Twisted homology and cohomology groups associated to the Wirtinger integral

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Dedicated to Professor Kazuo Okamoto on his sixtieth birthday.

Abstract. The first half of this paper deals with the structure of the twisted homology group associated to the Wirtinger integral. A basis of the first homology group is given, and the vanishing of the other homology groups is proved (Theorem 1). The second half deals with the structure of the twisted cohomology groups associated to the Wirtinger integral. The isomorphism between the twisted cohomology groups and the cohomology groups associated to a subcomplex of the de Rham complex is established, and a basis of the first cohomology group of this subcomplex (therefore, of the first twisted cohomology group) is given (Theorem 2).

0. Introduction.

In his paper [16], Wirtinger showed, 1902, that the composite function $F(\alpha,\beta,\gamma,\lambda(\tau))$ of the Gauss hypergeometric function $F(\alpha,\beta,\gamma,z)$ and the lambda function $z=\lambda(\tau)=\dfrac{\theta_1(0,\tau)^4}{\theta_3(0,\tau)^4}$ has the following integral representation

$$\begin{aligned}F(\alpha,\beta,\gamma,\lambda(\tau)) =&\frac{2\pi\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\gamma-\alpha)}\theta_1(0,\tau)^{2-2\gamma}\theta_2(0,\tau)^{2\gamma-2\alpha-2\beta}\theta_3(0,\tau)^{2\alpha+2\beta}\times\\ &\times\int_0^{\frac{1}{2}}\theta(u,\tau)^{2\alpha-1}\theta_1(u,\tau)^{2\gamma-2\alpha-1}\theta_2(u,\tau)^{2\beta-2\gamma+1}\theta_3(u,\tau)^{-2\beta+1}du\end{aligned}$$

(In this paper we follow Chandrasekharan's notation for theta functions. See [6]). The right-hand side is a function in τ, single-valued and holomorphic on the upper-half plane H, and we proposed in our paper [15] to call it *Wirtinger integral.* This integral representation seems to have been forgotten for a long while in the study on hypergeometric functions, whereas we gave recently in [15] a new derivation of the connection formulas for the Gauss hypergeometric function by exploiting the Wirtinger integral and Jacobi's imaginary transformations for theta functions. Our result suggests a possibility to reconstruct the theory of the Gauss hypergeometric function on the basis of the Wirtinger integral and theta functions, and to generalize the theory of the Gauss hypergeometric function from the viewpoint of the Wirtinger integral. In order to study the properties of the Wirtinger integral further, we give, in this paper, the computations of the twisted homology and cohomology groups associated to the Wirtinger integral. The computation of (co)homology groups consists of the proof of the vanishing of minor groups, and of the construction of a basis of non-vanishing groups. Thanks to the duality of the homology and cohomology, the vanishings of homology and cohomology groups are concluded simultaneously if the vanishing of either homology groups or cohomology groups is proved. The vanishing of cohomology groups on a compact Kähler manifold minus some cycle of

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real codimension two with local system coefficients was proved by Aomoto [2] with the aide of the Morse theory. The algebraic proof of the vanishing of cohomology groups on a complex projective space minus an algebraic divisor was given by Kita-Noumi [13] and Kita [12]. The construction of a basis of a non-vanishing homology group on a complex projective space minus hyperplanes with local system coefficients was given by Kita [10] and [11] (See also Aomoto [2] and Orlik-Terao [14], Chap.6). Some results were obtained by Aomoto [1], [3], [4] about the structure of a non-vanishing cohomology group on a complex projective space minus a divisor with local system coefficients (See also Deligne [7], Corollaire 6.11). In their papers [12] and [13], Kita and Noumi established the isomorphism between the cohomology groups with local system coefficients and the cohomology groups associated to the logarithmic complex on a complex projective space minus hyperplanes, and constructed a basis of a non-vanishing cohomology group associated to the logarithmic complex (See also [5]). Motivated by the works above, we give in §1 the computation (including the construction of a basis) of the twisted homology groups on the one-dimensional complex torus minus four points with the coefficients in the local system associated to the integrand of the Wirtinger integral (Theorem 1). In §2 we first establish the isomorphism between a non-vanishing cohomology group with local system coefficients and a cohomology group associated to the holomorphic 1-forms on the torus minus four points which may have poles at those four points of degree at most five. Next, we give a basis of the latter cohomology group (Theorem 2). This result on the cohomology groups suggests that a form having a pole (or poles) of degree more than one is needed in order to give a basis of a non-vanishing cohomology group. This situation is different from that of the case of a complex projective space minus hyperplanes.

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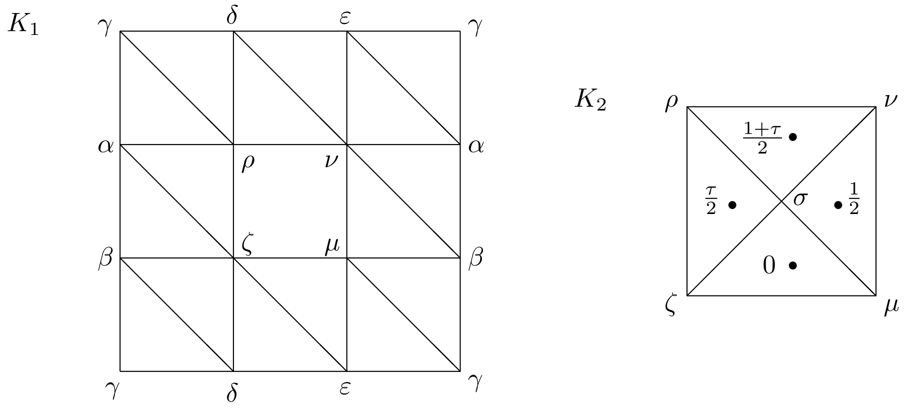
1. Twisted homology groups.

Let $\tau \in \mathbf{C}$ be such that $\mathrm{Im}(\tau) > 0$. We set $\Gamma = \mathbf{Z} + \mathbf{Z}\tau$, $D = \{0, \frac{1}{2}, \frac{\tau}{2}, \frac{1+\tau}{2}\}$, and $M = \mathbf{C}/\Gamma - D$, where $\mathbf{Z}$ denotes the additive group of integers, and $\mathbf{C}$ the additive group of complex numbers. Let p, q, r, s be complex numbers satisfying $p+q+r+s = 0$. *Throughout this paper we assume that p, q, r, s are not integers, and that the sum and the difference of any two of them are not integers either.* If we set $T(u) = \theta(u)^p \theta_1(u)^q \theta_2(u)^r \theta_3(u)^s$, where $\theta_i(u)$ means $\theta_i(u, \tau)$, then we have $T(u+1) = e^{-(p+q)\pi i} T(u)$ and $T(u+\tau) = e^{(p+r)\pi i} T(u)$. We set $\omega = d(\log T(u))$. We define a connection ∇ by $\nabla\varphi = d\varphi + \omega \wedge \varphi$. Then we have $\nabla\nabla = 0$ and $\nabla(1) = \omega$. Let $\mathcal{L}$ and $\check{\mathcal{L}}$ be the local systems on M defined by $T(u)^{-1}$ and $T(u)$, respectively: $\mathcal{L} = \mathbf{C}T(u)^{-1}$ and $\check{\mathcal{L}} = \mathbf{C}T(u)$, which are dual each other. In this section we compute the twisted homology groups $H_\bullet(M, \check{\mathcal{L}})$. Let us consider the following two-dimensional complex K_1 and one-dimensional complex K_2:



