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Linear differential relations satisfied by Wirtinger integrals

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Dedicated to Professor Masaaki Yoshida on his sixtieth birthday.

Abstract. We will derive linear differential relations satisfied by Wirtinger integrals by exploiting classical formulas of Jacobi's theta functions, forgetting that Wirtinger integrals are related to Gauss hypergeometric functions, although these linear differential relations are related to ones satisfied by Gauss hypergeometric functions.

0. Introduction.

In our recent paper [3] we computed twisted homology and cohomology groups with coefficients in local systems associated to a power product of Jacobi's four theta functions (see also Theorems 1.1 and 1.2 below). This result naturally leads us to consider the paring of non-vanishing homology and cohomology groups which is expressed as a definite integral by the twisted de Rham theory. Such an integral is identical with the one considered by Wirtinger [4]; we propose to call such an integral, including the original one obtained by him, *Wirtinger integral.* As is seen in [4] (see also [2]), Wirtinger integrals are the lifts of Gauss hypergeometric functions to the upper half plane. So one can expect that Wirtinger integrals satisfy linear differential relations which come from ones satisfied by Gauss hypergeometric differential equation. In this paper we will derive such linear differential relations for Wirtinger integrals by exploiting classical formulas of Jacobi's theta functions, forgetting that Wirtinger integrals are related to Gauss hypergeometric functions. Thus our method of derivation of the linear differential relations for Wirtinger integrals developped in this paper, with our previous ones [2] and [3], is regarded to form a part of the reconstruction of the theory of Gauss hypergeometric functions from the viewpoint of the theory of Jacobi's theta functions.

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1. Preliminaries and the main result.

In this paper we follow Chandrasekharan's notation for theta functions ([1]). For $\tau \in \mathbf{C}$ with $\mathrm{Im}(\tau) > 0$, we set $\mathit{\Gamma} = \mathbf{Z} + \mathbf{Z}\tau$, $D = \{0, \frac{1}{2}, \frac{\tau}{2}, \frac{1+\tau}{2}\}$, and $M = \mathbf{C}/\mathit{\Gamma} - D$. Let α, β, γ be complex parameters. Throughout this paper we assume the following conditions for α, β, γ:

$$\alpha \notin \frac{1}{2}\mathbf{Z},\ \beta \notin \frac{1}{2}\mathbf{Z},\ \gamma \notin \frac{1}{2}\mathbf{Z},\ \gamma - \alpha \notin \frac{1}{2}\mathbf{Z},\ \gamma - \beta \notin \frac{1}{2}\mathbf{Z},\ \gamma - \alpha - \beta \notin \frac{1}{2}\mathbf{Z},\ \text{and}\ \alpha - \beta \notin \frac{1}{2}\mathbf{Z},$$

where $\frac{1}{2}\mathbf{Z}$ denotes the group of integers and half integers. We set $T(u) = \theta(u)^{2\alpha}\theta_1(u)^{2\gamma-2\alpha-2}\theta_2(u)^{2\beta-2\gamma+2}\theta_3(u)^{-2\beta}$, where $\theta_i(u)$ means $\theta_i(u, \tau)$. We define

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a connection ∇ by $\nabla\varphi = d\varphi + \omega \wedge \varphi$ for a differential form φ, where d denotes the exterior differential with respect to u, and $\omega = d(\log T(u))$. Then we have $\nabla\nabla = 0$ and $\nabla(1) = \omega$. Let $\mathcal{L}$ and $\check{\mathcal{L}}$ be the local systems on M defined by $T(u)^{-1}$ and $T(u)$, respectively: $\mathcal{L} = \mathbf{C}T(u)^{-1}$ and $\check{\mathcal{L}} = \mathbf{C}T(u)$. They are dual to each other. For a non-negative integer k, let $C_k(M, \check{\mathcal{L}})$ be the group of twisted k-chains with coefficients in $\check{\mathcal{L}}$, and let ∂ be the boundary operator of the complex $C_\bullet(M, \check{\mathcal{L}})$. Let $\Omega^k(M)$ be the vector space of single-valued holomorphic k-forms on M. Then the connection ∇ induces a natural homomorphism $\nabla : \Omega^k(M) \to \Omega^{k+1}(M)$. The following theorem is fundamental:

