Spectral Area Estimates For Norms Of Commutators

By

Muneo Chō * And Takahiko Nakazi **

2000 Mathematics Subject Classification : Primary 47 A 20

Key words and phrases : subnormal, p-hyponormal, Putnam inequality

* This research is partially supported by Grant-in-Aid Scientific Research No.17540139
** This research is partially supported by Grant-in-Aid Scientific Research No.17540176

Abstract. Let A and B be commuting bounded linear operators on a Hilbert space. In this paper, we study spectral area estimates for norms of $A^*B - BA^*$ when A is subnormal or p-hyponormal.

§1. Introduction

Let $\mathcal{H}$ be a Hilbert space and $\mathcal{B}(\mathcal{H})$ the set of all bounded linear operators on $\mathcal{H}$. If T is a hyponormal operator in $\mathcal{B}(\mathcal{H})$ then C.R.Putnam [7] proved that $\| T^*T - TT^* \| \leq Area(\sigma(T))/\pi$ where $\sigma(T)$ is the spectrum of T. The second named author [5] has proved that if T is a hyponormal operator and K is in $\mathcal{B}(\mathcal{H})$ with $KT = TK$ then

$$\|T^*K - KT^*\| \leq 2\{Area(\sigma(T))/\pi\}^{1/2}\|K\|.$$

We don't know whether the constant 2 in the inequality is best possible for a hyponormal operator. In §2, we show that the constant is not best possible for a subnormal operator.

When T is a p-hyponormal operator in $\mathcal{B}(\mathcal{H})$, A.Uchiyama [10] generalized the Putnam inequality, that is,

$$\|T^*T - TT^*\| \leq \phi\left(\frac{1}{p}\right)\|T\|^{2(1-p)}\{Area(\sigma(T))/\pi\}^p.$$

This inequality gives the Putnam inequality when $p = 1$. In §3, we generalize the above inquality for the spectral area estimate of $\|T^*K - KT^*\|$ when $TK = KT$. H.Alexander [1] proved the following inequality for a uniform algebra A. If f is in A then

$$dist(\bar{f}, A) \leq \{Area(\sigma(f))/\pi\}^{1/2}.$$

The second named author [5] gave an operator version for the Alexander inequality. This was used in order to estimate $\|T^*K - KT^*\|$ when T is a hyponormal operator and $KT = TK$. We also give an Alexander inequality for a p-hyponormal and we use it to estimate $\|T^*K - KT^*\|$.

In §4, we try to estimate $\|T^*K - KT^*\|$ for arbitrary contraction. In §5, we show a few results about area estimates for p-quasihyponormal operators, restricted shifts and analytic Toeplitz operators.

For $0 < p \leq 1$, T is said to be p-hyponormal if $(T^*T)^p - (TT^*)^p \geq 0$. A 1-hyponormal operator is hyponormal. For an algebra $\mathcal{A}$ in $\mathcal{B}(\mathcal{H})$, let $lat\mathcal{A}$ be the lattice of all $\mathcal{A}$-invariant projections. For a compact subset X in $\mathbb{C}$, $rat(X)$ denotes the set of all rational functions on X.

§2. Subnormal operator

In order to prove Theorem 1, we use the original Alexander inequality.

Theorem 1. *Let T be a subnormal operator in $\mathcal{B}(\mathcal{H})$ and f a rational function on $\sigma(T)$ whose poles are not on it. Then*

$$\| T^*f(T) - f(T)T^* \| \leq \{Area(\sigma(T))/\pi\}^{1/2}\{Area(\sigma(f(T)))/\pi\}^{1/2}.$$

Proof. Suppose that $N \in \mathcal{B}(\mathcal{K})$ is a normal extension of $T \in \mathcal{B}(\mathcal{H})$ and P is an orthogonal projection from $\mathcal{K}$ to $\mathcal{H}$. Then $T = PN \mid \mathcal{H}$ and so

$$\begin{aligned}
&T^*f(T) - f(T)T^* \\
&\quad = \; PN^*Pf(N)P - Pf(N)PN^*P \\
&\quad = \; PN^*f(N)P - Pf(N)PN^*P \\
&\quad = \; Pf(N)N^*P - Pf(N)PN^*P \\
&\quad = \; Pf(N)(1-P)N^*P \\
&\quad = \; Pf(N)(1-P)\cdot(1-P)N^*P
\end{aligned}$$

because $f(N)P = Pf(N)P$ and $f(N)N^* = N^*f(N)$.