In the figure for K_1, let the 1-chains $\langle\gamma,\delta\rangle+\langle\delta,\varepsilon\rangle+\langle\varepsilon,\gamma\rangle$ and $\langle\gamma,\beta\rangle+\langle\beta,\alpha\rangle+\langle\alpha,\gamma\rangle$ in K_1 be homologous to the 1-chains defined by the periods 1 and τ of the torus $\mathbf{C}/\Gamma$, respectively, and let no 2-chain be contained inside the square $\zeta\mu\nu\rho\zeta$ of K_1. In the figure for K_2, we added the four points $0, \frac{1}{2}, \frac{\tau}{2}, \frac{1+\tau}{2}$ to the complex K_2 to indicate the configuration of chains of K_2. These points are not 0-chains of K_2. We set $K=K_1\cup K_2$ and $K'=K_1\cap K_2$. Since the complex K is homotopically equivalent to the surface M, the group $H_\bullet(M,\check{\mathcal{L}})$ is isomorphic to $H_\bullet(K,\check{\mathcal{L}})$. In order to compute $H_\bullet(K,\check{\mathcal{L}})$, we need the homology groups of the three subcomplexes K', K_1, K_2. Since $p+q+r+s=0$, the complex K' is homotopically equivalent to the circle S^1, from which we have immediately

Lemma 1.1. *$H_2(K',\check{\mathcal{L}})=0$, $H_1(K',\check{\mathcal{L}})\cong\mathbf{C}$, $H_0(K',\check{\mathcal{L}})\cong\mathbf{C}$.*

For the complex K_1, we have

Lemma 1.2. *$H_2(K_1,\check{\mathcal{L}})=0$, $H_1(K_1,\check{\mathcal{L}})\cong\mathbf{C}$, $H_0(K_1,\check{\mathcal{L}})=0$.*

Proof. Since there is no non-zero element $c\in C_2(K_1,\check{\mathcal{L}})$ such that $\partial c=0$, we have $Z_2(K_1,\check{\mathcal{L}})=0$, that is, $H_2(K_1,\check{\mathcal{L}})=0$. It is easy to see that two 1-chains $\langle\zeta,\mu\rangle+\langle\mu,\nu\rangle+\langle\nu,\rho\rangle+\langle\rho,\zeta\rangle$ and $\langle\gamma,\delta\rangle+\langle\delta,\varepsilon\rangle+\langle\varepsilon,\gamma\rangle+\langle\gamma,\beta\rangle+\langle\beta,\alpha\rangle+\langle\alpha,\gamma\rangle+\langle\gamma,\varepsilon\rangle+\langle\varepsilon,\delta\rangle+\langle\delta,\gamma\rangle+\langle\gamma,\alpha\rangle+\langle\alpha,\beta\rangle+\langle\beta,\gamma\rangle$ belong to $Z_1(K_1,\check{\mathcal{L}})$ but not to $B_1(K_1,\check{\mathcal{L}})$. Obviously they are homologous each other, so we have $H_1(K_1,\check{\mathcal{L}})\cong\mathbf{C}(\langle\zeta,\mu\rangle+\langle\mu,\nu\rangle+\langle\nu,\rho\rangle+\langle\rho,\zeta\rangle)$. Finally, since $\partial(\langle\gamma,\delta\rangle+\langle\delta,\varepsilon\rangle+\langle\varepsilon,\gamma\rangle)=(e^{-(p+q)\pi i}-1)\langle\gamma\rangle$, we have $\langle\gamma\rangle\in B_0(K_1,\check{\mathcal{L}})$. Similarly, we have $\langle\delta\rangle,\langle\varepsilon\rangle,\langle\alpha\rangle,\langle\beta\rangle\in B_0(K_1,\check{\mathcal{L}})$. Moreover, since $\langle\zeta\rangle=\langle\beta\rangle+\partial\langle\beta,\zeta\rangle$, we have $\langle\zeta\rangle\in B_0(K_1,\check{\mathcal{L}})$. Similarly, we have $\langle\mu\rangle,\langle\nu\rangle,\langle\rho\rangle\in B_0(K_1,\check{\mathcal{L}})$. Therefore we see that $Z_0(K_1,\check{\mathcal{L}})=B_0(K_1,\check{\mathcal{L}})$, that is, $H_0(K_1,\check{\mathcal{L}})=0$.

The result for the complex K_2 is as follows:

Lemma 1.3. *$H_2(K_2,\check{\mathcal{L}})=0$, $H_1(K_2,\check{\mathcal{L}})\cong\mathbf{C}^3$, $H_0(K_2,\check{\mathcal{L}})=0$.*

Proof. Since $\partial(\langle\zeta,\mu\rangle+\langle\mu,\sigma\rangle+\langle\sigma,\zeta\rangle)=(e^{2\pi ip}-1)\langle\zeta\rangle$, we have $\langle\zeta\rangle\in B_0(K_2,\check{\mathcal{L}})$. Similarly, we have $\langle\mu\rangle,\langle\nu\rangle,\langle\rho\rangle,\langle\sigma\rangle\in B_0(K_2,\check{\mathcal{L}})$. So we have $H_0(K_2,\check{\mathcal{L}})=0$. We define four 1-chains $c_p,c_q,c_r,c_s\in C_1(K_2,\check{\mathcal{L}})$ by $c_p=\langle\sigma,\zeta\rangle+\langle\zeta,\mu\rangle+\langle\mu,\sigma\rangle$, $c_q=\langle\sigma,\mu\rangle+\langle\mu,\nu\rangle+\langle\nu,\sigma\rangle$, $c_r=\langle\sigma,\rho\rangle+\langle\rho,\zeta\rangle+\langle\zeta,\sigma\rangle$, $c_s=\langle\sigma,\nu\rangle+\langle\nu,\rho\rangle+\langle\rho,\sigma\rangle$, where we assume that the restrictions of the branches of $T(u)$ to the four chains define the same germ at the common initial point σ of those chains. Then we have $\partial c_p = (e^{2\pi ip}-1)\langle\sigma\rangle$, $\partial c_q = (e^{2\pi iq}-1)\langle\sigma\rangle$, $\partial c_r = (e^{2\pi ir}-1)\langle\sigma\rangle$, $\partial c_s = (e^{2\pi is}-1)\langle\sigma\rangle$. For $k,l \in \{p,q,r,s\}$ $(k \neq l)$, we set $c_{kl} = \frac{1}{e^{2\pi ik}-1}c_k - \frac{1}{e^{2\pi il}-1}c_l$. Then we see that $c_{kl} \in Z_1(K_2, \check{\mathcal{L}})$, $c_{kl} = -c_{lk}$, $c_{pq}+c_{qs} = c_{ps}$, $c_{rp}+c_{pq} = c_{rq}$, $c_{pq}+c_{qs}+c_{sr}+c_{rp} = 0$. Let us consider the chain $c = \langle\zeta,\mu\rangle+\langle\mu,\nu\rangle+\langle\nu,\rho\rangle+\langle\rho,\zeta\rangle$. Since $p+q+r+s = 0$, we have $c \in Z_1(K_2, \check{\mathcal{L}})$. Here we assume that the restrictions of the branches of $T(u)$ to the chains c and c_p define the same germ at the common point ζ on those chains. Then we have $c = c_p + e^{2\pi ip}c_q + e^{2\pi i(p+q+s)}c_r + e^{2\pi i(p+q)}c_s$, from which it follows by simple calculation that $c = (e^{2\pi ip}-1)c_{pq} + (e^{2\pi i(p+q)}-1)c_{qs} + (e^{-2\pi ir}-1)c_{sr}$. Since c_{pq}, c_{qs}, c_{sr} are linearly independent, we have $Z_1(K_2, \check{\mathcal{L}}) = \mathbf{C}c_{pq} \oplus \mathbf{C}c_{qs} \oplus \mathbf{C}c_{sr}$. Therefore we have $H_1(K_2, \check{\mathcal{L}}) \cong Z_1(K_2, \check{\mathcal{L}}) \cong \mathbf{C}^3$.

