DISCRETE AND CONTINUOUS
DYNAMICAL SYSTEMS
Volume **38**, Number **6**, June **2018**

doi:10.3934/dcds.2018132

pp. **3033–3054**

A SINGULAR CAHN–HILLIARD–OONO PHASE-FIELD SYSTEM WITH HEREDITARY MEMORY

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(Communicated by Roger Temam)

Abstract. We consider a phase-field system modeling phase transition phenomena, where the Cahn–Hilliard–Oono equation for the order parameter is coupled with the Coleman–Gurtin heat law for the temperature. The former suitably describes both local and nonlocal (long-ranged) interactions in the material undergoing phase-separation, while the latter takes into account thermal memory effects. We study the well-posedness and longtime behavior of the corresponding dynamical system in the history space setting, for a class of physically relevant and singular potentials. Besides, we investigate the regularization properties of the solutions and, for sufficiently smooth data, we establish the strict separation property from the pure phases.

1. Introduction.

In the Nineties, G. Caginalp introduced the following phase-field system

$$\varphi_t - \Delta\mu = 0, \qquad \mu = -\Delta\varphi + \Psi'(\varphi) - \vartheta, \tag{1}$$

$$\vartheta_t - \Delta\vartheta = -\varphi_t, \tag{2}$$

2010 *Mathematics Subject Classification.* Primary: 35B41, 35K55, 35L05, 80A22.
Key words and phrases. Phase-field models, Cahn–Hilliard and Oono equations, memory effects, singular potential, strict separation property.

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in order to describe phase transition phenomena such as melting/solidification processes, see [3] and [4]. Here, φ is the order parameter of the material undergoing the transition process, ϑ is its (relative) temperature, and Ψ' is the derivative of a double-well potential of the form

$$\Psi(\varphi) = \frac{1}{4}(\varphi^2 - 1)^2. \tag{3}$$

The model consists of the coupling of the Cahn–Hilliard equation, introduced in [5] and [6], with the heat equation, assuming the usual Fourier law for heat conduction (for simplicity, we have set all physical constants equal to 1 here). If we consider instead a linearized version of the Coleman–Gurtin law (see [7]) which accounts for (past) memory effects, we end up with the following equation for the relative temperature

$$\vartheta_t - k_d \Delta\vartheta - \int_0^\infty k(s)\Delta\vartheta(t-s)\,\mathrm{d}s = -\varphi_t, \tag{4}$$

where k is a nonnegative and summable memory kernel and $k_d > 0$ (we will take $k_d = 1$ in what follows). Systems (1)-(2) and (1)-(4) are known as conserved phase-field models, in the sense that, when endowed with Neumann boundary conditions, the spatial average of the order parameter is a conserved quantity. They both have been much studied in the literature, see, e.g., [1, 2, 21], [12, 14, 16, 17] and the references therein, mainly for *regular* potentials Ψ of the form (3). However, it is well known since the pioneering papers by J.W. Cahn and J.E. Hilliard that, in this context, logarithmic potentials of the form

$$\Psi(\varphi) = \frac{\Theta}{2}\Big((1+\varphi)\ln(1+\varphi) + (1-\varphi)\ln(1-\varphi)\Big) - \frac{\Theta_0}{2}\varphi^2, \qquad \text{where } \Theta_0 > \Theta > 0, \tag{5}$$

are physically relevant, and provide a better description of transition processes. This motivates the present paper, where the goal is to investigate the conserved phase-field system with memory in presence of a general class of *singular* potentials (see below), including the logarithmic prototype. Actually, we also generalize (1) by considering the Cahn–Hilliard–Oono equation

$$\varphi_t - \Delta\mu + \alpha\varphi = 0, \qquad \alpha \ge 0, \tag{6}$$

introduced in [24] (see also [28]) to account for long-ranged (i.e. nonlocal) interactions. This equation was studied in [20] for regular potentials and in [15] for logarithmic potentials.

In order to deal with (4) coupled with (6) as a dynamical system, it is convenient to exploit the past history formulation proposed in [10]. Namely, we introduce the auxiliary variable

$$\eta^t(s) = \int_0^s \vartheta(t-y)\,\mathrm{d}y,$$

so that, setting $g = -k'$, we (formally) arrive at the following problem:

$$\begin{cases} \varphi_t - \Delta\mu + \alpha\varphi = 0, \\ \mu = -\Delta\varphi + \Psi'(\varphi) - \vartheta, \\ \vartheta_t - \Delta\vartheta - \int_0^\infty g(s)\Delta\eta(s)\,\mathrm{d}s = -\varphi_t, \\ \eta_t = -\eta_s + \vartheta, \end{cases} \tag{7}$$

in $\Omega \times (0, \infty)$, where Ω is the domain occupied by the material, with boundary Γ. We couple the system with the Neumann boundary conditions

$$\partial_{\mathbf{n}}\varphi = \partial_{\mathbf{n}}\mu = \partial_{\mathbf{n}}\vartheta = 0 \quad \text{and} \quad \int_0^\infty g(s)\partial_{\mathbf{n}}\eta(s)\,\mathrm{d}s = 0 \quad \text{on } \Gamma \times [0, \infty), \tag{8}$$

and the initial conditions

$$\varphi(0) = \varphi_0, \quad \vartheta(0) = \vartheta_0, \quad \eta^0 = \eta_0 \quad \text{in } \Omega. \tag{9}$$

Our aim in this paper is to study the well-posedness and the asymptotic behavior, in terms of attractors, of (7)-(9) with singular potentials. This is a quite challenging problem: indeed, as pointed out in [15] for the sole Cahn–Hilliard–Oono equation, the occurrence of singular potentials combined with the Oono term introduces essential difficulties. Furthermore, the memory effects yield a lack of regularization on the temperature, making the longterm analysis delicate.

We also study the (strict) separation of the order parameter from the pure phases. The latter is a very sensitive property in the context of the Cahn–Hilliard (see [23]) and Cahn–Hilliard–Oono (see [15] and [22]) equations and is only valid in one and two space dimensions in general. In our case, due to the aforementioned lack of regularization, we can prove this property for more regular initial temperatures only.

2. **Preliminaries.**

2.1. **Assumptions.**

We assume that Ψ is a quadratic perturbation of a singular (strictly) convex function in $[-1, 1]$, namely

$$\Psi(s) = F(s) - \frac{\Theta_0}{2}s^2,$$

where $F \in \mathcal{C}([-1, 1]) \cap \mathcal{C}^2(-1, 1)$ fulfils

$$\lim_{s \to -1} F^{'}(s) = -\infty, \quad \lim_{s \to 1} F^{'}(s) = +\infty,$$

and there exists $\Theta > 0$ such that

$$F^{''}(s) \geq \Theta, \quad \forall\, s \in (-1, 1). \tag{10}$$

We assume that

$$\Theta_0 - \Theta > 0,$$

so that Ψ has a double-well shape as in the prototype model (5). We also extend F by $F(s) = +\infty$ for any $s \notin [-1, 1]$. Note that the above assumptions imply that there exists $s_0 \in (-1, 1)$ such that $F'(s_0) = 0$. Without loss of generality, we assume that $s_0 = 0$ and that $F(s_0) = 0$ as well. In particular, this entails that $F(s) \geq 0$ for all $s \in [-1, 1]$.

Concerning the memory kernel, we suppose that k is a nonnegative summable function of total mass equal to 1, having the explicit form

$$k(s) = \int_s^\infty g(y)\,\mathrm{d}y,$$

where $g \in L^1(\mathbb{R}^+)$ is a nonincreasing, nonnegative, absolutely continuous function satisfying, for some $\delta > 0$,

$$g'(s) + \delta g(s) \leq 0, \quad \forall s > 0. \tag{11}$$

We agree to denote $\int_0^\infty g(s)\,\mathrm{d}s = k(0) = \kappa$.

2.2. **Functional spaces.** Let $N \leq 3$ and $\Omega \subset \mathbb{R}^N$ be a smooth bounded domain. We denote by $(\mathrm{H}, \langle\cdot,\cdot\rangle, \|\cdot\|)$ the space $L^2(\Omega)$ (or $[L^2(\Omega)]^N$ according to the context) endowed with the standard scalar product and norm. Besides, we denote by

$$\mathrm{H}^\sigma = H^\sigma(\Omega) = W^{\sigma,2}(\Omega), \quad \sigma > 0,$$

the standard Sobolev spaces, with scalar products $\langle\cdot,\cdot\rangle_\sigma$ and norms $\|\cdot\|_{\mathrm{H}^\sigma}$. We will use the notation $V = \mathrm{H}^1$, equipped with the norm

$$\|u\|_V^2 = \|\nabla u\|^2 + \|u\|^2,$$

and we indicate by V' the dual space of V, by $\|\cdot\|_{V'}$ its norm and again by $\langle\cdot,\cdot\rangle$ the duality product $\langle\cdot,\cdot\rangle_{V',V}$. Denoting by

$$\langle u\rangle = \frac{1}{|\Omega|}\int_\Omega u(x)\,\mathrm{d}x$$

the average of any measurable function u over Ω, let us recall that

$$\|u\|_V^2 \leq C\left(\|\nabla u\|^2 + |\langle u\rangle|^2\right), \quad \forall\, u \in V.$$

We introduce the space of zero-mean functions and its dual space

$$V_0 = \{v \in V : \langle v\rangle = 0\}, \quad V_0' = \{f \in V' : \langle f, 1\rangle = 0\}.$$

We then consider the linear operator $A : V \to V'$ defined by

$$\langle Au, v\rangle = \int_\Omega \nabla u \cdot \nabla v \;\mathrm{d}x, \quad \forall\, u, v \in V,$$

which is an isomorphism from V_0 onto V_0'. Its inverse map

$$\mathcal{N} : V_0' \to V_0$$

satisfies

$$\langle Au, \mathcal{N}f\rangle = \langle f, u\rangle, \quad \forall\, u \in V,\ \forall\, f \in V_0',$$
$$\langle f, \mathcal{N}g\rangle = \langle g, \mathcal{N}f\rangle = \int_\Omega \nabla(\mathcal{N}f)\cdot\nabla(\mathcal{N}g)\,\mathrm{d}x, \quad \forall\, f, g \in V_0'.$$

Moreover,

$$\|f\|_* := \|\nabla\mathcal{N}f\| = \langle f, \mathcal{N}f\rangle^{1/2}$$

is an equivalent norm in V_0', and

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u(t)\|_*^2 = \langle u_t(t), \mathcal{N}u(t)\rangle, \quad \text{for a.e. } t \in (0,T),\ \forall\, u \in H^1(0,T;V_0').$$

We denote by

$$\|f\|_{V'}^2 := \|f - \langle f, 1\rangle\|_*^2 + |\langle f, 1\rangle|^2$$

the corresponding (equivalent) norm in V'.