Let F be a rational function in $rat(\sigma(T))$. Put $\mathcal{B}_F$ = the norm closure of $\{g(F(N))\ ;\ g \in rat(\sigma(F(N))\}$ then P belongs to $lat\mathcal{B}_F$. Hence

$$\begin{aligned}
&\| (1-P)F(N)^*P \| \\
&\quad \leq \; dist(F(N)^*, \mathcal{B}_F) \leq dist(\bar{z}, rat(\sigma(F(N)))) \\
&\quad \leq \; \{Area(\sigma(F(N)))/\pi\}^{1/2}
\end{aligned}$$

by the Alexander's theorem [1]. Hence, applying F to $F = z$ or $F = f$

$$\begin{aligned}
&\| T^*f(T) - f(T)T^* \| \\
&\quad \leq \; \| (1-P)f(N)^*P \| \cdot \| (1-P)N^*P \| \\
&\quad \leq \; \{Area(\sigma(f(N)))/\pi\}^{1/2}\{Area(\sigma(N))/\pi\}^{1/2} \\
&\quad \leq \; \{Area(\sigma(f(T)))/\pi\}^{1/2}\{Area(\sigma(T))/\pi\}^{1/2}.
\end{aligned}$$

If T is a cyclic subnormal operator and $KT = TK$ then using a theorem of T.Yoshino [12] we can prove that

$$\|T^*K - KT^*\| \leq \{Area(\sigma(T))/\pi\}^{1/2}\{Area(\sigma(K))/\pi\}^{1/2}.$$

The proof is almost same to one of Theorem 1.

§3. p-hyponormal

In order to prove Theorem 2, we use an operator version of the Alexander inequality for a p-hyponormal operator. Unfortunately Lemma 3 is not best possible for $p = 1$ (see [5]). Lemma 1 is due to W.Arveson [2, Lemma 2] and Lemma 2 is due to A.Uchiyama [11, Theorem 3].

We need the following notation to give Theorem 2 and Proposition 1. Let ϕ be a positive function on $(0, \infty)$ such that

$$\phi(t) = \begin{cases} t & \text{if } t \text{ is an integer} \\ t+2 & \text{if } t \text{ is not an integer.} \end{cases}$$

We write $\ell^2 \otimes \mathcal{H}$ for the Hilbert space direct sum $\mathcal{H} \oplus \mathcal{H} \oplus \cdots$, and $1 \otimes T$ denotes the operator $T \oplus T \oplus \cdots \in \mathcal{B}(\ell^2 \otimes \mathcal{H})$ for each operator $T \in \mathcal{B}(\mathcal{H})$.

Lemma 1. *Let $\mathcal{A}$ be an arbitrary ultra-weakly closed subalgebra of $\mathcal{B}(\mathcal{H})$ containing 1, and let $T \in \mathcal{B}(\mathcal{H})$. Then*

$$dist(T, \mathcal{A}) = \sup\{\|(1-P)(1 \otimes T)P\| \ ; \ P \in lat(1 \otimes \mathcal{A})\}.$$

Lemma 2. *If T is a p-hyponormal operator, then*

$$\|T^*T - TT^*\| \leq \phi\left(\frac{1}{p}\right) \|T\|^{2(1-p)} \{Area(\sigma(T))/\pi\}^p.$$

Lemma 3. *If T is a p-hyponormal operator then*

$$dist(T^*, \mathcal{A}) \leq \sqrt{2\phi\left(\frac{1}{p}\right) \|T\|^{1-p} \{Area(\sigma(T))/\pi\}^{p/2}}$$

where $\mathcal{A}$ is the strong closure of $\{f(T) \ ; \ f \in rat(\sigma(T))\}$.

