 3.15]. For the other containment the case $m \geqslant \lfloor \frac{-k_0}{2} \rfloor$ follows as in the simple case. If $m < \lfloor \frac{-k_0}{2} \rfloor$ then we show by induction on m that the restriction θ_i of an element θ of $\mathcal{C}(\Lambda, m, \beta)$ is an element of $\mathcal{C}(\Lambda^i, m, \beta_i)_{BKS}$. By induction hypothesis $\theta_i|_{H^{m+2}(\beta_i, \Lambda^i)}$ is an element of $\mathcal{C}(\Lambda^i, m+1, \beta_i)_{BKS}$ and the axioms for θ imply the factorization condition for θ_i. Thus θ_i is an element of $\mathcal{C}(\Lambda^i, m, \beta_i)$ because it is normalized by $\mathfrak{n}(\Lambda^i_{E_i})$ (because $\theta_i|_{H^{m+2}(\beta_i, \Lambda^i)}$ and $\theta_i|_{\tilde{\mathrm{U}}^{m+1}(\Lambda^i_{E_i})}$ are) and because $\theta_i|_{H^{m+2}(\beta_i, \Lambda^i)}$ is an element of $\mathcal{C}(\Lambda^i, m+1, \beta_i)$ by the simple case. Thus θ_i is an element of $\mathcal{C}(\Lambda^i, m, \beta_i)_{BKS}$ again by the simple case. This finishes the proof. □

Let us recall the intertwining formula for a semisimple character:

Proposition 9.8 ([Ste05, 3.22], [BK93, (3.3.2)]). *The intertwining of a semisimple character $\theta \in \mathcal{C}(\Lambda, m, \beta)$ is given by $S(\beta) B_\beta^\times S(\beta)$ where*

$$S(\beta) = S(\Lambda, m, \beta) = 1 + \mathfrak{m}_{-k_0-m} + \mathfrak{j}^{\lfloor \frac{-k_0+1}{2} \rfloor}(\beta, \Lambda).$$

Reading the proofs of [BK93] from (3.2.1) to (3.5.10) we see that all statements are true for semisimple characters (after replacing $\mathfrak{b}_{\beta,t}\mathfrak{n}_l$ by $\mathfrak{a}_t \cap \mathfrak{n}_{t+l}$ throughout) except the statements (3.3.17) and (3.5.1). However, there are obvious modifications of (3.3.17) and (3.5.1) which are still true.

Proposition 9.9 (see [BK93, (3.5.1)]). *Let $[\Lambda, q, m, \beta]$ and $[\Lambda, q, m, \beta']$ be semisimple strata such that $\mathcal{C}(\Lambda, m, \beta) \cap \mathcal{C}(\Lambda, m, \beta')$ is non-empty. Then there is a bijection $\tau : I \to I'$ such that $1^i \equiv 1^{\tau(i)} \pmod{\mathfrak{a}_1}$, and:*

(i) $k_0(\beta, \Lambda) = k_0(\beta', \Lambda)$;

(ii) *the field extensions $E_i|F$ and $E'_{\tau(i)}|F$ have the same inertia degree and the same ramification index;*

(iii) *the dimensions of* V^i *and* $V^{\tau(i)}$ *as* F*-vector spaces coincide;*
(iv) *there is an element* g *of* $S(\beta)$ *such that* gV^i *is equal to* $V^{\tau(i)}$*. In fact, the element* $g = \sum_i 1^{\tau(i)}1^i$ *is an example.*

Proof. The existence of τ follows from the two descriptions of the intertwining of an element $\theta \in \mathcal{C}(\Lambda, m, \beta) \cap \mathcal{C}(\Lambda, m, \beta')$,

$$I(\theta) = S(\beta)B_\beta S(\beta) = S(\beta')B_{\beta'}S(\beta'),$$

together with Lemma 7.17. We now follow the proof of [BK93, (3.5.1)] to get that $\mathfrak{b}_{\beta,0}/\mathfrak{b}_{\beta,1}$ is isomorphic to $\mathfrak{b}_{\beta',0}/\mathfrak{b}_{\beta',1}$, by an isomorphism of κ_F-algebras which maps 1^i to $1^{\tau(i)}$. We also have that $(1^i\mathfrak{a}_0 1^i)/\mathfrak{a}_1 = (1^{\tau(i)}\mathfrak{a}_0 1^{\tau(i)})/\mathfrak{a}_1$ and thus, as in the proof of [BK93, (2.1.4)], we get the desired equalities.

The equality of the additive closures of the intertwining set $I(\theta) \cap \tilde{\mathrm{U}}(\Lambda)$ in terms of β and β' implies that, for each $i \in I$, we can write $1^{\tau(i)} = (1+u)b(1+v)$ with $(1+u), (1+v) \in S(\beta)$ and $b \in B_\beta$. By Lemma 7.13 applied with $\mathfrak{k}_0 = \mathfrak{a}_0 \cap B_\beta$ and $\mathfrak{k}_r = (S(\beta)-1) \cap \mathfrak{a}_r$, there is an idempotent e in B_β which is congruent to b (mod $S(\beta)-1$). Since, in particular, $e \equiv b \equiv 1^{\tau(i)} \equiv 1^i \pmod{\mathfrak{a}_1}$, Lemma 7.16 implies that e is a central idempotent in B_β, in particular a sum of primitive central idempotents of B_β. Since $e \equiv 1^i \pmod{\mathfrak{a}_1}$, we see that in fact $e = 1^i$. Thus in fact $1^i \equiv 1^{\tau(i)}$ (mod $S(\beta)-1$) and we deduce that $g = \sum_i 1^{\tau(i)}1^i$ is an element of $S(\beta)$ with the required property. ☐

Remark 9.10. Recall that a semisimple character θ is called *simple* if there is a simple stratum $[\Lambda, q, m, \beta]$ such that $\theta \in \mathcal{C}(\Lambda, m, \beta)$; then, by Proposition 9.9, every semisimple stratum $[\Lambda, q, m, \beta']$ such that $\theta \in \mathcal{C}(\Lambda, m, \beta')$ has to be simple.

Proposition 9.11 (see [BK93, (3.3.17)])**.** *For every element of* θ *of* $\mathcal{C}(\Lambda, m, \beta)$ *the intersection of the normalizer* $\mathfrak{n}(\theta)$ *of* θ *with* $\mathfrak{n}(\Lambda)$ *is the set* $S(\beta)\mathfrak{n}(\Lambda_E)$*. In particular the intersection of* $\mathfrak{n}(\theta)$ *with* $\tilde{\mathrm{U}}_1(\Lambda)$ *is equal to*

$$1 + \mathfrak{b}_1 + \mathfrak{m}_{-k_0-m}(\beta, \Lambda) + \mathfrak{J}^{\lfloor \frac{-k_0+1}{2} \rfloor}(\beta, \Lambda) \tag{9.12}$$

and $(\tilde{\mathrm{U}}_1(\Lambda) \cap \mathfrak{n}(\theta)) - 1$ *is a* $\mathfrak{b}_0$*-bimodule and is closed under multiplication.*

Proof. An element of $\mathfrak{n}(\theta) \cap \mathfrak{n}(\Lambda)$ intertwines θ and normalizes Λ thus it is contained in $S(\beta)\mathfrak{n}(\Lambda_E)$ by the intertwining formula. The latter set is contained in the normalizer of θ because $S(\beta)$ and $\mathfrak{n}(\Lambda_E)$ are. This finishes the proof of the first statement. If we intersect the set $\mathfrak{n}(\theta) \cap \mathfrak{n}(\Lambda)$ further with $\tilde{\mathrm{U}}^1(\Lambda)$ then we obtain the formula (9.12) and $(\tilde{\mathrm{U}}_1(\Lambda) \cap \mathfrak{n}(\theta)) - 1$ is a $\mathfrak{b}_0$-bi-module and closed under multiplication by [BK93, (3.1.10)]. ☐

By [BK93, Theorem 3.5.8], a non-trivial intersection of $\mathcal{C}(\Lambda, m, \beta)$ with $\mathcal{C}(\Lambda, m, \beta')$ implies equality of the sets. We will generalize this theorem to a block-wise version. If $V = \bigoplus_k V^k$ is a splitting for V which splits a semisimple stratum $[\Lambda, q, m, \beta]$, we write θ_k for the restriction of θ to $H^{m+1}(\beta_k, \Lambda^k)$.