In order to handle the convolution integral, we define as usual the so-called *memory spaces*

$$\mathcal{M}^\sigma = L_g^2(\mathbb{R}^+; \mathrm{H}^\sigma), \quad \langle\eta, \xi\rangle_{\mathcal{M}^\sigma} = \int_0^\infty g(s)\langle\eta(s), \xi(s)\rangle_\sigma \mathrm{d}s, \qquad \sigma \geq 0,$$

where $\langle\cdot,\cdot\rangle_\sigma$ is the scalar product in H^σ (we omit σ in the notation when $\sigma = 0$). Then, we introduce the operator $T : \mathcal{M}^1 \to \mathcal{M}^1$

$$T\eta = -\eta', \quad \mathfrak{D}(T) = \left\{\eta \in \mathcal{M}^1 : \eta' \in \mathcal{M}^1,\ \eta(0) = 0\right\}$$

(where η' denotes the derivative of η with respect to the internal variable s), which is the infinitesimal generator of the strongly continuous semigroup of right translations on the memory space $\mathcal{M}^1$. We now recall some well-established facts, see e.g. [9].

The representation formula. If $\vartheta \in L^1(0,T;\mathrm{H}^2)$, then, for every $\eta_0 \in \mathcal{M}^2$, the Cauchy problem

$$\begin{cases} \eta_t = T\eta + \vartheta, \\ \eta^0 = \eta_0, \end{cases} \tag{12}$$

has a unique (mild) solution $\eta \in \mathcal{C}([0,T],\mathcal{M}^2)$ which has the explicit representation formula

$$\eta^t(s) = \begin{cases} \int_0^s \vartheta(t-y)\,\mathrm{d}y, & 0 < s \le t, \\ \eta_0(s-t) + \int_0^t \vartheta(t-y)\,\mathrm{d}y, & s > t. \end{cases}$$

Besides, $\eta_t + \eta_s \in L^2(0,T;\mathcal{M}^2)$ and

$$\langle \eta_t(t) + \eta_s(t), \psi \rangle_{\mathcal{M}^2} = \langle \vartheta, \psi \rangle_{\mathcal{M}^2},$$

for every $\psi \in \mathcal{M}^2$ and a.e. $t \in [0,T]$.

Compact embeddings. Despite the compact inclusions $\mathrm{H}^{\sigma+1} \Subset \mathrm{H}^{\sigma}$ ($\sigma \ge 0$), the embedding $\mathcal{M}^{\sigma+1} \subset \mathcal{M}^{\sigma}$ is not compact, see [26]. To recover compactness, the following Banach spaces are introduced:

$$\mathcal{K}^{\sigma+1} = \{\eta \in \mathcal{M}^{\sigma+1},\, \eta_s \in \mathcal{M}^{\sigma},\, \eta(0) = 0,\, \sup_{x \ge 1} x\mathbb{T}(x;\eta) < \infty\},$$

where

$$\mathbb{T}(x;\eta) = \int_x^\infty g(s)\|\eta(s)\|^2_{\mathrm{H}^\sigma}\,\mathrm{d}s,$$

and the norm is given by

$$\|\eta\|^2_{\mathcal{K}^{\sigma+1}} = \|\eta\|^2_{\mathcal{M}^{\sigma+1}} + \|\eta_s\|^2_{\mathcal{M}^{\sigma}} + \sup_{x \ge 1} x\mathbb{T}(x;\eta).$$

It is well-known (see e.g. [13, Proposition 5.4]) that its closed balls are closed in $\mathcal{M}^\sigma$ and the embedding

$$\mathcal{K}^{\sigma+1} \Subset \mathcal{M}^{\sigma}$$

is compact.

We also recall the following result, useful to handle the norm in $\mathcal{K}^{\sigma+1}$ (see [9], Lemma 3.3 and Lemma 3.4).

Lemma 2.1. *Assume that η satisfies the Cauchy problem*

$$\begin{cases} \eta_t = T\eta + \vartheta, \\ \eta^0 = 0, \end{cases}$$

on $(0,T)$, for some $T > 0$. Then, for every $t \in (0,T)$, we have $\eta^t(0) = 0$ and

$$\|\eta_s\|^2_{\mathcal{M}^\sigma} + \sup_{x \ge 1} x\mathbb{T}(x;\eta) \le c\|\vartheta\|^2_{\mathrm{H}^\sigma}.$$

Finally, we denote by

$$\mathcal{H}^\sigma = \mathrm{H}^{\sigma+1} \times \mathrm{H}^{\sigma+1} \times \mathcal{M}^{\sigma+2}, \qquad \sigma \ge 0,$$

the hierarchy of the *extended spaces*, endowed with their natural scalar product. Again, we omit the superscript σ whenever it equals zero. In particular,

$$\mathcal{M} = L^2_g(\mathbb{R}^+;\mathrm{H}),$$

$$\mathcal{H} = V \times V \times L^2_g(\mathbb{R}^+; \mathrm{H}^2) \qquad \text{and} \qquad \mathcal{H}^1 = \mathrm{H}^2 \times \mathrm{H}^2 \times L^2_g(\mathbb{R}^+; \mathrm{H}^3).$$

By the above discussion, we learn that

$$\mathcal{W} = \mathrm{H}^2 \times \mathrm{H}^2 \times \mathcal{K}^3 \subset \mathcal{H}^1$$

is compactly embedded in $\mathcal{H}$.

Notation. Throughout the paper $c > 0$ denotes a generic constant and $\mathcal{Q}(\cdot) > 0$ is a generic increasing function, only depending on the structural parameters of the problem. Moreover, given any measurable function u, we set $\bar{u} = u - \langle u \rangle$.

3. Weak solutions and their basic properties.

3.1. **Definition.** Let $(\varphi_0, \vartheta_0, \eta_0) \in \mathcal{H}$ with $\Psi(\varphi_0) \in L^1(\Omega)$ and $T > 0$ be given. A triplet $(\varphi, \vartheta, \eta)$ is called *weak solution* to the system (7)-(9) on $[0, T]$ if

$$(\varphi, \vartheta, \eta) \in L^\infty(0, T; \mathcal{H}),$$

$$\varphi \in L^\infty(\Omega \times (0, T)), \quad \text{with} \quad |\varphi(x, t)| < 1 \quad \text{a.e. } (x, t) \in \Omega \times (0, T),$$
$$\varphi \in H^1(0, T; V') \cap L^2(0, T; \mathrm{H}^2),$$
$$\vartheta \in H^1(0, T; V') \cap L^2(0, T; \mathrm{H}^2),$$

with

$$\mu = -\Delta\varphi + \Psi'(\varphi) - \vartheta \in L^2(0, T; V).$$

Besides, for a.e. $t \in [0, T]$ and every $w \in V$,

$$\langle \varphi_t(t), w \rangle + \langle \nabla\mu(t), \nabla w \rangle + \alpha\langle \varphi(t), w \rangle = 0,$$
$$\langle \vartheta_t(t) + \varphi_t(t), w \rangle + \langle \nabla\vartheta(t), \nabla w \rangle + \int_0^\infty g(s)\langle \nabla\eta^t(s), \nabla w \rangle \, \mathrm{d}s = 0,$$

and η is a mild solution of the Cauchy problem (12) on $\mathcal{M}^2$. Moreover, $\varphi|_{t=0} = \varphi_0$ and $\vartheta|_{t=0} = \vartheta_0$ a.e. in Ω.

We associate to each weak solution z its (finite) energy

$$\mathcal{E}(z) = \frac{1}{2}\|z\|^2_{\mathcal{H}} + \int_\Omega \Psi(\varphi) \, \mathrm{d}x.$$

3.2. **Continuity of the weak solutions.** It is apparent from the definition that, since $\varphi \in H^1(0, T; V') \cap L^2(0, T; \mathrm{H}^2)$, then $\varphi \in \mathcal{C}([0, T], \mathrm{H})$. Indeed, it is possible to prove by a standard argument (see e.g. [8]) that $\varphi \in \mathcal{C}([0, T], V)$. On the other hand, if we consider the third equation of (7) written for $H = \varphi + \vartheta$, namely

$$H_t = \Delta\vartheta + \int_0^\infty g(s)\Delta\eta(s) \, \mathrm{d}s,$$

we learn by comparison that $H_t \in L^2(0, T; \mathrm{H})$. This, together with $H \in L^2(0, T; \mathrm{H}^2)$ yields $H \in \mathcal{C}([0, T], V)$, and, in turn, $\vartheta \in \mathcal{C}([0, T], V)$.

3.3. **Conservation laws.** Integrating $(7)_1$ over Ω, we deduce that

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle \varphi \rangle + \alpha\langle \varphi \rangle = 0, \tag{13}$$

yielding

$$\langle \varphi(t) \rangle = \langle \varphi_0 \rangle e^{-\alpha t}, \quad t \geq 0. \tag{14}$$

Besides, we learn from the third equation of (7) that

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle H\rangle = 0,$$

whence the conservation law

$$\langle H(t)\rangle = \langle(\varphi+\vartheta)(t)\rangle = \langle \varphi_0+\vartheta_0\rangle =: \langle H_0\rangle. \tag{15}$$

The existence of a weak solution will be proved in Section 5 via an approximation procedure involving auxiliary problems in which the singular potential F is substituted by a family of globally Lipschitz potentials F_λ converging (in a suitable sense) to F as $\lambda \to 0$. This is done in the next two sections.

4. **Approximating problems.**

Let us recall (see [11]) that, given F as in Section 2.1, for every $\lambda > 0$, there exists

$$F_\lambda : \mathbb{R} \to \mathbb{R}$$

such that

- (i) F_λ is convex and $F_\lambda(s) \nearrow F(s)$, for all $s \in \mathbb{R}$, as $\lambda \to 0$.
- (ii) F'_λ is Lipschitz continuous on $\mathbb{R}$, with constant $\frac{1}{\lambda}$, and $F''_\lambda(s)$ is nonnegative for all $s \in \mathbb{R}$.
- (iii) $|F'_\lambda(s)| \nearrow |F'(s)|$ for $s \in (-1,1)$ and F'_λ converges uniformly to F' on any compact set $M \subset (-1,1)$.
- (iv) $F_\lambda(0) = F'_\lambda(0) = 0$, for all $\lambda > 0$.
- (v) For any $0 < \overline{\lambda} \leq 1$, there exists $\overline{C} > 0$ such that

$$F_\lambda(s) \geq \frac{1}{4\overline{\lambda}} s^2 - \overline{C}, \quad \forall\, s \in \mathbb{R},\ \forall\, \lambda \in (0, \overline{\lambda}]. \tag{16}$$

Accordingly, for any given $\lambda > 0$, we define

$$\Psi_\lambda(s) = F_\lambda(s) - \frac{\Theta_0}{2} s^2,$$

and we consider the following approximated problems (P_λ) in the unknown $z = (\varphi, \vartheta, \eta)$:

$$\begin{cases} \varphi_t - \Delta\mu + \alpha\varphi = 0, \\ \mu = -\Delta\varphi + \Psi'_\lambda(\varphi) - \vartheta, \\ \vartheta_t - \Delta\vartheta - \int_0^\infty g(s)\Delta\eta(s)\,\mathrm{d}s = -\varphi_t, \\ \eta_t = T\eta + \vartheta, \end{cases} \tag{17}$$

subject to the boundary and initial conditions (8)-(9), with the corresponding energy functionals

$$\mathcal{E}_\lambda(z) = \frac{1}{2}\|z\|^2_{\mathcal{H}} + \int_\Omega \Psi_\lambda(\varphi)\,\mathrm{d}x.$$

Note that the solutions to the approximating problems satisfy the analogous conservation laws established in (14) and (15).