Stokes' Theorem. For $\sigma \in C_k(M, \check{\mathcal{L}})$ and $\varphi \in \Omega^{k-1}(M)$, we have

$$\int_\sigma T(u) \cdot \nabla\varphi = \int_{\partial\sigma} T(u) \cdot \varphi.$$

Let $Z_k(M, \check{\mathcal{L}})$ be the group of twisted k-cycles with coefficients in $\check{\mathcal{L}}$, and let $B_k(M, \check{\mathcal{L}})$ be the group of twisted k-boundaries with coefficients in $\check{\mathcal{L}}$. We set $H_k(M, \check{\mathcal{L}}) = Z_k(M, \check{\mathcal{L}})/B_k(M, \check{\mathcal{L}})$: the k-th twisted homology group with coefficients in $\check{\mathcal{L}}$. Concerning the twisted homology groups, we have obtained in our paper [3] the following:

Theorem 1.1. We have $H_2(M, \check{\mathcal{L}}) = H_0(M, \check{\mathcal{L}}) = 0$, $H_1(M, \check{\mathcal{L}}) \cong \mathbf{C}c_1 \oplus \mathbf{C}c_2 \oplus \mathbf{C}c_3 \oplus \mathbf{C}c_4$. Here c_1, c_2, c_3, c_4 denote the homology classes of the twisted cycles $\sigma_1, \sigma_2, \sigma_3, \sigma_4$, respectively, which are given by

$$\begin{aligned}
\sigma_1 =& \frac{1}{(1 - e^{4\pi i\alpha})(1 - e^{4\pi i(\gamma-\alpha)})} \left(\frac{1}{2}+, 0+, \frac{1}{2}-, 0-\right), \\
\sigma_2 =& \frac{1}{(1 - e^{-4\pi i\beta})(1 - e^{4\pi i(\gamma-\alpha)})} \left(\frac{1+\tau}{2}+, \frac{1}{2}+, \frac{1+\tau}{2}-, \frac{1}{2}-\right), \\
\sigma_3 =& \frac{1}{(1 - e^{-4\pi i\beta})(1 - e^{4\pi i(\beta-\gamma)})} \left(\frac{\tau}{2}+, \frac{1+\tau}{2}+, \frac{\tau}{2}-, \frac{1+\tau}{2}-\right), \\
\sigma_4 =& \frac{1}{(1 - e^{-2\pi i\gamma})(1 - e^{4\pi i\alpha})} (l, 0+, -l, 0-),
\end{aligned}$$

where l and $-l$ are global cycles of $\mathbf{C}/\Gamma$ defined by the curves $l(s) = -\frac{\tau}{4} + s$ and $(-l)(s) = -\frac{\tau}{4} - s$ $(0 \le s \le 1)$, respectively.

Let D be the effective divisor on $\mathbf{C}/\Gamma$ given by $D = 2[0] + [\frac{1}{2}] + [\frac{\tau}{2}] + [\frac{1+\tau}{2}]$. Let Ω_D be the sheaf of meromorphic 1-forms on $\mathbf{C}/\Gamma$ which are multiples of the divisor $-D$. Let $H^k(M, \mathcal{L})$ be the k-th twisted cohomology group with coefficients in $\mathcal{L}$. Our result concerning the twisted cohomology groups is as follows ([3]):

Theorem 1.2. We have $H^0(M, \mathcal{L}) = H^2(M, \mathcal{L}) = 0$, $H^1(M, \mathcal{L}) \cong H^0(\mathbf{C}/\Gamma, \Omega_D)/\nabla(\mathbf{C}) = \mathbf{C}[\varphi_1] \oplus \mathbf{C}[\varphi_2] \oplus \mathbf{C}[\varphi_3] \oplus \mathbf{C}[\varphi_4]$. Here $[\varphi]$ denotes the image of an element φ of $H^0(\mathbf{C}/\Gamma, \Omega_D)$ by the natural map $H^0(\mathbf{C}/\Gamma, \Omega_D) \to H^0(\mathbf{C}/\Gamma, \Omega_D)/\nabla(\mathbf{C})$, and $\varphi_1, \varphi_2, \varphi_3, \varphi_4$ are elements of $H^0(\mathbf{C}/\Gamma, \Omega_D)$ given by

$$\varphi_1 = \pi\theta_3^2 du,\ \varphi_2 = \pi\theta_1^2 \frac{\theta_2(u)^2}{\theta(u)^2} du,\ \varphi_3 = \pi\theta_3^2 \frac{\theta_1(u)\theta_2(u)}{\theta(u)\theta_3(u)} du,\ \varphi_4 = \pi\theta_3^2 \frac{\theta(u)\theta_3(u)}{\theta_1(u)\theta_2(u)} du,$$

where θ_i denotes the theta constant $\theta_i(0)$.