Proof. Let $S = 1 \otimes T$. Then S is p-hyponormal. In order to prove the lemma, by Lemma 1 it is enough to estimate $\sup\{\|(1-P)SP\| \ ; \ P \in lat(1 \otimes \mathcal{A})\}$. If $P \in lat(1 \otimes \mathcal{A})$ then $SP = PSP$ and so

$$\begin{aligned} &\|(1-P)SP\|^2 \\ &= \|PSS^*P - PSPS^*P\| \\ &= \|PSS^*P - PS^*SP + PS^*SP - PSPS^*P\| \\ &\leq \|P(S^*S - SS^*)P\| + \|(PSP)^*(PSP) - (PSP)(PSP)^*\| \\ &\leq \|S^*S - SS^*\| + \|(PSP)^*(PSP) - (PSP)(PSP)^*\|. \end{aligned}$$

By [11, Lemma 4], PSP is p-hyponormal and so by Lemma 2 we have

$$\begin{aligned} &\|PSS^*P - PSPS^*P\|^2 \\ &\leq \phi\left(\frac{1}{p}\right) \|T\|^{2(1-p)} \{Area(\sigma(T))/\pi\}^p + \phi\left(\frac{1}{p}\right) \|PSP\|^{2(1-p)} \{Area(\sigma(PSP))/\pi\}^p \\ &\leq 2\phi\left(\frac{1}{p}\right) \|T\|^{2(1-p)} \{Area(\sigma(T))/\pi\}^p \end{aligned}$$

because $\|PSP\| \leq \|S\| = \|T\|$ and $\sigma(PSP) \subset \sigma(S) = \sigma(T)$. By Lemma 1,

$$dist(T^*, \mathcal{A}) \leq \sqrt{2\phi\left(\frac{1}{p}\right)}\|T\|^{1-p}\{Area(\sigma(T))/\pi\}^{p/2}.$$

Theorem 2. *If T is a p-hyponormal operator in $\mathcal{B}(\mathcal{H})$ and if K is in $\mathcal{B}(\mathcal{H})$ with $KT = TK$, then*

$$\|T^*K - KT^*\| \leq 2\sqrt{2\phi\left(\frac{1}{p}\right)}\|T\|^{1-p}\{Area(\sigma(T))/\pi\}^{p/2}\|K\|.$$

Proof. When $\mathcal{A}$ is the strong closure of $\{f(T)\ ;\ f \in rat(\sigma(T))\}$, for any $A \in \mathcal{A}$

$$\|T^*K - KT^*\| = \|(T^* - A)K + AK - KT^*\| \leq 2\|T^* - A\|\|K\|.$$

Now Lemma 3 implies the theorem.

In Theorem 2, if $p = 1$, that is, T is hyponormal then $\| T^*K - KT^*\| \leq 2\sqrt{2}\{Area(\sigma(T))/2\}^{1/2}\|K\|$. The constant $2\sqrt{2}$ is not best because the second author [5] proved that $\|T^*K - KT^*\| \leq 2\{Area(\sigma(T))/2\}^{1/2}\|K\|$. If $p = \dfrac{1}{2}$, that is, T is semi-hyponormal then $\|T^*K - KT^*\| \leq 4\|T\|^{1/2}\{Area(\sigma(T))/\pi\}^{1/4}\|K\|$.

§4. Norm estimates

In general, it is easy to see that $\|T^*T - TT^*\| \leq \|T\|^2$. By Theorem 1, if T is subnormal and f is an analytic polynomial then

$$\|T^*f(T) - f(T)T^*\| \leq \|T\|\|f(T)\|.$$

In this section, we will prove that $\|T^*T^n - T^nT^*\| \leq \|T\|^{n+1}$ for arbitrary T in $\mathcal{B}(\mathcal{H})$.

Theorem 3. *If T is a contraction on $\mathcal{H}$ and f is an analytic function on the closed unit disc $\bar{D}$ then $\| T^*f(T) - f(T)T^* \| \leq \sup_{z\in D} | f(z) |$.*