Let us now apply the Mayer-Vietoris exact sequence to the complexes K, K_1, K_2, K' (For the Mayer-Vietoris exact sequence, see [9]):

$$
\begin{aligned}
&0 \to H_2(K, \check{\mathcal{L}}) \to H_1(K', \check{\mathcal{L}}) \to H_1(K_1, \check{\mathcal{L}}) \oplus H_1(K_2, \check{\mathcal{L}}) \to H_1(K, \check{\mathcal{L}}) \\
&\quad \to H_0(K', \check{\mathcal{L}}) \to H_0(K_1, \check{\mathcal{L}}) \oplus H_0(K_2, \check{\mathcal{L}}) \to H_0(K, \check{\mathcal{L}}) \to 0.
\end{aligned} \tag{1.1}
$$

Since $H_0(K_1, \check{\mathcal{L}}) \oplus H_0(K_2, \check{\mathcal{L}}) = 0$ by Lemmas 1.2 and 1.3, we have $H_0(K, \check{\mathcal{L}}) = 0$. Furthermore, since the map $H_1(K', \check{\mathcal{L}}) \to H_1(K_1, \check{\mathcal{L}})$ is an isomorphism and the map $H_1(K', \check{\mathcal{L}}) \to H_1(K_2, \check{\mathcal{L}})$ is injective, the map $H_1(K', \check{\mathcal{L}}) \to H_1(K_1, \check{\mathcal{L}}) \oplus H_1(K_2, \check{\mathcal{L}})$ is also injective, from which it follows that $H_2(K, \check{\mathcal{L}}) = 0$. Therefore the exact sequence (1.1) is turned to

$$
0 \to H_1(K', \check{\mathcal{L}}) \to H_1(K_1, \check{\mathcal{L}}) \oplus H_1(K_2, \check{\mathcal{L}}) \to H_1(K, \check{\mathcal{L}}) \to H_0(K', \check{\mathcal{L}}) \to 0,
$$

from which it follows that

$$
0 \to \left(H_1(K_1, \check{\mathcal{L}}) \oplus H_1(K_2, \check{\mathcal{L}})\right)/H_1(K', \check{\mathcal{L}}) \to H_1(K, \check{\mathcal{L}}) \xrightarrow{\Delta} H_0(K', \check{\mathcal{L}}) \to 0. \tag{1.2}
$$

By abuse of notation we may think $\left(H_1(K_1, \check{\mathcal{L}}) \oplus H_1(K_2, \check{\mathcal{L}})\right)/H_1(K', \check{\mathcal{L}}) \cong \mathbf{C}c_{pq} \oplus \mathbf{C}c_{qs} \oplus \mathbf{C}c_{sr}$. Without loss of generality we may think that $H_0(K', \check{\mathcal{L}}) = \mathbf{C}\langle\zeta\rangle$. Let us construct an element $c_0 \in H_1(K, \check{\mathcal{L}})$ such that $\Delta(c_0) = \langle\zeta\rangle$. If we regard c_0 as an element in $Z_1(K, \check{\mathcal{L}})$, then we can write $c_0 = c_1 + c_2$ for some $c_i \in C_1(K_i, \check{\mathcal{L}})$ $(i = 1,2)$. So it is sufficient to construct $c_i \in C_1(K_i, \check{\mathcal{L}})$ $(i = 1,2)$ such that $\Delta(c_0) = \partial(c_1) = -\partial(c_2) = \langle\zeta\rangle$. Obviously, such elements c_1 and c_2 must satisfy the conditions $c_i \notin Z_1(K_i, \check{\mathcal{L}})$ $(i = 1,2)$. We define c_1 and c_2 by $c_1 = \frac{1}{e^{-\pi i(p+q)}-1}(\langle\zeta,\mu\rangle + \langle\mu,\beta\rangle + \langle\beta,\zeta\rangle)$ and $c_2 = \frac{1}{1-e^{2\pi ip}}(\langle\zeta,\mu\rangle+\langle\mu,\sigma\rangle+\langle\sigma,\zeta\rangle)$, where we assume that the restrictions of the branches of $T(u)$ to the chains c_1 and c_2 define the same germ at the common initial point ζ on those chains. We have $\partial(c_1) = \langle\zeta\rangle$ and $\partial(c_2) = -\langle\zeta\rangle$. Then the sum $c_0 = c_1 + c_2$ is the desired solution of the equation $\Delta(c_0) = \langle\zeta\rangle$. If we set $c_2' = \frac{1}{1-e^{2\pi ir}}(\langle\zeta,\sigma\rangle + \langle\sigma,\rho\rangle + \langle\rho,\zeta\rangle)$, the sum $c' = c_1 + c_2'$ is also a cycle satisfying the same equation. Since $c' = c_0 + c_{pr}$, we have $c' - c_0 \in \ker\Delta$. Furthermore, if we set $c_1' = \frac{1}{e^{\pi i(p+r)}-1}(\langle\zeta,\rho\rangle + \langle\rho,\delta\rangle + \langle\delta,\zeta\rangle)$, the sum $c'' = c_1' + c_2$ is also a cycle satisfying the same equation. Since $c'' - c_0 \sim \langle\zeta,\mu\rangle+\langle\mu,\nu\rangle+\langle\nu,\rho\rangle+\langle\rho,\zeta\rangle \in Z_1(K', \check{\mathcal{L}})$, regarding c_0 and c'' as their homology classes, we have $c''-c_0 \in \ker\Delta$. Therefore we see that the set of the elements of $H_1(K, \check{\mathcal{L}})$ mapped by Δ to $\langle\zeta\rangle$ coincides with $c_0 + \ker\Delta$. We note that $\mathbf{C}c_0 \cap \ker\Delta = 0$. If we define the map $\iota : H_0(K', \check{\mathcal{L}}) \to H_1(K, \check{\mathcal{L}})$ by $\iota(\langle\zeta\rangle) = c_0$, then we see that the exact sequence (1.2) is split. Namely we have $H_1(K, \check{\mathcal{L}}) \cong \left[\left(H_1(K_1, \check{\mathcal{L}}) \oplus H_1(K_2, \check{\mathcal{L}})\right)/H_1(K', \check{\mathcal{L}})\right] \oplus \iota\left(H_0(K', \check{\mathcal{L}})\right)$. Here we note that the map ι is an isomorphism. Since the complex K and the surface M are homotopically equivalent, we arrive at the following

Theorem 1. We have $H_2(M, \check{\mathcal{L}}) = H_0(M, \check{\mathcal{L}}) = 0$, $H_1(M, \check{\mathcal{L}}) \cong \mathbf{C}c_{pq} \oplus \mathbf{C}c_{qs} \oplus \mathbf{C}c_{sr} \oplus \mathbf{C}c_0$, where we regard $c_{pq},\ c_{qs},\ c_{sr},\ c_0$ as cycles on M by abuse of notation.

