SOME SIEGEL MODULAR STANDARD L-VALUES, AND SHAFAREVICH-TATE GROUPS

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ABSTRACT. We explain how the Bloch-Kato conjecture leads us to the following conclusion: a large prime dividing a critical value of the L-function of a classical Hecke eigenform f of level 1, should often also divide certain ratios of critical values for the standard L-function of a related genus two (and in general vector-valued) Hecke eigenform F. The relation between f and F (Harder's conjecture in the vector-valued case) is a congruence involving Hecke eigenvalues, modulo the large prime. In the scalar-valued case we prove the divisibility, subject to weak conditions. In two instances in the vector-valued case, we confirm the divisibility using elaborate computations involving special differential operators. These computations do not depend for their validity on any unproved conjecture.

1. INTRODUCTION

The Bloch-Kato conjecture [BK, Fo2] gives a conjectural formula for the leading term (up to units) of any motivic L-function at any integer point. When combined with other conjectures on orders of vanishing, it may be viewed as a great generalisation of Dirichlet's class number formula (about the Dedekind zeta function of a number field at $s = 0$) and the Birch and Swinnerton-Dyer conjecture (about the L-function of an elliptic curve at $s = 1$). In this paper, we shall be concerned only with critical values, the subject of [De]. For such values, Deligne's conjecture gives an interpretation of the L-value as an algebraic multiple of a certain period (which is in fact only defined up to an algebraic multiple). The Bloch-Kato conjecture is an integral refinement, giving a conjectural factorisation of the ratio of the L-value to the period, once choices have been made to fix the period.

The L-function $L(f,s)$ of a cuspidal Hecke eigenform $f = q + \sum_{n=2}^{\infty} a_n(f)q^n$ of weight k' for $SL_2(\mathbb{Z})$, is an example of an L-function to which the Bloch-Kato conjecture should apply, the critical values being at $s = 1, \dots, k'-1$. Choosing canonical periods to divide by, one obtains normalised L-values $L_{\mathrm{alg}}(f,t)$ for integers $1 \le t \le k'-1$. According to the Bloch-Kato conjecture, a sufficiently large prime λ dividing $L_{\mathrm{alg}}(f,t)$ should be the order of an element in some generalised Shafarevich-Tate group. This element will live in a group defined using the Galois cohomology of the t^{th} Tate twist of the λ-adic representation $\rho_{f,\lambda}$ of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, attached to f.

In the case that $k' = 2k-2$ with k even, and $t = k$ (or equivalently $t = k'-1-k = k-2$), Brown [Br] has shown how to construct such an element using Siegel modular forms of genus 2 and weight k for $\mathrm{Sp}(2,\mathbb{Z})$. There is such a form $\widehat{f}$, the Saito-Kurokawa lift of f. Its spinor L-function is $L(f,s)\zeta(s-(k-1))\zeta(s-(k-2))$, while its standard zeta function is $\zeta(s)L(f,s+k-1)L(f,s+k-2)$. It is a cuspidal

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Hecke eigenform, and the Hecke eigenvalue for $T(p)$ is given by $\lambda_{\hat{f}}(p) = a_p(f) + p^{k-1} + p^{k-2}$. Under certain conditions, it is possible to show [Br, Ka1] that there exists another cuspidal Hecke eigenform F of genus 2 and weight k for $Sp(2,\mathbb{Z})$, but which is not a Saito-Kurokawa lift, such that the Hecke eigenvalues of F and $\hat{f}$ are congruent modulo λ. To this F may be attached a 4-dimensional λ-adic Galois representation $\rho_{F,\lambda}$, by a theorem of Weissauer [We1]. Interpreting Hecke eigenvalues as eigenvalues of Frobenius, it follows from the congruence that if we reduce modulo λ then the composition factors of the reduced representation $\overline{\rho}_{F,\lambda}$ are $\overline{\rho}_{f,\lambda}$ (if we ensure it is irreducible) together with the twists $\mathbb{F}_\lambda(1-k)$ and $\mathbb{F}_\lambda(2-k)$ of the trivial representation. The required Galois cohomology class may be constructed using a non-trivial extension of $\mathbb{F}_\lambda(2-k)$ by $\overline{\rho}_{f,\lambda}$ inside $\overline{\rho}_{F,\lambda}$. This generalises Ribet's construction of elements in class groups of cyclotomic fields [R], which uses the Galois interpretation of congruences between classical Eisenstein series and cusp forms.

In this paper we exploit Brown's construction, together with an injection of $\overline{\rho}_{f,\lambda}(2-k)$ (i.e. $\overline{\rho}_{f,\lambda} \wedge \mathbb{F}_\lambda(2-k)$) into $\wedge^2\overline{\rho}_{F,\lambda}$, to construct a non-zero element of order λ in a Selmer group defined in terms of the Galois cohomology of an appropriate twist of $\wedge^2\rho_{F,\lambda}$. Although the standard L-function of F is not actually known to arise from a premotivic structure, it ought to, so assuming that it does we can see what consequence our construction should have, given that the L-function attached to the Galois representation $\wedge^2\rho_{F,\lambda}$ is $\zeta(s-(j+2k-3))L(F,s-(j+2k-3),St)$. The prediction we arrive at (the case $j=0$ of Conjecture 5.3) is that (under certain conditions) the ratio of $L(F,2,St)$ to (a power of π times) any other critical value, has a factor of λ in the numerator. (The trick of looking at a ratio of critical values has the effect of making unknown Deligne periods in the Bloch-Kato conjecture cancel out.) If we replace F by $\hat{f}$, the factor of λ arises because $L(f,k)$ is a factor of $L(\hat{f},2,St)$ (using $L(\hat{f},s,St) = \zeta(s)L(f,s+k-1)L(f,s+k-2)$). In §6, we show how this divisibility can be somehow transmitted across the congruence between $\hat{f}$ and F.

Brown's construction can be applied to other critical values $L(f,t)$ (not just $t=k$) if one accepts a conjecture of Harder [Ha, vdG]. In general, we write the weight of f as $k' = j+2k-2$, and look at large λ dividing $L_{alg}(f,j+k)$. So far we have only considered the case $j=0$. This time we must look at Siegel modular forms for $Sp(2,\mathbb{Z})$, of type $\det^k \otimes \mathrm{Sym}^j(\mathbb{C}^2)$, which are vector valued when $j>0$. Once $j>0$ there is no Saito-Kurokawa lift, but Harder's conjecture cuts out this intermediary, and asserts nonetheless the existence of a cuspidal eigenform F such that, for all primes p,

$$\lambda_F(p) \equiv a_p(f) + p^{j+k-1} + p^{k-2} \pmod{\lambda}.$$

Using $\rho_{F,\lambda}$ as before, we are led to Conjecture 5.3, on the ratio of $L(F,j+2,St)$ to other critical values. In particular, in the case $k=10, j=4$, for which the space of cusp forms is 1-dimensional, we predict that $\mathrm{ord}_{41}\left(\frac{\pi^6 L(F,6,St)}{L(F,8,St)}\right) > 0$.

In the case that $k'/2$ is odd, $L(f,k'/2)$ vanishes, and if f is ordinary at λ then using either a theorem of Skinner and Urban [SU] or a theorem of Nekovář [N], we get an element of order λ in a Selmer group associated to $\rho_{f,\lambda}(k'/2)$, which as before may be moved, using the supposed congruence, to a Selmer group for a twist of $\wedge^2\rho_{F,\lambda}$. We are then led to a conjecture (5.4) on the ratio of $L(F,(j/2)+1,St)$ to

other critical values. In particular, in the case $k = 11, j = 10$, for which the space of cusp forms is again 1-dimensional, we predict that $\mathrm{ord}_{97}\left(\frac{\pi^6 L(F,6,St)}{L(F,8,St)}\right) > 0$.

In these cases where $j > 0$, there is no Saito-Kurokawa lift with which to prove our predictions (assuming Harder's conjecture), but we may, without the need to assume any conjecture, seek to confirm our predictions numerically by calculating the standard L-values in question. Kozima [Koz] gave a formula for the pullback, to $\mathfrak{H}_2 \times \mathfrak{H}_2$, of a genus 4 Siegel-Eisenstein series, to which a certain composition of differential operators (due to Böcherer) had been applied to produce a vector-valued form. Choosing the operators appropriately, a desired critical value of the standard L-function of F appears in the coefficient of $F \otimes F$ on the right hand side. The case of the rightmost critical value was already in [BSY]. In principle, using knowledge of the Fourier coefficients of the Siegel-Eisenstein series [Ka2], one could hope to use this pullback formula to calculate the critical values we require. However, the differential operators are not easy to work with, so we replace them by certain differential operators introduced in [I1], which are known to be necessarily the same up to a multiplicative constant (which may be determined by applying both to a test function). Computing these operators is possible (just) in the cases at hand, and involves finding certain invariant pluriharmonic polynomials, one of which takes two pages just to write down.

Section 2 introduces the Bloch-Kato conjecture in the case of critical values of $L(f, s)$. In Section 3 we state Katsurada's version of the theorem on congruences of Hecke eigenvalues between Saito-Kurokawa lifts and non-lifts, and also Harder's conjecture on the analogous congruence in the vector-valued case. Section 4 gives a summary of Brown's construction of elements in Selmer groups. In Section 5 we exploit this as outlined above to make our conjectures about ratios of standard L-values. Section 6 contains the proof of the scalar valued case, while Sections 7 and 8 report on the two big computations confirming our specific predictions involving $\ell = 41$ and $\ell = 97$.

1.1. **Definitions and notation.** Let $\mathfrak{H}_r$ be the Siegel upper half plane of r by r complex symmetric matrices with positive-definite imaginary part. Let $\Gamma_r := \mathrm{Sp}(r, \mathbb{Z}) = \mathrm{Sp}_{2r}(\mathbb{Z}) = \{M \in \mathrm{GL}_{2r}(\mathbb{Z}) : {}^tMJM = J\}$, where $J = \begin{pmatrix} 0_r & I_r \\ -I_r & 0_r \end{pmatrix}$. For $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \Gamma_r$ and $Z \in \mathfrak{H}_r$, let $M(Z) := (AZ + B)(CZ + D)^{-1}$ and $J(M, Z) := CZ + D$. Let V be the space of a finite-dimensional representation ρ of $\mathrm{GL}(r, \mathbb{C})$. A holomorphic function $f : \mathfrak{H}_r \to V$ is said to belong to the space $M_\rho = M_\rho(\Gamma_r)$ of Siegel modular forms of genus r and weight ρ if

$$f(M(Z)) = \rho(J(M, Z))f(Z) \quad \forall M \in \Gamma_r,\ Z \in \mathfrak{H}_r.$$

Such an f has a Fourier expansion

$$f(Z) = \sum_{S \geq 0} a(S)\mathbf{e}(\mathrm{Tr}(SZ)) = \sum_{S \geq 0} a(S, f)\mathbf{e}(\mathrm{Tr}(SZ)),$$

where the sum is over all positive semi-definite half-integral matrices, and $\mathbf{e}(z) := e^{2\pi i z}$.

The Siegel operator Φ on $M_\rho(\Gamma_r)$ is defined by

$$\Phi f(z) = \lim_{t \to \infty} f\left(\begin{bmatrix} z & 0 \\ 0 & it \end{bmatrix}\right) \quad \text{for } z \in \mathfrak{H}_{r-1}, t \in \mathbb{R}.$$

The kernel of Φ, denoted S_ρ, is the space of Siegel cusp forms of genus r and weight ρ. When ρ is of the special form $\det^k \otimes \mathrm{Sym}^j(\mathbb{C}^r)$ (where $\mathbb{C}^r$ is the standard representation of $GL_r(\mathbb{C})$), we put $M_{k,j}$ and $S_{k,j}$ for M_ρ and S_ρ, and we let $M_k := M_{k,0}$, $S_k := S_{k,0}$. For $S_{k,j}$, the Petersson inner product and Hecke operators will be as in §2 of [Koz] and §2 of [Ar], respectively. For a Hecke eigenform F, the spinor and standard L-functions $L(F, s, \mathrm{spin})$ and $L(F, s, \mathrm{St})$ may be defined in terms of Satake parameters as in §20 of [vdG].

2. The Bloch Kato conjecture for critical values of modular L-functions

Let $f = \sum_{n=1}^{\infty} a_n(f) q^n \in S_{k'}(\Gamma_1)$ be a normalised Hecke eigenform. Attached to f is its L-function $L(f, s)$, defined by the Dirichlet series $\sum_{n=1}^{\infty} a_n(f) n^{-s}$ for $\Re(s) > \frac{k'+1}{2}$, but having an analytic continuation to the whole complex plane. Also attached to f is a "premotivic structure" M_f over $\mathbb{Q}$ with coefficients in K, any number field (considered as a subfield of $\mathbb{C}$) containing $\mathbb{Q}(f)$, the extension of $\mathbb{Q}$ generated by the $a_n(f)$. Thus there are 2-dimensional K-vector spaces $M_{f,B}$ and $M_{f,dR}$ (the Betti and de Rham realisations) and, for each finite prime λ of O_K, a 2-dimensional K_λ-vector space $M_{f,\lambda}$, the λ-adic realisation. These come with various structures and comparison isomorphisms, such as $M_{f,B} \otimes_K K_\lambda \simeq M_{f,\lambda}$. See 1.1.1 of [DFG] for the precise definition of a premotivic structure, and 1.6.2 of [DFG] for the construction of M_f. The λ-adic realisation $M_{f,\lambda}$ comes with a continuous linear action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. Let $\rho_{f,\lambda}$ be this representation. For each prime number p, the restriction to $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ may be used to define a local L-factor (which is in fact known to be independent of λ in this case), and the Euler product is precisely $L_f(s)$. In particular, $\rho_{f,\lambda}$ is unramified at all primes $p \neq \ell$, with

$$\mathrm{Tr}(\rho_{f,\lambda}(\mathrm{Frob}_p^{-1})) = a_p, \ \det(\rho_{f,\lambda}(\mathrm{Frob}_p^{-1})) = p^{k'-1},$$

where $\mathrm{Frob}_p \in \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ lifts the p-power map of $\mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p)$. As the L-function attached to a premotivic structure, its orders of vanishing and leading terms at integer points may be interpreted via the Bloch-Kato conjecture.

On $M_{f,B}$ there is an action of $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$, and the eigenspaces $M_{f,B}^{\pm}$ are 1-dimensional. On $M_{f,dR}$ there is a decreasing filtration, with F^t a 1-dimensional space precisely for $1 \leq t \leq k'-1$. The de Rham isomorphism $M_{f,B} \otimes_K \mathbb{C} \simeq M_{f,dR} \otimes_K \mathbb{C}$ induces isomorphisms between $M_{f,B}^{\pm} \otimes \mathbb{C}$ and $(M_{f,dR}/F) \otimes \mathbb{C}$, where $F := F^1 = \ldots = F^{k'-1}$. Define $\Omega^{\pm}$ to be the determinants of these isomorphisms. These depend on the choices of K-bases for $M_{f,B}^{\pm}$ and $M_{f,dR}/F$, so should be viewed as elements of $\mathbb{C}^\times/K^\times$. The Tate-twisted premotivic structures $M_f(t)$, for $1 \leq t \leq k'-1$, are *critical* (because the above maps *are* isomorphisms), and the Deligne period ("c^+", see [De]) of $M_f(t)$ is $(2\pi i)^t \Omega^{(-1)^t}$. Deligne's conjecture for $M_f(t)$, known in this case, asserts then that $L(f,t)/(2\pi i)^t \Omega^{(-1)^t}$ is an element of K.

If we choose K-bases for $M_{f,B}$ and $M_{f,dR}$, to pin down $\Omega^{\pm}$, then the Bloch-Kato conjecture predicts the prime factorisation of the element $L(f,t)/(2\pi i)^t \Omega^{(-1)^t}$ of K. In fact, we shall choose an O_K-submodule $\mathfrak{M}_{f,B}$, generating $M_{f,B}$ over K, but not necessarily free, and likewise an $O_K[1/S]$-submodule $\mathfrak{M}_{f,dR}$, generating $M_{f,dR}$ over K, where S is the set of primes dividing $k'!$. We take these as in 1.6.2 of [DFG]. They are part of the "S-integral premotivic structure" associated to f. With these

choices it is still natural to talk of an element "$L(f,t)/(2\pi i)^{t}\Omega^{(-1)^t}$" of the group of fractional ideals of $O_K[1/S]$, and the Bloch-Kato conjecture predicts its prime factorisation.

In order to define the various terms appearing in the conjecture, we shall need, for each prime λ of O_K, the element $\mathfrak{M}_{f,\lambda}$ of the S-integral premotivic structure (which includes also the crystalline realisations $\mathfrak{M}_{f,\lambda\text{-crys}}$ for each $\lambda \notin S$). We choose these as in 1.6.2 of [DFG]. For each λ, $\mathfrak{M}_{f,\lambda}$ is a $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-stable O_λ-lattice in $M_{f,\lambda}$. Let $A_\lambda := M_{f,\lambda}/\mathfrak{M}_{f,\lambda}$, and let $A[\lambda]$ be the λ-torsion subgroup of A_λ. Let $\check{A}_\lambda := \check{M}_{f,\lambda}/\check{\mathfrak{M}}_{f,\lambda}$, where $\check{M}_{f,\lambda}$ and $\check{\mathfrak{M}}_{f,\lambda}$ are the vector space and O_λ-lattice dual to $M_{f,\lambda}$ and $\mathfrak{M}_{f,\lambda}$ respectively, with the natural $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-actions. Let $A := \oplus_\lambda A_\lambda$, etc. Let $\overline{\rho}_{f,\lambda}$ denote the representation on $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on $A[\lambda]$.