Thanks to the Lipschitz regularity of F_λ, exploiting a standard Galerkin method and a basic energy estimate (see below), it is a standard matter to prove the following existence result.

Theorem 4.1. *Let $z_0 = (\varphi_0, \vartheta_0, \eta_0) \in \mathcal{H}$. Then, for every $\lambda > 0$ and every $T > 0$, (P_λ) has a weak solution $z = (\varphi, \vartheta, \eta)$ on $[0, T]$.*

In the next section we will prove several energy estimates for the solutions to (P_λ) which are *uniform with respect to* λ. This is the main step allowing to pass to the limit as $\lambda \to 0$ in order to find a solution to the original problem (7).

5. **Energy estimates and existence of a weak solution.** In what follows, α belongs to a given compact subset $J \subset [0, \infty)$ and λ is fixed. Let

$$z_0 = (\varphi_0, \vartheta_0, \eta_0) \in \mathcal{H} \ \text{ be such that } \ F(\varphi_0) \in L^1(\Omega) \ \text{ and } \ |\langle \varphi_0 \rangle| = m < 1, \tag{18}$$

and let

$$z_\lambda(t) = z(t) = (\varphi(t), \vartheta(t), \eta^t), \qquad t \geq 0,$$

be any global solution to (P_λ) with initial datum z_0. Throughout the section, the generic constant $Q > 0$ depends on J, $|\langle \varphi_0 \rangle|$ and $\langle H_0 \rangle = \langle \varphi_0 + \vartheta_0 \rangle$, but is independent of λ and of the specific initial datum.

First, we prove some energy estimates in the less regular space

$$\tilde{\mathcal{H}} = V \times \mathrm{H} \times \mathcal{M}^1,$$

involving the weaker energy functionals

$$\tilde{\mathcal{E}}(z) = \frac{1}{2}\|z\|^2_{\tilde{\mathcal{H}}} + \int_\Omega \Psi(\varphi)\,\mathrm{d}x, \qquad \tilde{\mathcal{E}}_\lambda(z) = \frac{1}{2}\|z\|^2_{\tilde{\mathcal{H}}} + \int_\Omega \Psi_\lambda(\varphi)\,\mathrm{d}x.$$

Lemma 5.1. *There exist* $\tilde{\omega} > 0$ *and* $\overline{\lambda} > 0$ *such that, for any* $0 < \lambda \leq \overline{\lambda}$,

$$\tilde{\mathcal{E}}_\lambda(z(t)) + \|z(t)\|^2_{\tilde{\mathcal{H}}} + \int_t^{t+1} \Big(\|\nabla \mu(\tau)\|^2 + \|\nabla \vartheta(\tau)\|^2 \Big)\,\mathrm{d}\tau \leq c\tilde{\mathcal{E}}(z_0)\mathrm{e}^{-\tilde{\omega} t} + Q, \tag{19}$$

for every $t \geq 0$.

Proof. First note that

$$\bar{\varphi}_t - \Delta\mu + \alpha\bar{\varphi} = 0. \tag{20}$$

Multiplying (20) by μ, $(17)_2$ by $\bar{\varphi}_t$ and $\bar{\varphi}$, we have

$$\begin{aligned}
&\langle \mu, \bar{\varphi}_t \rangle + \|\nabla\mu\|^2 + \alpha\langle \mu, \bar{\varphi}\rangle = 0,\\
&\langle \mu, \bar{\varphi}_t \rangle = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\|\nabla\varphi\|^2 + \int_\Omega \Psi_\lambda(\varphi)\,\mathrm{d}x \right) - \langle\varphi\rangle_t \int_\Omega \Psi'_\lambda(\varphi)\,\mathrm{d}x - \langle \vartheta, \bar{\varphi}_t\rangle,\\
&\langle \mu, \bar{\varphi} \rangle = \|\nabla\varphi\|^2 + \langle \Psi'_\lambda(\varphi), \varphi\rangle - \langle\varphi\rangle \int_\Omega \Psi'_\lambda(\varphi)\,\mathrm{d}x - \langle \vartheta, \bar{\varphi}\rangle.
\end{aligned} \tag{21}$$

Collecting the above estimates, in view of (13), we deduce that

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\|\nabla\varphi\|^2 + \int_\Omega \Psi_\lambda(\varphi)\,\mathrm{d}x \right) + \|\nabla\mu\|^2 + \alpha\|\nabla\varphi\|^2 + \alpha\langle \Psi'_\lambda(\varphi), \varphi\rangle\\
=&\langle \vartheta, \varphi_t\rangle + \alpha\langle \vartheta, \varphi\rangle.
\end{aligned}$$

Next we multiply $(17)_3$ by ϑ and $(17)_4$ by $-\Delta\eta$ in $\mathcal{M}$ to obtain

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\|\vartheta\|^2 \right) + \|\nabla\vartheta\|^2 - \int_0^\infty g(s)\langle \Delta\eta(s), \vartheta\rangle\,\mathrm{d}s = -\langle \varphi_t, \vartheta\rangle,\\
&\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\int_0^\infty g(s)\|\nabla\eta(s)\|^2\,\mathrm{d}s \right) - \frac{1}{2}\int_0^\infty g'(s)\|\nabla\eta(s)\|^2\,\mathrm{d}s = -\int_0^\infty g(s)\langle \Delta\eta(s), \vartheta\rangle\,\mathrm{d}s,
\end{aligned}$$

leading to

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\|\vartheta\|^2 + \frac{1}{2}\int_0^\infty g(s)\|\nabla\eta(s)\|^2\,\mathrm{d}s\right) + \|\nabla\vartheta\|^2 - \frac{1}{2}\int_0^\infty g'(s)\|\nabla\eta(s)\|^2\,\mathrm{d}s = -\langle\varphi_t,\vartheta\rangle.$$

Hence the functional

$$\tilde{E}_\lambda(z) = \frac{1}{2}\left(\|\nabla\varphi\|^2 + \|\vartheta\|^2 + \int_0^\infty g(s)\|\nabla\eta(s)\|^2\,\mathrm{d}s\right) + \int_\Omega \Psi_\lambda(\varphi)\,\mathrm{d}x$$

satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t}\tilde{E}_\lambda + \alpha\|\nabla\varphi\|^2 + \alpha\langle\Psi_\lambda'(\varphi),\varphi\rangle + \|\nabla\vartheta\|^2 - \frac{1}{2}\int_0^\infty g'(s)\|\nabla\eta(s)\|^2\,\mathrm{d}s + \|\nabla\mu\|^2 = \alpha\langle\vartheta,\varphi\rangle.$$

The conservation law (15) gives

$$\langle\vartheta\rangle\langle\varphi\rangle = \frac{1}{2}\left(\langle\vartheta_0+\varphi_0\rangle^2 - \langle\vartheta\rangle^2 - \langle\varphi\rangle^2\right).$$

Then the definition of Ψ_λ and (11) entail

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\tilde{E}_\lambda + \alpha\|\nabla\varphi\|^2 + \frac{\alpha}{2}|\Omega|\langle\varphi\rangle^2 \\ &\quad + \|\nabla\vartheta\|^2 + \frac{\alpha}{2}|\Omega|\langle\vartheta\rangle^2 + \frac{\delta}{2}\int_0^\infty g(s)\|\nabla\eta(s)\|^2\,\mathrm{d}s + \|\nabla\mu\|^2 \\ &\quad + \alpha\left(\langle F_\lambda'(\varphi),\varphi\rangle - \Theta_0\|\varphi\|^2 - \langle\overline{\vartheta},\varphi\rangle\right) \le \frac{\alpha}{2}|\Omega|\langle\vartheta_0+\varphi_0\rangle^2.\end{aligned}$$

Owing to the convexity of F_λ and exploiting (16), for any $\lambda \in (0,\overline{\lambda}]$, since $0 < \overline{\lambda} \le 1$, then

$$\begin{aligned}\langle F_\lambda'(\varphi),\varphi\rangle - \Theta_0\|\varphi\|^2 - \langle\bar{\vartheta},\varphi\rangle &\ge \frac{1}{4\overline{\lambda}}\|\varphi\|^2 - \Theta_0\|\varphi\|^2 - c\|\nabla\vartheta\|\|\varphi\| - c \\ &\ge \left(\frac{1}{4\overline{\lambda}} - \Theta_0 - \alpha c\right)\|\varphi\|^2 - \frac{1}{4\alpha}\|\nabla\vartheta\|^2 - c \\ &\ge -\frac{1}{4\alpha}\|\nabla\vartheta\|^2 - c,\end{aligned}$$

provided that $\overline{\lambda} = \overline{\lambda}(\alpha) > 0$ is small enough (note that in the case $\alpha = 0$ no restriction on $\overline{\lambda}$ is needed). Thus,

$$\frac{\mathrm{d}}{\mathrm{d}t}\tilde{E}_\lambda + \alpha\|\nabla\varphi\|^2 + \frac{3}{4}\|\nabla\vartheta\|^2 + \frac{\delta}{2}\int_0^\infty g(s)\|\nabla\eta(s)\|^2\,\mathrm{d}s + \|\nabla\mu\|^2 \le \frac{\alpha}{2}|\Omega|\langle\vartheta_0+\varphi_0\rangle^2 + \alpha c.$$

Adding to both sides $\langle\vartheta\rangle^2$, where

$$\langle\vartheta\rangle^2 = \langle\vartheta_0+\varphi_0\rangle^2 - 2\langle\vartheta_0+\varphi_0\rangle\langle\varphi_0\rangle e^{-\alpha t} + \langle\varphi_0\rangle^2 e^{-2\alpha t} \le 2\langle\vartheta_0+\varphi_0\rangle^2 + 2\langle\varphi_0\rangle^2 e^{-2\alpha t},$$

we are lead to

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\tilde{E}_\lambda + \alpha\|\nabla\varphi\|^2 + \frac{3}{4}\|\nabla\vartheta\|^2 + \langle\vartheta\rangle^2 + \frac{\delta}{2}\int_0^\infty g(s)\|\nabla\eta(s)\|^2\,\mathrm{d}s + \|\nabla\mu\|^2 \\ &\le c(1+\alpha)\langle\vartheta_0+\varphi_0\rangle^2 + 2\langle\varphi_0\rangle^2 + \alpha c =: Q.\end{aligned} \tag{22}$$

We now go back to (21) written as

$$\|\nabla\varphi\|^2 + \langle F_\lambda'(\varphi),\bar{\varphi}\rangle = \langle\mu,\bar{\varphi}\rangle + \langle\vartheta,\bar{\varphi}\rangle + \Theta_0\langle\varphi,\bar{\varphi}\rangle,$$

noticing that, owing to the convexity of F_λ,

$$\|\nabla\varphi\|^2 + \langle F_\lambda'(\varphi), \bar\varphi\rangle \geq \|\nabla\varphi\|^2 + \int_\Omega F_\lambda(\varphi)\,\mathrm{d}x - \int_\Omega F_\lambda(\langle\varphi\rangle)\,\mathrm{d}x,$$

while

$$\langle\mu, \bar\varphi\rangle + \langle\vartheta, \bar\varphi\rangle + \Theta_0\langle\varphi, \bar\varphi\rangle \leq \frac{1}{2}\|\nabla\varphi\|^2 + c\left(\|\nabla\mu\|^2 + \|\nabla\vartheta\|^2 + \Theta_0^2\|\varphi\|^2\right).$$