Lemma 9.13. *Suppose that $V = \bigoplus_k V^k$ is a splitting which refines the associated splittings of two semisimple strata $[\Lambda, q, m, \beta]$ and $[\Lambda, q, m, \beta']$. Suppose further that the sets $\mathcal{C}(\Lambda, m+1, \beta)$ and $\mathcal{C}(\Lambda, m+1, \beta')$ coincide and that $\mathcal{C}(\Lambda^k, m, \beta_k)$ is equal to $\mathcal{C}(\Lambda^k, m, \beta'_k)$, for all k. Let $a \in \mathfrak{a}_{-m-1} \cap \prod_k A^{k,k}$, $\theta \in \mathcal{C}(\Lambda, m, \beta)$ and $\theta' \in \mathcal{C}(\Lambda, m, \beta')$ be given such that θ_k coincides with $\theta'_k \psi_{a_k}$ for all indices k. Then $[\Lambda, q, m, \beta' + a]$ is equivalent to a semisimple stratum with the same associated splitting as β, and the sets $\mathcal{C}(\Lambda, m, \beta)$ and $\psi_a \mathcal{C}(\Lambda, m, \beta')$ coincide and both contain $\theta = \theta' \psi_a$.*

Proof. The group $H^{m+1}(\beta, \Lambda)$ is the same as $H^{m+1}(\beta', \Lambda)$ by [BK93, (3.5.9)]. We show that $[\Lambda, q, m, \beta' + a]$ is equivalent to a semisimple stratum $[\Lambda, q, m, \beta'']$ which is split by $V = \bigoplus_k V^k$. Let s' be a tame corestriction with respect to γ', a parameter for a first member of a defining sequence for $[\Lambda, q, m, \beta']$. Since $\theta'_k \psi_{-a_k} \in \mathcal{C}(\Lambda^k, m, \beta'_k)$, we have that the coset $s'(a_k) + \mathfrak{b}'_{\gamma'_k, -m}$ is intertwined by the centralizer of γ'_k and thus $s'(a_k)$ is congruent to an element of $F[\gamma'_k]$ modulo $\mathfrak{a}_{-m}$. Further, the stratum $[\Lambda^k, m+1, m, s'(\beta'_k - \gamma'_k)]$ is equivalent to a simple stratum because $[\Lambda^k, q, m, \beta'_k]$ is simple. Thus the stratum $[\Lambda^k, m+1, m, s'(\beta'_k + a_k - \gamma'_k)]$ is equivalent to a simple stratum and it follows that $[\Lambda^k, q, m, \beta'_k + a_k]$ is equivalent to a simple stratum by Corollary 6.15. By coarsening the splitting $\bigoplus_k V^k$ we find a semisimple stratum $[\Lambda, q, m, \beta'']$ equivalent to $[\Lambda, q, m, \beta' + a]$ and by Theorem 6.16 we can choose the desired stratum to be spilt by $\bigoplus_k V^k$.

Since $[\Lambda, q, m, \beta' + a]$ is equivalent to the semisimple stratum $[\Lambda, q, m, \beta'']$, it follows that $\mathcal{C}(\Lambda, m, \beta'')$ is equal to $\mathcal{C}(\Lambda, m, \beta')\psi_a$ and intersects $\mathcal{C}(\Lambda, m, \beta)$ non-trivially, i.e. they coincide by [BK93, Theorem 3.5.8].

It remains to show that β and β'' in fact have the same associated splittings. The corresponding idempotents $(1^i)_i$ and $(1^{i'})_{i'}$ of the associated splittings commute because the idempotents $(1^k)_k$ commute with all of them and are a refinement of both. By Proposition 9.9 there is a bijection τ from I to I' such that $1^{\tau(i)}$ is congruent to 1^i modulo $\mathfrak{a}_1$. The product $1^i 1^{i'}$ is congruent to zero for $i' \neq \tau(i)$ and thus it is zero, because they commute (take powers). Thus 1^i and $1^{\tau(i)}$ coincide, that is, the splittings coincide. □

Corollary 9.14. *Suppose that $V = \bigoplus_k V^k$ is a splitting which refines the two splittings associated to the semisimple strata $[\Lambda, q, m, \beta]$ and $[\Lambda, q, m, \beta']$, and suppose that there are semisimple characters $\theta \in \mathcal{C}(\Lambda, m, \beta)$ and $\theta' \in \mathcal{C}(\Lambda, m, \beta')$ such that for every index k the characters θ_k and θ'_k coincide. Then $\mathcal{C}(\Lambda, m, \beta) = \mathcal{C}(\Lambda, m, \beta')$ and both contain $\theta = \theta'$.*

Proof. This follows inductively from Lemma 9.13 with $a = 0$. □

9.2. The transfer principle for $\tilde{G}$.

We would like to be able to get an analogue of strata induction for semisimple characters, for which we need

- the "translation principle" initially introduced for simple characters in [BK94, 2.11], and

- a result on "derived characters" (see Proposition 9.17 below).

From now on the element γ can be arbitrary, i.e. we free γ from the requirements of the beginning of the previous section.

In the following we use the notation $\mathfrak{m}(\Delta)$ for the set $\mathfrak{m}_{-(k_0(\beta,\Lambda)+m+1)}(\beta,\Lambda)$ for a stratum $\Delta = [\Lambda, q, m, \beta]$. Equivalent strata Δ and Δ' give coinciding sets $\mathfrak{m}(\Delta) = \mathfrak{m}(\Delta')$.

Lemma 9.15. *Suppose $\Delta = [\Lambda, q, m, \beta]$ is a semisimple stratum split by $V = \bigoplus_i V^i$ and $[\Lambda, q, m, \beta']$ is a semisimple stratum equivalent to Δ. Then there is an element u of $1 + \mathfrak{m}(\Delta)$ such that $u\beta' u^{-1}$ is split by $V = \bigoplus_i V^i$.*

Proof. By the intertwining formula, taking the intersection with $\tilde{\mathrm{U}}(\Lambda)$ and then the additive closure, we get

$$\mathfrak{m}(\Delta) + \mathfrak{b}_\beta = \mathfrak{m}(\Delta) + \mathfrak{b}_{\beta'}.$$

Thus for every index i there is an element $\alpha_i \in \mathfrak{b}_{\beta'}$ congruent to the idempotent 1^i modulo $\mathfrak{m}(\Delta)$. Corollary 7.14 provides idempotents $1'^i \in \mathfrak{b}_{\beta'}$ congruent to 1^i which sum to 1. The element $u = \sum_i 1^i 1'^i$ has the desired property. □

Theorem 9.16. *Let $\Delta := [\Lambda, q, m+1, \gamma]$ and $\Delta' := [\Lambda, q, m+1, \gamma']$ be semisimple strata with the same associated splitting $V = \bigoplus_j V^j$ such that*

$$\mathcal{C}(\Lambda, m+1, \gamma) = \mathcal{C}(\Lambda, m+1, \gamma').$$

Let $[\Lambda, q, m, \beta]$ be a semisimple stratum with associated splitting $V = \bigoplus_{i \in I} V^i$ such that Δ is equivalent to $[\Lambda, q, m+1, \beta]$ and γ is an element of $\prod_{i\in I} A^{i,i}$. Then there exist a semisimple stratum $[\Lambda, q, m, \beta']$ with associated splitting $V = \bigoplus_{i'\in I'} V'^{i'}$ and an element $u \in (1+\mathfrak{m}(\Delta')) \cap \prod_j A^{j,j}$ such that $[\Lambda, q, m+1, \beta']$ is equivalent to Δ', with $u\gamma' u^{-1} \in \prod_{i' \in I'} A^{i',i'}$ and