Following [BK] (Section 3), for $p \neq \ell$ (including $p = \infty$, where $\lambda \mid \ell$) let

$$H^1_f(\mathbb{Q}_p, M_{f,\lambda}(t)) = \ker\big(H^1(D_p, M_{f,\lambda}(t)) \to H^1(I_p, M_{f,\lambda}(t))\big).$$

Here D_p is a decomposition subgroup at a prime above p, I_p is the inertia subgroup, and $M_{f,\lambda}(t)$ is a Tate twist of $M_{f,\lambda}$, etc. The cohomology is for continuous cocycles and coboundaries. For $p = \ell$ let

$$H^1_f(\mathbb{Q}_\ell, M_{f,\lambda}(t)) = \ker\big(H^1(D_\ell, M_{f,\lambda}(t)) \to H^1(D_\ell, M_{f,\lambda}(t) \otimes_{\mathbb{Q}_\ell} B_{\mathrm{crys}})\big).$$

(See §1 of [BK] or §2 of [Fo1] for the definition of Fontaine's ring B_{crys}.) Let $H^1_f(\mathbb{Q}, M_{f,\lambda}(t))$ be the subspace of those elements of $H^1(\mathbb{Q}, M_{f,\lambda}(t))$ that, for all primes p, have local restriction lying in $H^1_f(\mathbb{Q}_p, M_{f,\lambda}(t))$. There is a natural exact sequence

$$0 \longrightarrow \mathfrak{M}_{f,\lambda}(t) \longrightarrow M_{f,\lambda}(t) \xrightarrow{\pi} A_\lambda(t) \longrightarrow 0.$$

Let $H^1_f(\mathbb{Q}_p, A_\lambda(t)) = \pi_* H^1_f(\mathbb{Q}_p, M_{f,\lambda}(t))$. Define the λ-Selmer group $H^1_f(\mathbb{Q}, A_\lambda(t))$ to be the subgroup of elements of $H^1(\mathbb{Q}, A_\lambda(t))$ whose local restrictions lie in $H^1_f(\mathbb{Q}_p, A_\lambda(t))$ for all primes p. Note that the condition at $p = \infty$ is superfluous unless $\ell = 2$. Define the Shafarevich-Tate group

$$\tilde{Ш}(t) = \bigoplus_\lambda \frac{H^1_f(\mathbb{Q}, A_\lambda(t))}{\pi_* H^1_f(\mathbb{Q}, M_{f,\lambda}(t))}.$$

Conjecture 2.1 (Case of Bloch-Kato)**.** *Suppose that* $1 \leq t \leq k'-1$. *Then we have the following equality of fractional ideals of* $O_K[1/S]$*:*

$$\frac{L(f,t)}{(2\pi i)^{t}\Omega^{(-1)^t}} = \frac{\prod_{p \leq \infty} \tilde{c}_p(t)\ \#\tilde{Ш}(t)}{\#H^0(\mathbb{Q}, A(t))\#H^0(\mathbb{Q}, \check{A}(1-t))}.$$

We omit the definition of the Tamagawa factors $\tilde{c}_p(t)$, but note that $\tilde{c}_\infty(t)$ is at worst a power of 2, that for $\lambda \mid \ell$ with $\ell \neq p$ the λ-part of $\tilde{c}_p(t)$ is trivial (a simple consequence of $M_{f,\lambda}$ being unramified at all $p \neq \ell$) and that even the λ-part of $\tilde{c}_\ell(t)$ is trivial as long as $\ell > k'$ (a consequence of Theorem 4.1(iii) of [BK]). See §2.4 of [DFG], or §11 of [Fo2], for precise definitions.

If the λ-part of $H^0(\mathbb{Q}, A(t))$ is non-trivial, then $A[\lambda]$ has a $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-submodule isomorphic to $\mathbb{F}_\lambda(-t)$, with quotient isomorphic to $\mathbb{F}_\lambda(1-k'+t)$ (so that the determinant is $\mathbb{F}_\lambda(1-k')$). Evaluating at Frob_p^{-1}, and taking the trace, we find that $a_p(f) \equiv p^t + p^{k'-1-t} \pmod{\lambda}$, for all $p \neq \ell$. A straightforward generalisation of Lemma 8 of [SD] shows that this is only possible if $\ell < k'$ or if $\mathrm{ord}_\ell(B_{k'}) > 0$

(in which case $t = k' - 1$). Noting also that $\check{A}(1-t) \simeq A(k'-t)$, we have the following.

Lemma 2.2. *For some $1 \leq t \leq k'-1$, suppose that $\mathrm{ord}_\lambda\left(\frac{L(f,t)}{(2\pi i)^t \Omega^{(-1)^t}}\right) > 0$, with $\lambda \mid \ell$ and $\ell > k'$. If $t = 1$ or $k'-1$, suppose also that $\mathrm{ord}_\ell(B_{k'}) = 0$. The Bloch-Kato conjecture predicts that the λ-part of $Ш(t)$ is non-trivial, hence also that $H^1_f(\mathbb{Q}, A_\lambda(t))$ is non-trivial.*

3. Congruences between Saito-Kurokawa lifts and non-lifts, and Harder's conjecture

Let $f \in S_{k'}(\Gamma_1)$ be as above. For $1 \leq t \leq k'-1$, define $L_{\mathrm{alg}}(f,t) := \frac{L(f,t)}{(2\pi i)^t \Omega^{(-1)^t}}$. If $t > k'/2$, we choose $j, k \geq 0$ such that $t = j + k$ and $k' = j + 2k - 2$. In other words, $k = k' + 2 - t$, $j = 2t - 2 - k'$. Note that $t = k' - (k-2)$ is paired with $k-2$ by the functional equation relating $L_f(s)$ and $L_f(k'-s)$.

First we consider the case $t = (k'/2)+1$, the critical point immediately right-of-centre. In this case, $t = k$, $k' = 2k-2$ and $j = 0$. We suppose that k is even. For any quadratic character χ_D associated to a fundamental discriminant $D < 0$, define $L_{\mathrm{alg}}(f, k-1, \chi_D) := \frac{L(f,k-1,\chi_D)}{(2\pi i)^{k-1}\tau(\chi_D)\Omega^+}$, where $L(f,s,\chi_D) = \sum_{n=1}^\infty \chi_D(n) a_n(f) n^{-s}$ and $\tau(\chi_D)$ is a Gauss sum. Associated with f is a Hecke eigenform $\hat{f} \in S_k(\Gamma_2)$, its Saito-Kurokawa lift. This is only defined up to scaling. It is related to f by its standard L-function

$$L(\hat{f}, s, \mathrm{St}) = \zeta(s)L(f, s+k-1)L(f, s+k-2)$$

and its spinor L-function

$$L(\hat{f}, s, \mathrm{spin}) = \zeta(s-(k-1))\zeta(s-(k-2))L(f,s).$$

Related to the latter is the following, for any prime p:

$$\mu_{\hat{f}}(p) = p^{k-1} + p^{k-2} + a_p(f),$$

where $T(p)\hat{f} = \mu_{\hat{f}}(p)\,\hat{f}$. Let $\mathbb{Q}(f)$ be the field generated by the Hecke eigenvalues of f. Likewise, for any Hecke eigenform $F \in S_k(\Gamma_2)$, let $\mathbb{Q}(F)$ be the field generated by the Hecke eigenvalues of F. The following is (a consequence of) Theorem 6.1 of [Ka1]. Theorem 6.5 of [Br] is also closely related. It is essentially part of what is proved in §6 below.

Theorem 3.1. *Let $f = \sum_{n=1}^\infty a_n(f) q^n \in S_{k'}(\Gamma_1)$ be a normalised Hecke eigenform, with $k' = 2k-2$ and k even. Let $\lambda' \mid \ell$, with $\ell > 2k$, be a prime of $\mathbb{Q}(f)$, such that $\mathrm{ord}_{\lambda'} L_{\mathrm{alg}}(f,k) > 0$. Suppose that*

1. *λ' is not a congruence prime for f in $S_{k'}(\Gamma_1)$, i.e. there does not exist another normalised Hecke eigenform $g \in S_{k'}(\Gamma_1)$, and a prime λ of $\mathbb{Q}(f)\mathbb{Q}(g)$, dividing λ', such that $a_n(f) \equiv a_n(g) \pmod{\lambda}$ for all $n \geq 1$.*
2. *There exists a fundamental discriminant $D < 0$ such that $\mathrm{ord}_{\lambda'}(|D|^{k-1} L_{\mathrm{alg}}(f, k-1, \chi_D)) = 0$.*
3. *There exists even m such that $2 < m < k-2$ and*

$$\mathrm{ord}_{\lambda'}(L_{\mathrm{alg}}(m+k-2)L_{\mathrm{alg}}(m+k-1)\zeta(1-m)) = 0.$$

Then there exists a Hecke eigenform $F \in S_k(\Gamma_2)$*, not a Saito-Kurokawa lift from* $S_{k'}$*, and a prime* $\lambda \mid \lambda'$ *in (any field containing)* $\mathbb{Q}(f)\mathbb{Q}(F)$*, such that for all primes* p*,*

$$\mu_F(p) \equiv \mu_{\hat{f}}(p) \pmod{\lambda} \text{ and } \mu_F(p^2) \equiv \mu_{\hat{f}}(p^2) \pmod{\lambda}.$$

In particular, for all primes p*,*

$$\mu_F(p) \equiv p^{k-2} + p^{k-1} + a_p(f) \pmod{\lambda}.$$

The conditions (1)–(3) are very weak.

In the case $j > 0$ there is no Saito-Kurokawa lift with which to prove such a theorem. The following, due to Harder, is Conjecture 3 in §26 of [vdG]. Special cases are discussed in [Ha].

Conjecture 3.2. *Let* $f = \sum a_n(f)q^n \in S_{k'}(\Gamma_1)$ *be a normalised eigenform, and suppose that a "large" prime* λ' *of* $\mathbb{Q}(f)$ *divides* $L_{\text{alg}}(f,t)$*, with* $(k'/2) < t \le k'-1$*. As above, let* $k = k' + 2 - t, j = 2t - 2 - k'$*. In the case* $j > 0$*, there exists an eigenform* $F \in S_{k,j}(\Gamma_2)$*, and a prime* $\lambda \mid \lambda'$ *in (any field containing)* $\mathbb{Q}(f)\mathbb{Q}(F)$ *such that, for all primes* p*,*

$$\mu_F(p) \equiv p^{k-2} + p^{j+k-1} + a_p(f) \pmod{\lambda}.$$

Numerical evidence obtained by Faber and van der Geer [vdG] supports the conjecture in the following cases (where the subscript on f is the weight k'):

$$41 \mid L_{\text{alg}}(f_{22}, 14),\ 43 \mid L_{\text{alg}}(f_{26}, 23),\ 97 \mid L_{\text{alg}}(f_{26}, 21),\ 29 \mid L_{\text{alg}}(f_{26}, 19),$$

(and in some other cases with $k' \le 38$). The corresponding spaces $S_{10,4}(\Gamma_2)$, $S_{5,18}(\Gamma_2)$, $S_{7,14}(\Gamma_2)$ and $S_{9,10}(\Gamma_2)$ are all 1-dimensional.

Note that if one tries to allow $j = 0$ in this conjecture (the case to which Theorem 3.1 applies), one must exclude the case that k is odd. For example, when $k' = 48$ (so $k = 25$), $\text{ord}_{\lambda'} L_{\text{alg}}(f, k) > 0$, for $\lambda' \mid \ell = 7025111$ (obtained from [St]), but $S_k = \{0\}$ for odd $k < 35$. We also note that a variant of Harder's conjecture for Siegel modular forms of half-integral weight is proposed in [I2], directly connected to the integral weight case through a conjectural Shimura type correspondence.

4. Brown's construction of elements in Selmer groups

To a Hecke eigenform $F \in S_{k,j}(\Gamma_2)$ may be associated a cuspidal automorphic form $\Phi_F \in L^2_0(Z(\mathbb{A}_\mathbb{Q})GSp_4(\mathbb{Q})\backslash GSp_4(\mathbb{A}_\mathbb{Q}))$. This adelic interpretation is described in detail in §3 of [AS] (§3.1 for the scalar-valued case, §3.5 for the vector-valued case). Let Π_F be any irreducible constituent of the unitary representation of $GSp_4(\mathbb{A}_\mathbb{Q})$ generated by right translates of Φ_F, as in 3.4 of [AS]. They are all isomorphic, in fact this unitary representation is expected to be irreducible already. To such a Π_F we shall shortly apply (with our special choice of λ) the following theorem, which is part of Theorem I of [We1].

Theorem 4.1 (Weissauer)**.** *Suppose that* Π *is a unitary, irreducible, automorphic representation of* $GSp_4(\mathbb{A}_\mathbb{Q})$ *for which* Π_∞ *belongs to the discrete series of weight* (k_1, k_2)*. Let* S *denote the set of ramified places of the representation* Π*. Put* $w = k_1 + k_2 - 3$*. Then there exists a number field* E *such that*

(1) *for any prime* $p \notin S$*, if* $L_p(p^{-s}) = L_p(\Pi_p, s - w/2)$ *is the local factor in the spinor* L*-function, then* $L_p(X)^{-1} \in E[X]$*;*

(2) *for any prime* λ *of* O_E*, there exists a finite extension* K *of* E *(and* K_λ *of* E_λ*), and a* 4*-dimensional semisimple Galois representation*

$$\rho_{\Pi,\lambda} : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_4(K_\lambda),$$

unramified outside $S \cup \{\ell\}$ *(where* $\lambda \mid \ell$*), such that for each prime* $p \notin S \cup \{\ell\}$*,*

$$L_p(\Pi_p, s - w/2) = \det(I - \rho_{\Pi,\lambda}(\mathrm{Frob}_p^{-1})p^{-s})^{-1}.$$

These Galois representations are found (when Π is neither CAP nor a weak endoscopic lift) in the third ℓ-adic cohomology (in general with non-trivial coefficients) of an inverse system of Siegel modular threefolds. They were studied by Taylor [T], who deduced a list of possibilities, but he was not able to narrow it down enough to prove the existence of a 4-dimensional representation (or in that case to prove such a strong statement about the set of primes where the L-factors match). To prove Theorem 4.1 required trace formula methods. The main theorems in [We1] depend on hypotheses (A and B), whose proofs have now appeared in [We2].

Now recall the situation of §3, where $F \in S_{k,j}(\Gamma_2)$ is a Hecke eigenform such that, for all primes p,

$$\mu_F(p) \equiv p^{k-2} + p^{j+k-1} + a_p(f) \pmod{\lambda}, \tag{1}$$

where λ is a "large" prime divisor (in any field K containing $\mathbb{Q}(f)\mathbb{Q}(F)$) of $L_{\mathrm{alg}}(f, j+k)$. Recall that, in the case $j = 0$, k even, the existence of such a non-Saito-Kurokawa lift F is given by Theorem 3.1, assuming weak hypotheses, while in the case $j > 0$ we assume Harder's conjecture.

Proposition 4.2. *Let* F *and* λ *be as above. Suppose that* $k \geq 3$*,* $\ell > k' + 1$ *and* $\mathrm{ord}_\ell(B_{k'}) = 0$*. Then the following hold.*

(1) *If* K *is sufficiently large then there exists a* 4*-dimensional semisimple Galois representation*

$$\rho_{F,\lambda} : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}(4, K_\lambda),$$

unramified outside $\{\ell\}$*, such that for each prime* $p \neq \ell$*,* $\det(I - \rho_{F,\lambda}(\mathrm{Frob}_p^{-1})p^{-s})^{-1}$ *is the local factor in the spinor* L *function of* F*.*

(2) *Choose a* $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$*-invariant* O_λ*-lattice* T'_λ *in* V'_λ *(the space of* $\rho_{F,\lambda}$*) and consider the representation* $\overline{\rho}_{F,\lambda}$ *of* $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ *on* $T'_\lambda/\lambda T'_\lambda$*. Then the composition factors of* $\overline{\rho}_{F,\lambda}$ *are* $\mathbb{F}_\lambda(2-k)$*,* $\mathbb{F}_\lambda(1-j-k)$ *and* $\overline{\rho}_{f,\lambda}$*.*

Proof. (1) This is a direct consequence of Theorem 4.1, applied to Π_F. Note that here $k_1 = j + k$, $k_2 = k$, $w = j + 2k - 3$, and the condition $k \geq 3$ is necessary to ensure that Π_∞ is discrete series. Also Π_F is unramified at all primes p, since F is for the full modular group Γ_2.

(2) The congruence (1), with conclusion (1), implies that $\mathrm{tr}(\overline{\rho}_{F,\lambda}(\mathrm{Frob}_p^{-1})) = \mathrm{tr}((\chi^{2-k} \oplus \chi^{1-j-k} \oplus \overline{\rho}_{f,\lambda})(\mathrm{Frob}_p^{-1}))$, where χ is the (mod ℓ) cyclotomic character. It remains to observe that $\overline{\rho}_{f,\lambda}$ is (absolutely) irreducible, a consequence of $\ell > k' + 1$ and $\ell \nmid B_{k'}$, by Lemma 8 of [SD].

□

The following is a very straightforward generalisation of Theorem 8.4 of [Br], which is the case $j = 0$. In the case $j > 0$, Harder [Ha] clearly recognised this consequence of his conjecture.