Summing up the two inequalities we obtain

$$\frac{1}{2}\|\nabla\varphi\|^2 + \int_\Omega F_\lambda(\varphi)\,\mathrm{d}x \leq c\left(\|\nabla\mu\|^2 + \|\nabla\vartheta\|^2 + \Theta_0^2\|\varphi\|^2\right) + \int_\Omega F_\lambda(\langle\varphi\rangle)\,\mathrm{d}x,$$

so that, owing to (16) and choosing $\overline{\lambda}$ small enough, we get

$$\frac{1}{2}\left(\frac{1}{2}\|\nabla\varphi\|^2 + \int_\Omega F_\lambda(\varphi)\,\mathrm{d}x\right) \leq c\left(\|\nabla\mu\|^2 + \|\nabla\vartheta\|^2\right) + \int_\Omega F_\lambda(\langle\varphi\rangle)\,\mathrm{d}x + c. \tag{23}$$

Adding together (22) and β times (23), provided that $\beta > 0$ is small enough, we arrive at the differential inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}\tilde E_\lambda(z) + \tilde\omega\tilde E_\lambda(z) + \frac{1}{4}\left(\|\nabla\mu\|^2 + \|\nabla\vartheta\|^2\right) \leq Q + \int_\Omega F_\lambda(\langle\varphi\rangle)\,\mathrm{d}x + c, \tag{24}$$

for some $\tilde\omega > 0$. (Note that $\tilde\omega$ is independent of α). In light of property (i) we have

$$F_\lambda(s) \leq F(s) \leq \max_{[-1,1]} F < \infty, \quad \forall\, s \in [-1,1]. \tag{25}$$

Besides, by (14), $|\langle\varphi(t)\rangle| \leq |\langle\varphi_0\rangle| < 1$. Hence, $\int_\Omega F_\lambda(\langle\varphi\rangle)\,\mathrm{d}x \leq c$ and $\tilde E_\lambda(z_0) \leq \tilde{\mathcal{E}}(z_0)$. Then, the Gronwall Lemma yields

$$\tilde E_\lambda(z(t)) \leq \tilde{\mathcal{E}}(z_0)e^{-\tilde\omega t} + c(Q+1), \quad t \geq 0.$$

Now, appealing again to (16) and possibly reducing $\overline{\lambda}$, we have

$$\tilde E_\lambda(z) \geq \frac{1}{2}\left(\|\nabla\varphi\|^2 + \|\varphi\|^2 + \|\vartheta\|^2 + \|\nabla\eta\|^2_{\mathcal{M}}\right) - c,$$

whence

$$\|\varphi(t)\|_V^2 + \|\vartheta(t)\|^2 + \|\nabla\eta^t\|^2_{\mathcal{M}} \leq 2\tilde{\mathcal{E}}(z_0)e^{-\tilde\omega t} + c(Q+1). \tag{26}$$

In order to complete the norm of the memory variable, we multiply $(17)_4$ by η in $\mathcal{M}$ yielding

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\eta^t\|^2_{\mathcal{M}} + \frac{\delta}{2}\|\eta^t\|^2_{\mathcal{M}} \leq c\|\vartheta(t)\|^2.$$

On account of (26) we deduce that

$$\|\eta^t\|^2_{\mathcal{M}} \leq \|\eta_0\|^2_{\mathcal{M}}e^{-\delta t/2} + c\tilde{\mathcal{E}}(z_0)e^{-\tilde\omega t} + c(Q+1),$$

and, for a possibly smaller $\tilde\omega$, the required inequality for $\tilde{\mathcal{E}}_\lambda(z)$ and $\|z\|^2_{\tilde{\mathcal{H}}}$ follows. In light of this, a further integration of (24) over $[t, t+1]$ completes the proof. □

In what follows, we assume that $\lambda \in (0, \overline{\lambda}]$.

Lemma 5.2. *We have*

$$\int_t^{t+1}\left(\|\Delta\varphi(\tau)\|^4 + \|F_\lambda'(\varphi(\tau))\|^2_{L^1(\Omega)} + |\langle\mu(\tau)\rangle|^2\right)\mathrm{d}\tau \leq c\Big(\tilde{\mathcal{E}}(z_0)\mathrm{e}^{-\tilde\omega t} + Q\Big)^2, \tag{27}$$

for every $t \geq 0$.

Proof. The product of $(17)_2$ by $-\Delta\varphi$ gives

$$\langle \nabla\mu, \nabla\varphi\rangle = \|\Delta\varphi\|^2 + \langle \Psi''_\lambda(\varphi)\nabla\varphi, \nabla\varphi\rangle - \langle \nabla\vartheta, \nabla\varphi\rangle.$$

Since

$$\langle \Psi''_\lambda(\varphi)\nabla\varphi, \nabla\varphi\rangle \geq -\Theta_0\|\nabla\varphi\|^2,$$

we conclude that

$$\|\Delta\varphi\|^2 \leq c\|\nabla\varphi\|^2 + c\|\nabla\varphi\|(\|\nabla\mu\| + \|\nabla\vartheta\|). \tag{28}$$

Next, integrating $(17)_2$ over Ω, we see that, owing to (14) and the enthalpy's mean conservation,

$$\begin{aligned}\langle\mu\rangle &= \langle \Psi'_\lambda(\varphi)\rangle - \langle\vartheta\rangle = \langle F'_\lambda(\varphi)\rangle - \Theta_0\langle\varphi\rangle - \langle\vartheta_0 + \varphi_0\rangle + \langle\varphi\rangle \\ &= \langle F'_\lambda(\varphi)\rangle + (1-\Theta_0)\langle\varphi_0\rangle e^{-\alpha t} - \langle\vartheta_0 + \varphi_0\rangle.\end{aligned}$$

Although $\langle\varphi\rangle$ is not preserved by the evolution, it holds $|\langle\varphi(t)\rangle| = |\langle\varphi_0\rangle| e^{-\alpha t} \leq |\langle\varphi_0\rangle|$, whence we can still argue as in [8, 11], establishing the existence of $K > 0$ (depending on $|\langle\varphi_0\rangle|$ and diverging at $+\infty$ as $|\langle\varphi_0\rangle| \to 1$) such that

$$\|F'_\lambda(\varphi)\|_{L^1(\Omega)} \leq K(1 + |\langle F'_\lambda(\varphi), \bar\varphi\rangle|),$$

uniformly in λ. This yields

$$\langle F'_\lambda(\varphi), \bar\varphi\rangle \leq \langle\mu, \bar\varphi\rangle + \Theta_0\langle\varphi, \bar\varphi\rangle + \langle\vartheta, \bar\varphi\rangle \leq c\|\nabla\varphi\|^2 + c\|\nabla\varphi\|(\|\nabla\mu\| + \|\nabla\vartheta\|).$$

Thus

$$|\langle\mu\rangle| \leq c\|\nabla\varphi\|^2 + c\|\nabla\varphi\|(\|\nabla\mu\| + \|\nabla\vartheta\|) + c. \tag{29}$$

Collecting (28) and (29), the proof is done in light of Lemma 5.1. □

We now provide estimates for φ_t and ϑ_t.

Lemma 5.3. *We have*

$$\int_t^{t+1} \big(\|\varphi_t(\tau)\|^2_{V'} + \|\vartheta_t(\tau)\|^2_{V'}\big)\,\mathrm{d}\tau \leq c\Big(\tilde{\mathcal{E}}(z_0)\mathrm{e}^{-\tilde\omega t} + Q\Big). \tag{30}$$

Proof. Notice first that, by comparison in (20),

$$\|\bar\varphi_t\|_* \leq c(\|\nabla\mu\| + \|\bar\varphi\|_*).$$

Then, since on account of (13) we have $\langle\varphi_t\rangle = -\alpha\langle\varphi\rangle$, we infer that

$$\|\varphi_t\|_{V'} \leq c(\|\nabla\mu\| + \|\varphi\|_{V'}),$$

and the conclusion follows by (19). Analogously, we obtain the result for ϑ_t by comparison in the third equation. □

Finally we have

Lemma 5.4. *There exists $\nu > 0$ such that, for any $t \geq 0$,*

$$\|\vartheta(t)\|^2_V + \|\eta^t\|^2_{\mathcal{M}^2} + \int_t^{t+1} \|\Delta\vartheta(\tau)\|^2\,\mathrm{d}\tau \leq \mathcal{Q}(\|z_0\|_{\mathcal{H}})\mathrm{e}^{-\nu t} + Q. \tag{31}$$

Proof. Multiply equation $(17)_3$ written as

$$H_t - \Delta H - \int_0^\infty g(s)\Delta\eta^t(s)\,\mathrm{d}s = -\Delta\varphi$$

by $-\Delta H$ and then $(17)_4$ written as

$$\eta_t = T\eta + H - \varphi$$

by $\Delta^2\eta$ in $\mathcal{M}$, getting

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\nabla H\|^2 + \|\Delta\eta\|^2_{\mathcal{M}}\right) + \|\Delta H\|^2 + \frac{\delta}{2}\|\Delta\eta\|^2_{\mathcal{M}} \\ &\le \langle \Delta H, \Delta\varphi\rangle - \int_0^\infty g(s)\langle \Delta\eta(s), \Delta\varphi\rangle\,\mathrm{d}s \\ &\le \frac{1}{2}\|\Delta H\|^2 + \frac{\delta}{4}\|\Delta\eta\|^2_{\mathcal{M}} + c\|\Delta\varphi\|^2. \end{aligned} \tag{32}$$

Note that, in light of (28),

$$\begin{aligned} \int_t^{t+1}\|\Delta\varphi(\tau)\|^2 d\tau &\le c\int_t^{t+1}\left(\|\nabla\varphi(\tau)\|^2 + \|\nabla\mu(\tau)\|^2 + \|\nabla\vartheta(\tau)\|^2\right) d\tau \\ &\le c\left(\tilde{\mathcal{E}}(z_0)\mathrm{e}^{-\tilde{\omega}t} + Q\right). \end{aligned}$$

Hence, by the well-known inequality (see e.g. [14]).

$$\sup_{t\ge 0}\int_0^t \mathrm{e}^{-\omega(t-s)}h(s)\mathrm{d}s \le \frac{\mathrm{e}^{\omega}}{1-\mathrm{e}^{-\omega}}\sup_{t\ge 0}\int_t^{t+1} h(y)\mathrm{d}y, \tag{33}$$

valid for every nonnegative locally summable function h and every $\omega > 0$, we obtain from the Gronwall Lemma

$$\|\nabla H\|^2 + \|\Delta\eta\|^2_{\mathcal{M}} \le (\|H_0\|^2_V + \|\eta_0\|^2_{\mathcal{M}^2})\mathrm{e}^{-\delta t/4} + c\Big(\tilde{\mathcal{E}}(z_0)\mathrm{e}^{-\tilde{\omega}t} + Q\Big).$$

The desired conclusion for $\|\vartheta\|^2_V$ now follows recalling that $\vartheta = H - \varphi$ and (19). A final integration in time of (32) yields the integral control for $\|\Delta\vartheta\|^2$. □