Remark. The homology groups of M with integral coefficients are given by $H_2(M, \mathbf{Z}) = 0$, $H_1(M, \mathbf{Z}) \cong \mathbf{Z}^5$, $H_0(M, \mathbf{Z}) \cong \mathbf{Z}$. We see that the Euler number of the homology with integral coefficients is equal to that of the homology with the local system coefficients.

2. Twisted cohomology groups.

Let Ω^k $(k = 0, 1)$ be the sheaf of holomorphic k-forms on M. We have the exact sequence $0 \to \mathbf{C} \to \Omega^0 \xrightarrow{d} \Omega^1 \to 0$. Since the local system $\mathcal{L} = \mathbf{C}T(u)^{-1}$ is locally constant and without torsion, the tensor functor $\otimes_{\mathbf{C}}\mathcal{L}$ is exact. Namely we have the exact sequence $0 \to \mathcal{L} \to \Omega^0 \otimes_{\mathbf{C}} \mathcal{L} \xrightarrow{d} \Omega^1 \otimes_{\mathbf{C}} \mathcal{L} \to 0$. Let us define the isomorphism between Ω^k and $\Omega^k \otimes_{\mathbf{C}} \mathcal{L}$ by $\Omega^k \ni \varphi \mapsto T(u)\varphi \in \Omega^k \otimes_{\mathbf{C}} \mathcal{L}$, then we have $d(T(u)\varphi) = T(u)\nabla\varphi$, which means that the following diagram is commutative:

$$\begin{array}{ccc} \Omega^k & \xrightarrow{\nabla} & \Omega^{k+1} \\ \downarrow & & \downarrow \\ \Omega^k \otimes_{\mathbf{C}} \mathcal{L} & \xrightarrow[d]{} & \Omega^{k+1} \otimes_{\mathbf{C}} \mathcal{L}, \end{array}$$

where the vertical arrows represent isomorphisms. Combining this commutative diagram and the preceding exact sequence for $\mathcal{L}$, we have the exact sequence $0 \to \mathcal{L} \to \Omega^0 \xrightarrow{\nabla} \Omega^1 \to 0$, from which it follows by the standard procedure that the following exact sequence holds:

$$\begin{aligned} 0 \to & H^0(M, \mathcal{L}) \to H^0(M, \Omega^0) \xrightarrow{\nabla} H^0(M, \Omega^1) \to H^1(M, \mathcal{L}) \to H^1(M, \Omega^0) \\ \xrightarrow{\nabla} & H^1(M, \Omega^1) \to 0. \end{aligned}$$

Then we have

Lemma 2.1. $H^0(M, \mathcal{L}) = 0,\ H^1(M, \mathcal{L}) \cong H^0(M, \Omega^1)/\nabla(H^0(M, \Omega^0))$.

Proof. By definition we have $H^0(M, \mathcal{L}) = \{f \in \Gamma(M, \Omega^0) \mid \nabla f = 0\}$, where $\Gamma(M, \Omega^0)$ denotes the vector space of single-valued holomorphic functions on M. Since the function f satisfying the equation $\nabla f = 0$ is of the form $f(u) = c\theta(u)^{-p}\theta_1(u)^{-q}\theta_2(u)^{-r}\theta_3(u)^{-s}$ for some constant c, which is in general multivalued, we have $H^0(M, \mathcal{L}) = 0$. It is well-known that $H^1(U, \Omega^0) = 0$ for any open Riemann surface U (e.g. [8]). Then we have the short exact sequence $0 \to H^0(M, \Omega^0) \xrightarrow{\nabla} H^0(M, \Omega^1) \to H^1(M, \mathcal{L}) \to 0$, from which it follows that $H^1(M, \mathcal{L}) \cong H^0(M, \Omega^1)/\nabla(H^0(M, \Omega^0))$.

Let Ω^k_{mer} $(k = 0, 1)$ be the sheaf of meromorphic k-forms on $\mathbf{C}/\Gamma$ which are holomorphic on M and have poles only at $u = 0, \frac{1}{2}, \frac{\tau}{2}, \frac{1+\tau}{2}$ if they exist. The restriction of Ω^k_{mer} to M is a subsheaf of Ω^k. We have a subcomplex $0 \to \mathcal{L} \to \Omega^0_{\text{mer}} \xrightarrow{\nabla} \Omega^1_{\text{mer}} \to 0$ of the complex $0 \to \mathcal{L} \to \Omega^0 \xrightarrow{\nabla} \Omega^1 \to 0$. The natural map of sheaf complexes,

$\iota : (\Omega^\bullet_{\mathrm{mer}}, \nabla) \to (\Omega^\bullet, \nabla)$, induces the natural homomorphism of de Rham cohomologies: $\iota_* : H^1_{\mathrm{DR}}(\Omega^\bullet_{\mathrm{mer}}, \nabla) \to H^1_{\mathrm{DR}}(\Omega^\bullet, \nabla)$, where $H^1_{\mathrm{DR}}(\Omega^\bullet_{\mathrm{mer}}, \nabla)$ and $H^1_{\mathrm{DR}}(\Omega^\bullet, \nabla)$ denote $H^0(M, \Omega^1_{\mathrm{mer}})/\nabla(H^0(M, \Omega^0_{\mathrm{mer}}))$ and $H^0(M, \Omega^1)/\nabla(H^0(M, \Omega^0))$, respectively. In fact we have

Lemma 2.2. ι_* is an isomorphism.

It is well-known that this lemma is proved by the Grothendieck-Deligne comparison theorem ([7], II, §6). Nevertheless we give here a direct proof by the technique of the complex analysis because our proof tells us what subcomplex of the de Rham complex $H^0(M, \Omega^\bullet_{\mathrm{mer}})$ is suitable to take for establishing the isomorphism between the group $H^1_{\mathrm{DR}}(\Omega^\bullet_{\mathrm{mer}}, \nabla)$ and a group whose structure is clearer. This will be explained in detail later.