$$\mathcal{C}(\Lambda, m, \beta) = \mathcal{C}(\Lambda, m, \beta').$$

Proof.
- (i) Let us first remark that, given a semisimple stratum $\Delta'' = [\Lambda, q, m+1, \gamma'']$ with the same associated splitting as γ, once we know the assertion for (Δ, Δ') and for $(u.\Delta', \Delta'')$, for some $u \in (1 + \mathfrak{m}(\Delta')) \cap \prod_j A^{j,j}$, then we know it also for (Δ, Δ'').
- (ii) The case where Δ and Δ' are equivalent follows directly from Lemma 9.15 applied to Δ, Δ', taking u to be as there and $\beta' = u\beta u^{-1}$.
- (iii) We now reduce to the case that Δ is simple so assume that the result has already been proven in this case. Then, applying this to the simple strata $[\Lambda^j, q, m, \beta_j]$, for each j, we can find $[\Lambda^j, q, m, \beta'_j]$ and elements $u_j \in 1 + \mathfrak{m}(\Delta)$ such that $C(\Lambda^j, m, \beta_j)$ is equal to $C(\Lambda^j, m, \beta'_j)$ and $u_j \gamma'_j u_j^{-1}$ is split by the associated splitting of β'_j. Moreover, by conjugating β_j with an element of $S(\beta_j)$ we can assume that β_j and β'_j have the same associated splitting, see Proposition 9.9(iv). Then the semisimple strata $[\Lambda, q, m, \beta]$ and $[\Lambda, q, m, \beta']$ have the same associated splitting, where $\beta' = \sum_j \beta'_j$.

Take $\theta \in \mathcal{C}(\Lambda, m, \beta)$. Although we know that, for each i, the restriction of θ to block i lies in $C(\Lambda^i, m, \beta'_i)$, it does not follow that $\theta \in \mathcal{C}(\Lambda, m, \beta')$, since the definition of semisimple character entails a certain compatibility between the blocks. However, there is an element $a \in \prod_i A^{i,i} \cap \mathfrak{a}_{-m-1}$ such that $\theta\psi_{-a}$ is an element of $\mathcal{C}(\Lambda, m, \beta')$. Then, by Lemma 9.13, the stratum $[\Lambda, q, m, \beta' + a]$ is equivalent to a semisimple stratum with the same associated splitting as β' with the same set of semisimple characters as the stratum with entry β. This finishes the proof of this case.

(iv) Finally, we assume that Δ is simple., so that Δ' is simple too, by Proposition 9.9. By [BK93, (3.5.9)] there is a simple stratum $[\Lambda, q, m+1, \gamma'']$, equivalent to Δ', such that $\mathcal{C}(\Lambda, m, \gamma)$ is equal to $\mathcal{C}(\Lambda, m, \gamma'')$. Thus by (i) and (ii) we can assume $\gamma' = \gamma''$. As in [BK94, 5.2(iii)], we take two tame corestrictions s and s' for γ and γ' such that $s(x) \equiv s'(x) \pmod{\mathfrak{a}_{t+1}}$, for all elements x of $\mathfrak{a}_t$ and all integers t.

We put $c = \beta - \gamma$. Then $[\Lambda, m+1, m, s(c)]$ is equivalent to a semisimple stratum by Corollary 6.15. As in [BK94, 5.3], the fact that $[\Lambda, m+1, m, s(c)]$ is fundamental implies that $[\Lambda, m+1, m, s'(c)]$ is fundamental too; however, we need that the latter stratum also satisfies the criterion on the maps $m_{n,m+1,s'(c)}$ of Proposition 6.11. Note that the same proposition implies that the maps $m_{n,m+1,s(c)}$ do satisfy this criterion.

The tame corestrictions s and s' are surjective as maps from $\mathfrak{a}_t$ to $\mathfrak{b}_{\gamma,t}$ and to $\mathfrak{b}_{\gamma',t}$, respectively, and thus we obtain an isomorphism of κ_F-vector spaces ϕ_t from $\mathfrak{b}_{\gamma,t}/\mathfrak{b}_{\gamma,t+1}$ to $\mathfrak{b}_{\gamma',t}/\mathfrak{b}_{\gamma',t+1}$, for all integers t, by sending the class of $s(x)$ to that of $s'(x)$; note that this is well defined by the choice of s and s'. Then $m_{n,m+1,s'(c)}$ is equal to $\phi_{n-m-1} \circ m_{n,m+1,s(c)} \circ \phi_n^{-1}$ and thus, varying n, the maps $m_{n,m+1,s'(c)}$ satisfy the additional criterion of Proposition 6.11.

The arguments after [BK94, (5.4)] show that the algebras $\mathcal{R}([\Lambda, m+1, m, s(c)])$ and $\mathcal{R}([\Lambda, m+1, m, s'(c)])$ are isomorphic, which implies, by Proposition 6.11, that $[\Lambda, m+1, m, s'(c)]$ is equivalent to a semisimple stratum, say with associated splitting $(1^{i'})$. By Corollary 6.15 the stratum $[\Lambda, q, m, \gamma' + \sum_{i'} 1^{i'} c 1^{i'}]$ is equivalent to a semisimple stratum $[\Lambda, q, m, \beta'']$ with the same splitting and by Proposition 7.6 there is an element u of $1 + \mathfrak{m}(\Delta)$ such that $u\beta''u^{-1}$ is equivalent to $\gamma' + c$ modulo $\mathfrak{a}_{-m}$. Thus, setting $\beta' = u\beta''u^{-1}$, the stratum $[\Lambda, q, m, \beta']$ satisfies the desired properties, as

$$\mathcal{C}(\Lambda, m, \beta) = \psi_c\mathcal{C}(\Lambda, m, \gamma) = \psi_c\mathcal{C}(\Lambda, m, \gamma') = \mathcal{C}(\Lambda, m, \beta'),$$

where the last equality follows from [BK93, (3.3.20)(i)].

□

Proposition 9.17. *Suppose $m < q-1$ and let $[\Lambda, q, m, \beta]$ and $[\Lambda, q, m, \beta']$ be semisimple strata which have defining sequences with a common first element $[\Lambda, q, m+1, \gamma]$. Suppose $\theta \in \mathcal{C}(\Lambda, m, \beta)$ and $\theta' \in \mathcal{C}(\Lambda, m, \beta')$ are semisimple characters which agree on restriction to $H^{m+2}(\gamma, \Lambda)$, so that we can write $\theta' = \theta_0\psi_{\beta'-\gamma}$ and $\theta = \theta_0\psi_{\beta-\gamma+c}$, for some $\theta_0 \in \mathcal{C}(\Lambda, m, \gamma)$ and $c \in \mathfrak{a}_{-(m+1)}$. Let s_γ be a tame corestriction with respect to γ.*

(i) *For any $g \in I(\theta, \theta')$ there are elements $x, y \in S(\gamma)$ and $g' \in B_\gamma$ such that $g = xg'y$; moreover, g' intertwines $\psi_{s_\gamma(\beta-\gamma+c)}$ with $\psi_{s_\gamma(\beta'-\gamma)}$.*

(ii) *For any $g' \in I_{B_\gamma^\times}(\psi_{s_\gamma(\beta-\gamma+c)}, \psi_{s_\gamma(\beta'-\gamma)})$, there are elements x, y of $1 + \mathfrak{m}_{-k_0(\gamma,\Lambda)-m-1}$ such that $xg'y$ intertwines θ with θ'.*

(iii) *If the characters $\psi_{s(\beta-\gamma+c)}$ and $\psi_{s(\beta'-\gamma)}$ are equal, then there is $z \in 1+\mathfrak{m}_{-k_0(\gamma,\Lambda)-m-1}$ such that $\theta^z = \theta'$.*

Remark 9.18. The strategy of the proof of Proposition 9.17(ii) is as follows: we take x and y such that $xg'y$ intertwines the stratum $[\Lambda, q, m, \beta']$ with $[\Lambda, q, m, \beta + c]$ (see also Proposition 7.6) and prove that $xg'y$ intertwines θ with θ'. Thus, if c is an element of $\prod_i A^{i,i}$ and g' maps the splitting associated to β' to that of β, then we can choose $x \in \prod_i A^{i,i}$ and $y \in \prod_{i'} A^{i'i'}$ which satisfy the assertions of Proposition 9.17(ii).