Existence of a weak solution. Let $z_\lambda = (\varphi_\lambda, \vartheta_\lambda, \eta_\lambda)$, $\lambda \in (0, \overline{\lambda}]$, be a family of solutions to (P_λ) departing from the same initial datum $z_0 = (\varphi_0, \vartheta_0, \eta_0)$ as in (18). Collecting Lemmata 5.1-5.4 we know that

$$\|z_\lambda(t)\|^2_{\mathcal{H}} + \int_t^{t+1}\Big(\|\varphi_\lambda(\tau)\|^4_{\mathrm{H}^2} + \|\vartheta_\lambda(\tau)\|^2_{\mathrm{H}^2} + \|\mu_\lambda(\tau)\|^2_V + \|\varphi_{\lambda,t}(\tau)\|^2_{V'} + \|\vartheta_{\lambda,t}(\tau)\|^2_{V'}\Big)\,\mathrm{d}\tau \le C,$$

for every $t \ge 0$ and for some $C > 0$ independent of λ. Hence, we can pass to the limit $\lambda \to 0$, ending up with a triplet $z = (\varphi, \vartheta, \eta)$ such that (up to subsequences)

$$\begin{aligned} &\varphi_\lambda \to \varphi \text{ weakly star in } L^\infty(0,T;V), \\ &\varphi_\lambda \to \varphi \text{ weakly in } L^2(0,T;\mathrm{H}^2), \\ &\varphi_{\lambda,t} \to \varphi_t \text{ weakly in } L^2(0,T;V'). \end{aligned}$$

Note that by the classical Aubin-Lions Theorem, we also have

$$\varphi_\lambda \to \varphi \text{ strongly in } L^2(0,T;V)\cap\mathcal{C}([0,T],\mathrm{H}),$$

and the pointwise convergence

$$\varphi_\lambda(x,t) \to \varphi(x,t) \quad \text{a.e. } (x,t) \text{ in } \Omega\times(0,T).$$

Then, the uniform convergence of F'_λ to F' on any compact set in $(-1,1)$ yields

$$F'_\lambda(\varphi_\lambda) \to F'(\varphi) \quad \text{a.e. } (x,t)\in\Omega\times(0,T).$$

Note that the very same convergences hold true for $\vartheta_\lambda \to \vartheta$. Besides, we have

$$\mu_\lambda \to \mu \text{ weakly in } L^2(0,T;V)$$

and

$$\eta_\lambda \to \eta \text{ weakly star in } L^\infty(0,T;\mathcal{M}^2).$$

The next step is to show that the limit triplet z is a weak solution to (7). This is proved by passing to the limit in the weak formulation of (P_λ) in a standard way (see e.g. [14] for the memory component), the only nontrivial parts being that $|\varphi| < 1$ a.e. in $\Omega \times (0,T)$. But this can be done reasoning as in [8, 11] and we omit the details.

Finally, we observe that all the energy estimates proved in the first part of this section for z_λ pass to the limit $\lambda \to 0$, whence they hold true for any limit triplet z. Summing up we have proved the following.

Theorem 5.5. *Let z_0 be an initial datum as in (18). Then, there exists a global weak solution $z = (\varphi, \vartheta, \eta)$ to problem (7)-(9) such that*

$$z \in \mathcal{C}([0,\infty),\mathcal{H}).$$

Furthermore, there exists $\omega > 0$ such that, for every $t \geq 0$,

$$\mathcal{E}(z(t)) + \|z(t)\|^2_{\mathcal{H}} \leq \mathcal{Q}(\|z_0\|_{\mathcal{H}})\mathrm{e}^{-\omega t} + Q, \tag{34}$$

and

$$\int_t^{t+1} \Big(\|\varphi(\tau)\|^4_{\mathrm{H}^2} + \|\vartheta(\tau)\|^2_{\mathrm{H}^2} + \|\mu(\tau)\|^2_V + \|\varphi_t(\tau)\|^2_{V'} + \|\vartheta_t(\tau)\|^2_{V'} \Big)\,\mathrm{d}\tau \leq \mathcal{Q}(\|z_0\|_{\mathcal{H}})\mathrm{e}^{-\omega t} + Q. \tag{35}$$

6. **Uniqueness of the weak solution.** It is convenient to introduce the space

$$\mathcal{H}_* = V' \times V' \times \mathcal{M}.$$

Theorem 6.1. *For $i = 1,2$, let $z_{0i} = (\varphi_{0i}, \vartheta_{0i}, \eta_{0i}) \in \mathcal{H}$ be two initial data complying with (18). Then, calling $z_i = (\varphi_i, \vartheta_i, \eta_i)$ any two solutions to the Cauchy problem with initial data z_{0i}, we have*

$$\|z_1(t) - z_2(t)\|_{\mathcal{H}_*} \leq C_T\|z_1(0) - z_2(0)\|_{\mathcal{H}_*} + C_T|\langle\varphi_{01}\rangle - \langle\varphi_{02}\rangle|^{1/2},$$

for any $t \in [0,T]$, where $C_T > 0$ only depends on T, $\mathcal{E}(z_{0i})$, $|\langle\varphi_{0i}\rangle|$ and $\langle H_{0i}\rangle = \langle\vartheta_{0i}\rangle + \langle\varphi_{0i}\rangle$. In particular, the weak solution to problem (7)-(9) is unique.

Proof. We first write the system for the difference $z_1 - z_2 = (\varphi, \vartheta, \eta)$ as

$$\begin{cases} \varphi_t - \Delta\mu + \alpha\varphi = 0, \\ \mu = -\Delta\varphi + \Psi'(\varphi_1) - \Psi'(\varphi_2) - H + \varphi, \\ H_t - \Delta H - \int_0^\infty g(s)\Delta\eta(s)\,\mathrm{d}s = -\Delta\varphi, \\ \eta_t = T\eta + H - \varphi \end{cases} \tag{36}$$

(where $H = \vartheta + \varphi$), subject to the boundary conditions

$$\partial_{\mathbf{n}}\varphi = \partial_{\mathbf{n}}\mu = \partial_{\mathbf{n}}H = \int_0^\infty g(s)\partial_{\mathbf{n}}\eta(s) = 0 \quad \text{on } \Gamma \times [0,\infty),$$

and with the initial data

$$\varphi_0 = \varphi_{01} - \varphi_{02}, \quad H_0 = \vartheta_{01} - \vartheta_{02} + \varphi_0, \quad \eta_0 = \eta_{01} - \eta_{02} \quad \text{in } \Omega.$$

Now, taking the product by $\mathcal{N}\bar{\varphi}$ of the first equation written as

$$\bar{\varphi}_t - \Delta\mu + \alpha\bar{\varphi} = 0,$$

we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\bar\varphi\|_*^2 + \alpha\|\bar\varphi\|_*^2 + \langle \mu, \bar\varphi\rangle = 0,$$

where the third term reads

$$\begin{aligned}\langle \mu, \bar\varphi\rangle =& \|\nabla\varphi\|^2 + \langle \Psi'(\varphi_1) \\ & - \Psi'(\varphi_2), \varphi\rangle - \langle H, \bar\varphi\rangle + \|\bar\varphi\|^2 - |\Omega|\langle \Psi'(\varphi_1) - \Psi'(\varphi_2)\rangle\langle\varphi\rangle,\end{aligned}$$

as it is readily seen multiplying $(36)_2$ by $\bar\varphi$. Thus

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\bar\varphi\|_*^2 + \alpha\|\bar\varphi\|_*^2 + \|\bar\varphi\|_V^2 - (\Theta_0 - \Theta)\|\varphi\|^2 \le \langle H, \bar\varphi\rangle + |\Omega|\langle \Psi'(\varphi_1) - \Psi'(\varphi_2)\rangle\langle\varphi\rangle. \quad (37)$$

The subsequent products of $(36)_3$ with $\mathcal{N}\bar H$ and $(36)_4$ with η in $\mathcal{M}$ furnish

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\bar H\|_*^2 + \int_0^\infty g(s)\|\eta(s)\|^2\,\mathrm{d}s\right) + \|\bar H\|^2 - \frac{1}{2}\int_0^\infty g'(s)\|\eta(s)\|^2\,\mathrm{d}s \quad (38)\\ &= \langle \bar\varphi, \bar H\rangle + \langle H_0\rangle \int_0^\infty g(s)\int_\Omega \eta(s)\,\mathrm{d}x\,\mathrm{d}s - \int_0^\infty g(s)\langle \varphi, \eta(s)\rangle\,\mathrm{d}s.\end{aligned}$$

Adding together (37) with (38) multiplied by a suitable constant β, we deduce that

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\bar\varphi\|_*^2 + \beta\|\bar H\|_*^2 + \beta\int_0^\infty g(s)\|\eta(s)\|^2\,\mathrm{d}s\right)\\ &+ \alpha\|\bar\varphi\|_*^2 + \|\bar\varphi\|_V^2 + \beta\|\bar H\|^2 - \frac{\beta}{2}\int_0^\infty g'(s)\|\eta(s)\|^2\,\mathrm{d}s\\ &\le (1+\beta)\langle \bar\varphi, \bar H\rangle + \beta\langle H_0\rangle\int_0^\infty g(s)\int_\Omega \eta(s)\,\mathrm{d}x\,\mathrm{d}s - \beta\int_0^\infty g(s)\langle \eta(s), \varphi\rangle\,\mathrm{d}s\\ &\quad + (\Theta_0 - \Theta)\|\varphi\|^2 + |\Omega|\langle \Psi'(\varphi_1) - \Psi'(\varphi_2)\rangle\langle\varphi\rangle.\end{aligned}$$

Here

$$(1+\beta)\langle \bar\varphi, \bar H\rangle \le \frac{1}{2}\|\bar\varphi\|_V^2 + c\|\bar H\|_*^2.$$

Also, recalling that

$$\|\varphi\|^2 = \|\bar\varphi\|^2 + |\Omega|\langle\varphi\rangle^2 \le \|\bar\varphi\|_*\|\bar\varphi\|_V + |\Omega|\langle\varphi\rangle^2,$$

we have

$$\begin{aligned}&-\beta\int_0^\infty g(s)\langle \eta(s), \varphi\rangle\,\mathrm{d}s + (\Theta_0 - \Theta)\|\varphi\|^2\\ &\le \delta\frac{\beta}{4}\|\eta\|_{\mathcal{M}}^2 + (\beta c + \Theta_0 - \Theta)\|\varphi\|^2\\ &\le \delta\frac{\beta}{4}\|\eta\|_{\mathcal{M}}^2 + \frac{1}{2}\|\bar\varphi\|_V^2 + c\|\bar\varphi\|_*^2 + c\langle\varphi\rangle^2.\end{aligned}$$

The last memory term can be dealt with as follows:

$$\beta\langle H_0\rangle\int_0^\infty g(s)\int_\Omega \eta(s)\,\mathrm{d}x\,\mathrm{d}s \le c\beta\langle H_0\rangle^2 + \delta\frac{\beta}{4}\|\eta\|_{\mathcal{M}}^2.$$