Proof of Lemma 2.2. Let φ be an element in $H^0(M, \Omega^1)$. We set $\varphi = f(u)du$. Then the function $f(u)$ is single-valued and holomorphic on M, and may have isolated essential singularities at $u = 0, \frac{1}{2}, \frac{\tau}{2}, \frac{1+\tau}{2}$ where $f(u)$ is expanded in Laurent series. Let $P_0(u),\ P_1(u),\ P_2(u),\ P_3(u)$ be the principal parts of the Laurent expansions of $f(u)$ at $u = 0, \frac{1}{2}, \frac{\tau}{2}, \frac{1+\tau}{2}$, respectively. For a while, let us restrict ourselves to the case of the neighbourhood at $u = 0$. Let us find a function $Q_0(u)$ single-valued around $u = 0$ satisfying the equation $P_0(u)du = \nabla Q_0$, that is, $P_0(u) = \dfrac{dQ_0}{du} + Q_0 \dfrac{d}{du}(\log T(u))$. Here we may assume that $P_0(u) = \sum_{n \ge 1} a_{-n} u^{-n}$. By the quadrature we have the general solution of this equation: $Q_0 = T(u)^{-1}[\int T(u)P_0(u)du + C]$ for some constant C. Since Q_0 is single-valued, the condition $C = 0$ is necessary. Let us investigate the behaviour of the solution $Q_0(u)$ with $C = 0$ around $u = 0$. Since p is not an integer (§1), the multivaluedness of $T(u)$ around $u = 0$ comes from the factor u^p. Namely we can write $T(u) = u^p \times$ (single-valued holomorphic function) around $u = 0$. Moreover, since we can write $T(u)P_0(u) = \sum_{n=-\infty}^{n=+\infty} c_n u^{p+n}$ around $u = 0$, we have $\displaystyle\int T(u)P_0(u)du = \sum_{n=-\infty}^{n=+\infty} \frac{c_n}{p+n+1} u^{p+n+1}$, which is of the form $u^p \times$ (single-valued analytic function which may have an isolated singularity at $u = 0$) around $u = 0$. Consequently, the function $Q_0(u) = T(u)^{-1} \int T(u)P_0(u)du$ is a single-valued analytic function around $u = 0$ which may have an isolated singularity at $u = 0$, and therefore can be expanded in Laurent series at $u = 0$. We set $Q_0(u) = \sum_{n=-\infty}^{n=+\infty} b_n u^n$, the Laurent expansion at $u = 0$. Moreover we set $Q_{0-}(u) = \sum_{n \le 0} b_n u^n$ and $Q_{0+}(u) = \sum_{n \ge 1} b_n u^n$. Substituting $Q_0 = Q_{0-} + Q_{0+}$ into the original equation above, we have $P_0 = Q'_{0-} + Q'_{0+} + Q_{0-} \cdot (\log T(u))' + Q_{0+} \cdot (\log T(u))'$. Since $(\log T(u))'$ has a pole of order one at $u = 0$ and Q_{0+} has a zero of order one at $u = 0$, the product $Q_{0+} \cdot (\log T(u))'$ is holomorphic at $u = 0$, and so is Q'_{0+}. Consequently, we see that in the right-hand side of the preceding relation the sum $Q'_{0-} + Q_{0-} \cdot (\log T(u))'$ contributes to the principal part P_0. Therefore, setting $\nabla Q_{0-} = g(u)du$, we see that the principal part of the Laurent expansion of $g(u)$ at $u = 0$ is equal to P_0. By the similar argument, we obtain functions $Q_{1-}(u),\ Q_{2-}(u),\ Q_{3-}(u)$ from principal parts $P_1(u),\ P_2(u),\ P_3(u)$, respectively. We give Laurent expansions for $Q_{k-}(u)$ $(k = 0, 1, 2, 3)$ as follows: $Q_{0-}(u) = \sum_{n \le 0} b_n^{(0)} u^n$, $Q_{1-}(u) = \sum_{n \le 0} b_n^{(1)} (u - \frac{1}{2})^n$, $Q_{2-}(u) = \sum_{n \le 0} b_n^{(2)} (u - \frac{\tau}{2})^n$, $Q_{3-}(u) = \sum_{n \le 0} b_n^{(3)} (u - \frac{1+\tau}{2})^n$. We set $Q_{0*} = Q_{0-} - b_0^{(0)} - (b_{-1}^{(0)} + b_{-1}^{(1)} + b_{-1}^{(2)} + b_{-1}^{(3)}) u^{-1}$, $Q_{1*} = Q_{1-} - b_0^{(1)}$, $Q_{2*} = Q_{2-} - b_0^{(2)}$, $Q_{3*} = Q_{3-} - b_0^{(3)}$. We see that the residue of Q_{0*} at $u = 0$ is $-b_{-1}^{(1)} - b_{-1}^{(2)} - b_{-1}^{(3)}$, that of Q_{1*} at $u = \frac{1}{2}$ is $b_{-1}^{(1)}$, that of Q_{2*} at $u = \frac{\tau}{2}$ is

$b_{-1}^{(2)}$, and that of Q_{3*} at $u = \frac{1+\tau}{2}$ is $b_{-1}^{(3)}$. By Mittag-Leffler's theorem (e.g. see [8]), there exists a global function $Q_* \in H^0(M, \Omega^0)$ whose principal parts of the Laurent expansions at $u = 0, \frac{1}{2}, \frac{\tau}{2}, \frac{1+\tau}{2}$ coincide with Q_{0*}, Q_{1*}, Q_{2*} and Q_{3*}, respectively. We note that the 1-form $P_0(u)du - \nabla b_0^{(0)} - \nabla \dfrac{b_{-1}^{(0)} + b_{-1}^{(1)} + b_{-1}^{(2)} + b_{-1}^{(3)}}{u} - \nabla Q_{0*}$ is holomorphic at $u = 0$, and that the forms $P_k(u)du - \nabla b_0^{(k)} - \nabla Q_{k*}$ $(k = 1, 2, 3)$ are holomorphic at $u = \frac{1}{2}, \frac{\tau}{2}, \frac{1+\tau}{2}$, respectively. Then there exist a constant ξ and an Abelian 1-form η of third kind with poles at $u = 0, \frac{1}{2}, \frac{\tau}{2}, \frac{1+\tau}{2}$ if they exist, such that the principal part of the Laurent expansion at $u = 0$ of the 1-form $\xi\mathcal{P}(u)du + \eta$, where $\mathcal{P}(u)$ denotes the Weierstrass P-function with periods 1 and τ, coincides with that of $\nabla(\dfrac{1}{u})$, and that the 1-form $f(u)du - (b_0^{(0)} + b_0^{(1)} + b_0^{(2)} + b_0^{(3)})\nabla(1) - (b_{-1}^{(0)} + b_{-1}^{(1)} + b_{-1}^{(2)} + b_{-1}^{(3)})(\xi\mathcal{P}(u)du + \eta) - \nabla Q_*$, which we denote by ζ, is holomorphic on the whole torus $\mathbf{C}/\Gamma$. Here we note that $\nabla(1)$ $(= \omega)$ is an Abelian 1-form of third kind, and $\xi\mathcal{P}(u)du$ is an Abelian 1-form of second kind. Setting $\psi = (b_0^{(0)} + b_0^{(1)} + b_0^{(2)} + b_0^{(3)})\nabla(1) + (b_{-1}^{(0)} + b_{-1}^{(1)} + b_{-1}^{(2)} + b_{-1}^{(3)})(\xi\mathcal{P}(u)du + \eta) + \zeta$, we see that $\psi \in H^0(M, \Omega^1_{\mathrm{mer}})$ and $\varphi = \psi + \nabla Q_*$. From this result we can show the surjectivity of the map ι_* as follows. Let us take $[\varphi] \in H^1_{\mathrm{DR}}(\Omega^\bullet, \nabla)$ arbitrarily, where $\varphi \in H^0(M, \Omega^1)$. If we form $[\psi] \in H^1_{\mathrm{DR}}(\Omega^\bullet_{\mathrm{mer}}, \nabla)$ from the element $\psi \in H^0(M, \Omega^1_{\mathrm{mer}})$ whose existence is guaranteed above, then we have $\iota_*[\psi] = [\varphi]$, which proves the surjectivity of ι_*. The proof of the injectivity of ι_* is as follows. For $[\psi] \in H^1_{\mathrm{DR}}(\Omega^\bullet_{\mathrm{mer}}, \nabla)$, we set $\iota_*[\psi] = 0$. This equation is translated into the assertion that there exists a single-valued function $g \in H^0(M, \Omega^0)$ such that $\psi = \nabla g$. If we set $\psi = f(u)du$, we see that $f(u)$ is holomorphic on M and has poles at $u = 0, \frac{1}{2}, \frac{\tau}{2}, \frac{1+\tau}{2}$ if $f(u)$ is not holomorphic there. The equation is rewritten as $f(u) = \dfrac{dg}{du} + g(u)\dfrac{d}{du}(\log T(u))$, from which we have the solution $g(u) = T(u)^{-1} \int T(u)f(u)du$. By the same argument as when we constructed Q_0 from P_0 and investigated the behaviour of Q_0 at $u = 0$, we see that $g(u)$ is single-valued and holomorphic on M, and has poles at $u = 0, \frac{1}{2}, \frac{\tau}{2}, \frac{1+\tau}{2}$ if $g(u)$ is not holomorphic there. Therefore we conclude that $g(u) \in H^0(M, \Omega^0_{\mathrm{mer}})$, and $[\psi] = 0$ as the equality in $H^1_{\mathrm{DR}}(\Omega^\bullet_{\mathrm{mer}}, \nabla)$, which proves the injectivity of ι_*. Q.E.D.