Since $H_1(M, \check{\mathcal{L}})$ and $H^1(M, \mathcal{L})$ are dual to each other, we have a natural nondegenerate bilinear form $H_1(M, \check{\mathcal{L}}) \times H^1(M, \mathcal{L}) \to \mathbf{C}$. For $[\sigma] \in H_1(M, \check{\mathcal{L}})$ and $[\varphi] \in H^1(M, \mathcal{L})$ with

$\sigma \in Z_1(M, \check{\mathcal{L}})$ and $\varphi \in H^0(\mathbf{C}/\Gamma, \Omega_D)$, let $< [\sigma], [\varphi] >$ be the image by this bilinear form. By the standard procedure for regarding twisted cycles and cocycles as currents, we obtain the expression $< [\sigma], [\varphi] >= \int_\sigma T(u)\varphi$, which we call *Wirtinger integral.* Every Wirtinger integral is a single-valued and holomorphic function of τ on the upper half plane H. We set $\int_{\sigma_j} T(u)\varphi_i = I_{ij}$ $(i, j = 1, 2, 3, 4)$. It is easy to see that, for a fixed j, the four integrals $I_{1j}, I_{2j}, I_{3j}, I_{4j}$ are linearly independent over $\mathbf{C}$, and that, for a fixed i, $I_{i1}, I_{i2}, I_{i3}, I_{i4}$ are linearly independent over $\mathbf{C}$. As was shown by Wirtinger [4] (see also [2]), these functions I_{ij}'s are related to the Gauss hypergeometric function, which we denote by $F(\alpha, \beta, \gamma, z)$: in fact we have, for example,

$$I_{31} = \frac{\Gamma(\alpha)\Gamma(\gamma-\alpha)}{2\Gamma(\gamma)} \lambda^{\frac{\gamma-1}{2}} (1-\lambda)^{\frac{\alpha+\beta-\gamma+1}{2}} F(\alpha, \beta+1, \gamma, \lambda),$$

where λ denotes the lambda function: $\lambda = \dfrac{\theta_1^4}{\theta_3^4}$. The other integrals I_{ij} have analogous expressions, too. The main result to be proved in this paper is as follows:

Theorem 1.3. The 16 functions I_{ij} $(i, j = 1, 2, 3, 4)$ of the variable τ satisfy the following system of linear differential equations with coefficients invariant under the action of $\Gamma(2)$ the principal congruence subgroup of level 2:

$$\frac{i}{\pi\theta_3^4}\frac{d}{d\tau} \begin{bmatrix} I_{11} & I_{12} & I_{13} & I_{14} \\ I_{21} & I_{22} & I_{23} & I_{24} \\ I_{31} & I_{32} & I_{33} & I_{34} \\ I_{41} & I_{42} & I_{43} & I_{44} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 \\ 0 & 0 & b_{11} & b_{12} \\ 0 & 0 & b_{21} & b_{22} \end{bmatrix} \begin{bmatrix} I_{11} & I_{12} & I_{13} & I_{14} \\ I_{21} & I_{22} & I_{23} & I_{24} \\ I_{31} & I_{32} & I_{33} & I_{34} \\ I_{41} & I_{42} & I_{43} & I_{44} \end{bmatrix}, \tag{1.1}$$

where a_{ij}'s and b_{ij}' are given by

$$a_{11} = \frac{\alpha-\beta-1}{2}\frac{\theta_1^4}{\theta_3^4} + \frac{\gamma-1}{2}, \quad a_{12} = \frac{1-2\alpha}{2}, \quad a_{21} = \frac{2\gamma-2\beta-3}{2}\frac{\theta_1^4}{\theta_3^4},$$
$$a_{22} = \frac{\beta-\alpha+1}{2}\frac{\theta_1^4}{\theta_3^4} + \frac{1-\gamma}{2}, \quad b_{11} = \frac{(1+\alpha-\gamma)(\alpha+1)}{2(\beta-\alpha)} + \frac{\beta}{2}\frac{\theta_1^4}{\theta_3^4} + \frac{\alpha}{2}\frac{\theta_2^4}{\theta_3^4},$$
$$b_{12} = \frac{(1+\alpha-\gamma)(1+\beta-\gamma)}{\beta-\alpha}, \quad b_{21} = \frac{\alpha\beta}{\alpha-\beta},$$
$$b_{22} = \frac{\alpha(\alpha+\beta-2\gamma+2)}{2(\alpha-\beta)} + \frac{\gamma-\alpha-1}{2}\frac{\theta_2^4}{\theta_3^4} + \frac{\gamma-\beta-1}{2}\frac{\theta_1^4}{\theta_3^4}.$$