Proposition 4.3. *Let* $f = \sum_{n=1}^{\infty} a_n(f)q^n \in S_{k'}(\Gamma_1)$ *be a normalised Hecke eigenform. Suppose that* $k' = j + 2k - 2$*, with* $j \geq 0$ *and* $k \geq 3$*, and that* $F \in S_{k,j}(\Gamma_2)$ *is a Hecke eigenform such that, for all primes* p*,*

$$\mu_F(p) \equiv p^{k-2} + p^{j+k-1} + a_p(f) \pmod{\lambda},$$

where $\lambda \mid \ell$ *is a prime divisor (in any field* K *containing* $\mathbb{Q}(f)\mathbb{Q}(F)$*) of* $L_{\mathrm{alg}}(f, j+k)$*. Suppose that* $\ell > k' + 1$ *and* $\mathrm{ord}_\ell(B_{j+2}) = \mathrm{ord}_\ell(B_{k'}) = 0$*. Let* K *be large enough as in Proposition 4.2. With notation as in §2,* $H^1_f(\mathbb{Q}, A_\lambda(j+k))$ *is non-trivial, in accord with the Bloch-Kato conjecture (see Lemma 2.2).*

Proof. We merely sketch the proof. The isomorphism class of $\overline{\rho}_{F,\lambda}$ depends on the choice of T'_λ, though the set of composition factors is well-defined. We claim it is possible to choose T'_λ in such a way that $T'_\lambda/\lambda T'_\lambda$ (the space of $\overline{\rho}_{F,\lambda}$) has a 3-dimensional submodule, with a submodule $\overline{\rho}_{f,\lambda}$ and a quotient $\mathbb{F}_\lambda(2-k)$. Clearly it is possible to arrange for $\overline{\rho}_{f,\lambda}$ to be a submodule of $T'_\lambda/\lambda T'_\lambda$. If it is not possible to make $\mathbb{F}_\lambda(2-k)$ the "next factor up" then the quotient of $T'_\lambda/\lambda T'_\lambda$ by $\overline{\rho}_{f,\lambda}$ must be a non-trivial extension of $\mathbb{F}_\lambda(2-k)$ by $\mathbb{F}_\lambda(1-j-k)$, which gives a non-trivial extension of $\mathbb{F}_\lambda$ by $\mathbb{F}_\lambda(-j-1)$. As in §8 of [Br] (which is the case $j = 0$), the action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on this 2-dimensional representation factors through $\mathrm{Gal}(F/\mathbb{Q})$, where F is an extension of $\mathbb{Q}(\mu_\ell)$ which corresponds, by Class Field Theory, to a non-trivial quotient of the χ^{-1-j}-isotypical part of the ℓ-part of the class group of $\mathbb{Q}(\mu_\ell)$. But by Herbrand's theorem, this would contradict our assumption that $\ell \nmid B_{j+2}$.

We get then, inside $T'_\lambda/\lambda T'_\lambda$, an extension of $\mathbb{F}_\lambda(2-k)$ by $\overline{\rho}_{f,\lambda}$. For a different choice, T''_λ, of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-invariant O_λ-lattice, this extension is inside a 3-dimensional quotient of $T''_\lambda/\lambda T''_\lambda$. If it was a trivial extension (of $\mathbb{F}_\lambda[\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})]$-modules) then, applying the method of the proof of Proposition 2.1 of [R] (as in the proof of Proposition 8.3 of [Br], where all the sums should start at $n = 0$), we would get a quotient of rank 1 of T''_λ, which is not possible, as explained in the proof of Proposition 8.3 of [Br].

This non-trivial extension of $\mathbb{F}_\lambda(2-k)$ by $\overline{\rho}_{f,\lambda}$ gives, by twisting, a non-trivial extension of $\mathbb{F}_\lambda$ by $\overline{\rho}_{f,\lambda}(k-2)$, hence a non-zero element of $H^1(\mathbb{Q}, A[\lambda](k-2))$. (Recall that $A[\lambda]$ is the space of $\overline{\rho}_{f,\lambda}$.) One may show, just as in §8 of [Br], that its image in $H^1(\mathbb{Q}, A_\lambda(k-2))$ is a non-zero element of the Bloch-Kato Selmer group $H^1_f(\mathbb{Q}, A_\lambda(k-2))$. The proof of the local conditions at $p \neq \ell$ uses the fact that $\rho_{F,\lambda}$ is unramified at such p, while the proof of the local condition at ℓ uses the fact that $\rho_{F,\lambda}|_{\mathrm{Gal}(\overline{\mathbb{Q}}_\ell/\mathbb{Q}_\ell)}$ is crystalline (Theorem 3.2(ii) of [U], which refers to [Fa] and [CF]).

By the main result of [Kato], $H^1_f(\mathbb{Q}, V'_\lambda(r)) = 0$ for any integer $r \neq k'/2$ with $1 \leq r \leq k' - 1$. Hence $H^1_f(\mathbb{Q}, A_\lambda(k-2)) = Ш(k-2)$. Using [Fl], we may reflect across the central point $s = k'/2$ to get a non-zero element of λ-torsion in $Ш(j+k)$, and hence in $H^1_f(\mathbb{Q}, A_\lambda(j+k))$, as required. □

5. The Bloch-Kato conjecture for critical values of genus-two standard L-functions

5.1. **The conjecture.** Let $F \in S_{k,j}(\Gamma_2)$ be a cuspidal Hecke eigenform. There ought to exist an "L-admissible premotivic structure" (c.f. 1.1.1 of [DFG]) M' over $\mathbb{Q}$, with coefficients in some finite extension K of $\mathbb{Q}(F)$, such that $L(M', s) = L(F, s, \mathrm{spin})$. For each prime λ of O_E there would be a 4-dimensional representation of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, with coefficients in K_λ, arising from the λ-adic realisation M'_λ. In

particular $L(M'_\lambda, s) = L(F, s, \mathrm{spin})$. At least these Galois representations are known to exist, by Proposition 4.2. Strictly speaking, for each non-archimedean completion of $\mathbb{Q}(F)$ there is a representation with coefficients in some finite extension. Let's just imagine that these are all completions of a fixed K. Eventually we shall be concerned only with the particular prime λ of previous sections.

If $M := \wedge^2 M'$ then $L(M, s) = \zeta(s-(j+2k-3))L(F, s-(j+2k-3), \mathrm{St})$. M' should have Hodge-type $\{(0, j+2k-3), (k-2, j+k-1), (j+k-1, k-2), (j+2k-3, 0)\}$. (On this list, (p, q) appears $h^{p,q} = \dim H^{p,q}$ times, where $M'_B \otimes \mathbb{C} = \oplus H^{p,q}$ is the Hodge decomposition.) Consequently, M would have Hodge-type

$$\{(k-2, 2j+3k-4), (j+k-1, j+3k-5), (j+2k-3, j+2k-3), (j+2k-3, j+2k-3),$$
$$(j+3k-5, j+k-1), (2j+3k-4, k-2)\}.$$

Now $\dim {M'_B}^+$ and $\dim {M'_B}^-$ would both be 2 (since complex conjugation switches $H^{p,q}$ and $H^{q,p}$), from which would follow $\dim M_B^+ = 2$ and $\dim M_B^- = 4$. Then the right-of-centre critical points for M would be of the form $r = m + (j+2k-3)$, where m is even with $0 < m \leq k-2$. Note that r is chosen so that $M_B^{(-1)^r}$ and M_{dR}/F^r have the same dimension, 4 in this case. According to Deligne's conjecture, $L(M, m+(j+2k-3))/(2\pi i)^{4m+4j+8k-12}\omega^-(M)$ belongs to K. Here $\omega^-(M)$ is the determinant (w.r.t. K-bases of M_B^- and M_{dR}/F^r) of the isomorphism $M_B^- \otimes \mathbb{C} \simeq (M_{dR}/F^r) \otimes \mathbb{C}$. Different choices of bases result in it being scaled by some factor in $K^\times$.

Let V'_λ be the (space of the) 4-dimensional representation of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ that is supposed to be M'_λ. Let T'_λ be a choice of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-invariant O_λ-lattice in V'_λ, and $W'_\lambda = V'_\lambda/T'_\lambda$. Let $W'[\lambda]$ denote the λ-torsion in W'_λ. Let $V_\lambda = \wedge^2 V'_\lambda$, $T_\lambda = \wedge^2 T'_\lambda$ and $W_\lambda = V_\lambda/T_\lambda$. Then let $W := \oplus_\lambda W_\lambda$. Having made the choice of T_λ, and having chosen also a K-basis of M_{dR}, the factors appearing in the equation (2) below may be defined as in the case of M_f in §2.

According to the Bloch-Kato conjecture,

$$\frac{L(M, m+(j+2k-3))}{(2\pi i)^{4r}\omega^-(M)} = \frac{\prod_p c_p(r)\ \#Ш(r)}{\#H^0(\mathbb{Q}, W(r))\ \#H^0(\mathbb{Q}, \check{W}(1-r))}\ , \tag{2}$$

where $r := m+(j+2k-3)$, with m even and $0 < m \leq k-2$. We read the two sides of this equation as fractional ideals of K. Note that $L(M, s)$ would be the same thing as $\zeta(s-(j+2k-3))L(F, s-(j+2k-3), \mathrm{St})$.

We return now to the situation of §3, and direct our attention to the λ-part of the Bloch-Kato conjecture, for critical values of $L(M, s)$. We shall make a different choice of T'_λ from that used in §4. From now on T'_λ will be like the T''_λ of §4. So $T'_\lambda/\lambda T'_\lambda \supset B \supset C \supset \{0\}$, with $C \simeq \mathbb{F}_\lambda(1-j-k)$, $B/C \simeq \overline{\rho}_f$ and $(T'_\lambda/\lambda T'_\lambda)/B \simeq \mathbb{F}_\lambda(2-k)$.

5.2. **Construction of elements of Selmer groups for the standard L-function.**

Proposition 5.1. *Let* $f = \sum_{n=1}^\infty a_n(f)q^n \in S_{k'}(\Gamma_1)$ *be a normalised Hecke eigenform. Suppose that* $k' = j+2k-2$, *with* $j \geq 0$ *and* $k \geq 3$, *and that* $F \in S_{k,j}(\Gamma_2)$ *is a Hecke eigenform such that, for all primes* p,

$$\mu_F(p) \equiv p^{k-2} + p^{j+k-1} + a_p(f) \pmod{\lambda},$$

where $\lambda \mid \ell$ *is a prime divisor (in any field* K *containing* $\mathbb{Q}(f)\mathbb{Q}(F)$*) of* $L_{\mathrm{alg}}(f, j+k)$. *Suppose that* $\ell > 2j+2k-1$ *and* $\mathrm{ord}_\ell(B_{j+2}) = \mathrm{ord}_\ell(B_{k'}) = 0$. *Let* K *be large enough*

as in Proposition 4.2. Then, with notation as in §5.1, $H^1_f(\mathbb{Q}, W_\lambda(2j+2k-1))$ *is non-trivial.*

Note that $2j+2k-1 = (j+2)+(j+2k-3)$.

Proof. Since $\wedge^2(\mathbb{F}_\lambda(1-j-k)) = 0$, we see that $W[\lambda] = \wedge^2 W'[\lambda]$ has a submodule isomorphic to $\overline{\rho}_{f,\lambda}(1-j-k)$. Hence $W[\lambda](2j+2k-1)$ has a submodule isomorphic to $\overline{\rho}_{f,\lambda}(j+k) \simeq A[\lambda](j+k)$, and the inclusion map will induce a map in Galois cohomology.

From Proposition 4.3, we have a non-zero element c'' of $H^1(\mathbb{Q}, A[\lambda](j+k))$, whose image d'' in $H^1(\mathbb{Q}, A_\lambda(j+k))$ lies in $H^1_f(\mathbb{Q}, A_\lambda(j+k))$. Let c be the image of c'' in $H^1(\mathbb{Q}, W[\lambda](2j+2k-1))$, and let d be the image of c in $H^1(\mathbb{Q}, W_\lambda(2j+2k-1))$. Our goal is to show that d is a non-zero element of $H^1_f(\mathbb{Q}, W_\lambda(2j+2k-1))$.

First we show that it is non-zero. There are two 1-dimensional subfactors of $W[\lambda] = \wedge^2(W'[\lambda])$, both isomorphic to $\mathbb{F}_\lambda(3-2k-j)$. Hence the only 1-dimensional subfactors of $W[\lambda](2j+2k-1)$ are isomorphic to $\mathbb{F}_\lambda(2+j)$. Since $\ell > 3+j$, this is non-trivial. Hence $H^0(\mathbb{Q}, W[\lambda](2j+2k-1)) = 0$, from which it follows that $H^0(\mathbb{Q}, W_\lambda(2j+2k-1)) = 0$. Also, $H^0(\mathbb{Q}, W[\lambda](2j+2k-1)/A[\lambda](j+k)) = 0$. Hence $H^1(\mathbb{Q}, A[\lambda](j+k))$ injects into $H^1(\mathbb{Q}, W[\lambda](2j+2k-1))$, which injects into $H^1(\mathbb{Q}, W_\lambda(2j+2k-1))$, so d is indeed non-zero.

Next we show that $\mathrm{res}_p(d) \in H^1_f(\mathbb{Q}_p, W_\lambda(2j+2k-1))$ for any $p \neq \ell$. Since $W'[\lambda]$ is unramified at p, the image of c'' in $H^1(I_p, A[\lambda](j+k))$ is zero. It follows that the image of d in $H^1(I_p, W_\lambda(2j+2k-1))$ is zero. Since $W_\lambda(2j+2k-1)$ is unramified at p, this guarantees that $\mathrm{res}_p(d) \in H^1_f(\mathbb{Q}_p, W_\lambda(2j+2k-1))$ (see, for example, Lemma 7.4 of [Br]).

Finally we show that $\mathrm{res}_\ell(d) \in H^1_f(\mathbb{Q}_\ell, W_\lambda(2j+2k-1)$. In Lemma 4.4 of [BK], a cohomological functor $\{h^i\}_{i\geq 0}$ is constructed on the Fontaine-Lafaille category of filtered Dieudonné modules over $\mathbb{Z}_p$. $h^i(\mathcal{M}) = 0$ for all $i \geq 2$ and all $\mathcal{M}$, and $h^i(\mathcal{M}) = \mathrm{Ext}^i(1_{FD}, \mathcal{M})$ for all i and $\mathcal{M}$, where 1_{FD} is the "unit" filtered Dieudonné module.

Recall that $\rho_{F,\lambda}|_{\mathrm{Gal}(\overline{\mathbb{Q}}_\ell/\mathbb{Q}_\ell)}$ (whose space is V'_λ) is crystalline, so $V_\lambda = \wedge^2 V'_\lambda$ is also crystalline. Examination of the composition factors of $T_\lambda/\lambda T_\lambda$ shows that the Hodge-Tate weights of V_λ must be as expected, i.e.

$$k-2,\ j+k-1,\ j+2k-3,\ j+2k-3,\ j+3k-5,\ 2j+3k-4.$$

Meanwhile, the Hodge-Tate weights of $\rho_{f,\lambda}|_{\mathrm{Gal}(\overline{\mathbb{Q}}_\ell/\mathbb{Q}_\ell)}$ are 0 and $j+2k-3$. Let $\mathcal{E}$ and $\mathcal{D}$ be filtered Dieudonné modules over $\mathbb{Z}_\ell$ such that the associated representations of $\mathrm{Gal}(\overline{\mathbb{Q}}_\ell/\mathbb{Q}_\ell)$ are (on) T_λ and $\mathfrak{M}_{f,\lambda}$ respectively (viewed as representations with $\mathbb{Z}_\ell$ coefficients). The condition $\ell > 2j+2k-1$ ensures that these both exist, and that $\mathcal{E}\{2j+2k-1\}$ and $\mathcal{D}\{j+k\}$ both satisfy $\mathrm{Fil}^a\mathcal{M} = \mathcal{M}, \mathrm{Fil}^{a+\ell-1}\mathcal{M} = \{0\}$, with $a = -2j-k-1$. It is essentially the condition (*) in §4 of [BK].

By Lemma 4.5 (c) of [BK], (with the typo that substituted "e" for "f" corrected),

$$h^1(\mathcal{D}) \simeq H^1_f(\mathbb{Q}_\ell, \mathfrak{M}_{f,\lambda}),$$

(defined to be the inverse image in $H^1(\mathbb{Q}_\ell, \mathfrak{M}_{f,\lambda})$ of $H^1_f(\mathbb{Q}_\ell, M_{f,\lambda})$). Twists may be applied to both sides of this isomorphism.

Something like the exact sequence in the middle of page 366 of [BK] gives us a commutative diagram

$$\begin{array}{ccccc} h^1(\mathcal{D}(j+k)) & \xrightarrow{\pi} & h^1(\mathcal{D}(j+k)) & \longrightarrow & h^1(\mathcal{D}(k)/\lambda\mathcal{D}(j+k)) \\ \downarrow & & \downarrow & & \downarrow \\ H^1(\mathbb{Q}_\ell, \mathfrak{M}_{f,\lambda}(j+k)) & \longrightarrow & H^1(\mathbb{Q}_\ell, \mathfrak{M}_{f,\lambda}(j+k)) & \longrightarrow & H^1(\mathbb{Q}_\ell, A[\lambda](j+k)) \end{array}$$

Here π is a uniformiser at λ. The vertical arrows are all inclusions and we know that the image of $h^1(\mathcal{D}(j+k))$ in $H^1(\mathbb{Q}_\ell, \mathfrak{M}_{f,\lambda}(j+k))$ is exactly $H^1_f(\mathbb{Q}_\ell, \mathfrak{M}_{f,\lambda}(j+k))$. The top right horizontal map is surjective since $h^2(\mathcal{D}(j+k)) = 0$. In fact, Lemma 4.4 of [BK] gives a description of $\mathrm{Ext}^1(1_{\mathrm{FD}}, \mathcal{M})$ as a quotient of $\mathcal{M}$, namely $\mathcal{M}/(1-\phi^0)(\mathrm{Fil}^0\mathcal{M})$, from which the surjectivity is obvious.

The class $c'' \in H^1(\mathbb{Q}_\ell, A[\lambda](j+k))$ is in the image of $H^1_f(\mathbb{Q}_\ell, \mathfrak{M}_{f,\lambda}(j+k))$ and therefore is in the image of $h^1(\mathcal{D}(j+k)/\lambda\mathcal{D}(j+k))$. Recall that $W[\lambda](2j+2k-1)$ has a Galois submodule isomorphic to $A[\lambda](j+k)$. By the fullness of the Fontaine-Lafaille functor [FL] (see Theorem 4.3 of [BK]), $\mathcal{E}(2j+2k-1)/\lambda\mathcal{E}(2j+2k-1)$ has a subobject isomorphic to $\mathcal{D}(j+k)/\lambda\mathcal{D}(j+k)$.

It follows that the class $c \in H^1(\mathbb{Q}_\ell, W[\lambda](2j+2k-1))$ is in the image of $h^1(\mathcal{E}((2j+2k-1)/\lambda\mathcal{E}(2j+2k-1))$ by the vertical map in the exact sequence analogous to the above. Since the map from $h^1(\mathcal{E}(2j+2k-1))$ to $h^1(\mathcal{E}(2j+2k-1)/\lambda\mathcal{E}(2j+2k-1))$ is surjective, c lies in the image of $H^1_f(\mathbb{Q}_\ell, T_\lambda(2j+2k-1))$. From this it follows that $d \in H^1_f(\mathbb{Q}_\ell, W_\lambda(2j+2k-1))$, as desired.