Inspired by the proof of Lemma 2.2, we give the following formulation. Let D be an effective divisor on $\mathbf{C}/\Gamma$ given by $D = 2[0] + [\frac{1}{2}] + [\frac{\tau}{2}] + [\frac{1+\tau}{2}]$. Let Ω_D be the sheaf of meromorphic 1-forms on $\mathbf{C}/\Gamma$ which are multiples of the divisor $-D$. Then Ω_D is a subsheaf of Ω^1_{mer}. Let $\mathcal{O}_D$ be the sheaf of meromorphic functions on $\mathbf{C}/\Gamma$ which are multiples of the divisor $-D$. We introduce two complexes:

$$0 \to H^0(M, \Omega^0_{\mathrm{mer}}) \xrightarrow{\nabla} H^0(M, \Omega^1_{\mathrm{mer}}) \to 0, \qquad 0 \to \mathbf{C} \xrightarrow{\nabla} H^0(\mathbf{C}/\Gamma, \Omega_D) \to 0,$$

where the latter is a subcomplex of the former: $\mathbf{C} \subset H^0(M, \Omega^0_{\mathrm{mer}})$ and $H^0(\mathbf{C}/\Gamma, \Omega_D) \subset H^0(M, \Omega^1_{\mathrm{mer}})$, and $H^0(\mathbf{C}/\Gamma, \Omega_D) = \{\varphi : \text{holomorphic function on } M \mid \mathrm{ord}_p(\varphi) \geq -\mathrm{ord}_p(D)$ for $p \in \mathbf{C}/\Gamma\}$. Let us observe the structure of the vector space $H^0(\mathbf{C}/\Gamma, \Omega_D)$. First we have

Lemma 2.3. $\dim H^0(\mathbf{C}/\Gamma, \Omega_D) = 5$.

Proof. The Riemann-Roch formula for a compact Riemann surface X is given by $\dim H^0(X, \mathcal{O}_{-D}) - \dim H^0(X, \Omega_D) = 1 - g - \deg D$. In our case, since $X = \mathbf{C}/\Gamma$, $g = 1$, $\deg D = 5$, $H^0(X, \mathcal{O}_{-D}) = 0$, we have $\dim H^0(X, \Omega_D) = 5$.

Let $\mathcal{P}(u)$ be the Weierstrass P-function with periods 1 and τ. For $i, j \in \{1, 2, 3\}$

$(i \neq j)$, we define 1-forms $\omega_i,\ \omega_{ij}$ by

$$\omega_1 = \frac{1}{2} d\log\left(\mathcal{P}(u) - \mathcal{P}\left(\frac{1}{2}\right)\right) = d\log\theta_1(u) - d\log\theta(u) = -\pi\theta_1^2 \frac{\theta_2(u)\theta_3(u)}{\theta(u)\theta_1(u)} du,$$

$$\omega_2 = \frac{1}{2} d\log\left(\mathcal{P}(u) - \mathcal{P}\left(\frac{\tau}{2}\right)\right) = d\log\theta_2(u) - d\log\theta(u) = -\pi\theta_2^2 \frac{\theta_1(u)\theta_3(u)}{\theta(u)\theta_2(u)} du,$$

$$\omega_3 = \frac{1}{2} d\log\left(\mathcal{P}(u) - \mathcal{P}\left(\frac{1+\tau}{2}\right)\right) = d\log\theta_3(u) - d\log\theta(u) = -\pi\theta_3^2 \frac{\theta_1(u)\theta_2(u)}{\theta(u)\theta_3(u)} du,$$

$$\omega_{12} = d\log\theta_2(u) - d\log\theta_1(u) = \pi\theta_3^2 \frac{\theta(u)\theta_3(u)}{\theta_1(u)\theta_2(u)} du,$$

$$\omega_{13} = d\log\theta_3(u) - d\log\theta_1(u) = \pi\theta_2^2 \frac{\theta(u)\theta_2(u)}{\theta_1(u)\theta_3(u)} du,$$

$$\omega_{23} = d\log\theta_3(u) - d\log\theta_2(u) = -\pi\theta_1^2 \frac{\theta(u)\theta_1(u)}{\theta_2(u)\theta_3(u)} du.$$

Moreover we set $\omega_{ij} = -\omega_{ji}$. Then we have $\omega_1 + \omega_{12} = \omega_2$, $\omega_1 + \omega_{13} = \omega_3$, $\omega_2 + \omega_{23} = \omega_3$, $\omega_{12} + \omega_{23} = \omega_{13}$. Therefore we see that the maximal number of linearly independent 1-forms among ones defined above is three. The 1-form ω_1 has poles of order one at $u = \frac{1}{2}, 0$ with residues $+1, -1$, respectively, ω_2 has poles of order one at $u = \frac{\tau}{2}, 0$ with residues $+1, -1$, respectively, ω_3 has poles of order one at $u = \frac{1+\tau}{2}, 0$ with residues $+1, -1$, respectively, ω_{12} has poles of order one at $u = \frac{\tau}{2}, \frac{1}{2}$ with residues $+1, -1$, respectively, ω_{13} has poles of order one at $u = \frac{1+\tau}{2}, \frac{\tau}{2}$ with residues $+1, -1$, respectively, and ω_{23} has poles of order one at $u = \frac{1+\tau}{2}, \frac{\tau}{2}$ with residues $+1, -1$, respectively. Obviously, we have $\omega_i, \omega_{ij} \in H^0(\mathbf{C}/\Gamma, \Omega_D)$. Besides, we have $du,\ \mathcal{P}(u)du \in H^0(\mathbf{C}/\Gamma, \Omega_D)$. Therefore we have

Lemma 2.4. The five 1-forms: du, $\mathcal{P}(u)du$ and three linearly independent 1-forms among $\omega_i,\ \omega_{ij}$, form a basis of $H^0(\mathbf{C}/\Gamma, \Omega_D)$.

The inclusion map between the two complexes defined above induces a natural map $I : H^0(\mathbf{C}/\Gamma, \Omega_D)/\nabla(\mathbf{C}) \longrightarrow H^1_{\mathrm{DR}}(\Omega^\bullet_{\mathrm{mer}}, \nabla) = H^0(M, \Omega^1_{\mathrm{mer}})/\nabla H^0(M, \Omega^0_{\mathrm{mer}})$. We want to prove that I is an isomorphism.