The system (1.1) is splitted into the following two systems of differential equations:

$$\frac{i}{\pi\theta_3^4}\frac{d}{d\tau}\begin{bmatrix} I_{1j} \\ I_{2j} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\begin{bmatrix} I_{1j} \\ I_{2j} \end{bmatrix}$$

and

$$\frac{i}{\pi\theta_3^4}\frac{d}{d\tau}\begin{bmatrix} I_{3j} \\ I_{4j} \end{bmatrix} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}\begin{bmatrix} I_{3j} \\ I_{4j} \end{bmatrix},$$

which are equivalent to the Gauss hypergeometric differential equation. We prove Theorem 1.3 in the next section by making use of classical formulas of theta functions, forgetting that the integrals I_{ij}'s are related to the Gauss hypergeometric function.

Remark. Note that the differential operator $\dfrac{i}{\pi\theta_3^4}\dfrac{d}{d\tau}$ is invariant under the action of the group $\Gamma(2)$. In fact we have $\dfrac{i}{\pi\theta_3^4}\dfrac{d}{d\tau} = \lambda(\lambda-1)\dfrac{d}{d\lambda}$.

2. Proof of Theorem 1.3 : Derivation of differential relations.

To derive the 16 relations in (1.1), it suffices to prove the following

Proposition 2.1. The following formulas hold:

$$\frac{i}{\pi\theta_3^4}\frac{\partial}{\partial\tau}(T(u)\varphi_1) \equiv a_{11}T(u)\varphi_1 + a_{12}T(u)\varphi_2 \mod B^1(M,\mathcal{L}); \tag{2.1}$$

$$\frac{i}{\pi\theta_3^4}\frac{\partial}{\partial\tau}(T(u)\varphi_2) \equiv a_{21}T(u)\varphi_1 + a_{22}T(u)\varphi_2 \mod B^1(M,\mathcal{L}); \tag{2.2}$$

$$\frac{i}{\pi\theta_3^4}\frac{\partial}{\partial\tau}(T(u)\varphi_3) \equiv b_{11}T(u)\varphi_3 + b_{12}T(u)\varphi_4 \mod B^1(M,\mathcal{L}); \tag{2.3}$$

$$\frac{i}{\pi\theta_3^4}\frac{\partial}{\partial\tau}(T(u)\varphi_4) \equiv b_{21}T(u)\varphi_3 + b_{22}T(u)\varphi_4 \mod B^1(M,\mathcal{L}), \tag{2.4}$$

where $B^1(M,\mathcal{L})$ denotes the group of twisted 1-coboundaries with coefficients in $\mathcal{L}$.

In fact, the 16 relations in (1.1) follows immediately if we integrate each of the four relations in the proposition along suitable cycles.

Proof of Proposition 2.1. We prove the formula (2.3) only, since the other formulas are proved similarly. We have

$$\begin{aligned}\frac{\partial}{\partial\tau}(T(u)\varphi_3) = \Bigg[& 2\frac{\theta_{3\tau}}{\theta_3} + (2\gamma-2\alpha-1)\left(\frac{\theta_{1\tau}(u)}{\theta_1(u)} - \frac{\theta_{2\tau}(u)}{\theta_2(u)}\right) \\ &+ (2\beta-2\alpha+2)\left(\frac{\theta_{2\tau}(u)}{\theta_2(u)} - \frac{\theta_{3\tau}(u)}{\theta_3(u)}\right) + (1-2\alpha)\left(\frac{\theta_{3\tau}(u)}{\theta_3(u)} - \frac{\theta_\tau(u)}{\theta(u)}\right)\Bigg] T(u)\varphi_3,\end{aligned} \tag{2.5}$$