Proof. We have $H^{m+1}(\beta, \Lambda) = H^{m+1}(\beta', \Lambda) = H^{m+1}(\gamma, \Lambda)$ by [BK93, (3.1.9)], so we just write H^{m+1}.

(i) The decomposition $g = xg'y$ follows directly from Proposition 9.8. We remark also that, by [BK93, (3.6.2) and (3.1.15)(ii)], the elements x and y normalize H^{m+1}. Thus $g' \in I(\theta^x, \theta'^{y^{-1}})$. By [BK93, (3.3.9)] we have
$$\theta^x = \theta_0\psi_{x^{-1}\gamma x-\gamma}\psi_{\beta-\gamma+c}, \qquad \text{and} \qquad \theta'^{y^{-1}} = \theta_0\psi_{y\gamma y^{-1}-\gamma}\psi_{\beta'-\gamma}.$$
We have $\psi_{x^{-1}\gamma x-\gamma} = \psi_{a_\gamma(x)}$ and $\psi_{y\gamma y^{-1}-\gamma} = \psi_{-a_\gamma(y)}$ (as characters of H^{m+1}) and thus their restrictions to $\tilde{\mathrm{U}}^{m+1}(\Lambda) \cap B_\gamma^\times$ are trivial. Thus, on $\tilde{\mathrm{U}}^{m+1}(\Lambda) \cap B_\gamma^\times$, we have
$$\theta^x = \theta = \theta_0\psi_{\beta-\gamma+c} = \theta_0\psi_{s_\gamma(\beta-\gamma+c)},$$
and analogously for $\theta^{y^{-1}}$. Since g' intertwines θ^x with $\theta^{y^{-1}}$ and θ_0 with itself, it also intertwines $\psi_{s_\gamma(\beta-\gamma+c)}$ with $\psi_{s_\gamma(\beta'-\gamma)}$.

(ii) If some element $g' \in B_\gamma^\times$ intertwines $\psi_{s_\gamma(\beta-\gamma+c)}$ with $\psi_{s_\gamma(\beta'-\gamma)}$ then it intertwines the stratum $[\Lambda, m+1, m, s_\gamma(\beta'-\gamma)]$ with $[\Lambda, m+1, m, s_\gamma(\beta-\gamma+c)]$ and thus, by Proposition 7.6, there are elements x, y of $1+\mathfrak{m}_{-k_0(\gamma,\Lambda)-m-1}$ such that g' intertwines $[\Lambda, q, m, y\beta'y^{-1}]$ with $[\Lambda, q, m, x^{-1}(\beta+c)x]$; that is, g' is an element of $I(\psi_{x^{-1}(\beta+c)x}, \psi_{y\beta'y^{-1}})$. Now we have
$$\psi_{x^{-1}(\beta+c)x} = \psi_{\beta+c-\gamma}\psi_{x^{-1}\gamma x-\gamma}\psi_\gamma,$$
and an analogous equation for $\psi_{y\beta'y^{-1}}$. Since g' intertwines each of θ_0 and $\psi_\gamma|_{H^{m+1}}$ with themselves, we deduce that g' intertwines θ^x with $\theta'^{y^{-1}}$.

(iii) This follows immediately from (ii) applied to the identity element by putting $z = xy$, which normalizes H^{m+1}.

□

Corollary 9.19. *Let $[\Lambda, q, m, \beta]$ and $[\Lambda, q, m, \beta']$ be semisimple strata which have semisimple approximations $[\Lambda, q, m+1, \gamma]$ and $[\Lambda, q, m+1, \gamma']$ respectively with a common associated splitting $V = \bigoplus_j V^j$. Suppose that $H^{m+2}(\beta, \Lambda)$ is equal to $H^{m+2}(\beta', \Lambda)$ and let $\theta \in$*

$\mathcal{C}(\Lambda, m, \beta)$ and $\theta' \in \mathcal{C}(\Lambda, m, \beta')$ be two intertwining semisimple characters which coincide on $H^{m+2}(\beta, \Lambda)$. Then there is an element in $\prod_j (A^{j,j})^\times$ which intertwines θ with θ'.

Proof. By the translation principle, Theorem 9.16, there are $[\Lambda, q, m, \beta'']$, a semisimple stratum which has $[\Lambda, q, m+1, \gamma]$ as an approximation, and an element $u \in (1+\mathfrak{m}_\gamma) \cap \prod A^{j,j}$ such that $\mathcal{C}(\Lambda, m, \beta'') = \mathcal{C}(\Lambda, m, \beta')$ and $u\gamma u^{-1}$ is split by the associated splitting of β''. Replacing β' by $u^{-1}\beta'' u$, we reduce to the case where $\gamma' = \gamma$.

Now set $\theta_0 = \psi_{\gamma-\beta}\theta \in \mathcal{C}(\Lambda, m, \gamma)$. Then there is an element $c \in \bigoplus_j A^{j,j}$ such that $\theta' = \theta_0 \psi_{\beta'-\gamma+c}$, because $\theta_0\psi_{\beta'-\gamma}$ and θ' are trivial on the lower and upper unipotent parts of the Iwahori decomposition with respect to $V = \bigoplus_j V^j$. As we know that θ and θ' intertwine, Proposition 9.17(i) provides an element of $\prod_j (B^{j,j})^\times$ which intertwines the corresponding derived characters. Now fix a block j, then Proposition 9.17(ii) provides an element g_j of $(A^{j,j})^\times$ which intertwines θ_j with θ'_j. Thus $g = (g_j)$ intertwines θ with θ' because both characters are trivial on the unipotent parts of the Iwahori decomposition with respect to $V = \bigoplus_j V^j$. □

9.3. **Semisimple characters for G.** Suppose now that $[\Lambda, q, m, \beta]$ is a skew-semisimple stratum and continue with the notation of the previous subsection. The adjoint anti-involution σ of the signed hermitian form h acts on $\mathcal{C}(\Lambda, m, \beta)$ via

$$(\sigma \cdot \theta)(g) := \theta(\sigma(g^{-1})),\ \ g \in H^{m+1}(\beta, \Lambda).$$

Definition 9.20. We define the set of semisimple characters $\mathcal{C}_-(\Lambda, m, \beta)$ to be the set of all restrictions $\theta|_{H^{m+1}(\Lambda,\beta)\cap G}$ where θ run through all elements of $\mathcal{C}(\Lambda, m, \beta)^\sigma$, the set of σ-fixed points.

We call an element of $\mathcal{C}_-(\Lambda, m, \beta)$ a *semisimple character for G*.

Remark 9.21 ([Ste05, 3.6], [Ste01b, 2.5])**.**

(i) The restriction map from $\mathcal{C}(\Lambda, m, \beta)^\sigma$ to $\mathcal{C}_-(\Lambda, m, \beta)$ is bijective, in particular injective.
(ii) For two skew-semisimple strata $[\Lambda, q, m, \beta]$ and $[\Lambda, q, m, \beta']$, $g \in G$, and characters $\theta \in \mathcal{C}(\Lambda, m, \beta)^\sigma$ and $\theta' \in \mathcal{C}(\Lambda, m, \beta')^\sigma$ the following conditions are equivalent:
  - $g \in I_G(\theta, \theta')$;
  - $g \in I_G(\theta|_{H^{m+1}(\Lambda,\beta)\cap G}, \theta'|_{H^{m+1}(\Lambda,\beta)\cap G})$.

We have an analogous description to that of Proposition 9.8 of the intertwining of a semisimple character for G.

Proposition 9.22 ([Ste05, 3.27])**.** *For $\theta_- \in \mathcal{C}_-(\Lambda, m, \beta)$ a semisimple character of G, we have*

$$I_G(\theta_-) = (S(\beta) \cap G)(B_\beta \cap G)(S(\beta) \cap G).$$

For two skew-semisimple strata giving the same set of semisimple characters we have a stronger version of Proposition 9.9.