Lemma 2.5. I is injective.

Proof. It follows immediately from the fact $\nabla H^0(M, \Omega^0_{\mathrm{mer}}) \cap H^0(\mathbf{C}/\Gamma, \Omega_D) = \nabla(\mathbf{C})$.

The surjectivity of I follows immediately from the following

Lemma 2.6. For an arbitrary $\varphi \in H^0(M, \Omega^1_{\mathrm{mer}})$, there exist $\psi \in H^0(\mathbf{C}/\Gamma, \Omega_D)$ and $f \in H^0(M, \Omega^0_{\mathrm{mer}})$ such that $\varphi = \psi + \nabla f$.

Proof. The lemma holds if it is proved for the following two cases: (i) φ has only one pole of order 2 at $u = \frac{1}{2}$ or $u = \frac{\tau}{2}$ or $u = \frac{1+\tau}{2}$; (ii) φ has only one pole of order more that 2 at $u = 0$ or $u = \frac{1}{2}$ or $u = \frac{\tau}{2}$ or $u = \frac{1+\tau}{2}$.

(i) Without loss of generality, we may concentrate our attention to the case where $u = \frac{1}{2}$. The other cases are treated similarly. Let us compute

$$\nabla\left(\frac{\theta_2(u)\theta_3(u)}{\theta(u)\theta_1(u)}\right) = \frac{d}{du}\left(\frac{\theta_2(u)\theta_3(u)}{\theta(u)\theta_1(u)}\right) du + \frac{\theta_2(u)\theta_3(u)}{\theta(u)\theta_1(u)}\omega. \tag{2.1}$$

Here we have

$$\frac{d}{du}\left(\frac{\theta_2(u)\theta_3(u)}{\theta(u)\theta_1(u)}\right) = \frac{\theta_1(u)\theta_3(u)\{\theta(u)\theta_2'(u) - \theta'(u)\theta_2(u)\} + \theta(u)\theta_2(u)\{\theta_1(u)\theta_3'(u) - \theta_1'(u)\theta_3(u)\}}{\theta(u)^2\theta_1(u)^2}.$$

Applying the formulas $\{\theta'(u)\theta_2(u) - \theta(u)\theta_2'(u)\}\theta_1\theta_3 = \theta_1(u)\theta_3(u)\theta_2\theta'$, $\{\theta_3'(u)\theta_1(u) - \theta_3(u)\theta_1'(u)\}\theta_1\theta_3 = \theta_2(u)\theta(u)\theta_2\theta'$ and $\theta' = \pi\theta_1\theta_2\theta_3$ to the right-hand side of the preceding equality, we have

$$\frac{d}{du}\left(\frac{\theta_2(u)\theta_3(u)}{\theta(u)\theta_1(u)}\right) = -\pi\theta_2^2\frac{\theta_3(u)^2}{\theta(u)^2} + \pi\theta_2^2\frac{\theta_2(u)^2}{\theta_1(u)^2}. \tag{2.2}$$

From the relation $\omega = -p\omega_3 - q\omega_{13} - r\omega_{23}$, it follows that

$$\frac{\theta_2(u)\theta_3(u)}{\theta(u)\theta_1(u)}\omega = \left\{p\pi\theta_3^2\frac{\theta_2(u)^2}{\theta(u)^2} - q\pi\theta_2^2\frac{\theta_2(u)^2}{\theta_1(u)^2} + r\pi\theta_1^2\right\}du. \tag{2.3}$$

Substituting (2.2) and (2.3) into (2.1), we have

$$\nabla\left(\frac{\theta_2(u)\theta_3(u)}{\theta(u)\theta_1(u)}\right) = \left\{-\pi\theta_2^2\frac{\theta_3(u)^2}{\theta(u)^2} + (1-q)\pi\theta_2^2\frac{\theta_2(u)^2}{\theta_1(u)^2} + p\pi\theta_3^2\frac{\theta_2(u)^2}{\theta(u)^2} + r\pi\theta_1^2\right\}du. \tag{2.4}$$

Here we note that $\mathcal{P}\left(u+\frac{1}{2}\right) - \mathcal{P}\left(\frac{1+\tau}{2}\right) = \pi^2\theta_1^2\theta_2^2\frac{\theta_2(u)^2}{\theta_1(u)^2}$, $\mathcal{P}(u) - \mathcal{P}\left(\frac{1+\tau}{2}\right) = \pi^2\theta_1^2\theta_2^2\frac{\theta_3(u)^2}{\theta(u)^2}$, $\mathcal{P}(u) - \mathcal{P}\left(\frac{\tau}{2}\right) = \pi^2\theta_1^2\theta_3^2\frac{\theta_2(u)^2}{\theta(u)^2}$. Then the equality (2.4) means that, for $\varphi = \mathcal{P}(u+\frac{1}{2})du$, the lemma holds if we take $\psi = \mathcal{P}(u)du +$ (holomorphic 1-form).

(ii) Without loss of generality, we may assume that φ has only one pole of order ν (≥ 3) at $u = 0$. Moreover, we may assume that such a 1-form φ is written by $\varphi = \mathcal{P}(u)^k\mathcal{P}'(u)^l du$ ($2k + 3l = \nu \geq 3$, $k \geq 0$, $l \geq 0$). We prove by induction on ν. Let us first prove the lemma for $\varphi = \mathcal{P}'(u)du = -2\pi^3\theta_1^2\theta_2^2\theta_3^2\frac{\theta_1(u)\theta_2(u)\theta_3(u)}{\theta(u)^3}du$. We have

$$\begin{aligned}
\nabla\left(\pi^2\theta_1^2\theta_2^2\frac{\theta_3(u)^2}{\theta(u)^2}\right) &= \pi^2\theta_1^2\theta_2^2 d\left(\frac{\theta_3(u)^2}{\theta(u)^2}\right) + \pi^2\theta_1^2\theta_2^2\frac{\theta_3(u)^2}{\theta(u)^2}(-p\omega_3 - q\omega_{13} - r\omega_{23}) \\
&= \left\{-2\pi(\theta')^2\frac{\theta_1(u)\theta_2(u)\theta_3(u)}{\theta(u)^3} + \pi^2\theta_1^2\theta_2^2\frac{\theta_3(u)^2}{\theta(u)^2}\cdot p\pi\theta_3^2\frac{\theta_1(u)\theta_2(u)}{\theta(u)\theta_3(u)}\right. \\
&\quad \left. +\pi^2\theta_1^2\theta_2^2\frac{\theta_3(u)^2}{\theta(u)^2}\cdot(-q)\pi\theta_2^2\frac{\theta(u)\theta_2(u)}{\theta_1(u)\theta_3(u)} + \pi^2\theta_1^2\theta_2^2\frac{\theta_3(u)^2}{\theta(u)^2}\cdot r\pi\theta_1^2\frac{\theta(u)\theta_1(u)}{\theta_2(u)\theta_3(u)}\right\}du \\
&= \left\{(p-2)\pi^3\theta_1^2\theta_2^2\theta_3^2\frac{\theta_1(u)\theta_2(u)\theta_3(u)}{\theta(u)^3} - q\pi^3\theta_1^2\theta_2^4\frac{\theta_2(u)\theta_3(u)}{\theta(u)\theta_1(u)} + r\pi^3\theta_1^4\theta_2^2\frac{\theta_3(u)\theta_1(u)}{\theta(u)\theta_2(u)}\right\}du,
\end{aligned}$$