where $\theta_{3\tau}$ denotes $\dfrac{\partial\theta_3}{\partial\tau}(0,\tau)$, and $\theta_{1\tau}(u)$ denotes $\dfrac{\partial\theta_1}{\partial\tau}(u,\tau)$, etc. Since the four theta functions $\theta(u)$ and $\theta_i(u)$ satisfy the common partial differential equation $4\pi i\dfrac{\partial\Theta}{\partial\tau} = \dfrac{\partial^2\Theta}{\partial u^2}$, (2.5) is turned to

$$\begin{aligned}\frac{\partial}{\partial\tau}(T(u)\varphi_3) = \Bigg[& 2\frac{\theta_{3\tau}}{\theta_3} + \frac{2\gamma-2\alpha-1}{4\pi i}\left(\frac{\theta_1''(u)}{\theta_1(u)} - \frac{\theta_2''(u)}{\theta_2(u)}\right) \\ &+ \frac{2\beta-2\alpha+2}{4\pi i}\left(\frac{\theta_2''(u)}{\theta_2(u)} - \frac{\theta_3''(u)}{\theta_3(u)}\right) + \frac{1-2\alpha}{4\pi i}\left(\frac{\theta_3''(u)}{\theta_3(u)} - \frac{\theta''(u)}{\theta(u)}\right)\Bigg] T(u)\varphi_3,\end{aligned} \tag{2.6}$$

where $\theta_i''(u)$ denotes $\dfrac{\partial^2\theta_i}{\partial u^2}(u,\tau)$. Now we note the following formulas: $\theta_2''(u)\theta_1(u) - \theta_1''(u)\theta_2(u) = \pi\theta_3^2(\theta'(u)\theta_3(u) + \theta_3'(u)\theta(u))$, $\theta_2''(u)\theta_3(u) - \theta_3''(u)\theta_2(u) = \pi\theta_1^2(\theta'(u)\theta_1(u) + \theta_1'(u)\theta(u))$, $\theta''(u)\theta_3(u) - \theta_3''(u)\theta(u) = \pi\theta_3^2(\theta_1'(u)\theta_2(u) + \theta_2'(u)\theta_1(u))$, where $\theta_i'(u)$ denotes $\dfrac{\partial\theta_i}{\partial u}(u,\tau)$. Applying these formulas to the right-hand side of (2.6), we have

$$\begin{aligned}\frac{\partial}{\partial\tau}(T(u)\varphi_3) = \Bigg[& 2\frac{\theta_{3\tau}}{\theta_3} - \frac{2\gamma-2\alpha-1}{4i}\theta_3^2\frac{\theta(u)\theta_3(u)}{\theta_1(u)\theta_2(u)}\left(\frac{\theta'(u)}{\theta(u)} + \frac{\theta_3'(u)}{\theta_3(u)}\right) \\ &+ \frac{2\beta-2\alpha+2}{4i}\theta_1^2\frac{\theta(u)\theta_1(u)}{\theta_2(u)\theta_3(u)}\left(\frac{\theta'(u)}{\theta(u)} + \frac{\theta_1'(u)}{\theta_1(u)}\right) \\ &- \frac{1-2\alpha}{4i}\theta_3^2\frac{\theta_1(u)\theta_2(u)}{\theta(u)\theta_3(u)}\left(\frac{\theta_1'(u)}{\theta_1(u)} + \frac{\theta_2'(u)}{\theta_2(u)}\right)\Bigg] T(u)\varphi_3.\end{aligned} \tag{2.7}$$

Substituting the formulas:

$$\frac{\theta_3'(u)}{\theta_3(u)} = \frac{\theta_1'(u)}{\theta_1(u)} + \pi\theta_2^2\frac{\theta(u)\theta_2(u)}{\theta_1(u)\theta_3(u)} \quad \text{and} \quad \frac{\theta_2'(u)}{\theta_2(u)} = \frac{\theta'(u)}{\theta(u)} - \pi\theta_2^2\frac{\theta_1(u)\theta_3(u)}{\theta(u)\theta_2(u)}$$