Proposition 9.23. *Let $[\Lambda, q, m, \beta]$ and $[\Lambda, q, m, \beta']$ be skew-semisimple strata such that the intersection $\mathcal{C}_-(\Lambda, m, \beta) \cap \mathcal{C}_-(\Lambda, m, \beta')$ is non-empty.*

(i) *The sets $\mathcal{C}(\Lambda, m, \beta)$ and $\mathcal{C}(\Lambda, m, \beta')$ coincide.*

Let $\tau : I \to I'$ be the bijection given by Proposition 9.9, such that $1^i \equiv 1^{\tau(i)} \pmod{\mathfrak{a}_1}$.

(ii) *The spaces V^i and $V^{\tau(i)}$ are isomorphic as hermitian spaces, for all indices $i \in I$*
(iii) *There is an element of $\mathrm{U}^1(\Lambda)$ which normalizes every element of $\mathcal{C}(\Lambda, m, \beta)$ and sends V^i onto $V^{\tau(i)}$.*

Proof. We prove the first statement by induction on m. If $m = q$ then $\beta = \beta' = 0$ and both sets only contain the trivial character on $\tilde{\mathrm{U}}(\Lambda)$, i.e. they coincide, so we suppose $m < q$. Let γ and γ' be entries of a first member of defining sequences of the skew-semisimple strata $[\Lambda, q, m, \beta]$ and $[\Lambda, q, m, \beta']$ respectively. Then by the induction hypothesis we can assume that $\mathcal{C}(\Lambda, m+1, \gamma)$ and $\mathcal{C}(\Lambda, m+1, \gamma')$ coincide so, by [BK93, (3.1.9)(ii),(3.5.1)], we have

$$H^{m+1}(\beta, \Lambda) = H^{m+1}(\gamma, \Lambda) = H^{m+1}(\gamma', \Lambda) = H^{m+1}(\beta', \Lambda),$$

a group which we denote by H^{m+1}. There is, by Glauberman's correspondence, then a unique σ-invariant lift of an element $\theta_- \in \mathcal{C}_-(\Lambda, m, \beta) \cap \mathcal{C}_-(\Lambda, m, \beta')$ to H^{m+1}, and this lies in both $\mathcal{C}(\Lambda, m, \beta)$ and $\mathcal{C}(\Lambda, m, \beta')$. The result now follows by Corollary 9.14.

The second statement follows directly from Proposition 3.1 applied to the map $f : v \mapsto \sum_{i \in I} 1^{\tau(i)} 1^i v$, which lies in $\tilde{U}^1(\Lambda)$. We are left to prove the third statement. We write the map f as a tuple $f = (f_i)$ where $f_i = 1^{\tau(i)} 1^i$. We write $S(\beta)_i$ for $1^i S(\beta) 1^i$. Then $\sigma(f_i) f_i = 1^i 1^{\tau(i)} 1^i \in S(\beta)_i$ so the double coset $S(\beta')_{\tau(i)} f_i S(\beta)_i$ contains an isometry, by Corollary 3.2. We can write this isometry as $(1^{\tau(i)} + u_{\tau(i)})(1^i + v_i)$, since $f_i = 1^{\tau(i)} 1^i$ can be absorbed into the other terms. We define $g = \sum_i (1^{\tau(i)} + u_{\tau(i)})(1^i + v_i) \in G$ so we have to show that $g - 1$ is an element of $\mathfrak{n}(\theta) \cap \tilde{\mathrm{U}}^1(\Lambda)$.

By Proposition 9.11 the set $(\mathfrak{n}(\theta) \cap \tilde{\mathrm{U}}^1(\Lambda)) - 1$ is a $\mathfrak{b}_0$- and a $\mathfrak{b}_0'$-bimodule, and is closed under multiplication. Thus the products $u_{\tau(i)}(1^i + v_i)$, $u_{\tau(i)} 1^i$ and $1^{\tau(i)} v_i$ and $(1^{\tau(i)} - 1^i) 1^i$ are all elements of $(\mathfrak{n}(\theta) \cap \tilde{\mathrm{U}}^1(\Lambda)) - 1$, and g is an element of $\mathfrak{n}(\theta) \cap \tilde{\mathrm{U}}^1(\Lambda)$ as required. □

We also get an analogue to [BK93, 3.5.9] for semisimple characters for G.

Proposition 9.24. *Suppose $[\Lambda, q, m, \beta']$ and $[\Lambda, q, m, \beta]$ are skew-semisimple strata with the same associated splitting, such that $m > 0$ and*

$$\mathcal{C}(\Lambda, m, \beta) = \mathcal{C}(\Lambda, m, \beta').$$

Then $H^m(\beta) = H^m(\beta')$ and there is a skew-semisimple stratum $[\Lambda, q, m, \beta'']$ equivalent to $[\Lambda, q, m, \beta]$, with the same associated splitting, such that

$$\mathcal{C}(\Lambda, m-1, \beta'') = \mathcal{C}(\Lambda, m-1, \beta').$$

Proof. The same proof as in the first part of [BK93, 3.5.9] shows that $H^m(\beta) = H^m(\beta')$. Now we take a character θ in $\mathcal{C}(\Lambda, m-1, \beta)^\sigma$ and a skew-symmetric element b of $\mathfrak{a}_{-m}(\Lambda)$ in $\prod_i A^{i,i}$ such that $\theta\psi_b$ is an element of $\mathcal{C}(\Lambda, m-1, \beta')$. The same proof as in the second part of [BK93, 3.5.9] shows that there is a semisimple stratum $[\Lambda, n, m-1, \beta'']$ equivalent to $[\Lambda, n, m-1, \beta+b]$ such that $\beta'' \in \prod_i A^{i,i}$. Since $\beta + b$ is skew-symmetric, β'' can be chosen skew-symmetric, by [Ste01b, 1.10]. Then

$$\mathcal{C}(\Lambda, m-1, \beta'') = \mathcal{C}(\Lambda, m-1, \beta)\psi_b$$

has a non-trivial intersection with $\mathcal{C}(\Lambda, m-1, \beta')$, and thus they are equal by the analogue of [BK93, Theorem 3.5.8]. □

Next we obtain an analogue of the translation principle, Theorem 9.16, for which we need the following lemma.

Lemma 9.25. *Suppose $\Delta = [\Lambda, q, m, \beta]$ and $[\Lambda, q, m, \beta']$ are equivalent skew-simple strata and suppose that Δ is split by the orthogonal sum $\bigoplus_i V^i$. Then there is an element u of $(1 + \mathfrak{m}(\Delta)) \cap G$ such that $u\beta' u^{-1}$ is an element of $\prod_i A^{i,i}$.*

Proof. As in the proof of Lemma 9.15 we find elements $\alpha_i \in \mathfrak{b}_{\beta'}$ congruent to 1^i modulo $\mathfrak{m}(\Delta)$. We can replace α_i by $(\alpha_i + \sigma(\alpha_i))/2$ to ensure that the elements α_i are symmetric. By Corollary 7.14 we obtain pairwise orthogonal symmetric idempotents $1'^i$ congruent to α_i who sum up to 1. As in the proof of Proposition 9.23(iii) we see that $(1'^i + \mathfrak{m}(\Delta_i))(1^i + \mathfrak{m}(\Delta_i))$ has a σ-fixed element, say $u_i u_i'$. Then $g := \sum_i u_i' u_i$ has the desired property. □

Theorem 9.26. *Let $\Delta = [\Lambda, q, m+1, \gamma]$ and $\Delta' = [\Lambda, q, m+1, \gamma']$ be skew-semisimple strata with the same associated splitting $V = \bigoplus_j V^j$ such that*

$$\mathcal{C}(\Lambda, m+1, \gamma) = \mathcal{C}(\Lambda, m+1, \gamma').$$

Let $[\Lambda, q, m, \beta]$ be a skew-semisimple stratum, with associated splitting $V = \bigoplus_{i\in I} V^i$, such that $[\Lambda, q, m+1, \beta]$ is equivalent to Δ and γ is an element of $\prod_{i\in I} A^{i,i}$. Then, there exists a skew-semisimple stratum $[\Lambda, q, m, \beta']$, with splitting $V = \bigoplus_{i'\in I'} V'^{i'}$ and an element $u \in (1 + \mathfrak{m}(\Delta')) \cap \prod_j A^{j,j} \cap G$, such that $[\Lambda, q, m+1, \beta']$ is equivalent to Δ', with $u\gamma' u^{-1} \in \prod_{i'\in I'} A^{i'i'}$ and

$$\mathcal{C}(\Lambda, m, \beta) = \mathcal{C}(\Lambda, m, \beta').$$

Proof. The proof is analogous to the proof of Theorem 9.16, following the same four steps. Step (i) is the same (with the added requirement that $u \in G$), while step (ii), the case where Δ, Δ' are equivalent, follows from Lemma 9.25. Step (iii), the reduction from the semisimple to the simple case, is line by line the same because we can take the element a block-wise skew and Theorem 6.16 ensures that the stratum $[\Lambda, q, m, \beta' + a]$ is equivalent to a skew-semisimple stratum with the same associated splitting as β'; the splitting of β_j is conjugate in G to that of β'_j by Proposition 9.23(iii).