□

Proposition 5.2. *Let $f = \sum_{n=1}^{\infty} a_n(f)q^n \in S_{k'}(\Gamma_1)$ be a normalised Hecke eigenform, with $k'/2$ odd. Suppose that $k' = j+2k-2$, with $j \geq 0$ and $k \geq 3$, and that $F \in S_{k,j}(\Gamma_2)$ is a Hecke eigenform such that, for all primes p,*

$$\mu_F(p) \equiv p^{k-2} + p^{j+k-1} + a_p(f) \pmod{\lambda},$$

where $\lambda \mid \ell$ is a prime divisor (in any field K containing $\mathbb{Q}(f)\mathbb{Q}(F)$) of $L_{\mathrm{alg}}(f, j+k)$. Suppose that $\ell > 2j+2k-1$ and $\mathrm{ord}_\ell(B_{k'}) = 0$, and that f is ordinary at λ (i.e. $\lambda \nmid a_\ell$). Let K be large enough as in Proposition 4.2. Then, with notation as in §5.1, $H^1_f(\mathbb{Q}, W_\lambda((k'/2)+j+k-1))$ is non-trivial.

Note that $(k'/2)+j+k-1 = (3j/2)+2k-2 = (j/2)+1+(j+2k-3)$.

Proof. The sign in the functional equation of $L(f,s)$ is $(-1)^{k'/2} = -1$. Applying either Théorème A of [SU] or the main theorem of §12 of [N] (both require the condition that f is ordinary at λ), $H^1_f(\mathbb{Q}, M_{f,\lambda}(k'/2))$ is non-trivial, from which one easily deduces that $H^1_f(\mathbb{Q}, \mathfrak{M}_{f,\lambda}(k'/2))$ and $H^1_f(\mathbb{Q}, A[\lambda](k'/2))$ are non-trivial. Taking non-zero $c'' \in H^1_f(\mathbb{Q}, A[\lambda](k'/2))$, one proceeds as above. This time, the only 1-dimensional subfactors of $W[\lambda]((k'/2)+j+k-1)$ are isomorphic to $\mathbb{F}_\lambda(1+(j/2))$. □

Note that the condition that f is ordinary at ℓ is, conjecturally, not necessary for the non-triviality of $H^1_f(\mathbb{Q}, M_{f,\lambda}(k'/2))$. By an analogue of the Birch and Swinnerton-Dyer conjecture, vanishing of $L(f, k'/2)$ should suffice. (See the "conjectures" $C_r(M)$ in §1 of [Fo2], and $C^i_\lambda(M)$ in §6.5 of [Fo2].)

5.3. **Conjectural consequences.** Suppose we are in the situation of Proposition 5.1 or Proposition 5.2. Recall that the only 1-dimensional composition factors of $W[\lambda]$ are isomorphic to $\mathbb{F}_\lambda(3-2k-j)$. It follows that none of the global torsion terms appearing in (2) for critical r could have a non-trivial λ-part, since $\ell > k-1$. For $p \neq \ell$, the λ-part of $c_p(r)$ is trivial, as in the case of M_f. If we choose the basis for the conjecturally existing M_{dR} in such a way that $\mathbb{V}(\mathfrak{M}) = T_\lambda$, where $\mathfrak{M}$ is the O_λ-lattice in $M_{dR} \otimes K_\lambda$ spanned by the basis and $\mathbb{V}$ is the Fontaine-Lafaille functor, then the λ-part of $c_\ell(r)$ is also trivial. The (conjecturally existing) period $\omega^-(M)$ depends on our choices, but will cancel when we consider ratios of critical values of $L(M,s)$. The Selmer groups attached to these (non-central) critical points are conjecturally finite, so should be equal to the corresponding Shafarevich-Tate groups. Recalling that $L(M,s) = \zeta(s-(j+2k-3))L(F,s-(j+2k-3),St)$, we are led, by the Bloch-Kato conjecture (2) and Propositions 5.1 and 5.2, to the following. (In any particular example it seems unlikely that the λ-part of $Ш(m+(j+2k-3))$ could be non-trivial, though strictly speaking there might be cases which would have to be excluded from the conjectures.)

Conjecture 5.3. *Let $f = \sum_{n=1}^{\infty} a_n(f)q^n \in S_{k'}(\Gamma_1)$ be a normalised Hecke eigenform. Suppose that $k' = j+2k-2$, with $j \geq 0$ and $k \geq 4$, and that $F \in S_{k,j}(\Gamma_2)$ is a Hecke eigenform such that, for all primes p,*

$$\mu_F(p) \equiv p^{k-2} + p^{j+k-1} + a_p(f) \pmod{\lambda},$$

where $\lambda \mid \ell$ is a prime divisor (in any field K containing $\mathbb{Q}(f)\mathbb{Q}(F)$) of $L_{alg}(f, j+k)$. Suppose that $\ell > 2j+2k-1$ and $\mathrm{ord}_\ell(B_{j+2}) = \mathrm{ord}_\ell(B_{k'}) = 0$. Suppose that $j \leq k-4$ (so that $0 < j+2 \leq k-2$). Take any even m with $0 < m \leq k-2$ but $m \neq j+2$. Then

$$\mathrm{ord}_\lambda\left(\frac{\pi^{3(m-(j+2))}L(F,j+2,St)}{L(F,m,St)}\right) > 0.$$

Under mild conditions, we shall prove the case $j = 0$ in §6 below, using the Saito-Kurokawa lift, but in the vector-valued case we have to resort to computation. In the case that f is a normalised generator of $S_{22}(\Gamma_1)$ and F is a generator of the 1-dimensional space $S_{10,4}(\Gamma_2)$, there is good numerical evidence for Harder's conjecture, with $\ell = 41$ dividing $L_{alg}(f, 14)$ [FvdG, vdG]. In this case $j+2 = 6$, which is in the required range, and in §7 below we shall confirm that

$$\mathrm{ord}_{41}\left(\frac{\pi^6\, L(F,6,St)}{L(F,8,St)}\right) > 0.$$

Conjecture 5.4. *Let $f = \sum_{n=1}^{\infty} a_n(f)q^n \in S_{k'}(\Gamma_1)$ be a normalised Hecke eigenform, with $k'/2$ odd. Suppose that $k' = j+2k-2$, with $j \geq 0$ and $k \geq 3$, and that $F \in S_{k,j}(\Gamma_2)$ is a Hecke eigenform such that, for all primes p,*

$$\mu_F(p) \equiv p^{k-2} + p^{j+k-1} + a_p(f) \pmod{\lambda},$$

where $\lambda \mid \ell$ is a prime divisor (in any field K containing $\mathbb{Q}(f)\mathbb{Q}(F)$) of $L_{alg}(f, j+k)$. Suppose that $\ell > 2j+2k-1$ and $\mathrm{ord}_\ell(B_{k'}) = 0$. Suppose that $(j/2)$ is odd, and that $j \leq 2k-6$, so that $(j/2)+1$ is even, with $0 < (j/2)+1 \leq k-2$. Suppose also that $\mathrm{ord}_\ell(B_{(j/2)+1}) = 0$. Take any even m with $0 < m \leq k-2$ but $m \neq (j/2)+1$. Then

$$\mathrm{ord}_\lambda\left(\frac{\pi^{3(m-((j/2)+1))}L(F,(j/2)+1,St)}{L(F,m,St)}\right) > 0.$$

The requirement that $(j/2)+1$ is even rules out $j=0$, so there is nothing to try to prove in the scalar-valued case here. In the case that f is one of the Galois-conjugate pair of normalised eigenforms spanning $S_{30}(\Gamma_1)$, and F is an appropriate Hecke eigenform in the 1-dimensional space $S_{11,10}(\Gamma_2)$, there is good numerical evidence for Harder's conjecture, with $\lambda \mid \ell = 97$ dividing $L_{\mathrm{alg}}(f,21)$ [vdG]. In this case $(j/2)+1=6$, which is in the required range (while $j+2$ fails to be $\leq k-2$), and in §8 below we shall confirm that

$$\mathrm{ord}_\lambda\left(\frac{\pi^6\, L(F,6,\mathrm{St})}{L(F,8,\mathrm{St})}\right) > 0.$$

6. The scalar-valued case

First we shall investigate the orders at λ of (normalised) standard L-values for the Saito-Kurokawa lift $\hat{f}$, then we shall note the occurrence of these values, as well as standard L-values for non-lifts, in a pullback formula. This will then be used to prove what we need about the standard L-values of the non-lift F to which $\hat{f}$ is congruent $\pmod{\lambda}$.

Let $f \in S_{2k-2}(\Gamma_1)$ be a normalised Hecke eigenform, with k even. Let K be a number field containing $\mathbb{Q}(f)$. Let $\tilde{f} = \sum c(n)q^n \in S_{k-1/2}(\Gamma_0(4))^+$ be a Hecke eigenform in the Kohnen plus-space, corresponding to f under the Kohnen-Shimura correspondence. Though $\tilde{f}$ is only defined up to scalar multiples, we may (and shall) assume that its Fourier coefficients belong to K. (This follows from the fact that $S_{k-1/2}(\Gamma_0(4))^+$ has a basis consisting of forms with rational Fourier coefficients [Koh1], together with the fact that the eigenvalues of the Hecke operators $T^+_{k-1/2}(p^2)$ (with p odd) on $\tilde{f}$ are the same as those of $T_{2k-2}(p)$ on f.) We define the Saito-Kurokawa lift to be the image of $\tilde{f}$ under a natural linear map from $S_{k-1/2}(\Gamma_0(4))^+$ to $S_k(\Gamma_2)$, as in [EZ] (passing through Jacobi cusp forms of weight k and index 1 on the way). The scaling of $\tilde{f}$ then determines the scaling of $\hat{f}$, and $\hat{f}$ also has Fourier coefficients in K. Note also that $\mathbb{Q}(\hat{f}) = \mathbb{Q}(f)$. By Kohnen and Skoruppa [KS],

$$\frac{\Gamma(k)L(f,k)}{(2\pi)^k} = 3\cdot 2^{3-k}\,\frac{\langle \hat{f},\hat{f}\rangle}{\langle \tilde{f},\tilde{f}\rangle}. \tag{3}$$

By Kohnen and Zagier [KZ],

$$\frac{c(|D|)^2}{\langle \tilde{f},\tilde{f}\rangle} = \frac{\Gamma(k-1)|D|^{k-3/2}L(f,k-1,\chi_D)}{\pi^{k-1}\,\langle f,f\rangle}, \tag{4}$$

where $D<0$ is a fundamental discriminant. Combining (3) and (4) gives

$$\langle \hat{f},\hat{f}\rangle = \frac{(k-1)}{2^3 3\pi}\cdot\frac{c(|D|)^2}{|D|^{k-3/2}}\cdot\frac{L(k,f)}{L(k-1,f,\chi_D)}\langle f,f\rangle.$$

Calculating as in (5.18) of [Hi] (and using Lemma 5.1.6 of [De], and the latter part of 1.5.1 of [DFG]), one finds that, up to S-units (where S is the set of primes dividing $k'!$),

$$\frac{\langle f,f\rangle}{i\Omega^+\Omega^-} = c(f),$$

where $c(f)$ is a certain "cohomology congruence ideal", which is integral. Take now an even integer $0<m\leq k-2$. Then $L(\hat{f},m,\mathrm{St})$ is a critical value. Combining the

previous two equations, and recalling that $L(\hat{f}, s, St) = \zeta(s)L(f, s+k-1)L(f, s+k-2)$, we arrive at (up to S-units)

$$\frac{L(\hat{f}, m, St)}{\pi^{2k+3m-3}\langle \hat{f}, \hat{f}\rangle} = \tag{5}$$

$$\frac{\zeta(m)}{\pi^m} L_{alg}(f, m+k-1)L_{alg}(f, m+k-2) \cdot \frac{|D|^{k-1}}{c(|D|)^2} \cdot \frac{L_{alg}(f, k-1, \chi_D)}{L_{alg}(f, k)c(f)}.$$

(We have pretty much followed [Br] or [Ka1].)

Let $\{F_1, \dots, F_d\}$ be a basis for $S_k(\Gamma_2)$, consisting of Hecke eigenforms. Let $\mathbb{Q}(F_i)$ be the field generated by the Hecke eigenvalues of F_i, and let K be the compositum of the $\mathbb{Q}(F_i)$. Let

$$F_i(Z) = \sum_A a_i(A) \exp(2\pi i \mathrm{tr}(AZ)),$$

where A runs over positive definite, half-integral, symmetric matrices, be the Fourier expansion of F_i. We may (and shall) assume that these Fourier coefficients belong to $\mathbb{Q}(F_i)$. This follows from the fact that there exists a basis for $S_k(\Gamma_2)$ consisting of forms with rational Fourier coefficients [Ba]. For a positive definite, half-integral, symmetric matrix $A = \begin{bmatrix} u & v/2 \\ v/2 & w \end{bmatrix}$, define the content $\mathrm{cont}A := \gcd(u, v, w)$, and let $D_A := v^2 - 4uw$. The following is a special case of Lemma 5.1 of [Ka1].

Lemma 6.1. *Let $\{F_1, \dots, F_d\}$ and K be as above. Suppose that $G \in S_k(\Gamma_2)$, with Fourier expansion $G = \sum_A a_G(A) \exp(2\pi i \mathrm{tr}(AZ))$. Let λ be a prime ideal of the ring of integers of K, dividing a rational prime ℓ. Assume that*

1. *all $a_G(A) \in K$, with $\mathrm{ord}_\lambda(a_G(A)) \geq 0$ for all A and, for some A_1, $\mathrm{ord}_\lambda(a_{F_1}(A_1)) = 0$;*
2. *there exist $c_1, \dots, c_d \in K$ such that $\mathrm{ord}_\lambda(c_1) < 0$ and*

$$G = \sum_{i=1}^{d} c_i F_i.$$

Then there exists $i \neq 1$ such that, for all primes p,

$$\mu_{F_1}(T(p)) \equiv \mu_{F_i}(T(p)) \pmod{\lambda} \quad \text{and} \quad \mu_{F_1}(T(p^2)) \equiv \mu_{F_i}(T(p^2)) \pmod{\lambda}.$$

If F is a Hecke eigenform in $S_k(\Gamma_2)$, we shall need a certain multiple $\Lambda(F, m, St) = C_{k,m} \frac{L(F,m,St)}{\pi^{2k+3m-3}\langle F,F\rangle}$, as defined precisely in the next section. All we need to know here about the constant $C_{k,m}$ is that it is a rational number with $\mathrm{ord}_\ell(C_{k,m}) = 0$ for any prime $\ell > 2k-2$. According to Theorem 4.4 of [Ka1], for any even integer m with $0 < m < k-2$,

$$\mathcal{F}_{m+2,k;A_1}(Z) = \sum_{j=1}^{d} \Lambda(F_j, m, St) a_j(A_1) F_j(Z). \tag{6}$$

Here $\mathcal{F}_{m+2,k;A_1}(Z) \in S_k(\Gamma_2)$ has rational Fourier coefficients, with denominators divisible at worst by primes less than or equal to $2m-1$. It is a coefficient in a partial Fourier expansion of the pullback to $\mathfrak{H}_2 \times \mathfrak{H}_2$ of the result of applying a certain differential operator to the Siegel-Eisenstein series of degree 4 and weight $m+2$. Comparing (6) with [Ka1], note that $\overline{F_j(-\overline{Z})} = F_j(Z)$, since we have arranged for the Fourier coefficients of the F_j to belong to K, which is totally real.

Lemma 6.2. *Suppose that* $\lambda \mid \ell > k-2$*, and that*

(1) $\mathrm{ord}_\lambda L_{\mathrm{alg}}(f,t) \geq 0$ *for all* $1 \leq t \leq 2k-3$*;*
(2) *there exists a fundamental discriminant* $D < 0$ *with* $\mathrm{ord}_\lambda(|D|^{k-1} L_{\mathrm{alg}}(f, k-1, \chi_D)) = 0$*.*

Then

(1) *it is possible to scale* $\tilde{f}$ *in such a way that* $\mathrm{ord}_\lambda(c(|D|)) = 0$ *and, for all* n*,* $\mathrm{ord}_\lambda(c(n)) \geq 0$*;*
(2) *for the corresponding scaling of* $\hat{f}$*,* $\mathrm{ord}_\lambda(a_{\hat{f}}(A)) \geq 0$ *for all* A*. Furthermore, if we choose* A_1 *such that* $D_{A_1} = D$*, then* $\mathrm{ord}_\lambda(a_{\hat{f}}(A_1)) = 0$*.*

Proof. (1) Let $D' < 0$ be any fundamental discriminant. Using modular symbols, $L_{\mathrm{alg}}(f, k-1, \chi_{D'})$ may be expressed as a linear combination of the $L_{\mathrm{alg}}(f,t)$. See for example the formula (8.6) of [MTT] (together with the discussion in §2 of [MTT] for the reduction of the modular symbols). This formula has in its denominator a $(k-2)!$ and a power of the conductor of the character, but $\ell > k-2$, and the power of the conductor cancels with $|D'|^{k-1}$, so in our case the coefficients in the linear combination for $|D'|^{k-1} L_{\mathrm{alg}}(f, k-1, \chi_{D'})$ will be integral at ℓ (hence at λ). Given assumption (1), it follows that $\mathrm{ord}_\lambda(|D'|^{k-1} L_{\mathrm{alg}}(f, k-1, \chi_{D'})) \geq 0$. Given that $\mathrm{ord}_\lambda(|D|^{k-1} L_{\mathrm{alg}}(f, k-1, \chi_D)) = 0$, it follows from equation (4) that, if we fix any scaling of $\tilde{f}$ then, among fundamental discriminants $D' < 0$, $\mathrm{ord}_\lambda(c(|D|))$ is the minimum. Part (1) follows easily from this.

(2) This is a direct consequence of the formula

$$a(A) = \sum_{b \mid \mathrm{cont} A} b^{k-1} c\left(\frac{|D_A|}{b^2}\right),$$

which comes from Theorem 1 and Proposition 3 of [Koh2]. For the second part, note that $\mathrm{cont} A_1 = 1$.