into the right-hand side of (2.7) and making some calculation, we have

$$\begin{aligned}\frac{\partial}{\partial\tau}(T(u)\varphi_3) =& 2\frac{\theta_{3\tau}}{\theta_3}T(u)\varphi_3 + \frac{\theta_3^2}{4i}\Bigg[-(2\gamma-2\alpha-1)\pi\theta_3^2\frac{\theta(u)\theta_3(u)}{\theta_1(u)\theta_2(u)} \\ & + (2\beta-2\alpha+2)\pi\theta_1^2\,\frac{\theta(u)\theta_1(u)}{\theta_2(u)\theta_3(u)} - (1-2\alpha)\pi\theta_3^2\frac{\theta_1(u)\theta_2(u)}{\theta(u)\theta_3(u)}\Bigg] \\ & \times\left(\frac{\theta'(u)}{\theta(u)} + \frac{\theta_1'(u)}{\theta_1(u)}\right)\theta(u)^{2\alpha-1}\theta_1(u)^{2\gamma-2\alpha-1}\theta_2(u)^{2\beta-2\gamma+3}\theta_3(u)^{-2\beta-1}du \\ & + \frac{\pi^2\theta_2^2\theta_3^4}{4i}\left[-(2\gamma-2\alpha-1)\frac{\theta(u)\theta_2(u)}{\theta_1(u)\theta_3(u)} + (1-2\alpha)\frac{\theta_1(u)^3\theta_2(u)}{\theta(u)^3\theta_3(u)}\right]T(u)du.\end{aligned} \tag{2.8}$$

Here we note the equality

$$\begin{aligned}&\Bigg[-(2\gamma-2\alpha-1)\pi\theta_3^2\frac{\theta(u)\theta_3(u)}{\theta_1(u)\theta_2(u)} + (2\beta-2\alpha+2)\pi\theta_1^2\,\frac{\theta(u)\theta_1(u)}{\theta_2(u)\theta_3(u)} \\ &-(1-2\alpha)\pi\theta_3^2\frac{\theta_1(u)\theta_2(u)}{\theta(u)\theta_3(u)}\Bigg]\theta(u)^{2\alpha-1}\theta_1(u)^{2\gamma-2\alpha-1}\theta_2(u)^{2\beta-2\gamma+3}\theta_3(u)^{-2\beta-1} \\ &= \frac{d}{du}\{\theta(u)^{2\alpha-1}\theta_1(u)^{2\gamma-2\alpha-1}\theta_2(u)^{2\beta-2\gamma+3}\theta_3(u)^{-2\beta-1}\}.\end{aligned} \tag{2.9}$$

Applying (2.9) to the right-hand side of (2.8), we have

$$\begin{aligned}\frac{\partial}{\partial\tau}(T(u)\varphi_3) =& 2\frac{\theta_{3\tau}}{\theta_3}T(u)\varphi_3 \\ & + \frac{\theta_3^2}{4i}d\left[\left(\frac{\theta'(u)}{\theta(u)} + \frac{\theta_1'(u)}{\theta_1(u)}\right)\theta(u)^{2\alpha-1}\theta_1(u)^{2\gamma-2\alpha-1}\theta_2(u)^{2\beta-2\gamma+3}\theta_3(u)^{-2\beta-1}\right] \\ & + \frac{\theta_3^2}{4i}\left(\frac{\theta'(u)^2}{\theta(u)^2} - \frac{\theta''(u)}{\theta(u)} + \frac{\theta_1'(u)^2}{\theta_1(u)^2} - \frac{\theta_1''(u)}{\theta_1(u)}\right) \\ & \times\theta(u)^{2\alpha-1}\theta_1(u)^{2\gamma-2\alpha-1}\theta_2(u)^{2\beta-2\gamma+3}\theta_3(u)^{-2\beta-1}du \\ & + \frac{\pi^2\theta_2^2\theta_3^4}{4i}\left[-(2\gamma-2\alpha-1)\frac{\theta(u)\theta_2(u)}{\theta_1(u)\theta_3(u)} + (1-2\alpha)\frac{\theta_1(u)^3\theta_2(u)}{\theta(u)^3\theta_3(u)}\right]T(u)du.\end{aligned} \tag{2.10}$$

We note that

$$\begin{aligned}\frac{\theta'(u)^2}{\theta(u)^2} - \frac{\theta''(u)}{\theta(u)} &= -4\pi i\frac{\theta_{1\tau}}{\theta_1} + \pi^2\theta_2^2\theta_3^2\frac{\theta_1(u)^2}{\theta(u)^2}, \\ \frac{\theta_1'(u)^2}{\theta_1(u)^2} - \frac{\theta_1''(u)}{\theta_1(u)} &= -4\pi i\frac{\theta_{1\tau}}{\theta_1} + \pi^2\theta_2^2\theta_3^2\frac{\theta(u)^2}{\theta_1(u)^2},\end{aligned}$$

and

$$d\left[\left(\frac{\theta'(u)}{\theta(u)} + \frac{\theta_1'(u)}{\theta_1(u)}\right)\theta(u)^{2\alpha-1}\theta_1(u)^{2\gamma-2\alpha-1}\theta_2(u)^{2\beta-2\gamma+3}\theta_3(u)^{-2\beta-1}\right] \in B^1(M,\mathcal{L}).$$