Finally, having introduced $\Upsilon(t) = c(\|\Psi'(\varphi_1(t))\|_{L^1(\Omega)} + \|\Psi'(\varphi_2(t))\|_{L^1(\Omega)})$, we have

$$|\Omega|\langle \Psi'(\varphi_1) - \Psi'(\varphi_2)\rangle\langle\varphi\rangle \le \Upsilon|\langle\varphi\rangle|.$$

Collecting the above computations, we see that, properly choosing β, the functional

$$\mathcal{L}_*(t) = \|\bar\varphi(t)\|_*^2 + \beta\|\bar H(t)\|_*^2 + \beta\|\eta^t\|_{\mathcal{M}}^2$$

satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{L}_* \leq c\mathcal{L}_* + \Upsilon|\langle\varphi\rangle| + c|\langle\varphi\rangle|^2.$$

On account of

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\langle\varphi\rangle^2 + \alpha\langle\varphi\rangle^2 = 0,$$

recalling that $\langle H_t\rangle = 0$ and $\langle\varphi(t)\rangle = \mathrm{e}^{-\alpha t}\langle\varphi_0\rangle$, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{L} \leq c\mathcal{L} + (\Upsilon + c|\langle\varphi_0\rangle|)|\langle\varphi_0\rangle|, \quad \text{where} \quad \mathcal{L}(t) = \|\varphi(t)\|^2_{V'} + \beta\|H(t)\|^2_{V'} + \beta\|\eta^t\|^2_{\mathcal{M}}.$$

Therefore, owing to (27), the Gronwall Lemma applied on $[0,T]$ yields

$$\|(\varphi(t), H(t), \eta^t)\|^2_{\mathcal{H}_*} \leq C_T\|(\varphi_0, H_0, \eta_0)\|^2_{\mathcal{H}_*} + C_T|\langle\varphi_0\rangle|, \quad t \in [0,T].$$

The thesis is completed by recalling that $\vartheta = H - \varphi$. □

7. Partial regularization in finite time.

In this section we show that the system exhibits a partial regularization effect in finite time. In particular, we prove that the order parameter is bounded in H^2 as soon as $t > 0$. To this aim, let $z(t) = (\varphi(t), \vartheta(t), \eta^t)$, $t \geq 0$, be the solution departing from a given $z_0 = (\varphi_0, \vartheta_0, \eta_0)$ as in (18). In light of the estimates of Section 5, we have

$$\|\varphi(t)\|^2_V + \|\vartheta(t)\|^2_V + \|\eta^t\|^2_{\mathcal{M}^2} + \int_t^{t+1} \big(\|\mu(\tau)\|^2_V + \|\varphi_t(\tau)\|^2_{V'} + \|\bar{\vartheta}_t(\tau)\|^2_{V'}\big)\,\mathrm{d}\tau \leq C \tag{39}$$

for any $t \geq 0$, where, here and in what follows, the generic constant $C > 0$ may depend on $\mathcal{E}(z_0)$, $|\langle\varphi_0\rangle|$ and $\langle H_0\rangle = \langle\varphi_0\rangle + \langle\vartheta_0\rangle$.

We start with the following lemma.

Lemma 7.1. *For every $\sigma > 0$, there exists $C = C(\sigma)$ such that*

$$\sup_{t\geq\sigma}\|\mu(t)\|_V \leq C,$$

and, for every $t \geq \sigma$,

$$\int_t^{t+1}\|\varphi_t(\tau)\|^2_V\,\mathrm{d}\tau \leq C. \tag{40}$$

Proof. Multiplying $(7)_1$ by μ_t we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\|\nabla\mu\|^2 + \alpha\langle\varphi,\mu\rangle\right) + \langle\varphi_t,\mu_t\rangle = \alpha\langle\varphi_t,\mu\rangle.$$

Differentiating $(7)_2$ in time and multiplying the resulting equation by φ_t yield

$$\begin{aligned}\langle\varphi_t,\mu_t\rangle &= \|\nabla\varphi_t\|^2 + \langle\Psi''(\varphi)\varphi_t,\varphi_t\rangle - \langle\vartheta_t,\varphi_t\rangle\\ &\geq \|\nabla\varphi_t\|^2 - \Theta_0\|\varphi_t\|^2 - \langle\vartheta_t,\varphi_t\rangle\\ &= \|\nabla\varphi_t\|^2 - \Theta_0\|\bar{\varphi}_t\|^2 - \Theta_0|\Omega|\langle\varphi\rangle_t^2 - \langle\bar{\vartheta}_t,\bar{\varphi}_t\rangle + |\Omega|\langle\varphi\rangle_t^2\\ &\geq \frac{1}{2}\|\nabla\varphi_t\|^2 - C\|\bar{\varphi}_t\|^2_{V'} - \Theta_0|\Omega|\langle\varphi\rangle_t^2 - \langle\bar{\vartheta}_t,\bar{\varphi}_t\rangle,\end{aligned}$$

having observed that

$$\langle\vartheta_t,\varphi_t\rangle = \langle\bar{\vartheta}_t,\bar{\varphi}_t\rangle - |\Omega|\langle\varphi\rangle_t^2.$$

Therefore,

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\|\nabla\mu\|^2 + \alpha\langle\varphi,\mu\rangle\right) + \frac{1}{2}\|\nabla\varphi_t\|^2\\ &\leq C\|\bar{\varphi}_t\|^2_{V'} + \Theta_0|\Omega|\langle\varphi\rangle_t^2 + \alpha\langle\varphi_t,\mu\rangle + \langle\bar{\vartheta}_t,\bar{\varphi}_t\rangle,\end{aligned} \tag{41}$$

where $|\langle\varphi\rangle_t^2| \le C$ on account of (13) and (14). We estimate the right-hand side as follows:

$$\langle \bar\vartheta_t, \bar\varphi_t\rangle \le \|\bar\vartheta_t\|_{V'}\|\|\bar\varphi_t\|_V \le C\|\bar\vartheta_t\|^2_{V'} + \frac{1}{4}\|\nabla\varphi_t\|^2,$$

and

$$\alpha\langle\varphi_t,\mu\rangle \le C\|\varphi_t\|^2_{V'} + C\|\mu\|^2_V.$$

Exploiting the convexity of F, recalling that $\|\varphi\| + \|\vartheta\| \le C$ and $F \ge 0$, we obtain

$$\begin{aligned}\langle\mu,\varphi\rangle &= \|\nabla\varphi\|^2 + \langle F'(\varphi),\varphi\rangle - \Theta_0\|\varphi\|^2 - \langle\vartheta,\varphi\rangle \\ &\ge \int_\Omega F(\varphi)\,\mathrm{d}x - \Theta_0\|\varphi\|^2 - \|\vartheta\|\|\varphi\| \\ &\ge \int_\Omega F(\varphi)\,\mathrm{d}x - C \ge -C.\end{aligned}$$

We thus end up with

$$\frac{\mathrm{d}}{\mathrm{d}t}\Lambda + \frac{1}{4}\|\nabla\varphi_t\|^2 \le h(t), \tag{42}$$

where

$$\Lambda(t) = \frac{1}{2}\|\nabla\mu(t)\|^2 + \alpha\langle\varphi(t),\mu(t)\rangle + \tilde C$$

and

$$h(t) = C(1 + \|\mu(t)\|^2_V + \|\varphi_t(t)\|^2_{V'} + \|\bar\vartheta_t(t)\|^2_{V'}),$$

for any $\tilde C \ge 0$. Note that, provided that $\tilde C$ is properly chosen (depending on $\mathcal{E}(z_0)$, $\langle\varphi_0\rangle$ and $\langle H_0\rangle$), we have $\Lambda \ge \frac{1}{2}\|\nabla\mu\|^2$ and

$$\int_t^{t+1} \big(\Lambda(\tau) + h(\tau)\big)\mathrm{d}\tau \le C,$$

in light of (39) and the control

$$\Lambda \le \|\nabla\mu\|^2 + \alpha\|\varphi\|\|\mu\| + C \le C(\|\mu\|^2_V + 1).$$

As a consequence, the Uniform Gronwall Lemma allows to prove the desired bound for $\|\nabla\mu(t)\|$ for $t \ge \sigma$. A subsequent integration of (42) in time over $[t, t+1]$ yields

$$\int_t^{t+1} \|\nabla\varphi_t(\tau)\|^2\,\mathrm{d}\tau \le C, \qquad t \ge \sigma. \tag{43}$$

On account of (13) the control holds true for the whole norm of φ_t in V. □

Corollary 1. *For every $\sigma > 0$, there exists $C = C(\sigma) > 0$ such that*

$$\|\varphi\|_{L^\infty(\sigma,\infty;\mathrm{H}^2)} + \|F'(\varphi)\|_{L^\infty(\sigma,\infty;\mathrm{H})} \le C.$$

Proof. We multiply $(7)_2$ written as

$$-\Delta\varphi + F'(\varphi) = \mu^\star := \mu + \vartheta + \Theta_0\varphi \tag{44}$$

by $F'(\varphi)$, getting

$$\langle F''(\varphi)\nabla\varphi, \nabla\varphi\rangle + \|F'(\varphi)\|^2 = \langle\mu^\star, F'(\varphi)\rangle.$$

Thanks to (10) and the Young inequality, we have

$$\|F'(\varphi(t))\|^2 \le \|\mu^\star(t)\|^2 \le C, \quad t \ge \sigma,$$

on account of Lemma 7.1. Then the elliptic regularity applied to (44) allows to conclude. □

8. **The dissipative semigroup and its attractor.** Let $m \in [0,1)$ and $\ell \in \mathbb{R}$ be arbitrarily fixed. On account of Theorem 5.5 and Theorem 6.1, we can consider the family of solution operators defined via the rule

$$S(t)z_0 = z(t), \quad \forall\, t \ge 0,$$

where $z(t) = (\varphi(t), \vartheta(t), \eta^t)$ is the unique solution at time t to the Cauchy problem (7)-(9) with initial datum

$$z_0 \in \mathcal{H}_{m,\ell} = \{z = (\varphi, \vartheta, \eta) \in \mathcal{H} \,:\, F(\varphi) \in L^1(\Omega),\, |\langle\varphi\rangle| \le m,\, \langle H\rangle = \ell\}.$$

It turns out that

$$S(t) : \mathcal{H}_{m,\ell} \to \mathcal{H}_{m,\ell}$$

is a semigroup which is *closed* in light of the continuous dependence estimate in Theorem 6.1. Besides, $S(t)$ is *dissipative*. Indeed, in light of (34) it is apparent that the ball

$$B_0 = \{z \in \mathcal{H}_{m,\ell} : \|z\|_{\mathcal{H}} \le R_0\}$$

with R_0 sufficiently large (depending on m and ℓ) is an *absorbing set*, namely, for every bounded set $B \subset \mathcal{H}_{m,\ell}$ there exists an entering time $t_B > 0$ such that

$$S(t)B \subset B_0, \qquad \forall t \ge t_B.$$

Now consider the entering time of B_0 into itself, namely $t_0 = t_{B_0}$, and define

$$\mathcal{B}_0 = \bigcup_{t \ge t_0+1} S(t)B_0,$$

noticing that $\mathcal{B}_0$ is again an absorbing set and is invariant (namely $S(t)\mathcal{B}_0 \subset \mathcal{B}_0$ for every $t \ge 0$). Furthermore, according to Lemma 7.1 and Corollary 1, we have

$$\sup_{z \in \mathcal{B}_0} \sup_{t \ge 0} \left(\|\mu(t)\|_V + \|\varphi(t)\|_{\mathrm{H}^2} + \int_t^{t+1} \|\varphi_t(\tau)\|_V^2 \,\mathrm{d}\tau \right) \le C, \tag{45}$$

where, along the section, $C > 0$ is a generic constant depending on $\mathcal{B}_0$ but independent of the specific initial datum.