□

Let $\{f_1 = f, f_2, \ldots, f_r\}$ be a basis of normalised Hecke eigenforms in $S_{2k-2}(\Gamma_1)$. Order the basis $\{F_1, \ldots, F_d\}$ for $S_k(\Gamma_2)$ in such a way that $(F_1, \ldots, F_r) = (\hat{f_1}, \ldots, \hat{f_r})$. Recall that K is the compositum of the $\mathbb{Q}(F_i)$ for $1 \leq i \leq d$, and note that $\mathbb{Q}(f) \subset K$, since $\mathbb{Q}(\hat{f}) = \mathbb{Q}(f)$.

Theorem 6.3. *Suppose that* $\lambda \mid \ell > 2k-2$ *and that*

$$\mathrm{ord}_\lambda L_{\mathrm{alg}}(f,k) > 0,$$

with

(1) $\mathrm{ord}_\lambda L_{\mathrm{alg}}(f,t) \geq 0$ *for all* $1 \leq t \leq 2k-3$*;*
(2) *there exists a fundamental discriminant* $D < 0$ *with* $\mathrm{ord}_\lambda(|D|^{k-1} L_{\mathrm{alg}}(f, k-1, \chi_D)) = 0$*;*
(3) *there exists an even* m *such that* $2 < m < k-2$ *and* $\mathrm{ord}_\lambda(B_m L_{\mathrm{alg}}(m+k-1) L_{\mathrm{alg}}(m+k-2)) = 0$*;*
(4) *there does not exist* $1 < i \leq r$ *such that* $a_p(f) \equiv a_p(f_i)$ *for all primes* p*.*

Then

(1) *there exists a Hecke eigenform* $F \in S_k(\Gamma_2)$*, not a Saito-Kurokawa lift, such that*

(a) *for all primes* p,

$$\mu_F(p) \equiv \mu_{\hat{f}}(p) \equiv p^{k-2} + p^{k-1} + a_p(f) \pmod{\lambda} \text{ and } \mu_F(p^2) \equiv \mu_{\hat{f}}(p^2) \pmod{\lambda};$$

(b) *if we scale* F *to have Fourier coefficients integral at* λ*, then, for* m *as in (3),*

$$\mathrm{ord}_\lambda \Lambda(F, m, \mathrm{St}) < 0.$$

(2) *If* F *is unique (up to scaling) with the property (1a), and if we scale* F *so that* $\mathrm{ord}_\lambda(a_F(A)) \geq 0$ *for all* A *but* $\mathrm{ord}_\lambda(a_F(B)) = 0$ *for some* B*, then*

$$\mathrm{ord}_\lambda \Lambda(F, 2, \mathrm{St}) \geq 0.$$

Note that the $\Lambda(F_j, m, \mathrm{St})$ are solutions of linear equations with coefficients in K, arising from (6), so they do belong to K.

Proof. (1) (a) Given assumptions (1) and (2), we may scale $\hat{f}$ as in Lemma 6.2. Now we apply Lemma 6.1 to equation (6), with A_1 as in Lemma 6.2. We need $\mathrm{ord}_\lambda \Lambda(\hat{f}, m, \mathrm{St}) < 0$, but given assumption (3) and $\mathrm{ord}_\lambda L_{\mathrm{alg}}(f, k) > 0$, this follows from equation (5). If F were a Saito-Kurokawa lift, it is easy to see that assumption (4) would be contradicted.

(b) We can scale all the F_i to have Fourier coefficients integral at λ, and move to the left hand side of equation (6) any terms with $\mathrm{ord}_\lambda \Lambda(F_i, m, \mathrm{St}) \geq 0$, before applying Lemma 6.1.

(2) When $m = 2$, the $L_{\mathrm{alg}}(m + k - 2)$ in the numerator of equation (5) cancels the $L_{\mathrm{alg}}(f, k)$ in the denominator, so (again scaling as in Lemma 6.2) $\mathrm{ord}_\lambda \Lambda(\hat{f}, 2, \mathrm{St}) \geq 0$. Note that $\mathrm{ord}_\lambda(c(|D|)) = 0$, by Lemma 6.2, and if $\mathrm{ord}_\lambda(c(f)) > 0$ then assumption (4) would be contradicted. Consider again equation (6), with $m = 2$ and F_i scaled as above, and move to the left hand side any terms with $\mathrm{ord}_\lambda \Lambda(F_i, 2, \mathrm{St}) \geq 0$, including the $i = 1$ term. If it were not the case that $\mathrm{ord}_\lambda(F, 2, \mathrm{St}) \geq 0$ then we could apply Lemma 6.1 (with F in place of F_1) to deduce a congruence $\pmod{\lambda}$ of Hecke eigenvalues between F and another F_i (not $\hat{f}$), contradicting our assumption about the uniqueness of F.

□

This theorem may be illustrated by a numerical example in [Ka2], where $k = 22$ and $\ell = 1423$.

Corollary 6.4. *In the situation of Theorem 6.3, let* F *be as in (1a). Assuming that such an* F *is unique up to scaling, and taking* m *as in (3),*

$$\mathrm{ord}_\lambda \left(\frac{\pi^{3(m-2)} L(F, 2, \mathrm{St})}{L(F, m, \mathrm{St})} \right) > 0.$$

7. Computational support for Conjecture 5.3: $k = 10, j = 4, \ell = 41$.

First we review the pullback formula of the Siegel Eisenstein series following Böcherer [Bö], Böcherer, Satoh and Yamazaki [BSY], and Kozima [Koz]. For a $\mathbb{C}$-vector space V and non-negative integer m we denote by $V^{(m)}$ its m-th symmetric tensor product. We make the convention that $V^{(0)} = \mathbb{C}$. From now on we put $V_r = \mathbb{C}u_1 \oplus \cdots \oplus \mathbb{C}u_r$, and identify $V_r^{(m)}$ with the vector space of homogeneous

polynomials in $u_1, ..., u_r$ of degree m with coefficients in $\mathbb{C}$. Let ν be a non-negative integer. We then define the representation $\tau_{r;(\nu,m)} : GL_r(\mathbb{C}) \longrightarrow \mathrm{Aut}(V_r^{(m)})$ as

$$\tau_{r;(\nu,m)}(g) \cdot h(u) = (\det g)^{\nu} h(ug)$$

for $g \in GL_r(\mathbb{C})$ and $h \in V_r^{(m)}$. This is a realisation of $\det^{\nu} \otimes Sym(m)$, which will be fixed throughout this section. In particular, if r is even, we put $V_{r/2,1} = \mathbb{C}u_1 \oplus \cdots \oplus \mathbb{C}u_{r/2}$, and $V_{r/2,2} = \mathbb{C}u_{r/2+1} \oplus \cdots \oplus \mathbb{C}u_r$. We then regard $V_{r/2,1}^{(m)}$ and $V_{r/2,2}^{(m)}$ as subspaces of $V_r^{(m)}$ in a natural way. Let $M_{k,m}(\Gamma_r)$ (resp. $S_{k,m}(\Gamma_r)$) be the space of Siegel modular forms (resp. cusp forms) of weight $\det^k \otimes Sym(m)$ with respect to Γ_r. In particular we put $M_k(\Gamma_r) = M_{k,0}(\Gamma_r)$ and the others as usual. Then an element of $M_{k,m}(\Gamma_{2n})$ can be regarded as a function with values in $V_{2n}^{(m)}$, and an element of $M_{k,m}(\Gamma_n) \otimes M_{k,m}(\Gamma_n)$ can be regarded as a function with values in $V_{n,1}^{(m)} \otimes V_{n,2}^{(m)}$. Let $Z = (z_{ij})_{1 \le i,j \le 2n}$ be a matrix of variables with $z_{ij} = z_{ji}$, and we write $\partial_{ij} = \frac{(1+\delta_{ij})}{2} \frac{\partial}{\partial z_{ij}}$, and $\left(\frac{\partial}{\partial Z}\right) = (\partial_{ij})_{1 \le i,j \le 2n}$. We use the notation in [Bö] or [BS] and we put

$$\Delta(r,q) = \sum_{a+b=q} (-1)^b \binom{q}{b} z_2^{[a]} \partial_4^{[a]} \sqcup (1_n^{[r]} \sqcup z_2^{[b]} \partial_3^{[b]})(Ad^{[r+b]} \partial_1) \partial_2^{[r+b]},$$

and

$$\tilde{\mathcal{D}}_\alpha = \sum_{r+q=n} \binom{n}{q} C_q(-\alpha + n/2)^{-1} \Delta(r,q),$$

where $C_p(s) = s(s+1/2)...(s+(p-1)/2)$ for $s \in \mathbb{C}$. (Note that there are typos in [BS] or in [Bö] in the definition of $\Delta(r,q)$, e.g. in [BS], there appears $\binom{n}{b}$ but the above $\binom{q}{b}$ is correct.) Here the definition of the notation is complicated, so we do not repeat the details (cf. [Bö]), but we note that $A^{[0]} = 1$, $A^{[n]} = \det(A)$, $Ad^{[n]} A = 1$ and that if $0 < r < n$, then $A^{[r]}$ is a matrix such that each component is a homogeneous polynomial of components of A of positive order. We note that $\tilde{\mathcal{D}}_\alpha$ can be written as

$$\tilde{\mathcal{D}}_\alpha = \frac{(-1)^n}{C_n(\alpha - n + 1/2)} \sum_{r+q=n} \binom{n}{r} (-1)^r C_r(\alpha - n + 1/2) \Delta(r,q).$$

For non-negative integers ν and α, we define $\tilde{\mathcal{D}}_\alpha^\nu$ as

$$\tilde{\mathcal{D}}_\alpha^\nu = \tilde{\mathcal{D}}_{\alpha+\nu-1} \circ \cdots \tilde{\mathcal{D}}_{\alpha+1} \circ \tilde{\mathcal{D}}_\alpha.$$

The operator $\tilde{\mathcal{D}}_\alpha^\nu$ maps $C^\infty(\mathfrak{H}_{2n}, V_{2n}^{(m)})$ to $C^\infty(\mathfrak{H}_{2n}, V_{2n}^{(m)})$ for any non-negative integer ν. For non-negative integers m and $f \in C^\infty(\mathfrak{H}_{2n}, V_{2n}^{(m)})$, put

$$\tilde{D}f = U \left(\frac{\partial}{\partial Z}(f) \right) {}^t U$$

for $U = (u_1, ..., u_{2n})$. Then $\tilde{D}f$ belongs to $C^\infty(\mathfrak{H}_{2n}, V_{2n}^{(m+2)})$. We note that this $\tilde{D}f$ is $2\pi i$ times the Df defined in [BSY]. We also define two maps $\tilde{D}_\uparrow : C^\infty(\mathfrak{H}_{2n}, V_{2n}^{(m)}) \longrightarrow C^\infty(\mathfrak{H}_{2n}, V_{n,1}^{(m+2)})$ and $\tilde{D}_\downarrow : C^\infty(\mathfrak{H}_{2n}, V_{2n}^{(m)}) \longrightarrow C^\infty(\mathfrak{H}_{2n}, V_{n,2}^{(m+2)})$ by

$$\tilde{D}_\uparrow(f)(u_1, ..., u_n) = \tilde{D}(f)(u_1, ..., u_n, 0, ..., 0)$$

and

$$\tilde{D}_\downarrow(f)(u_{n+1}, ..., u_{2n}) = \tilde{D}(f)(0, ..., 0, u_{n+1}, ..., u_{2n}).$$

Furthermore let $\tilde{L}^{k,m}$ be the differential operator defined as follows:

$$\tilde{L}^{k,m} = \frac{1}{(k)_m} \sum_{\mu=0}^{[m/2]} \frac{1}{\mu!(m-2\mu)!(2-k-m)_\mu} (\tilde{D}_\uparrow \tilde{D}_\downarrow)^\mu (\tilde{D} - \tilde{D}_\uparrow - \tilde{D}_\downarrow)^{m-2\mu},$$

where $(k)_\mu = k(k+1)\cdots(k+\mu-1)$. For non-negative integers k, m we put

$$\mathcal{D}_{k-\nu,(k,m)}(f) = \tilde{L}^{k,m} \tilde{\mathcal{D}}^\nu_{k-\nu}(f)|_{\mathfrak{H}_n \times \mathfrak{H}_n}.$$

In particular, $\mathcal{D}_{k,(k,m)} = \tilde{L}^{k,m}$. Then $\mathcal{D}_{k-\nu,(k,m)}$ maps each element of $C^\infty(\mathfrak{H}_{2n}, \mathbb{C})$ to $C^\infty(\mathfrak{H}_n, V^{(m)}_{n,1}) \otimes C^\infty(\mathfrak{H}_n, V^{(m)}_{n,2})$. Furthermore it maps $M_{k-\nu}(\Gamma_{2n})$ to $M_{k,m}(\Gamma_n) \otimes M_{k,m}(\Gamma_n)$, and in particular its image is contained in $S_{k,m}(\Gamma_n) \otimes S_{k,m}(\Gamma_n)$ if $\nu > 0$. For an even positive integer l, we define the Siegel Eisenstein series $E_{2n,l}(Z,s)$ of degree $2n$ as

$$\begin{aligned} E_{2n,l}(Z,s) &= \zeta(1-l-2s) \prod_{i=1}^{n} \zeta(1-2l-4s+2i) \\ &\quad \times \sum_{M \in \Gamma_{2n,\infty} \backslash \Gamma_{2n}} j(M,Z)^{-l} (\det(\mathrm{Im}(M(Z))))^s \end{aligned}$$

($Z \in \mathfrak{H}_{2n}, s \in \mathbb{C}$), where $\zeta(*)$ is Riemann's zeta function, and $\Gamma_{2n,\infty} = \left\{ \begin{pmatrix} * & * \\ O_{2n} & * \end{pmatrix} \in \Gamma_{2n} \right\}$. This series converges for $2\mathrm{Re}(s) + l > 2n+1$ and is continued meromorphically to the whole plane as a function of s. Furthermore assume that $l \geq n+3$ or $l \geq n+1$ according as $n \equiv 1 \bmod 4$ or not. Then $E_{2n,l}(Z,0)$ is a holomorphic Siegel modular form of weight l as a function of Z (cf. [Sh]). From now on we assume that $E_{2n,l}(Z,0)$ is holomorphic as a function of Z, and write $E_{2n,l}(Z) = E_{2n,l}(Z,0)$. For an integer $k \geq l$ put

$$F_{l,(k,m)}(Z_1,Z_2) = \frac{1}{(2\pi i)^{n(k-l)+m}} \mathcal{D}_{l,(k,m)}(E_{2n,l}) \left(\begin{pmatrix} Z_1 & O \\ O & Z_2 \end{pmatrix} \right).$$

Now for $F \in S_{k,m}(\Gamma_n)$, let

$$\begin{aligned} &\Lambda(F,l,\mathrm{St}) = \\ &2^{n(2n+3)+2-2nk-l(n+1)-m}(-1)^{l(n+1)/2} \times \frac{\rho_{k,k-l-n}}{(k)_m m!} \\ &\times \prod_{j=1}^{n-1} \frac{\Gamma(2k+2j-2n-1)}{\Gamma(2k+j-n-2)} \times \frac{\Gamma(k+m/2-1)\Gamma(k+m/2-1/2)\Gamma(k-n)\Gamma(2k+m-n-1)}{\Gamma(k)\Gamma(k-1/2)\Gamma(k-1)\Gamma(2k+m-2)} \\ &\times \frac{L(F,l,\mathrm{St})}{\pi^{nk+m+l(n+1)-n(n+1)/2} < F,F >}, \end{aligned}$$

where

$$\rho_{k,\nu} = \prod_{i=0}^{\nu-1} \prod_{j=1}^{n} (-k+\nu+(j-1)/2-i).$$

Then the following result is a special case of the pullback formula for the Siegel Eisenstein series in [BSY] and [Koz]:

Proposition 7.1. *Assume that* $\dim S_{k,m}(\Gamma_n) = 1$ *and let* F *be a generator of* $S_{k,m}(\Gamma_n)$. *Let* l *be an integer such that* $l \equiv n \bmod 2$, *and* $n+3 \leq l < k$ *or* $n+1 \leq l < k$ *according as* $n \equiv 1 \bmod 4$ *or not. Then we have*

$$F_{l,(k,m)}(Z_1, Z_2) = \Lambda(F, l-n, \mathrm{St})(\overline{F(-\overline{Z_1})} \otimes F(Z_2)).$$

Furthermore assume that $\dim M_{k,m}(\Gamma_n) = \dim S_{k,m}(\Gamma_n) = 1$. *Then we have*

$$F_{k,(k,m)}(Z_1, Z_2) = \Lambda(F, k-n, \mathrm{St})(\overline{F(-\overline{Z_1})} \otimes F(Z_2)).$$

Here note that the right hand side does not depend on the choice of F. Also we have $\overline{F(-\overline{Z})} = F(Z)$ if F has real Fourier coefficients, which is always the case under the assumption of this proposition, for a suitably scaled generator.