Then the equality (2.10) is turned to

$$\begin{aligned}\frac{\partial}{\partial\tau}(T(u)\varphi_3) \equiv & 2\left(\frac{\theta_{3\tau}}{\theta_3}-\frac{\theta_{1\tau}}{\theta_1}\right)T(u)\varphi_3\\ & +\frac{\pi^2\theta_2^2\theta_3^4}{2i}\left[(1+\alpha-\gamma)\frac{\theta(u)\theta_2(u)}{\theta_1(u)\theta_3(u)}\right.\\ & \left.+(1-\alpha)\frac{\theta_1(u)^3\theta_2(u)}{\theta(u)^3\theta_3(u)}\right]T(u)du \mod B^1(M,\mathcal{L}).\end{aligned} \tag{2.11}$$

Substituting the equality

$$\frac{\theta_1(u)^3\theta_2(u)}{\theta(u)^3\theta_3(u)}=\frac{\theta_1^2\theta_1(u)\theta_2(u)\theta_3(u)}{\theta_3^2\theta(u)^3}-\frac{\theta_2^2\theta_1(u)\theta_2(u)}{\theta_3^2\theta(u)\theta_3(u)}$$

into the right-hand side of (2.11), we have

$$\begin{aligned}\frac{\partial}{\partial\tau}(T(u)\varphi_3) \equiv & 2\left(\frac{\theta_{3\tau}}{\theta_3}-\frac{\theta_{1\tau}}{\theta_1}\right)T(u)\varphi_3\\ & +\frac{\pi^2\theta_2^2\theta_3^4}{2i}\left[(1+\alpha-\gamma)\frac{\theta(u)\theta_2(u)}{\theta_1(u)\theta_3(u)}\right.\\ & \left.+(1-\alpha)\frac{\theta_1^2}{\theta_3^2}\frac{\theta_1(u)\theta_2(u)\theta_3(u)}{\theta(u)^3}-(1-\alpha)\frac{\theta_2^2}{\theta_3^2}\frac{\theta_1(u)\theta_2(u)}{\theta(u)\theta_3(u)}\right]T(u)du \mod B^1(M,\mathcal{L}).\end{aligned} \tag{2.12}$$

We need the following

Lemma 2.2. The following formulas hold:

$$\frac{\theta(u)\theta_2(u)}{\theta_1(u)\theta_3(u)}du \equiv \frac{\alpha}{\beta-\alpha}\frac{1}{\pi\theta_2^2}\varphi_3+\frac{\beta-\gamma+1}{\beta-\alpha}\frac{1}{\pi\theta_2^2}\varphi_4 \mod B^1(M,\mathcal{L}); \tag{2.13}$$

$$\begin{aligned}\frac{\theta_1(u)\theta_2(u)\theta_3(u)}{\theta(u)^3}du \equiv & \left[\frac{\beta}{1-\alpha}\frac{\theta_1^2}{\pi\theta_2^2\theta_3^2}+\frac{(\alpha-\gamma+1)\beta}{(1-\alpha)(\beta-\alpha)}\frac{\theta_3^2}{\pi\theta_1^2\theta_2^2}\right]\varphi_3\\ & +\frac{(\alpha-\gamma+1)(\beta-\gamma+1)}{(1-\alpha)(\beta-\alpha)}\frac{\theta_3^2}{\pi\theta_1^2\theta_2^2}\varphi_4 \mod B^1(M,\mathcal{L}).\end{aligned} \tag{2.14}$$

Proof of Lemma 2.2. The formula (2.13) follows immediately from the equality

$$\omega=(2\beta-2\gamma+2)\pi\theta_3^2\frac{\theta(u)\theta_3(u)}{\theta_1(u)\theta_2(u)}du-(2\beta-2\alpha)\pi\theta_2^2\frac{\theta(u)\theta_2(u)}{\theta_1(u)\theta_3(u)}du+2\alpha\pi\theta_3^2\frac{\theta_1(u)\theta_2(u)}{\theta(u)\theta_3(u)}du.$$