There is more to say in step (iv), the case where Δ is simple. We can modify γ' by Proposition 9.24 to assume $\mathcal{C}(\Lambda, m, \gamma) = \mathcal{C}(\Lambda, m, \gamma')$. We choose s a σ-equivariant tame corestriction relative to γ, and likewise s' relative to γ'. The proof of [BK94, 5.2(iii)] shows that there is $\lambda \in k_{F[\gamma]}^{\times}$ such that $s(x) + \mathfrak{a}_{t+1} = \lambda(s'(x) + \mathfrak{a}_{t+1})$, for all $x \in \mathfrak{a}_t$ and all integers t. Since s, s' are σ-equivariant, we deduce that λ is symmetric, i.e. $\bar{\lambda} = \lambda$. Then, choosing a symmetric lift $\hat{\lambda}$ of λ to $o_{F[\gamma]}^{\times}$ and replacing s' by $\hat{\lambda}s'$, we see that we may assume that $s(x) \equiv s'(x) \pmod{\mathfrak{a}_{t+1}}$, for all $x \in \mathfrak{a}_t$ and all integers t.

We put $c = \beta - \gamma$, so that the derived stratum $[\Lambda, m+1, m, s(c)]$ is equivalent to a skew-semisimple stratum by Corollary 6.15 and [Ste01b, 1.10]. In particular, denoting by $\phi_i \in k_{F[\gamma]}(X)$ the primary factors of its characteristic polynomial, we have $\sigma(\phi_i)(X) = \eta^{\deg(\phi_i)}\phi_i(\eta X)$, where $\eta = (-1)^{(2(m+1)+ee_0)/ge_0}$ for $e = e(\Lambda|\mathfrak{o}_{F[\gamma]}$, $e_0 = e(F[\gamma]|F[\gamma]_0)$ and $g = \gcd(m+1, e)$.

Now the strata $[\Lambda, m+1, m, s(c)]$ and $[\Lambda, m+1, m, s'(c)]$ have the same characteristic polynomial, by the choice of s, s', and the duality σ acts in the same way on the residue fields $k_{F[\gamma]}, k_{F[\gamma']}$, since they have the same image in $\mathfrak{a}_0/\mathfrak{a}_1$ by [BK94, 5.2]. Hence $[\Lambda, m+1, m, s'(c)]$ satisfies the hypotheses of Lemma 7.19 and is equivalent to a skew-semisimple stratum. The argument now finishes as in step (iv) of Theorem 9.16. □

Finally, we get an analogue of Proposition 9.17, with the same proof (replacing the reference to Proposition 9.8 by Proposition 9.22).

Proposition 9.27. *Suppose $m < q-1$ and let $[\Lambda, q, m, \beta]$, $[\Lambda, q, m, \beta']$ be skew-semisimple strata which have defining sequences with a common first element $[\Lambda, q, m+1, \gamma]$. Let $\theta \in \mathcal{C}(\Lambda, m, \beta)^{\sigma}$ and $\theta' \in \mathcal{C}(\Lambda, m, \beta')^{\sigma}$ be semisimple characters which agree on $H^{m+2}(\Lambda, \gamma)$, so that we can write $\theta' = \theta_0\psi_{\beta'-\gamma}$ and $\theta = \theta_0\psi_{\beta-\gamma+c}$, for some $\theta_0 \in \mathcal{C}(\Lambda, m, \gamma)^{\sigma}$ and $c \in \mathfrak{a}_{-(m+1),-}$. Let s_γ be a σ-equivariant tame corestriction with respect to γ.*

(i) *For any $g \in I_G(\theta, \theta')$ there are elements $x, y \in S(\gamma) \cap G$ and $g' \in B_\gamma \cap G$ such that $g = xg'y$; moreover, g' intertwines $\psi_{s_\gamma(\beta-\gamma+c)}$ with $\psi_{s_\gamma(\beta'-\gamma)}$.*
(ii) *For any $g' \in I_{B_\gamma \cap G}(\psi_{s_\gamma(\beta-\gamma+c)}, \psi_{s_\gamma(\beta'-\gamma)})$, there are $x, y \in (1 + \mathfrak{m}_{-k_0(\gamma,\Lambda)-m-1}) \cap G$ such that $xg'y$ intertwines θ with θ'.*
(iii) *If $\psi_{s_\gamma(\beta-\gamma+c)} = \psi_{s_\gamma(\beta'-\gamma)}$ then there is $z \in (1 + \mathfrak{m}_{-k_0(\gamma,\Lambda)-m-1}) \cap G$ such that $\theta^z = \theta'$.*

Corollary 9.28. *Let $[\Lambda, q, m, \beta]$ and $[\Lambda, q, m, \beta']$ be two skew-semisimple strata such that we can chose elements $[\Lambda, q, m+1, \gamma]$ and $[\Lambda, q, m+1, \gamma']$ in defining sequences (with skew strata) such that γ and γ' have a common associated splitting, say (V^j), and such that $H^{m+2}(\beta, \Lambda)$ is equal to $H^{m+2}(\beta', \Lambda)$. Let $\theta \in \mathcal{C}(\Lambda, m, \beta)^{\sigma}$ and $\theta' \in \mathcal{C}(\Lambda, m, \beta')^{\sigma}$ be two semisimple characters which are intertwine by an element of G and which coincide on $H^{m+2}(\beta, \Lambda)$. Then there is an element in $\prod_j (A^{j,j})^{\times} \cap G$ which intertwines θ with θ'.*

Proof. The proof is the same as that of Proposition 9.19, where we use Theorem 9.26 in place of Theorem 9.16 and Proposition 9.27 in place of Proposition 9.17. Note that the element c in the proof of Proposition 9.19 can be chosen to be skew-symmetric by Pontrjagin duality. □

10. Matching and conjugacy for semisimple characters

In this final section we prove that there is an analogue of the matching Proposition 7.1 for semisimple characters which intertwine. One might think that this matching could just come from that for the underlying semisimple strata, but these do not necessarily intertwine so this is not possible. Then the sufficient condition (8.4) for an "intertwining implies conjugacy" result for semisimple strata is also sufficient for semisimple characters, also in the case of semisimple characters for G.

10.1. **For general linear groups.** For a semisimple character $\theta \in \mathcal{C}(\Lambda, m, \beta)$, with decomposition $V = \bigoplus_{i\in I} V^i$ associated to $[\Lambda, q, 0, \beta]$, we write θ_i for the restriction of θ to $H^{m+1}(\beta_i, \Lambda^i) = H^{m+1}(\beta, \Lambda) \cap A^{i,i}$, for each index $i \in I$.

Theorem 10.1. *Let $\theta \in \mathcal{C}(\Lambda, m, \beta)$ and $\theta' \in \mathcal{C}(\Lambda', m, \beta')$ be semisimple characters which intertwine and suppose that Λ and Λ' have the same period. Then there is a unique bijection $\zeta : I \to I'$ such that there is an element $g \in \tilde{G}$ with*

(i) *$gV^i = V'^{\zeta(i)}$, for all $i \in I$;*
(ii) *$\theta_i^{g^{-1}}$ and $\theta'_{\zeta(i)}$ intertwine, for all $i \in I$.*

Moreover, all elements of $\tilde{G}$ which satisfy (i) also satisfy (ii).