which proves the lemma for $\varphi = \mathcal{P}'(u)du$. Next we proceed to the general case. Since $\mathcal{P}(u)$ satisfies the differential equation $(\mathcal{P}'(u))^2 = 4\mathcal{P}(u)^3 - g_2\mathcal{P}(u) - g_3$ (g_2, g_3 are constants), without loss of generality, we may assume that the general 1-form φ is of the form $\varphi = \left(\frac{\theta_3(u)}{\theta(u)}\right)^{2N}\left(\frac{\theta_1(u)\theta_2(u)\theta_3(u)}{\theta(u)^3}\right)^M du$ ($N \geq 1$, $M = 0$ or 1). We have already proved the case where $\nu = 2N + 3M \leq 3$. So we assume that $\nu \geq 4$. Let us compute

$$\nabla\left(\frac{\theta_1(u)\theta_2(u)\theta_3(u)^{2N-3}}{\theta(u)^{2N-1}}\right) = d\left(\frac{\theta_1(u)\theta_2(u)\theta_3(u)^{2N-3}}{\theta(u)^{2N-1}}\right) + \frac{\theta_1(u)\theta_2(u)\theta_3(u)^{2N-3}}{\theta(u)^{2N-1}}\omega. \tag{2.5}$$

We have

$$\begin{aligned}
&\frac{d}{du}\left(\frac{\theta_1(u)\theta_2(u)\theta_3(u)^{2N-3}}{\theta(u)^{2N-1}}\right)\\
=&\frac{\theta_1'(u)\theta_2(u)\theta_3(u)^{2N-3}}{\theta(u)^{2N-1}}+\frac{\theta_1(u)\theta_2'(u)\theta_3(u)^{2N-3}}{\theta(u)^{2N-1}}+(2N-3)\frac{\theta_1(u)\theta_2(u)\theta_3(u)^{2N-4}\theta_3'(u)}{\theta(u)^{2N-1}}\\
&-(2N-1)\frac{\theta_1(u)\theta_2(u)\theta_3(u)^{2N-3}\theta'(u)}{\theta(u)^{2N}}\\
=&\frac{\theta_2(u)\theta_3(u)^{2N-3}\{\theta(u)\theta_1'(u)-\theta'(u)\theta_1(u)\}}{\theta(u)^{2N}}+\frac{\theta_1(u)\theta_3(u)^{2N-3}\{\theta(u)\theta_2'(u)-\theta'(u)\theta_2(u)\}}{\theta(u)^{2N}}\\
&+(2N-3)\frac{\theta_1(u)\theta_2(u)\theta_3(u)^{2N-4}\{\theta(u)\theta_3'(u)-\theta'(u)\theta_3(u)\}}{\theta(u)^{2N}}.
\end{aligned}$$

Applying the formula $\{\theta'(u)\theta_1(u)-\theta(u)\theta_1'(u)\}\theta_2\theta_3=\theta_2(u)\theta_3(u)\theta_1\theta'$ and several similar ones to the right-hand side of the preceding equality, we have

$$\begin{aligned}
\frac{d}{du}\left(\frac{\theta_1(u)\theta_2(u)\theta_3(u)^{2N-3}}{\theta(u)^{2N-1}}\right)=&-\pi\theta_1^2\frac{\theta_2(u)^2\theta_3(u)^{2N-2}}{\theta(u)^{2N}}-\pi\theta_2^2\frac{\theta_1(u)^2\theta_3(u)^{2N-2}}{\theta(u)^{2N}}\\
&-(2N-3)\pi\theta_3^2\frac{\theta_1(u)^2\theta_2(u)^2\theta_3(u)^{2N-4}}{\theta(u)^{2N}}.
\end{aligned}\tag{2.6}$$

Moreover we have

$$\begin{aligned}
\frac{\theta_1(u)\theta_2(u)\theta_3(u)^{2N-3}}{\theta(u)^{2N-1}}\omega=&\frac{\theta_1(u)\theta_2(u)\theta_3(u)^{2N-3}}{\theta(u)^{2N-1}}(-p\omega_3-q\omega_{13}-r\omega_{23})\\
=&\left\{p\pi\theta_3^2\frac{\theta_1(u)^2\theta_2(u)^2\theta_3(u)^{2N-4}}{\theta(u)^{2N}}-q\pi\theta_2^2\frac{\theta_2(u)^2\theta_3(u)^{2N-4}}{\theta(u)^{2N-2}}\right.\\
&\left.+r\pi\theta_1^2\frac{\theta_1(u)^2\theta_3(u)^{2N-4}}{\theta(u)^{2N-2}}\right\}du.
\end{aligned}\tag{2.7}$$

The result of the substitution of (2.6) and (2.7) into (2.5) means that the lemma holds for $\varphi=\left(\frac{\theta_3(u)}{\theta(u)}\right)^{2N}du$. Finally, we have

$$\begin{aligned}
\nabla\left(\frac{\theta_3(u)^{2N}}{\theta(u)^{2N}}\right)=&d\left(\frac{\theta_3(u)^{2N}}{\theta(u)^{2N}}\right)+\frac{\theta_3(u)^{2N}}{\theta(u)^{2N}}(-p\omega_3-q\omega_{13}-r\omega_{23})\\
=&\left\{-2N\pi\theta_3^2\frac{\theta_1(u)\theta_2(u)\theta_3(u)^{2N-1}}{\theta(u)^{2N+1}}+p\pi\theta_3^2\frac{\theta_1(u)\theta_2(u)\theta_3(u)^{2N-1}}{\theta(u)^{2N+1}}\right.\\
&\left.-q\pi\theta_2^2\frac{\theta_2(u)\theta_3(u)^{2N-1}}{\theta(u)^{2N-1}\theta_1(u)}+r\pi\theta_1^2\frac{\theta_1(u)\theta_3(u)^{2N-1}}{\theta(u)^{2N-1}\theta_2(u)}\right\}du,
\end{aligned}$$

which means that the lemma holds for $\varphi=\left(\frac{\theta_3(u)}{\theta(u)}\right)^{2N-2}\frac{\theta_1(u)\theta_2(u)\theta_3(u)}{\theta(u)^3}du$. Therefore Lemma 2.6 is proved completely.

Combining everything above, we have

Theorem 2. We have $H^0(M,\mathcal{L})=H^2(M,\mathcal{L})=0,\ H^1(M,\mathcal{L})\cong H^0(\mathbf{C}/\Gamma,\Omega_D)/\nabla(\mathbf{C})=\mathbf{C}[du]\oplus\mathbf{C}[\mathcal{P}(u)du]\oplus\mathbf{C}[\omega^{(1)}]\oplus\mathbf{C}[\omega^{(2)}]$, where $\omega^{(1)}$ and $\omega^{(2)}$ denote linearly independent vectors in the subspace generated by ω_i and ω_{ij} in $H^0(\mathbf{C}/\Gamma,\Omega_D)$ and $[\varphi]$ denotes the image of an element φ in $H^0(\mathbf{C}/\Gamma,\Omega_D)$ by the natural map $H^0(\mathbf{C}/\Gamma,\Omega_D)\to H^0(\mathbf{C}/\Gamma,\Omega_D)/\nabla(\mathbf{C})$.

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