The main result of this section is the following.

Theorem 8.1. *There exists*

$$\mathcal{K} \subset \mathcal{B}_0 \cap \mathcal{W}$$

bounded in $\mathcal{W}$ which is a compact attracting set for $S(t)$, namely, for every bounded set $B \subset \mathcal{H}_{m,\ell}$,

$$\lim_{t \to 0} \mathrm{dist}_{\mathcal{H}}(S(t)B, \mathcal{K}) = 0.$$

Proof. Let $\mathcal{B}_0$ be the above absorbing set and let $z = (\varphi_0, \vartheta_0, \eta_0) \in \mathcal{B}_0$ be arbitrarily given. We decompose the solution departing from z as follows:

$$S(t)z = (\varphi(t), \vartheta(t), \eta^t) = (0, \theta_*(t), \psi^t) + (\varphi(t), \theta(t), \xi^t),$$

where

$$\begin{cases} (\theta_*)_t - \Delta\theta_* - \int_0^\infty g(s)\Delta\psi(s)\,\mathrm{d}s = 0, \\ \psi_t = T\psi + \theta_*, \\ \theta_*(0) = \overline{\vartheta}_0, \quad \psi^0 = \eta_0, \end{cases} \qquad \begin{cases} \theta_t - \Delta\theta - \int_0^\infty g(s)\Delta\xi(s)\,\mathrm{d}s = -\varphi_t, \\ \xi_t = T\xi + \theta, \\ \theta(0) = \langle\vartheta_0\rangle, \quad \xi^0 = 0. \end{cases}$$

We already know that

$$\|\varphi(t)\|_{\mathrm{H}^2} \le C.$$

Then, by standard computations, we readily get the exponential decay

$$\|\theta_*(t)\|_V + \|\psi^t\|_{\mathcal{M}^2} \leq C\mathrm{e}^{-ct},$$

for some $c > 0$. This in turn gives

$$\|\theta(t)\|_V + \|\xi^t\|_{\mathcal{M}^2} \leq C, \qquad \forall t \geq 0.$$

As a matter of fact, we have the higher-order estimate

$$\|\theta(t)\|_{\mathrm{H}^2} + \|\xi^t\|_{\mathcal{M}^3} \leq C, \qquad \forall t \geq 0. \tag{46}$$

Indeed, a multiplication of the equation for θ by $\Delta^2\theta$ and of the equation for ξ by $-\Delta^3\xi$ in $\mathcal{M}$ yields

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\big(\|\Delta\theta\|^2 + \|\nabla\Delta\xi\|^2_{\mathcal{M}}\big) + \|\nabla\Delta\theta\|^2 + \frac{\delta}{2}\|\nabla\Delta\xi\|^2_{\mathcal{M}} \leq \langle\nabla\varphi_t, \nabla\Delta\theta\rangle.$$

On account of the control

$$\langle\nabla\varphi_t, \nabla\Delta\theta\rangle \leq c\|\nabla\varphi_t\|^2 + \frac{1}{2}\|\nabla\Delta\theta\|^2,$$

we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\Lambda + \nu\Lambda \leq c\|\nabla\varphi_t\|^2 \quad \text{ with } \quad \Lambda(t) = \|\Delta\theta(t)\|^2 + \|\nabla\Delta\xi^t\|^2_{\mathcal{M}}.$$

In light of (45), the conclusion follows by the Gronwall Lemma together with (33).

Owing to (46) we can now exploit Lemma 2.1 to infer that

$$\|\xi^t\|_{\mathcal{K}^3} \leq C,$$

and we can conclude

$$\|(\varphi(t), \theta(t), \xi^t)\|_{\mathcal{W}} \leq C.$$

This proves that the set

$$\mathcal{K} = \{z \in \mathcal{B}_0 \cap \mathcal{W} \,:\, \|z\|_{\mathcal{W}} \leq \varrho\},$$

for any ϱ sufficiently large, (exponentially) attracts the absorbing ball $\mathcal{B}_0$, namely

$$\mathrm{dist}_{\mathcal{H}}(S(t)\mathcal{B}_0, \mathcal{K}) \leq C\mathrm{e}^{-ct}, \qquad \forall t \geq 0.$$

Besides, $\mathcal{K}$ is compact due to the compact embedding $\mathcal{W} \subset \mathcal{H}$. Finally, since $\mathcal{K}$ exponentially attracts the absorbing ball $\mathcal{B}_0$, it is a standard matter to verify that $\mathcal{K}$ attracts (exponentially) every bounded set $B \subset \mathcal{H}$. This concludes the proof. □

In light of Theorem 8.1, the existence of a unique (compact and connected) global attractor $\mathcal{A}_{m,\ell}$ for $S(t)$ on $\mathcal{H}_{m,\ell}$ follows from standard results (see e.g. [25] and [27]). Besides, since the attractor is contained in any closed attracting set, then

$$\mathcal{A}_{m,\ell} \subset \mathcal{B}_0 \quad \text{and} \quad \mathcal{A}_{m,\ell} \text{ is a bounded set of } \mathcal{H}^1.$$

9. **Strict separation property.** In this section we consider solutions $S(t)z_0 = (\varphi(t), \vartheta(t), \eta(t))$ departing from initial data z_0 on the global attractor, and we study the validity of the strict separation property of $\varphi(t)$ from the pure phases ± 1. To this aim, we assume that the singular potential F satisfies the following additional assumption:

$$F'' \text{ is convex and } F''(x) \leq \mathrm{e}^{K|F'(x)|+K}, \quad \forall\, x \in (-1,1), \tag{47}$$

for some $K > 0$. Note that this includes the logarithmic potential. Besides, we have to assume that the space dimension $N \leq 2$. Indeed, the proof is obtained by exploiting the novel technique recently developed in [8, 11, 15], which relies on the

Trudinger–Moser inequality in dimension two. Accordingly, throughout the section we let $N = 2$, $m \in [0, 1)$ and $\ell \in \mathbb{R}$ be fixed, and let

$$z_0 \in \mathcal{A}_{m,\ell}.$$

Then, our main result reads as follows.

Theorem 9.1. *Let (47) hold and let $\sigma > 0$. Then, there exists $\delta = \delta(\sigma) > 0$ such that*

$$\|\varphi(t)\|_{L^\infty(\Omega)} \le 1 - \delta, \quad \forall\, t \ge 2\sigma. \tag{48}$$

Here, δ depends on m, ℓ and the size of the attractor in $\mathcal{H}_{m,\ell}$, but is independent of the specific initial datum.

The proof is based on the lemma below, whose proof can be found in [8, 11, 15] (see also [23]), based on the Trudinger–Moser inequality in dimension two. It is worth noticing that here the control $\mu^\star = \mu + \vartheta + \Theta_0\varphi \in L^\infty(0, \infty; V)$ plays a crucial role.

Lemma 9.2. *Let (47) hold. For every $\sigma > 0$ and $p \ge 2$, there exists $C = C(\sigma, p)$ such that*

$$\|F''(\varphi)\|_{L^p(t,t+1;L^p(\Omega))} \le C, \quad \forall\, t \ge \sigma.$$

The subsequent step consists in proving the following lemma.

Lemma 9.3. *Let (47) hold. For every $\sigma > 0$ there exists $C = C(\sigma)$ such that*

$$\|\varphi_t\|_{L^\infty(t,\infty;\mathrm{H})} + \|\varphi_t\|_{L^2(t,t+1;\mathrm{H}^2)} \le C, \quad \forall\, t \ge 2\sigma.$$

Proof. Let us first notice that, on account of (45), we learn by comparison in the equation for ϑ that

$$\int_t^{t+1} \|\vartheta_t(\tau)\|^2\, \mathrm{d}\tau \le C, \qquad \forall t \ge 0. \tag{49}$$

Now, given $h > 0$ and any function u, we introduce the quotient

$$\partial_t^h u := \frac{1}{h}[u(t+h) - u(t)].$$

Then, setting $v = \partial_t^h \varphi$, we have

$$v_t - \Delta\partial_t^h \mu + \alpha v = 0.$$

Testing the above equation by v, we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|v\|^2 + \alpha\|v\|^2 = \langle \Delta\partial_t^h \mu, v\rangle. \tag{50}$$

Note that

$$\begin{aligned}\langle \Delta\partial_t^h \mu, v\rangle &= \langle \partial_t^h \mu, \Delta v\rangle \\ &= -\|\Delta v\|^2 + \Theta_0\|\nabla v\|^2 + \langle \frac{1}{h}[F'(\varphi(t+h)) - F'(\varphi(t))], \Delta v\rangle - \langle \partial_t^h \vartheta, \Delta v\rangle.\end{aligned}$$

Reasoning as in [8, Lemma 7.3] in order to estimate the term involving F' in the right-hand side, we get

$$\begin{aligned}&\Big|\langle \frac{1}{h}[F'(\varphi(t+h)) - F'(\varphi(t))], \Delta v\rangle\Big| \\ &\quad \le \frac{1}{8}\|\Delta v\|^2 + C\Big(\|F''(\varphi(t+h))\|^2_{L^3(\Omega)} + \|F''(\varphi(t))\|^2_{L^3(\Omega)}\Big)\|v\|^2_{L^6(\Omega)}.\end{aligned}$$

Hence, since,

$$\|v\|_{L^6(\Omega)}^2 \leq c\|\nabla v\|^2 + c|\langle v\rangle|^2 \leq c\|v\|\|\Delta v\| + c|\langle v\rangle|^2,$$

we end up with the differential inequality

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|v\|^2 + \frac{1}{4}\|\Delta v\|^2 + \alpha\|v\|^2 \leq \Upsilon\|v\|^2 + \Upsilon^{1/2}|\langle v\rangle|^2 + \|\partial_t^h \vartheta\|^2, \tag{51}$$

where

$$\Upsilon(t) = C\Big(1 + \|F''(\varphi(t+h))\|_{L^3(\Omega)}^4 + \|F''(\varphi(t))\|_{L^3(\Omega)}^4\Big).$$

Note that by Lemma 9.2

$$\int_t^{t+1} \Upsilon(\tau)\,\mathrm{d}\tau \leq C, \quad \forall\, t \geq \sigma.$$

Besides, recalling that

$$\|\partial_t^h u\|_{L^2(t,t+1;\mathrm{H})} \leq \|u_t\|_{L^2(t,t+1;\mathrm{H})}$$

for every u, owing to (40) and (49) we have

$$\int_t^{t+1} \Big(\|v(\tau)\|^2 + \|\partial_t^h \vartheta(\tau)\|^2\Big)\,\mathrm{d}\tau \leq C, \quad \forall\, t \geq \sigma.$$

An application of the Uniform Gronwall Lemma and a final passage to the limit $h \to 0$ complete the proof. □