Proof. Assume that $\dim S_{k,m}(\Gamma_n) = 1$. Then $F_{l,(k,m)}(Z_1, Z_2)$ can be expressed as

$$F_{l,(k,m)}(Z_1, Z_2) = d\overline{F(-\overline{Z_1})} \otimes F(Z_2)$$

with some constant d. Thus we have

$$\langle F(Z_2), F_{l,(k,m)}(-\overline{Z_1}, Z_2)\rangle = \bar{d}F(Z_1)\langle F, F\rangle.$$

On the other hand, by the formula (4.1) in [Koz], we have

$$\langle F(Z_2), F_{l,(k,m)}(-\overline{Z_1}, Z_2)\rangle = cF(Z_1)$$

with

$$c = (2\pi i)^{l-k}\zeta(1-l)\prod_{i=1}^{n}\zeta(1-2l+2i)$$
$$\times \frac{\rho_{k,k-l}}{(k)_m m!} 2^{n(n+1-k)-m+1} i^{nk} \pi^{n(n+1)/2} \prod_{j=1}^{n-1} \frac{\Gamma(2k+2j-2n-1)}{\Gamma(2k+j-n-2)}$$
$$\times \frac{\Gamma(k+m/2-1)\Gamma(k+m/2-1/2)\Gamma(k-n)\Gamma(2k+m-n-1)}{\Gamma(k)\Gamma(k-1/2)\Gamma(k-1)\Gamma(2k+m-2)}$$
$$\times (\zeta(l)\prod_{j=1}^{n}\zeta(2l-2j))^{-1} L(F, l-n, \mathrm{St}).$$

Thus we have $d = \bar{c}\langle F, F\rangle^{-1}$. By a simple calculation we can show that $c = \Lambda(F, l-n, \mathrm{St})\langle F, F\rangle$. We note that c is a real number, and therefore we have $d = \Lambda(F, l-n, \mathrm{St})$. This proves the first assertion. Similarly the second assertion holds. □

The differential operators described above are very useful to get the arithmetic properties of the standard-L-values. However it does not seem so easy to get exact standard L-values by using them. But in [I1] we have another general characterization of differential operators which behave well under the restrictions of the domains equivariant with the action of the real symplectic group on both domains. These differential operators contain as a part of their formulation the restriction to the locus $Z_{12} = 0$ after the action of the above Böcherer's operators, and besides they are easier to handle. So we use this formulation below. We extract what we need from the theorem in [I1]. We take a positive integer l and put $d = 2l$. Let $X = (x_{rs})$ be an $n \times d$ matrix of variable components and for $1 \leq i, j \leq n$, we put $\Delta_{ij} = \sum_{s=1}^{d} \frac{\partial^2}{\partial x_{is}\partial x_{js}}$. A polynomial $P(X)$ in the entries of X is called pluriharmonic if $\Delta_{ij}P = 0$ for any $1 \leq i, j \leq n$. We fix non-negative integers ν and m and take

a polynomial mapping $P(X_1, X_2)$ from $M_{n,d}(\mathbb{C}) \times M_{n,d}(\mathbb{C})$ to $V_{n,1}^{(m)} \otimes V_{n,2}^{(m)}$ such that
D-1. $P(X_1, X_2)$ is pluriharmonic in each X_i $(i = 1, 2)$.
D-2. $P(X_1 g, X_2 g) = P(X_1, X_2)$ for any $g \in O(d)$, where $O(d)$ is the orthogonal group of degree $d = 2l$.
D-3. $P(a_1 X_1, a_2 X_2) = (\tau_{n;(\nu,m)}(a_1) \otimes \tau_{n;(\nu,m)}(a_2))P(X_1, X_2)$ for $a_1, a_2 \in GL_n(\mathbb{C})$.
Assume that $l \geq n$. Then there exists a unique polynomial mapping $Q(\mathfrak{S})$ from $\mathcal{S}_{2n}(\mathbb{C})$ to $V_{n,1}^{(m)} \otimes V_{n,2}^{(m)}$ such that $P(X_1, X_2) = Q\left(\begin{pmatrix} X_1 {}^t X_1 & X_1 {}^t X_2 \\ X_2 {}^t X_1 & X_2 {}^t X_2 \end{pmatrix}\right)$, where $\mathcal{S}_{2n}(\mathbb{C})$ denotes the set of symmetric matrices of degree $2n$ with entries in $\mathbb{C}$. We note that Q is homogeneous of degree $n\nu + m$. For any holomorphic function f on $\mathfrak{H}_{2n}$, we define $\mathcal{D}_Q(f)$ and $\tilde{\mathcal{D}}_Q(f)$ by

$$\mathcal{D}_Q(f) = Q\left(\frac{\partial}{\partial Z}\right)(f)$$

and

$$\tilde{\mathcal{D}}_Q(f) = \mathcal{D}_Q(f)\big|_{Z_{12}=0},$$

where we write $Z = \begin{pmatrix} Z_1 & Z_{12} \\ {}^t Z_{12} & Z_2 \end{pmatrix}$ with $Z_1, Z_2 \in \mathfrak{H}_n$ and $Z_{12} \in M_n(\mathbb{C})$. On the other hand, for $i = 1, 2$, take $g_i = \begin{pmatrix} A_i & B_i \\ C_i & D_i \end{pmatrix} \in Sp(n, \mathbb{R})$ and put

$$\iota(g_1, g_2) = \begin{pmatrix} A_1 & 0 & B_1 & 0 \\ 0 & A_2 & 0 & B_2 \\ C_1 & 0 & D_1 & 0 \\ 0 & C_2 & 0 & D_2 \end{pmatrix} \in Sp(2n, \mathbb{R}).$$

Let $\mathcal{D}$ be a $V_{n,1}^{(m)} \otimes V_{n,2}^{(m)}$ -valued linear holomorphic differential operator with constant coefficient. We consider the following condition on $\mathcal{D}$.
D-0. For any holomorphic function $f(Z)$ on $\mathfrak{H}_{2n}$ and any g_1, $g_2 \in Sp(n, \mathbb{R})$, we have

$$\begin{aligned} &\left(\mathcal{D}\left(f(\iota(g_1, g_2)Z) \det\left(\begin{pmatrix} C_1 & 0 \\ 0 & C_1 \end{pmatrix} Z + \begin{pmatrix} D_1 & 0 \\ 0 & D_2 \end{pmatrix}\right)^{-l}\right)\right)\Big|_{Z_{12}=0} \\ = \quad & \tau_{n,(\nu,m)}((C_1 Z_1 + D_1)^{-1}) \otimes \tau_{n,(\nu,m)}((C_2 Z_2 + D_2)^{-1})(\mathcal{D}f)\begin{pmatrix} g_1 Z_1 & 0 \\ 0 & g_2 Z_2 \end{pmatrix} \end{aligned}$$

Theorem 7.2 ([I1]). *We fix a positive integer $d = 2l \geq n$ and non-negative integers ν and m.*

(1) Notation being as above, there exists a polynomial P that satisfies D-1 to D-3. It, and the associated Q, are unique up to constant multiples.

(2) For Q as in (1), $\mathcal{D}_Q$ satisfies the condition D-0.

(3) Any linear holomorphic differential operator with constant coefficients, satisfying D-0, is of the form $\mathcal{D}_Q$ for some Q associated with a P satisfying D-1 to D-3.

The effect of the action of $\tilde{\mathcal{D}}_Q$ on the Fourier expansion is easily described as far as Q is explicitly given. We consider the action of the above operator on the Siegel Eisenstein series. We denote by $\mathcal{H}_m(\mathbb{Z})$ the set of half-integral matrices of

degree m over $\mathbb{Z}$. Furthermore we denote by $\mathcal{H}_m(\mathbb{Z})_{>0}$ (resp. $\mathcal{H}_m(\mathbb{Z})_{\geq 0}$) the subset of $\mathcal{H}_m(\mathbb{Z})$ consisting of positive definite (resp semi-positive definite) matrices. Let

$$E_{2n,l}(Z) = \sum_A c_{2n,l}(A)(\mathrm{tr}(AZ))$$

be the Fourier expansion of the Siegel Eisenstein series. Put $G_{l,Q}(Z_1, Z_2) := \tilde{\mathcal{D}}_Q(E_{2n,l})(Z_1, Z_2)$. Then, $G_{l,Q}(Z_1, Z_2)$ belongs to $S_{l+\nu,m}(\Gamma_n) \otimes S_{l+\nu,m}(\Gamma_n)$, and we have

$$G_{l,Q}(Z_1, Z_2) = (2\pi i)^{n\nu} \sum_{A_1, A_2 \in \mathcal{H}_n(\mathbb{Z})_{>0}} \exp(2\pi i \mathrm{tr}(A_1 Z_1 + A_2 Z_2))$$
$$\times \sum_{R \in M_n(\mathbb{Z})} Q\left(\begin{pmatrix} A_1 & \frac{1}{2}R \\ \frac{1}{2}{}^tR & A_2 \end{pmatrix}\right) c_{2n,l}\left(\begin{pmatrix} A_1 & \frac{1}{2}R \\ \frac{1}{2}{}^tR & A_2 \end{pmatrix}\right).$$

By the claim (3) of the above theorem, we have

Proposition 7.3. *Under the above notation and the assumption, we have*

$$\mathcal{D}_{l,(l+\nu,m)} = d_Q \tilde{\mathcal{D}}_Q,$$

where d_Q *is a non-zero constant. Therefore we have*

$$F_{l,(l+\nu,m)}(Z_1, Z_2) = d_Q G_{l,Q}(Z_1, Z_2).$$

When $\nu = 0$, for general m, the polynomial P is obtained using the classical Gegenbauer polynomials and when $n = 2$ and $m = 0$, a generating function of P is given (cf. in [I1] p.114). When both ν and m are positive, it is not so easy to find a polynomial $P(X_1, X_2)$ or $Q(\mathfrak{S})$ satisfying the above conditions. Here we give two examples. Let $\mathfrak{S}$ be a 4×4 symmetric matrix of variables and $U = (u_1, u_2, u_3, u_4)$ a vector of variables. We divide U into $u := (u_1, u_2)$ and $v := (u_3, u_4)$. We also divide $\mathfrak{S}$ as $\mathfrak{S} = \begin{pmatrix} R & T \\ {}^tT & S \end{pmatrix}$ with R, S symmetric 2×2 matrices and T a 2×2 matrix. First define a polynomial $\phi_{k,(k,4)}(\mathfrak{S}, U)$ as follows:

$$\phi_{k,(k,4)}(\mathfrak{S}, U) = \frac{k(k-1)}{6}\left(4(k+1)(k+2)s^4 - 12(k+1)s^2 m_0 + 3m_0^2\right),$$

where

$$\begin{aligned}
m_1 &= m_1(R, u) = (u_1, u_2) R \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \\
&= r_{11}u_1^2 + 2r_{12}u_1u_2 + r_{22}u_2^2 \\
m_2 &= m_2(S, v) = (u_3, u_4) S \begin{pmatrix} u_3 \\ u_4 \end{pmatrix} \\
&= s_{11}u_3^2 + 2s_{12}u_3u_4 + s_{22}u_4^2 \\
m_0 &= m_0(R, S, T, u, v) \\
&= m_1 m_2 = (r_{11}u_1^2 + 2r_{12}u_1u_2 + r_{22}u_2^2)(s_{11}u_3^2 + 2s_{12}u_3u_4 + s_{22}u_4^2) \\
s &= s(R, S, T, u, v) = (u_1, u_2) T \begin{pmatrix} u_3 \\ u_4 \end{pmatrix} \\
&= t_{11}u_1u_3 + t_{12}u_1u_4 + t_{21}u_2u_3 + t_{22}u_2u_4
\end{aligned}$$

for $R = \begin{pmatrix} r_{11} & r_{12} \\ r_{12} & r_{22} \end{pmatrix}, S = \begin{pmatrix} s_{11} & s_{12} \\ s_{12} & s_{22} \end{pmatrix}$ and $T = \begin{pmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{pmatrix}$. Then this is associated with P satisfying D-1 to D-3 for $\nu = 0$, $d = 2k$ and $m = 4$. This has been

already known in [I1] p. 114. Next we treat the case $d = 2l = 2k-4$, $\nu = 2$, and $m = 4$. This case is more complicated. First we explain an outline, then we give an explicit solution. Inside the space of polynomials in the entries of R, S, T, we seek subspaces realising the representation $\tau_{2,(2,4)} \otimes \tau_{2,(2,4)}$ of $GL(2) \times GL(2)$. Since $(A, B) \in GL(2) \times GL(2)$ acts on R, S, T by $AR{}^tA$, $BS{}^tB$ and $AT{}^tB$, if we denote each degree with respect to entries of R, S or T by a, b, $2c$, then, considering degrees in the entries of A and in the entries of B, we have $2a + 2c = 2b + 2c = n\nu + m = 8$ (hence c *is* an integer). Hence $a = b$, and the total degree in the entries of R, S and T is $a+b+2c = 8$. Calculating the characters, we can easily see the following facts. As a representation space of $GL(2) \times GL(2)$, the space of degree a polynomials in the entries of R decomposes into

$$\sum_{\nu=0}^{[a/2]} \tau_{2,(2\nu,2a-4\nu)} \otimes \tau_{2,(0,0)}$$

where $\tau_{2,(0,0)}$ is the trivial representation of $GL(2)$, and the space of degree a polynomials in the entries of S decomposes in the same way, where the left and the right of the tensor are transposed. As a representation space of $GL(2) \times GL(2)$, the space of polynomials of degree $2c$ in the entries of T decomposes as

$$\sum_{\nu=0}^{c} \tau_{2,(\nu,2c-2\nu)} \otimes \tau_{2,(\nu,2c-2\nu)}.$$

The space of homogenous polynomials of total degree 8 in the entries of R, S and T, with $a = b$, is a sum over $a + c = 4$ of tensor products of these three spaces. The irreducible decomposition of tensor products of symmetric tensor representations is known by Clebsch-Gordan. So we can easily count the multiplicity of $\tau_{2,(2,4)} \otimes \tau_{2,(2,4)}$, and it is 15. Now we consider polynomials $P(R, S, T, u, v)$, homogeneous of total degree 8 in the entries of R, S, T, and homogeneous of degree 4 in u_1, u_2 and in u_3, u_4, where we put $u = (u_1, u_2)$, $v = (u_3, u_4)$, such that

$$P(AR{}^tA, BS{}^tB, AT{}^tB, u, v). = \det(AB)^2 P(R, S, T, uA, vB).$$

Then the coefficients of such a P as a polynomial in the u_i give a basis of a representation space of $\tau_{2,(2,4)} \otimes \tau_{2,(2,4)}$. So the first task is to give 15 linearly independent such polynomials. The second task is to find, among their linear combinations, a polynomial pluri-harmonic with respect to each of X_1 and X_2, which is assured to exist uniquely up to constants. Proceeding along these lines, we define a polynomial $\phi_{k-2,(k,4)}(\mathfrak{S}, U)$ by

$$\begin{aligned}
&\phi_{k-2,(k,4)}(\mathfrak{S}, U) = \\
&\quad 4(d+6)(d+8)(d-2)(d+3)(d+4)P_0\, s^4 + 4(d+6)(d+8)(d+3)(d+4)P_1\, s^4 \\
&\quad +8(d+6)(d+8)(11d-12)P_2 s^4 - 8(d+6)(d+8)(d-3)(d+3)\det(T)Q_0\, s^2 \\
&\quad -48(d+6)(d+8)(d-3)Q_1 s^2 - 8(d+6)(d+3)(5d^2+4d-36)P_0 s^2 m_0 \\
&\quad +(60d^3+156d^2-648d-576)P_0\, m_0^2 - 72(d+6)(d^2+5d-10)P_1\, s^2 m_0 \\
&\quad -48(5d+6)(d+6)P_2\, m_0 s^2 + 24(d-3)(d+6)(d+1)\det(T)Q_0\, m_0 \\
&\quad +48(d-3)(d+6)Q_2 m_0 - 48(d+6)(d+8)(d-3)Q_2 s^2 \\
&\quad +12(5d^2+15d-48)P_1\, m_0^2 + 48(d-3)(d+6)Q_1 m_0 + 72(d+4)P_2 m_0^2,
\end{aligned}$$

where m_i $(i = 0, 1, 2)$ and s are as before and

$$\begin{aligned}
P_0 &= P_0(R,S,T) = \det(T)^2\\
P_2 &= P_2(R,S,T) = \det(RS)\\
P_1 &= P_1(R,S,T) = -r_{11}t_{21}^2s_{22} + 2r_{11}t_{21}t_{22}s_{12} - r_{11}t_{22}^2s_{11}\\
&\quad +2r_{12}t_{21}t_{11}s_{22} - 2r_{12}t_{21}t_{12}s_{12} - 2r_{12}t_{22}t_{11}s_{12}\\
&\quad +2r_{12}t_{22}t_{12}s_{11} - r_{22}t_{11}^2s_{22} + 2t_{11}r_{22}t_{12}s_{12} - r_{22}t_{12}^2s_{11}
\end{aligned}$$

$$\begin{aligned}
Q_r &= Q_r(R,T,v)\\
&= (r_{11}t_{21}^2 - 2r_{12}t_{11}t_{21} + r_{22}t_{11}^2)u_3^2\\
&\quad +2(r_{11}t_{21}t_{22} - r_{12}(t_{11}t_{22} + t_{12}t_{21}) + r_{22}t_{11}t_{12})u_3u_4\\
&\quad +(r_{11}t_{22}^2 - 2r_{12}t_{12}t_{22} + r_{22}t_{12}^2)u_4^2\\
Q_s &= Q_s(S,T,u)\\
&= (s_{11}t_{12}^2 - 2s_{12}t_{11}t_{12} + s_{22}t_{11}^2)u_1^2\\
&\quad +2(s_{11}t_{12}t_{22} - s_{12}(t_{11}t_{22} + t_{12}t_{21}) + s_{22}t_{11}t_{21})u_1u_2\\
&\quad +(s_{11}t_{22}^2 - 2s_{12}t_{21}t_{22} + s_{22}t_{21}^2)u_2^2\\
Q_1 &= Q_1(R,S,T,u,v) = Q_r m_1 \det(S)\\
Q_2 &= Q_2(R,S,T,u,v) = Q_s m_2 \det(R)
\end{aligned}$$

$$\begin{aligned}
Q_0 &= Q_0(R,S,T,u,v) = (u_1^2, u_1u_2, u_2^2)\Big(q_{ij}\Big)_{1\le i,j\le 3}\begin{pmatrix} u_3^2\\ u_3u_4\\ u_4^2\end{pmatrix}\\
&= q_{11}u_1^2u_3^2 + q_{12}u_1^2u_3u_4 + q_{13}u_1^2u_4^2\\
&\quad +q_{21}u_1u_2u_3^2 + q_{22}u_1u_2u_3u_4 + q_{23}u_1u_2u_4^2\\
&\quad +q_{31}u_2^2u_3^2 + q_{32}u_2^2u_3u_4 + q_{33}u_2^2u_4^2,
\end{aligned}$$

where

$$\begin{aligned}
q_{11} &= 2s_{11}(r_{11}t_{11}t_{22} + r_{11}t_{12}t_{21} - 2r_{12}t_{11}t_{12}) - 4t_{11}s_{12}(-t_{11}r_{12} + r_{11}t_{21})\\
q_{21} &= 4s_{11}(r_{11}t_{21}t_{22} - r_{22}t_{12}t_{11}) - 4s_{12}(r_{11}t_{21}^2 - r_{22}t_{11}^2)\\
q_{31} &= 2s_{11}(2r_{12}t_{21}t_{22} - r_{22}t_{11}t_{22} - r_{22}t_{12}t_{21}) - 4s_{12}t_{21}(r_{12}t_{21} - t_{11}r_{22})\\
q_{12} &= 4r_{11}s_{11}t_{12}t_{22} - 4r_{12}s_{11}t_{12}^2 - 4r_{11}s_{22}t_{11}t_{21} + 4r_{12}s_{22}t_{11}^2\\
q_{22} &= 4r_{11}s_{11}t_{22}^2 - 4r_{22}t_{12}^2s_{11} - 4r_{11}s_{22}t_{21}^2 + 4r_{22}s_{22}t_{11}^2\\
q_{32} &= 4r_{12}s_{11}t_{22}^2 - 4r_{22}s_{11}t_{12}t_{22} - 4r_{12}s_{22}t_{21}^2 + 4r_{22}s_{22}t_{11}t_{21}\\
q_{13} &= 4r_{11}s_{12}t_{12}t_{22} - 4r_{12}s_{12}t_{12}^2 - 2r_{11}s_{22}t_{11}t_{22} - 2r_{11}s_{22}t_{12}t_{21} + 4r_{12}s_{22}t_{11}t_{12}\\
q_{23} &= 4r_{11}s_{12}t_{22}^2 - 4r_{22}s_{12}t_{12}^2 - 4r_{11}s_{22}t_{21}t_{22} + 4r_{22}s_{22}t_{11}t_{12}\\
q_{33} &= 4r_{12}s_{12}t_{22}^2 - 4r_{22}s_{12}t_{12}t_{22} - 4r_{12}s_{22}t_{21}t_{22} + 2r_{22}s_{22}t_{11}t_{22} + 2r_{22}s_{22}t_{12}t_{21}
\end{aligned}$$