Proof. First we prove the uniqueness of ζ under the assumption that the existence statement is proven. If there are two bijections from I to I' satisfying the assertions of the theorem then there are indices $i_1, i_2 \in I$ and $i' \in I'$ such that θ_{i_1} and θ_{i_2} intertwine with $\theta'_{i'}$. By (i), we can conjugate $\theta'_{i'}$ to V^{i_1} and to V^{i_2}, and afterwards $\theta'_{i'} \otimes \theta'_{i'}$ is the Levi-part (under an Iwahori decomposition) of a simple character, which intertwines with $\theta_{i_1} \otimes \theta_{i_2}$. The index set of the latter two semisimple characters have different cardinalities and we obtain a contradiction.

To prove the final assertion of the statement let us assume that g' is another element of $\tilde{G}$ which satisfies (i). Then by the uniqueness of ζ the characters $\theta_i^{g^{-1}}$ and $\theta_i^{g'^{-1}}$ are conjugate by the restriction of $g'g^{-1}$ to $V'^{\zeta(i)}$. Thus $\theta_i^{g'^{-1}}$ and $\theta'_{\zeta(i)}$ intertwine, because $\theta_i^{g^{-1}}$ and $\theta'_{\zeta(i)}$ do.

We now turn to the existence proof. First we reduce to the case of lattice chains, in fact to the case where both lattice chains are block-wise principal lattice chains – that is, for each index i the dimension $\dim_{\kappa_F} \Lambda^i_k/\Lambda^i_{k+1}$ is independent of k. For that we repeat the †-construction $\Lambda^\dagger := \oplus_{j=0}^{e-1}(\Lambda - j)$, where e is the period of Λ (which we assume coincides with that of Λ'), and $\Lambda'^\dagger$ similarly. Let us remark that $\Lambda^\dagger$ is the direct sum of the $(\Lambda^i)^\dagger$. We will also need to use the notion of *endo-equivalence* of simple characters, for which we refer the reader to [BH96] and [BSS12].

By assumption, θ and θ' intertwine and thus $\theta^\dagger$ and $\theta'^\dagger$ intertwine. Assume that we have proven the existence of ζ for the case of block-wise principal lattice chains. In particular we find an element g which maps, for each index i, the vector space $(V^i)^\dagger$ to $(V'^{\zeta(i)})^\dagger$, and then $(\theta_i^\dagger)^{g^{-1}}$ and $\theta'^\dagger_{\zeta(i)}$ intertwine. In particular, this implies that $V^{\zeta(i)}$ and V^i have the same dimension and that $\theta_i^\dagger$ and $\theta'^\dagger_{\zeta(i)}$ are endo-equivalent. (More precisely, they are realizations of endo-equivalent ps-characters.) Thus there is an isomorphism $g_i : V^i \to V'^{\zeta(i)}$ and, for any such, the simple characters $\theta_i^{g_i^{-1}}$ and $\theta'_{\zeta(i)}$ intertwine, since they are realizations of endo-equivalent ps-characters on the same space. Thus the element $\sum_{i\in I} g_i$ has all the required properties. This finishes the proof of the reduction to the block-wise principal case.

Now we assume we are in the block-wise principal case and prove the existence of ζ. We proceed via induction on m, with the case $m \geqslant \lfloor \frac{q}{2} \rfloor$ following directly from Proposition 7.1. For $m < \lfloor \frac{q}{2} \rfloor$, let $[\Lambda, q, m+1, \gamma]$ be a semisimple stratum equivalent to $[\Lambda, q, m+1, \beta]$ with $\gamma \in \prod_i A^{i,i}$, and similarly for $[\Lambda', q, m+1, \gamma']$. We write J for the index set of the splitting of $[\Lambda, q, m+1, \gamma]$, and similarly J'. We have the character $\theta_\gamma = \theta|_{H^{m+2}(\gamma,\Lambda)} \in \mathcal{C}(\Lambda, m+1, \gamma)$, and similarly $\theta'_{\gamma'}$, and these characters intertwine. In particular, by induction, there are a bijection $\zeta_\gamma : J \to J'$ and $g \in \tilde{G}$ such that $gV^j = V'^{\zeta_\gamma(j)}$ and $\theta_{\gamma,j}^{g^{-1}}$ intertwines $\theta'_{\gamma',\zeta_\gamma(j)}$ for all $j \in J$, where $\theta_{\gamma,j} = \theta_\gamma|_{H^{m+2}(\gamma_j,\Lambda^j)}$. Since $g\Lambda^j$ and $\Lambda'^{\zeta_\gamma(j)}$ are then principal lattice chains of the same period in the same space, they are conjugate so, changing g, we may assume they are equal; that is, $g \in \tilde{\mathrm{U}}(\Lambda)$.

In particular, conjugating everything by g, we may assume that the strata $[\Lambda, q, m+1, \gamma]$ and $[\Lambda, q, m+1, \gamma']$ have the same associated splitting and $\theta_{\gamma,j}$ intertwines $\theta'_{\gamma',j}$. Since Λ^j and Λ'^j are again principal lattice chains of the same period in the same space, they are conjugate, and we can assume $\Lambda^j = \Lambda'^j$ for all indexes j. Then [BK93, Theorem 3.5.11] implies that $\theta_{\gamma,j}$ and $\theta'_{\gamma',j}$ are conjugate by an element of $\tilde{\mathrm{U}}(\Lambda^j)$ so, by conjugating, we can assume they are equal. By Corollary 9.14 this implies that θ_γ is equal to $\theta_{\gamma'}$. Corollary 9.19 provides an intertwiner for θ and θ' which preserves every V^j. Now, since we can then prove the existence of ζ separately for each block V^j, we may assume that $\theta_\gamma = \theta_{\gamma'}$ is simple. By [BK93, Theorem 3.5.8] we then have that

$$H^{m+1}(\beta, \Lambda) = H^{m+1}(\gamma, \Lambda) = H^{m+1}(\gamma', \Lambda) = H^{m+1}(\beta', \Lambda).$$

Thus we abbreviate H^{m+1}, and similarly H^{m+2}. By the translation principle Theorem 9.16, we can find a semisimple stratum $[\Lambda, q, m, \beta'']$ with splitting $V = \bigoplus_{i\in I''} V''^{i''}$ and an element $u \in (1 + \mathfrak{m}_{\gamma'}) \cap \prod_j A^{j,j}$ such that

- $[\Lambda, q, m+1, \beta'']$ is equivalent to $[\Lambda, q, m+1, \gamma']$;
- $\mathcal{C}(\Lambda, m, \beta'') = \mathcal{C}(\Lambda, m, \beta)$ and
- $u\gamma' u^{-1} \in \prod_{i''} A^{i''i''}$.

Note that $[\Lambda, q, m+1, \beta'']$ is then also equivalent to $[\Lambda, q, m+1, u\gamma' u^{-1}]$. Since $\mathcal{C}(\Lambda, m, \beta'') = \mathcal{C}(\Lambda, m, \beta)$, Proposition 9.9 implies that we have a bijection $\tau : I \to I''$ and $y \in S(\beta)$ such that $yV^i = V''^{\tau(i)}$. Moreover, the element y normalizes θ, thus $\theta_i^{y^{-1}} = \theta_{\tau(i)}$. In particular, we may replace the pair (β, γ) by $(\beta'', u\gamma' u^{-1})$, since we can then compose the bijection $\zeta : I'' \to I$ that we obtain with τ (and right multiply the element g we obtain by y). Thus we may assume that $\gamma = u\gamma' u^{-1}$. Now conjugating back with u (that is, replacing (β,γ,θ) by $(u^{-1}\beta u,\ u^{-1}\gamma u,\theta^u)$), we may assume that $\gamma=\gamma'$.