Proof of Theorem 9.1. We preliminary show that

$$\|F'(\varphi)\|_{L^p(\Omega)} \leq |\Omega|\|\mu^\star\|_{L^\infty(\Omega)}, \quad \forall\, p \geq 2,$$

where, again, $\mu^\star = \mu + \vartheta + \Theta_0\varphi$. Indeed, reasoning as in Corollary 1, given $p > 2$, we test the elliptic equation (44) by $|F'(\varphi)|^{p-2}F'(\varphi)$, yielding

$$\langle -\Delta\varphi, |F'(\varphi)|^{p-2}F'(\varphi)\rangle + \langle F'(\varphi), |F'(\varphi)|^{p-2}F'(\varphi)\rangle = \langle \mu^\star, F'(\varphi), |F'(\varphi)|^{p-2}F'(\varphi)\rangle.$$

(Note that this multiplication is formal, but can be rigorously justified working with suitable cut-off functions as in [8].) Since F'' is positive, we have

$$\langle -\Delta\varphi, |F'(\varphi)|^{p-2}F'(\varphi)\rangle = (p-1)\langle |F'(\varphi)|^{p-2}F''(\varphi)\nabla\varphi \cdot \chi_{[-1+\frac{1}{k},1-\frac{1}{k}]}, \nabla\varphi\rangle \geq 0,$$

which in turn gives

$$\|F'(\varphi)\|_{L^p(\Omega)}^p = \langle F'(\varphi), |F'(\varphi)|^{p-2}F'(\varphi)\rangle \leq \langle \mu^\star, |F'(\varphi)|^{p-2}F'(\varphi)\rangle.$$

Since, by the Hölder inequality,

$$\langle \mu^\star, |F'(\varphi)|^{p-2}F'(\varphi)\rangle \leq \|F'(\varphi)\|_{L^p(\Omega)}^{p-1}\|\mu^\star\|_{L^p(\Omega)},$$

we obtain the desired inequality

$$\|F'(\varphi)\|_{L^p(\Omega)} \leq \|\mu^\star\|_{L^p(\Omega)} \leq |\Omega|\|\mu^\star\|_{L^\infty(\Omega)}.$$

Now, thanks to Lemma 9.3, arguing by comparison in the first equation of (7) we infer that

$$\mu \in L^\infty(2\sigma,\infty;\mathrm{H}^2) \subset L^\infty(2\sigma,\infty;L^\infty(\Omega)).$$

Since on the attractor $\varphi, \vartheta \in L^\infty(2\sigma,\infty;L^\infty(\Omega))$, we conclude that

$$\|\mu^\star\|_{L^\infty(2\sigma,\infty;L^\infty(\Omega))} \leq C.$$

This implies

$$\|F'(\varphi)\|_{L^\infty(\Omega\times(t,t+1))} \leq C, \quad \forall\, t \geq 2\sigma,$$

so that, since F' is singular at ± 1, we deduce that

$$\|\varphi\|_{L^\infty(\Omega\times(t,t+1))} \leq 1-\delta, \quad \forall\, t \geq 2\sigma,$$

for some $\delta > 0$. Since $\varphi \in L^\infty(0,T;L^\infty(\Omega))$, the thesis immediately follows. □

Final remarks. By a standard result in the theory of dynamical systems (see e.g. [19]), the global attractor of a semigroup $\mathcal{S}(t) : X \to X$ has the form

$$\mathcal{A} = \big\{ z(t_0) : z \text{ is a CBT of } \mathcal{S}(t) \big\}, \tag{52}$$

for any arbitrarily fixed $t_0 \in \mathbb{R}$, where a complete bounded trajectory (CBT) of $\mathcal{S}(t)$ is a function $z \in \mathcal{C}(\mathbb{R}, X)$, bounded on $\mathbb{R}$ and satisfying

$$z(\tau) = \mathcal{S}(t) z(\tau - t), \quad \forall t \geq 0,\ \forall \tau \in \mathbb{R}.$$

Accordingly, as a consequence of Theorem 9.1 we deduce that the order parameter for all CBT's to our system is uniformly away from the pure phases *for every time* $t \in \mathbb{R}$.

Corollary 2. *Let (47) hold. Then, there exists $\delta_* > 0$ depending only on m, ℓ and the size of the attractor in $\mathcal{H}_{m,\ell}$ such that, for any $\hat{z} = (\hat{\varphi}, \hat{\vartheta}, \hat{\eta})$ CBT of $S(t)$ on $\mathcal{H}_{m,\ell}$, we have*

$$\sup_{\tau\in\mathbb{R}} \|\hat{\varphi}(\tau)\|_{L^\infty(\Omega)} \leq 1 - \delta_*.$$

Proof. Invoking Theorem 9.1, let $\delta_* = \delta(1)$. Then, for any CBT $\hat{z}$ of $S(t)$ on $\mathcal{H}_{m,\ell}$, we know that

$$\sup_{\tau\geq 2} \|\hat{\varphi}(\tau)\|_{L^\infty(\Omega)} \leq 1 - \delta_*.$$

Indeed, for any $\tau > 0$, we have $\hat{z}(\tau) = S(\tau)\hat{z}(0)$, where $\hat{z}(0) \in \mathcal{A}_{m,\ell}$, owing to (52). We are left to prove that the same inequality holds for any $\tau < 2$. To this aim, let $\hat{z}$ be a fixed CBT and let us consider the solution $z(t) = (\varphi(t), \vartheta(t), \eta^t)$, $t \geq 0$, departing at time $t = 0$ from the initial datum $z(0) = \hat{z}(\tau - 2)$. Since $\hat{z}(\tau-2) \in \mathcal{A}_{m,\ell}$ by (52), in light of Theorem 9.1 again we have

$$\|\varphi(t)\|_{L^\infty(\Omega)} \leq 1 - \delta_*, \qquad \forall t \geq 2. \tag{53}$$

But, by the very definition of a CBT,

$$\hat{z}(\tau) = S(2)\hat{z}(\tau - 2) = z(2),$$

and, in particular, $\hat{\varphi}(\tau) = \varphi(2)$. This, combined with (53) for $t = 2$, yields the desired result. □

Acknowledgments. This work has been supported by the 2016 GNAMPA project "Regolarità e comportamento asintotico di soluzioni di equazioni paraboliche" financed by INdAM.

REFERENCES

[1] D. Brochet, Maximal attractor and inertial sets for some second and fourth order phase field models, in *Pitman Res. Notes Math. Ser.*, Longman Sci. Tech., Harlow, **296** (1993), 77–85.

[2] D. Brochet, D. Hilhorst and A. Novick-Cohen, Maximal attractor and inertial sets for a conserved phase field model, *Adv. Diff. Eqns.*, **1** (1996), 547–578.

[3] G. Caginalp, A conserved phase field system; implications for kinetic undercooling, *Phys. Rev. B*, **38** (1988), 789–791.

[4] G. Caginalp, The dynamics of a conserved phase-field system: Stefan-like, Hele-Shaw and Cahn-Hilliard models as asymptotic limits, *IMA J. Appl. Math.*, **44** (1990), 77–94.

[5] J. W. Cahn, On spinodal decomposition, *Acta Metall.*, **9** (1961), 795–801.

[6] J. W. Cahn and J. E. Hilliard, Free energy of a nonuniform system I. Interfacial free energy, *J. Chem. Phys.*, **28** (1958), 258–267.
[7] B. D. Coleman and M. E. Gurtin, Equipresence and constitutive equations for rigid heat conductors, *Z. Angew. Math. Phys.*, **18** (1967), 199–208.
[8] M. Conti and A. Giorgini, The three-dimensional Cahn-Hilliard-Brinkman system with unmatched viscosities, preprint.
[9] M. Conti, V. Pata and M. Squassina, Singular limit of differential systems with memory, *Indiana Univ. Math. J.*, **55** (2006), 169–215.
[10] C. M. Dafermos, Asymptotic stability in viscoelasticity, *Arch. Rational Mech. Anal.*, **37** (1970), 297–308.
[11] C. G. Gal, A. Giorgini and M. Grasselli, The nonlocal Cahn-Hilliard equation with singular potential: Well-posedness, regularity and strict separation property, *J. Differential Equations,* **263** (2017), 5253–5297.
[12] S. Gatti, M. Grasselli and V. Pata, Exponential Attractors for a conserved phase-field system with memory, *Phys. D*, **189** (2004), 31–48.
[13] S. Gatti, A. Miranville, V. Pata and S. Zelik, Continuous families of exponential attractors for singularly perturbed equations with memory, *Proc. Roy. Soc. Edinburgh Sect. A*, **140** (2010), 329–366.
[14] C. Giorgi, M. Grasselli and V. Pata, Uniform attractors for a phase-field model with memory and quadratic nonlinearity, *Indiana Univ. Math. J.*, **48** (1999), 1395–1445.
[15] A. Giorgini, M. Grasselli and A. Miranville, The Cahn-Hilliard-Oono equation with singular potential, *Math. Models Methods Appl. Sci.*, **27** (2017), 2485–2510.
[16] M. Grasselli and V. Pata, Attractors of phase-field systems with memory, in *Mathematical methods and models in phase transitions*, (eds. A. Miranville), Nova Science Publishers, 2005, 157–175.
[17] M. Grasselli, V. Pata and F. M. Vegni, Longterm dynamics of a conserved phase-field model with memory, *Asymptot. Anal.*, **33** (2003), 261–320.
[18] M. E. Gurtin and A. C. Pipkin, A general theory of heat conduction with finite wave speeds, *Arch. Rational Mech. Anal.*, **31** (1968), 113–126.
[19] A. Haraux, *Systèmes Dynamiques Dissipatifs et Applications*, Recherches en Mathématiques Appliquées, Vol. **17**, Masson, Paris, 1991.
[20] A. Miranville, Asymptotic behavior of the Cahn-Hilliard-Oono equation, *J. Appl. Anal. Comput.*, **1** (2011), 523–536.
[21] A. Miranville, On the conserved phase-field system, *J. Math. Anal. Appl.*, **400** (2013), 143–152.
[22] A. Miranville, The Cahn-Hilliard equation and some of its variants, *AIMS Math.*, **2** (2017), 479–544.
[23] A. Miranville and S. Zelik, Robust exponential attractors for Cahn-Hilliard type equations with singular potentials, *Math. Methods Appl. Sci.*, **27** (2004), 545–582.
[24] Y. Oono and S. Puri, Computationally efficient modeling of ordering of quenched phases, *Phys. Rev. Lett.*, **58** (1987), 836–839.
[25] V. Pata and S. Zelik, A result on the existence of global attractors for semigroups of closed operators, *Commun. Pure Appl. Anal.*, **6** (2007), 481–486.
[26] V. Pata and A. Zucchi, Attractors for a damped hyperbolic equation with linear memory, *Adv. Math. Sci. Appl.*, **11** (2001), 505–529.
[27] R. Temam, *Infinite-dimensional Dynamical Systems in Mechanics and Physics*, 2^{nd} edition, Applied Mathematical Sciences, Vol. **68**, Springer-Verlag, New York, 1997.
[28] S. Villain-Guillot, *Phases Modulées et Dynamique de Cahn-Hilliard*, Habilitation thesis, Université Bordeaux I, 2010.

Received for publication September 2017.

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