Here we note that

$$\begin{aligned} P_i(AR^tA, BS^tB, AT^tB) &= \det(AB)^2 P_i(R,S,T), \\ m_0(AR^tA, BS^tB, AT^tB, u, v) &= m_0(R,S,T,uA,vB), \\ s(AR^tA, BS^tB, AT^tB, u, v) &= s(R,S,T,uA,vB), \\ Q_r(AR^tA, AT^tB, v) &= \det(A)^2 Q_r(R,T,vB) \\ Q_s(BS^tB, AT^tB, u) &= \det(B)^2 Q_s(S,T,uA) \\ Q_0(AR^tA, BS^tB, AT^tB, u, v) &= \det(AB)^2 Q_0(R,S,T,uA,vB) \end{aligned}$$

So the 15 terms in $\phi_{k-2,(k,4)}$ give the isobaric components associated with the representation $\tau_{2,(2,4)} \otimes \tau_{2,(2,4)}$ of $GL(2) \times GL(2)$ with multiplicity 15 (if they are linearly independent). The condition of pluri-harmonicity determines the coefficients given by the polynomials in $d = 2k-4$. Since there is no ready-made program suitable for this calculation, this part is a fairly elaborate hand calculation with the aid of Maple.

Then for $v = 0$ or 2 put

$$\Phi_{k-v,(k,4)} = \phi_{k-v,(k,4)}\left(\frac{\partial}{\partial Z}, U\right)\Big|_{Z_{12}=O_2}.$$

We note that $\Phi_{k-v,(k,4)}$ is a polynomial in $\frac{\partial}{\partial Z}$ and U. Then by a direct but long and elaborate calculation with the aid of Maple we have

Proposition 7.4. *Assume that* $k \geq 4$, *and let* $l = k$ *or* $k-2$. *Define a polynomial* $P_{l,(k,4)}(X_1, X_2)$ *by*

$$P_{l,(k,4)}(X_1, X_2) = \phi_{l,(k,4)}\left(\begin{pmatrix} X_1\,{}^tX_1 & X_1\,{}^tX_2 \\ X_2\,{}^tX_1 & X_2\,{}^tX_2 \end{pmatrix}, U\right)$$

for $X_1, X_2 \in M_{2,2l}(\mathbb{C})$. *Then* $P_{l,(k,4)}(X_1, X_2)$ *is a polynomial mapping from* $M_{2,2l}(\mathbb{C}) \times M_{2,2l}(\mathbb{C})$ *to* $V_{2,1}^{(4)} \otimes V_{2,2}^{(4)}$, *and satisfies the conditions D-1~D-3 stated above for the representation* $\tau_{2;k-l,4}$. *Therefore we have*

$$\mathcal{D}_{l,(k,4)} = c_{l,(k,4)} \Phi_{l,(k,4)}$$

for some non-zero rational number $c_{l,(k,4)}$.

Now we shall consider the prime factors of $c_{l,(k,4)}$ more precisely.

Lemma 7.5. *Let* $u = (u_1, u_2)$ *and* $v = (u_3, u_4)$ *be vectors of independent variables, and* $W = (z_{ij})_{1 \leq i \leq 2, 3 \leq j \leq 4}$ *be a* 2×2 *matrix with entries in variables.*

(1) *Define the differential operators* $\left|\frac{\partial}{\partial W}\right|$ *and* $\mathcal{E}$ *as*

$$\left|\frac{\partial}{\partial W}\right| = \det\left(\frac{1}{2}\frac{\partial}{\partial z_{ij}}\right)_{1 \leq i \leq 2, 3 \leq j \leq 4},$$

and

$$\mathcal{E} = \frac{1}{2}\sum_{i=1}^{2}\sum_{j=3}^{4} z_{ij}\frac{\partial}{\partial z_{ij}}.$$

Then we have

$$\left|\frac{\partial}{\partial W}\right|((uW^tv)^m (\det W)^\sigma) = \frac{\sigma(\sigma+m+1)}{4}(uW^tv)^m (\det W)^{\sigma-1}$$

and

$$\mathcal{E}((uW^t v)^m (\det W)^{\sigma-1}) = \frac{2\sigma+m-2}{2}(uW^t v)^m (\det W)^{\sigma-1}$$

for any non-negative integer m *and positive integer* σ.

(2) *We have* $\tilde{D} - \tilde{D}_\uparrow - \tilde{D}_\downarrow = u\left(\frac{\partial}{\partial z_{ij}}\right)_{1\le i\le 2, 3\le j\le 4}{}^t v$ *and*

$$(\tilde{D} - \tilde{D}_\uparrow - \tilde{D}_\downarrow)((uW^t v)^m) = m(uW^t v)^{m-1}\langle u,u\rangle\langle v,v\rangle,$$

and

$$(\tilde{D}_\uparrow \tilde{D}_\downarrow)(uW^t v)^m) = 0,$$

for any non-negative integers m, *where* $\langle u,u\rangle = u_1^2 + u_2^2$, $\langle v,v\rangle = u_3^2 + u_4^2$.

Proof. The assertions can be proved directly from the definitions of the differential operators in question. □

Corollary 7.6. *Let* $W = \begin{pmatrix} z_{13} & z_{14} \\ z_{23} & z_{24} \end{pmatrix}$, *and* $u = (u_1, u_2), v = (u_3, u_4)$.

(1) *We have*

$$\tilde{\mathcal{D}}_\alpha^\nu((uW^t v)^m (\det W)^\sigma) = d_{\alpha,(\alpha+\nu,m),\sigma}(uW^t v)^m (\det W)^{\sigma-\nu}$$

for any positive integers α *and* ν *and non-negative integers* σ *and* m, *where*

$$d_{\alpha,(\alpha+\nu,m),\sigma} = \frac{\prod_{i=0}^{\nu-1}(\sigma-i)(\sigma-i+m+1)(\sigma+i+2\alpha-4)(\sigma+i+m+2\alpha-3)}{2^{4\nu}\prod_{i=0}^{\nu-1} C_2(\alpha+i-3/2)}.$$

(2) *We have*

$$\tilde{L}^{k,m}((uW^t v)^m) = \frac{1}{(k)_m}\langle u,u\rangle^m \langle v,v\rangle^m$$

for any positive integer k *and non-negative integer* m.

Proof. By definition we have

$$\tilde{D}_\alpha = C_2(\alpha-3/2)^{-1}\sum_{r=0}^{2}\binom{2}{r} C_r(\alpha-3/2)(1_2^{[r]} \sqcup z_2^{[2-r]}\partial_3^{[2-r]})\det \partial_2 + F\left(Z, \frac{\partial}{\partial Z}\right),$$

where $F(Z, \frac{\partial}{\partial Z})$ is a polynomial in Z and $\frac{\partial}{\partial Z}$ whose degree with respect to ∂_1 and ∂_4 is greater than or equal to 1 and hence acts as zero. Thus we have

$$\tilde{D}_\alpha((uW^t v)^m (\det W)^\sigma) = C_2(\alpha-3/2)^{-1}$$

$$\times \sum_{r=0}^{2}\binom{2}{r} C_r(\alpha-3/2)(1_2^{[r]} \sqcup z_2^{[2-r]}\partial_3^{[2-r]})\det \partial_2((uW^t v)^m (\det W)^\sigma).$$

Using the definitions in [Bö], we have

$$\sum_{r=0}^{2}\binom{2}{r} C_r(\alpha-3/2)(1_2^{[r]} \sqcup z_2^{[2-r]}\partial_3^{[2-r]})\det \partial_2((uW^t v)^m (\det W)^\sigma)$$

$$= \left(\det W\left|\frac{\partial}{\partial W}\right|^2 + (\alpha-3/2)\mathcal{E}\left|\frac{\partial}{\partial W}\right| + (\alpha-3/2)(\alpha-1)\left|\frac{\partial}{\partial W}\right|\right)((uW^t v)^m (\det W)^\sigma),$$

and by (1) of Lemma 7.5 we have

$$\tilde{\mathcal{D}}_\alpha((uW^t v)^m (\det W)^\sigma) = d_{\alpha,(\alpha+1,m),\sigma}(uW^t v)^m (\det W)^{\sigma-1}.$$

Thus the assertion (1) can be proved by repeated application of this formula. The assertion (2) follows directly from (2) of Lemma 7.5. □

Proposition 7.7. *Assume that* $k \geq 4$, *and let* $l = k$ *or* $k-2$. *Let* $c_{l,(k,4)}$ *be as in Proposition 7.4. Then* $c_{l,(k,4)}$ *is a* p*-unit for any prime number* $p > 2k-1$.

Proof. We note that both the operators $\mathcal{D}_{l,(k,4)}$ and $\Phi_{l,(k,4)}$ are polynomials in $\frac{\partial}{\partial Z}$ and they can be also regarded as maps from $C^\infty(\mathfrak{H}_4, V_4^{(8)})$ to $C^\infty(\mathfrak{H}_2, V_{2,1}^{(8)}) \otimes C^\infty(\mathfrak{H}_2, V_{2,2}^{(8)})$. Thus to prove the assertion we apply these two differential operators to the function $(uW^t v)^4(\det W)^{k-l}$, where $W = \begin{pmatrix} z_{13} & z_{14} \\ z_{23} & z_{24} \end{pmatrix}$ and $u = (u_1, u_2), v = (u_3, u_4)$. By Corollary 7.6 we have

$$\mathcal{D}_{l,(k,4)}((uW^t v)^4(\det W)^{k-l}) = \frac{d_{l,(k,4),k-l}}{(k)_4}\langle u, u\rangle^4\langle v, v\rangle^4.$$

Now we apply $\Phi_{l,(k,4)}$ to $(uW^t v)^4(\det W)^{k-l}$. First write the polynomial $\phi_{k,(k,4)}(S, U)$ as

$$\phi_{k,(k,4)}(S, U) = \frac{2k(k-1)(k+1)(k+2)s^4}{3} + \psi(S, U).$$

Then it is easily seen that

$$\psi\left(\frac{\partial}{\partial Z}, U\right)((uW^t v)^4)|_{\mathfrak{H}_2\times\mathfrak{H}_2} = 0.$$

Thus we have

$$\Phi_{k,(k,4)}((uW^t v)^4) = \frac{2k(k-1)(k+1)(k+2)}{3}\left(u\left(\frac{1}{2}\frac{\partial}{\partial z_{ij}}\right)_{i=1,2,j=3,4}{}^t v\right)^4((uW^t v)^4)|_{\mathfrak{H}_2\times\mathfrak{H}_2}$$

$$= \frac{2k(k-1)(k+1)(k+2)}{3}\frac{4!}{2^4}\langle u, u\rangle^4\langle v, v\rangle^4$$

by repeated application of (2) of Lemma 7.5, and hence $c_{k,(k,4)} = ((k-1)k^2(k+1)^2(k+2)^2(k+3))^{-1}$ is p-unit for any prime number $p > 2k-1$ if $k \geq 4$. Similarly the terms except for those coming from $P_0 s^4$ acts as zero on $(uW^t v)^4(\det W)^2$, so we have

$$\Phi_{k-2,(k,4)}((uW^t v)^4(\det W)^2) = 2^3\cdot 3^2\cdot 7k(k+1)(k+2)(k-3)(2k-1)\langle u, u\rangle^4\langle v, v\rangle^4.$$

We also have

$$\begin{aligned}\mathcal{D}_{k-2,(k,4)}((uW^t v)^4(\det(W)^2)) &= \frac{d_{k-2,(k,4),2}}{(k)_4}\langle u, u\rangle^4\langle v, v\rangle^4 \\ &= \frac{21}{4}\frac{2k-1}{(k+1)(k+2)(k+3)(2k-7)}\langle u, u\rangle^4\langle v, v\rangle^4\end{aligned}$$

Hence $c_{k-2,(k,4)} = (96(2k-7)(k-2)(k-3)k(k+1)^2(k+2)^2(k+3))^{-1}$ is a p-unit for any prime number $p > 2k-1$, if $k \geq 4$. □

To get exact standard L-values of Siegel modular forms, we need an explicit formula for the Fourier coefficients of $E_{4,l}$. Let A be an element of $\mathcal{H}_m(\mathbb{Z})_{>0}$ with m even. We then denote by ξ_A the Kronecker character corresponding to the

extension $\mathbb{Q}(\sqrt{(-1)^{m/2}\det A})/\mathbb{Q}$. Let $A = \begin{pmatrix} a_{11} & a_{12}/2 \\ a_{12}/2 & a_{22} \end{pmatrix} \in \mathcal{H}_2(\mathbb{Z})_{\geq 0}$. First assume $\mathrm{rank}(A) = 1$. Then we define

$$F_p^{(1)}(A, X) = \sum_{i=0}^{\mathrm{ord}_p(\mathrm{cont} A)} (pX)^i.$$

Next assume $A > 0$. Then we can write $D_A = \mathfrak{d}_A f_A^2$ with $\mathfrak{d}_A$ a fundamental discriminant and $f_A \in \mathbb{Z}_{>0}$. Then we define

$$F_p^{(2)}(A, X) = \sum_{i=0}^{\mathrm{ord}_p(\mathrm{cont} A)} (p^2X)^i \sum_{j=0}^{\mathrm{ord}_p(f_A)-i} (p^3X^2)^j$$

$$-\xi_A(p)pX \sum_{i=0}^{\mathrm{ord}_p(\mathrm{cont} A)} (p^2X)^i \sum_{j=0}^{\mathrm{ord}_p(f_A)-i-1} (p^3X^2)^j.$$

Then we have the following (cf. [Ka2].)

Proposition 7.8. *Let* $A = \begin{pmatrix} A_1 & R/2 \\ {}^tR/2 & A_2 \end{pmatrix}$ *be an element of* $\mathcal{H}_4(\mathbb{Z})_{\geq 0}$ *of rank* m *with* $A_1, A_2 \in \mathcal{H}_2(\mathbb{Z})_{>0}, R \in M_2(\mathbb{Z})$. *Fix a prime number* p_0. *Assume that* $2A_1 \in GL_2(\mathbb{Z}_p)$ *for any prime number* $p \neq p_0$, *and* $2A_2 \in GL_2(\mathbb{Z}_{p_0})$. *Then we have* $m \geq 3$ *and the* A*-th Fourier coefficient* $c_{4,l}(A)$ *of* $E_{4,l}$ *is given by*

$$c_{4,l}(A) = 2^2 F_{p_0}^{(m-2)}\left(A_1 - \frac{1}{4}RA_2^{-1}{}^tR, \xi_{A_2}(p_0)p_0^{l-m}\right)$$

$$\times \prod_{p \neq p_0} F_p^{(m-2)}\left(A_2 - \frac{1}{4}{}^tRA_1^{-1}R, \xi_{A_1}(p)p^{l-m}\right) \times \begin{cases} L(3-l, \xi_A) & \text{if } m = 4 \\ \zeta(5-2l) & \text{if } m = 3 \end{cases},$$

Now put $G_{l,(k,4)}(Z_1, Z_2) = \Phi_{l,(k,4)}(E_{4,l})\left(\begin{pmatrix} Z_1 & O \\ O & Z_2 \end{pmatrix}\right)$. Assume that $G_{l,(k,4)}(Z_1, Z_2)$ belongs to belongs to $S_{k,4}(\Gamma_2) \otimes S_{k,4}(\Gamma_2)$. Then, by the remark before Proposition 7.3, $G_{l,(k,4)}(Z_1, Z_2)$ can be written as

$$G_{l,(k,4)}(Z_1, Z_2) = \sum_{A,B \in \mathcal{H}_2(\mathbb{Z})_{>0}} \epsilon_{l,(k,4)}(A_1, A_2; U) \exp(2\pi i \mathrm{tr}(A_1Z_1 + A_2Z_2)),$$

where

$$\epsilon_{l,(k,4)}(A_1, A_2; U) = \sum_{R \in M_2(\mathbb{Z})} c_{4,l}\left(\begin{pmatrix} A_1 & R/2 \\ {}^tR/2 & A_2 \end{pmatrix}\right) \phi_{l,(k,4)}\left(\begin{pmatrix} A_1 & R/2 \\ {}^tR/2 & A_2 \end{pmatrix}, U\right).$$

Now we have computed $\epsilon_{l,(k,4)}$ exactly in some cases with Mathematica.