Now let s_γ be a tame corestriction with respect to γ. We write θ and θ' as in Proposition 9.17,

$$\theta = \theta_0\psi_{\beta-\gamma+c}, \qquad \text{and} \qquad \theta' = \theta_0\psi_{\beta'-\gamma},$$

with $\theta_0 \in \mathcal{C}(\Lambda,m,\gamma)$ and $c \in \mathfrak{a}_{-(m+1)}$. Moreover, by Remark 9.18, we can assume that c is decomposed by the splitting $V=\bigoplus_{i\in I} V^i$. Since θ_0 and $\psi_c\theta_0$ are both elements of $\mathcal{C}(\Lambda,m,\gamma)$, both are intertwined by every element of $B_\gamma^\times$; in particular, we deduce that the derived stratum $[\Lambda,m+1,m,s_\gamma(c)]$ is intertwined by every element of $B_\gamma^\times$ and thus $s_\gamma(c)$ is an element of $F[\gamma]+\mathfrak{b}_{\gamma,-m}$, by [BK93, Lemma 2.4.11]. Then, since c,β are both decomposed by the splitting $V=\bigoplus_{i\in I} V^i$, there is a semisimple stratum $[\Lambda,m+1,m,\delta]$ equivalent to $[\Lambda,m+1,m,s(\beta-\gamma+c)]$ with splitting $V=\bigoplus_{i\in I} V^i$. Similarly, there is a semisimple stratum $[\Lambda,m+1,m,\delta']$ equivalent to $[\Lambda,m+1,m,s(\beta'-\gamma)]$ with splitting $V=\bigoplus_{i\in I'} V^{i'}$. By Proposition 9.17 there is an element of B_γ which intertwines $[\Lambda,m+1,m,s(\beta'-\gamma)]$ with $[\Lambda,m+1,m,s(\beta-\gamma+c)]$, so intertwines the semisimple strata $[\Lambda,m+1,m,\delta]$ and $[\Lambda,m+1,m,\delta']$. Then the matching for semisimple strata, Proposition 7.1, implies that there is $g\in B_\gamma^\times$ which matches their splittings; indeed, since we are in the block-wise principal case, we may choose such $g\in \tilde{\mathrm{U}}(\Lambda)\cap B_\gamma$. In particular, conjugating by this element (which centralizes γ), we may assume that $I=I'$ and the strata $[\Lambda,m+1,m,\delta]$ and $[\Lambda,m+1,m,\delta']$ are intertwined by an element of $B_\gamma\cap\prod_i A^{i,i}$. But then, by Proposition 9.17 again, θ_i intertwines with θ_i' for all $i\in I$, which finishes the proof. □

Theorem 10.2. *Let $\theta\in\mathcal{C}(\Lambda,m,\beta)$ and $\theta'\in\mathcal{C}(\Lambda,m,\beta')$ be semisimple characters which intertwine, let $\zeta: I\to I'$ be the matching given by Theorem 10.1, and suppose that condition (8.4) holds. Then θ is conjugate to θ' by an element of $\tilde{\mathrm{U}}(\Lambda)\cap\prod_i A^{i,\zeta(i)}$.*

Proof. We first remark that the result is transitive: that is, if the hypotheses are also satisfied for a pair (θ',θ'') of semisimple characters then the same is true of the pair (θ,θ'') and, similarly, the conclusion for the pairs (θ,θ') and (θ',θ'') implies that for (θ,θ''). Similarly, if (θ'',β'') is conjugate to (θ',β') then the result for (θ,θ') is equivalent to that for (θ,θ'').

We need to consider three steps.

(i) Suppose first that θ is equal to θ'. Then, by Proposition 9.9(iv) we can find an element of $S(\beta)$ which maps V^i onto $V^{\zeta(i)}$. This element normalizes θ.

(ii) Suppose $q>m$ and that $[\Lambda,q,m,\beta]$ and $[\Lambda,q,m,\beta']$ have *simple* strata $[\Lambda,q,m+1,\gamma]$ and $[\Lambda,q,m+1,\gamma]$ in their defining sequences, respectively, and suppose that $\theta|_{H^{m+2}(\gamma,\Lambda)}$ and $\theta'|_{H^{m+2}(\gamma',\Lambda)}$ coincide; in particular the sets of simple characters for the strata for γ and γ' coincide. Then the translation principle Proposition 9.16 provides a semisimple stratum with element β'', such that $\beta''-\gamma$ is an element of $\mathfrak{a}_{-m-1}$ and $\mathcal{C}(\Lambda,m,\beta'')=\mathcal{C}(\Lambda,m,\beta')$, and an element u of $1+\mathfrak{m}_\gamma$ such that $u\gamma u^{-1}$ is split by the associated splitting of β''.

Now we can apply part (i) to (θ', β') and (θ', β'') so, by transitivity, we reduce to the case where $\gamma' = u\gamma u^{-1}$. Then, by conjugating by u, we reduce to the case where $\gamma = \gamma'$. Now writing θ, θ' as in Proposition 9.17, we get that the derived strata intertwine so, by Theorem 8.3, are conjugate by elements of $\tilde{\mathrm{U}}(\Lambda) \cap B_\gamma$. But then Proposition 9.17(iii) gives us an element of $\tilde{\mathrm{U}}(\Lambda)$ which conjugates θ to θ'. Part (i) enables us to modify the conjugating element such that V^i is mapped to $V^{\zeta(i)}$ for all indices i.

(iii) We now prove the general case by induction on m. If the strata are null strata ($m = q$), then we can take the identity as the conjugating element. Suppose now $m < q$. Take for the strata first members of the defining sequences with entries γ and γ', respectively. By the induction hypothesis $\theta|_{H^{m+2}(\gamma,\Lambda)}$ is conjugate to $\theta'|_{H^{m+2}(\gamma',\Lambda)}$ by an element which conjugates the splittings; thus, by conjugating, we may assume that these restrictions are equal, and further that γ and γ' have the same associated splitting, say $V = \bigoplus_j V^j$. Now Corollary 9.19 provides an intertwiner of θ with θ' which preserves V^j, for all indices j. We apply Part (ii) for each j to obtain an element $g = (g_j)$ of $\tilde{\mathrm{U}}(\Lambda) \cap \prod_i A^{i,\zeta(i)}$ which conjugates θ_j to θ'_j for all indices j. Finally, Corollary 9.14 applied to $\theta^{g^{-1}}$ and θ' and the splitting $\bigoplus_i V^{\zeta(i)}$ gives that $\theta^{g^{-1}}$ and θ' coincide, and the element g is as required.

□

10.2. **For classical groups.** If two characters $\theta_- \in \mathcal{C}_-(\Lambda, m, \beta)$ and $\theta'_- \in \mathcal{C}_-(\Lambda, m, \beta')$ intertwine then their lifts $\theta \in \mathcal{C}(\Lambda, m, \beta)^\sigma$ and $\theta' \in \mathcal{C}(\Lambda, m, \beta')^\sigma$ intertwine and we get a matching $\zeta : I \to I'$ from Theorem 10.1. Let us state the main theorem:

Theorem 10.3. *Let $\theta_- \in \mathcal{C}_-(\Lambda, m, \beta)$ and $\theta'_- \in \mathcal{C}_-(\Lambda, m, \beta')$ be two semisimple characters of G, which intertwine in G, and assume that their matching satisfies (8.4). Then, θ_- and θ'_- are $\mathrm{U}(\Lambda) \cap (\prod_i A^{i,i})$-conjugate.*

Proof. The proof is completely the same as for Theorem 10.2 by using σ-fixed lifts of the characters and the relevant results for G in place of those for $\tilde{G}$. Specifically: In step (i), we use Proposition 9.23(iii). In step (ii), we use the translation principle for G, Theorem 9.26, to reduce to the case of a common γ and we use Proposition 9.27(i) to reduce to the derived strata; the case of minimal strata is done in Theorem 8.7. In step (iii), we use Corollary 9.28 to reduce to the case where the stratum with γ is simple. □

We also conjecture a more natural version of the Matching Theorem 10.1 for G.

Conjecture 10.4. *Let $[\Lambda, q, m, \beta]$ and $[\Lambda', q, m, \beta']$ be skew-semisimple strata and $\theta \in \mathcal{C}(\Lambda, m, \beta)^\sigma$ and $\theta' \in \mathcal{C}(\Lambda', m, \beta')^\sigma$ two semisimple characters which are intertwined by an element of G. Let $\zeta : I \to I'$ be the matching from Theorem 10.1. Then, there is an element $g \in G$ such that $gV^i = V'^{\zeta(i)}$, for all $i \in I$.*

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