Now we see that

$$\begin{aligned}\nabla\left(\frac{\theta_2(u)^2}{\theta(u)^2}\right)= & -2\pi\theta_2^2\frac{\theta_1(u)\theta_2(u)\theta_3(u)}{\theta(u)^3}du+\frac{\theta_2(u)^2}{\theta(u)^2}\left[2\alpha\pi\theta_2^2\frac{\theta_1(u)\theta_3(u)}{\theta(u)\theta_2(u)}\right.\\ & \left.+(2\alpha+2\beta-2\gamma+2)\pi\theta_3^2\frac{\theta(u)\theta_3(u)}{\theta_1(u)\theta_2(u)}-2\beta\pi\theta_2^2\frac{\theta(u)\theta_2(u)}{\theta_1(u)\theta_3(u)}\right]du\\ = & (\alpha-1)2\pi\theta_2^2\frac{\theta_1(u)\theta_2(u)\theta_3(u)}{\theta(u)^3}du+(\alpha+\beta-\gamma+1)2\pi\theta_3^2\frac{\theta_2(u)\theta_3(u)}{\theta(u)\theta_1(u)}du\\ & -2\beta\pi\theta_2^2\frac{\theta_2(u)^3}{\theta(u)\theta_1(u)\theta_3(u)}du,\end{aligned}$$

from which it follows immediately that

$$\theta_2^2\frac{\theta_1(u)\theta_2(u)\theta_3(u)}{\theta(u)^3}du \equiv \frac{\alpha+\beta-\gamma+1}{1-\alpha}\theta_3^2\frac{\theta_2(u)\theta_3(u)}{\theta(u)\theta_1(u)}du - \frac{\beta}{1-\alpha}\theta_2^2\frac{\theta_2(u)^3}{\theta(u)\theta_1(u)\theta_3(u)}du \mod B^1(M,\mathcal{L}). \tag{2.15}$$

Substituting the equality

$$\frac{\theta_2^2\theta_2(u)^3}{\theta(u)\theta_1(u)\theta_3(u)} = \frac{\theta_3^2\theta_2(u)\theta_3(u)}{\theta(u)\theta_1(u)} - \frac{\theta_1^2\theta_1(u)\theta_2(u)}{\theta(u)\theta_3(u)}$$

into the right-hand side of (2.15), we have

$$\theta_2^2\frac{\theta_1(u)\theta_2(u)\theta_3(u)}{\theta(u)^3}du \equiv \frac{\alpha-\gamma+1}{1-\alpha}\theta_3^2\frac{\theta_2(u)\theta_3(u)}{\theta(u)\theta_1(u)}du + \frac{\beta}{1-\alpha}\theta_1^2\frac{\theta_1(u)\theta_2(u)}{\theta(u)\theta_3(u)}du \mod B^1(M,\mathcal{L}). \tag{2.16}$$

Now we have

$$\omega = (2\beta-2\gamma+2)\pi\theta_3^2\frac{\theta(u)\theta_3(u)}{\theta_1(u)\theta_2(u)}du - (2\beta-2\alpha)\pi\theta_1^2\frac{\theta_2(u)\theta_3(u)}{\theta(u)\theta_1(u)}du + 2\beta\pi\theta_3^2\frac{\theta_1(u)\theta_2(u)}{\theta(u)\theta_3(u)}du,$$

from which it follows immediately that

$$\frac{\theta_2(u)\theta_3(u)}{\theta(u)\theta_1(u)}du \equiv \frac{\beta-\gamma+1}{\beta-\alpha}\frac{\theta_3^2}{\theta_1^2}\frac{\theta(u)\theta_3(u)}{\theta_1(u)\theta_2(u)}du + \frac{\beta}{\beta-\alpha}\frac{\theta_3^2}{\theta_1^2}\frac{\theta_1(u)\theta_2(u)}{\theta(u)\theta_3(u)}du \mod B^1(M,\mathcal{L}). \tag{2.17}$$

Substituting (2.17) into the right-hand side of (2.16), we have the desired equality (2.14), which proves Lemma 2.2.

Let us return to our proof of Proposition 2.1. Substituting the two equalities (2.13) and (2.14) into the right-hand side of (2.12), we have the desired equality (2.3). Proposition 2.1 is thereby proved.

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