Theorem 7.9. *With the above notation, we have*

$$\epsilon_{8,(10,4)}(\mathbb{A}_2, 1_2; U) = 10391040(u_1^2 + u_1u_2 + u_2^2)^2 \otimes (u_3^4 - 9u_3^2u_4^2 + u_4^4)$$

and

$$\epsilon_{10,(10,4)}(\mathbb{A}_2, 1_2; U) = -17^{-1} \cdot 10886400(u_1^2 + u_1u_2 + u_2^2)^2 \otimes (u_3^4 - 9u_3^2u_4^2 + u_4^4),$$

where $\mathbb{A}_2 = \begin{pmatrix} 1 & 1/2 \\ 1/2 & 1 \end{pmatrix}$ *and* $1_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

Now we note that $\dim M_{10,4}(\Gamma_2) = \dim S_{10,4}(\Gamma_2) = 1$. Fix a Hecke eigenform F of $S_{10,4}(\Gamma_2)$. Then by Propositions 7.1 and 7.4, for $l = 8, 10$, we have

$$G_{l,(10,4)}(Z_1, Z_2) = c_{l,(10,4)}^{-1} \Lambda(F, l-2, St)(\overline{F(-\overline{Z_1})} \otimes F(Z_2)).$$

Thus we have

Corollary 7.10. *We have*

$$\Lambda(F, 6, St) = 10391040\ c_{8,(10,4)}, \text{ and } \Lambda(F, 8, St) = -17^{-1} \cdot 10886400\ c_{10,(10,4)}.$$

We note that we have

$$10391040/(17^{-1} \cdot 10886400) = 2 \cdot 3^{-3} \cdot 5^{-1} \cdot 7^{-1} \cdot 11 \cdot 17 \cdot 41.$$

Thus by Proposition 7.7 we have

Theorem 7.11. *With the above notation, we have*

$$\mathrm{ord}_{41}(\Lambda(F, 6, St)/\Lambda(F, 8, St)) = 1.$$

We note that

$$\Lambda(F, 6, St)/\Lambda(F, 8, St) = -\rho_{10,2}/\rho_{10,0} 2^6 \pi^6 L(F, 6, St)/L(F, 8, St).$$

Thus we have

Corollary 7.12. *We have*

$$\mathrm{ord}_{41}(\pi^6 L(F, 6, St)/L(F, 8, St)) = 1.$$

8. Computational support for Conjecture 5.4: $k = 11, m = 10, \ell = 97$.

The method of the following calculation is the same as we outlined in the last section. First, letting $d = 2k-2$, we define a polynomial $\phi_{k-1,(k,10)}(\mathfrak{S}, U)$ as

$$\begin{aligned}\phi_{k-1,(k,10)}(\mathfrak{S}, U) = \big(&-2(d+10)(d+8)(d+18)(d+16)(d+14)(d+12)s^{10}\\ &+20(d+16)(d+14)(5d+36)(d+12)(d+10)s^8 m_0\\ &-540(d+14)(d+12)(d+10)(3d+20)s^6 m_0^2 + 5040(d+10)(2d+13)(d+12)s^4 m_0^3\\ &-3150(d+10)(7d+48)s^2 m_0^4 + 11340(d+8)m_0^5\big)\det(T)\\ &+(5(d+10)(d+14)(d+18)(d+12)(d+16)s^8\\ &-180(d+10)(d+14)(d+12)(d+16)s^6 m_0\\ &+1890(d+10)(d+14)(d+12)s^4 m_0^2 - 6300(d+12)(d+10)s^2 m_0^3\\ &+4725(d+10)m_0^4)Q_0.\end{aligned}$$

Next, letting now $d = 2k-6$, we define a polynomial $\phi_{k-3,(k,10)}(\mathfrak{S}, U)$ by

$$\phi_{k-3,(k,10)}(\mathfrak{S}, U) = \sum_{\nu=1}^{51} w_\nu f_\nu,$$

where

$$
\begin{aligned}
w_1 &= 4725(d+14)(d+21)(11d^2+143d+18)\\
w_2 &= -18900(d+14)(d+21)(3d^2+45d+20)(d+16)\\
w_3 &= 13230(d+14)(d+21)(d^2+17d+18)(d+18)(d+16)\\
w_4 &= -180(d+14)(d+20)(d+21)(5d^2+95d+216)(d+18)(d+16)\\
w_5 &= 15(d+14)(20+d)(d+11)(d+10)(d+22)(d+21)(d+18)(d+16)\\
w_6 &= 198450(d+14)(d+10)(d+21)\\
w_7 &= -37800(d+14)(d+21)(9d+64)(d+16)\\
w_8 &= 3780(d+14)(37d+162)(d+21)(d+18)(d+16)\\
w_9 &= -1080(d+14)(20+d)(d+21)(17d+40)(d+18)(d+16)\\
w_{10} &= 30(d+14)(20+d)(23d+22)(d+22)(d+21)(d+18)(d+16)
\end{aligned}
$$

$$
\begin{aligned}
w_{11} &= 37800(d+14)(d+21)(d+16)(d-3)\\
w_{12} &= -37800(d+14)(d+21)(d+18)(d-3)(d+16)\\
w_{13} &= 7560(d+14)(20+d)(d-3)(d+21)(d+18)(d+16)\\
w_{14} &= -360(d+14)(20+d)(d+22)(d+21)(d-3)(d+18)(d+16)\\
w_{15} &= 37800(d+14)(d+21)(d+16)(d-3)\\
w_{16} &= -37800(d+14)(d+21)(d+18)(d-3)(d+16)\\
w_{17} &= 7560(d+14)(20+d)(d-3)(d+21)(d+18)(d+16)\\
w_{18} &= -360(d+14)(20+d)(d+22)(d+21)(d-3)(d+18)(d+16)\\
w_{19} &= 3780(d+21)(44d^3+957d^2+4993d-210)\\
w_{20} &= -9450(d+14)(d+21)(39d^3+831d^2+3614d-1720)
\end{aligned}
$$

$$
\begin{aligned}
w_{21} &= 52920(d+14)(d+21)(3d^3+70d^2+317d-258)(d+16)\\
w_{22} &= -180(d+14)(115d^3+3015d^2+16034d-11184)(d+21)(d+18)(d+16)\\
w_{23} &= 420(d+14)(20+d)(d+10)(d+21)(2d^2+39d+7)(d+18)(d+16)\\
w_{24} &= -6(d+14)(20+d)(d+11)(d+10)(d+12)(d+21)(d+22)(d+18)(d+16)\\
w_{25} &= 476280(d+10)(d+12)(d+21)\\
w_{26} &= -132300(d+21)(d+14)(9d+70)(d+10)\\
w_{27} &= 211680(d+14)(d+21)(4d+27)(d+6)(d+16)\\
w_{28} &= -1080(d+14)(d+21)(209d^2+2020d+3252)(d+18)(d+16)\\
w_{29} &= 120(d+14)(20+d)(d+21)(187d^2+1614d-64)(d+18)(d+16)\\
w_{30} &= -36(d+14)(20+d)(d+10)(19d-22)(d+22)(d+21)(d+18)(d+16)
\end{aligned}
$$

$$
\begin{aligned}
w_{31} &= 132300(d+21)(d+10)(d+14)(d-3)\\
w_{32} &= -151200(d+14)(d+21)(2d+15)(d-3)(d+16)\\
w_{33} &= 7560(d+14)(d+21)(17d+130)(d+18)(d-3)(d+16)\\
w_{34} &= -1440(d+14)(11d+95)(d-3)(20+d)(d+21)(d+18)(d+16)\\
w_{35} &= 540(d+14)(20+d)(d+22)(d+10)(d-3)(d+21)(d+18)(d+16)\\
w_{36} &= 132300(d+21)(d+10)(d+14)(d-3)\\
w_{37} &= -151200(d+14)(d+21)(2d+15)(d-3)(d+16)\\
w_{38} &= 7560(d+14)(d+21)(17d+130)(d+18)(d-3)(d+16)\\
w_{39} &= -1440(d+14)(11d+95)(d-3)(20+d)(d+21)(d+18)(d+16)\\
w_{40} &= 540(d+14)(20+d)(d+22)(d+10)(d-3)(d+21)(d+18)(d+16)
\end{aligned}
$$

$$
\begin{aligned}
w_{41} &= 14175(d+14)(d+21)(d^3+3d^2+4d+116)\\
w_{42} &= -18900(d+14)(d+21)(d^3+7d^2+8d+68)(d+16)\\
w_{43} &= 5670(d+14)(d+21)(d^3+11d^2+20d-4)(d+18)(d+16)\\
w_{44} &= -540(d+14)(20+d)(d+10)(d+21)(d^2+5d-10)(d+18)(d+16)\\
w_{45} &= 15(d+14)(20+d)(d+11)(d+10)(d+22)(d+21)(d-2)(d+18)(d+16)\\
w_{46} &= 3780(d+21)(8d^4+129d^3+361d^2+1530d+12600)\\
w_{47} &= -9450(d+14)(d+21)(5d^4+101d^3+530d^2+1288d+5888)\\
w_{48} &= 17640(d+14)(d+21)(d^4+24d^3+173d^2+426d+648)(d+16)\\
w_{49} &= -180(d+14)(d+10)(d+21)(13d^3+227d^2+876d+144)(d+18)(d+16)\\
w_{50} &= 60(d+14)(20+d)(d+11)(d+10)(2d^2+19d-4)(d+21)(d+18)(d+16)\\
w_{51} &= -2(d+14)(20+d)(d+11)(d+10)d(d+12)(d+21)(d+22)(d+18)(d+16)
\end{aligned}
$$

and

$$
\begin{aligned}
&f_1 = Q_0P_1m_0^4, f_2 = Q_0P_1m_0^3s^2, f_3 = Q_0P_1m_0^2s^4, f_4 = Q_0P_1m_0s^6, f_5 = Q_0P_1s^8,\\
&f_6 = Q_0P_2m_0^4, f_7 = Q_0P_2m_0^3s^2, f_8 = Q_0P_2m_0^2s^4, f_9 = Q_0P_2m_0s^6, f_{10} = Q_0P_2s^8,\\
&f_{11} = Q_0Q_1m_0^3, f_{12} = Q_0Q_1m_0^2s^2, f_{13} = Q_0Q_1m_0s^4, f_{14} = Q_0Q_1s^6, f_{15} = Q_0Q_2m_0^3,\\
&f_{16} = Q_0Q_2m_0^2s^2, f_{17} = Q_0Q_2m_0s^4, f_{18} = Q_0Q_2s^6, f_{19} = \det(T)P_1m_0^5, f_{20} = \det(T)P_1m_0^4s^2,\\
&f_{21} = \det(T)P_1m_0^3s^4, f_{22} = \det(T)P_1m_0^2s^6, f_{23} = \det(T)P_1m_0s^8, f_{24} = \det(T)P_1s^{10},,\\
&f_{25} = \det(T)P_2m_0^5, f_{26} = \det(T)P_2m_0^4s^2, f_{27} = \det(T)P_2m_0^3s^4, f_{28} = \det(T)P_2m_0^2s^6,\\
&f_{29} = \det(T)P_2m_0s^8, f_{30} = \det(T)P_2s^{10}, f_{31} = \det(T)Q_1m_0^4, f_{32} = \det(T)Q_1m_0^3s^2,\\
&f_{33} = \det(T)Q_1m_0^2s^4, f_{34} = \det(T)Q_1m_0s^6, f_{35} = \det(T)Q_1s^8, f_{36} = \det(T)Q_2m_0^4,\\
&f_{37} = \det(T)Q_2m_0^3s^2, f_{38} = \det(T)Q_2m_0^2s^4, f_{39} = \det(T)Q_2m_0s^6, f_{40} = \det(T)Q_2s^8,\\
&f_{41} = \det(T)^2Q_0m_0^4, f_{42} = \det(T)^2Q_0m_0^3s^2, f_{43} = \det(T)^2Q_0m_0^2s^4, f_{44} = \det(T)^2Q_0m_0s^6,\\
&f_{45} = \det(T)^2Q_0s^8, f_{46} = \det(T)^3m_0^5, f_{47} = \det(T)^3m_0^4s^2, f_{48} = \det(T)^3m_0^3s^4,\\
&f_{49} = \det(T)^3m_0^2s^6, f_{50} = \det(T)^3m_0s^8, f_{51} = \det(T)^3s^{10}.
\end{aligned}
$$

For $\nu = 1, 3$ put

$$\Phi_{k-\nu,(k,10)} = \phi_{k-\nu,(k,10)}\left(\frac{\partial}{\partial Z}, U\right)\Big|_{Z_{12}=O_2}.$$

Then by a more elaborate calculation than that in the previous section, we can show the following.

Proposition 8.1. *Assume that* $k \geq 4$, *and let* $l = k-1$ *or* $k-3$. *Define a polynomial* $P_{l,(k,10)}(X_1, X_2)$ *by*

$$P_{l,(k,10)}(X_1, X_2) = \phi_{l,(k,10)}\left(\begin{pmatrix} X_1\,{}^tX_1 & X_1\,{}^tX_2 \\ X_2\,{}^tX_1 & X_2\,{}^tX_2 \end{pmatrix}, U\right)$$

for $X_1, X_2 \in M_{2,2l}(\mathbb{C})$. *Then* $P_{l,(k,10)}(X_1, X_2)$ *is a polynomial mapping from* $M_{2,2l}(\mathbb{C}) \times M_{2,2l}(\mathbb{C})$ *to* $V_{2,1}^{(10)} \otimes V_{2,2}^{(10)}$, *and satisfies the conditions D-1~D-3 stated above for the representation* $\tau_{2;k-l,10}$.

We can calculate $c_{l,(k,10)}$ similarly as before and we have

$$\begin{aligned} c_{k-1,(k,10)} &= -\frac{2^3}{10!} \times \frac{1}{(k-2)(k+4)_5(k)_{10}} \\ c_{k-3,(k,10)} &= -\frac{2}{10!} \times \frac{1}{(2k-9)(2k+15)(k-4)_3(k+4)_5(k)_{10}} \end{aligned}$$

Thus similarly to Proposition 7.6 we have

Proposition 8.2. *Assume that* $k \geq 4$, *and let* $l = k-1$ *or* $k-3$. *Then we have*

$$\mathcal{D}_{l,(k,10)} = c_{l,(k,10)}\Phi_{l,(k,10)}$$

with a non-zero rational number $c_{l,(k,10)}$, *and in particular* $c_{l,(k,10)}$ *is a* p*-unit for a prime number* $p > 2k+15$.

Now for $l = k-1, k-3$, put $G_{l,(k,10)}(Z_1, Z_2) = \Phi_{l,(k,10)}\left(E_{4,l}\left(\begin{pmatrix} Z_1 & O \\ O & Z_2 \end{pmatrix}\right)\right)$.
Then, similarly to Section 7, $G_{k,l,m}(Z_1, Z_2)$ can be written as

$$G_{l.(k,10)}(Z_1, Z_2) = \sum_{A,B} \epsilon_{l,(k,10)}(A_1, A_2; U) \exp(2\pi i \mathrm{tr}(A_1 Z_1 + A_2 Z_2)),$$

where

$$\epsilon_{l,(k,10)}(A_1, A_2; U) = \sum_{R \in M_2(\mathbf{Z})} c_{4,l}\left(\begin{pmatrix} A_1 & R/2 \\ {}^tR/2 & A_2 \end{pmatrix}\right)$$
$$\times \phi_{l,(k,10)}\left(\begin{pmatrix} A_1 & R/2 \\ {}^tR/2 & A_2 \end{pmatrix}, U\right).$$

Then similarly to Theorem 7.8, we have

Theorem 8.3. *Let the notation be as in Theorem 7.9. Then*

$$\epsilon_{8,(11,10)}(\mathbb{A}_2, 1_2; U) = 61498907532000 u_1 u_2 (u_1^2 - u_2^2)(2u_1 + u_2)(u_1 + 2u_2)(u_1^2 + u_1 u_2 + u_2^2)^2$$
$$\otimes u_3 u_4 (u_3^4 - u_4^4)(5u_3^4 - 8u_3^2 u_4^2 + 5u_4^4),$$

and

$$\epsilon_{10,(11,10)}(\mathbb{A}_2, 1_2; U) = -276449241600 u_1 u_2 (u_1^2 - u_2^2)(2u_1 + u_2)(u_1 + 2u_2)(u_1^2 + u_1 u_2 + u_2^2)^2$$
$$\otimes u_3 u_4 (u_3^4 - u_4^4)(5u_3^4 - 8u_3^2 u_4^2 + 5u_4^4).$$

Note that $\dim S_{11,10}(\Gamma_2) = 1$. Fix a Hecke eigenform G of $S_{11,10}(\Gamma_2)$. Then by Propositions 7.1 and 7.4, for $l = 8, 10$, we have

$$G_{l,(11,10)}(Z_1, Z_2) = c_{l,(11,10)}^{-1} \Lambda(G, l-2, \mathrm{St})(\overline{G(-\overline{Z_1})} \otimes F(Z_2)).$$

Thus we have

Corollary 8.4.

$$\Lambda(G, 6, \mathrm{St}) = 61498907532000\, c_{8,(11,10)} \text{ and } \Lambda(G, 8, \mathrm{St}) = -276449241600\, c_{10,(11,10)}.$$

Note that

$$61498907532000/276449241600 = 2^{-4} \cdot 5 \cdot 7^{-1} \cdot 17^2 \cdot 37 \cdot 61 \cdot 97 \cdot 12697^{-1}.$$

Thus we have

Theorem 8.5. *With the above notation,*

$$\mathrm{ord}_{97}(\Lambda(G, 6, \mathrm{St})/\Lambda(G, 8, \mathrm{St})) = 1.$$

Corollary 8.6.

$$\mathrm{ord}_{97}(\pi^6 L(G, 6, \mathrm{St})/L(G, 8, \mathrm{St})) = 1.$$

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