Proof. By a theorem of Sz.-Nagy [6], there exists a unitary operator U on $\mathcal{K}$ such that $\mathcal{K}$ is a Hilbert space with $\mathcal{K} \supseteq \mathcal{H}$ and $T^n = PU^n \mid \mathcal{K}$ for $n \geq 0$ where P is an orthogonal projection from $\mathcal{K}$ to $\mathcal{H}$. Then it is known that $U^*P = PU^*P$ and $f(T) = Pf(U) \mid \mathcal{H}$. Hence

$$\begin{aligned} & T^*f(T) - f(T)T^* \\ = \ & PU^*Pf(U)P - Pf(U)PU^*P \\ = \ & PU^*Pf(U)P - Pf(U)U^*P \\ = \ & PU^*(I - P)f(U)P \end{aligned}$$

because $U^*P = PU^*P$ and $f(U)U^* = U^*f(U)$. Therefore

$$\begin{aligned} &\| T^*f(T) - f(T)T^* \| \\ &= \| PU^*(I-P)f(U)P \| \leq \sup_{z\in D} | f(z) | . \end{aligned}$$

Corollary 1. *If T is in $\mathcal{B}(\mathcal{H})$ then for any $n \geq 1$* $\| T^*T^n - T^nT^* \| \leq \| T \|^{n+1}$.

Proof. Put $A = T/\|T\|$ then A is a contraction and so by Theorem 2 $\|A^*A^n - A^nA^*\| \leq 1$ and so $\|T^*T^n - T^nT^*\| \leq \|T\|^{n+1}$.

§5. Remarks

In this section, we give spectral area estimates for p-quasihyponomal operators, restricted shifts and analytic Toeplitz operators.

For $0 < p \leq 1, T$ is said to be p-quasihyponormal if $T^*\{(T^*T)^p - (TT^*)^p\}T \geq 0$. A 1-quasihyponormal operator is called quasihyponormal.

Lemma 4. *Let T be p-quasihyponormal and P be a projection such that $TP = PTP$. Then PTP is also p-quasihyponormal.*

Proof. Since T is p-quaihyponormal, $T^*(T^*T)^pT \geq T^*(TT^*)^pT$. Hence, we have

$$PT^*(T^*T)^pTP \geq PT^*(TT^*)^pTP.$$

Since by the Hansen's inequality [4]

$$\begin{aligned} PT^*(T^*T)^pTP &= (PTP)^*P(T^*T)^pP(PTP) \\ &\leq (PTP)^*(PT^*TP)^p(PTP) \\ &= (PTP)^*\{(PTP)^*(PTP)\}^p(PTP) \end{aligned}$$

and by $0 < p < 1$

$$\begin{aligned} PT^*(TT^*)^pTP &\geq (PT^*P)(TPT^*)^p(PTP) \\ &= (PTP)^*\{(PTP)(PTP)^*\}^p(PTP), \end{aligned}$$

we have

$$(PTP)^*\{(PTP)^*(PTP)\}^p \geq (PTP)^*\{(PTP)(PTP)^*\}^p(PTP).$$

Hence, PTP is p-quasihyponormal.

Proposition 1. *If T is a p-quasihyponormal operator in $\mathcal{B}(\mathcal{H})$ and if K is in $\mathcal{B}(\mathcal{H})$ with $KT = TK$, then*

$$\|T^*K - KT^*\| \leq 4\left[\phi\left(\frac{1}{p}\right)\right]^{1/4} \|T\|^{1-p/2}\{Area(\sigma(T))/\pi\}^{p/4}\|K\|.$$

In particular, if T is quasihyponormal then

$$\|T^*K - KT^*\| \le 4\|T\|^{1/2}\{Area(\sigma(T))/\pi\}^{1/4}\|K\|.$$

Proof. We can prove it as in the proof of Theorem 2. By [11, Theorem 6], $\|T^*T - TT^*\| \le 2\|T\|^{2-p}\sqrt{\phi(\frac{1}{p})}\{Area(\sigma(T))/\pi\}^{p/2}$. Hence by Lemma 4

$$dist(T^*, \mathcal{A}) \le 2\|T\|^{1-\frac{p}{2}}\phi\left(\frac{1}{p}\right)^{\frac{1}{4}}\{Area(\sigma(T))/\pi\}^{p/4}.$$

This implies the proposition.

Let H^2 and H^∞ be the usual Hardy spaces on the unit circle and z the coordinate function. M denotes an invariant subspace of H^2 under the multiplication by z. By the well known Beurling theorem, $M = qH^2$ for some inner function. Suppose N is the orthogonal complement of M in H^2. For a function ϕ in H^∞, S_ϕ is an operator on N such that $S_\phi f = P(\phi f)$ $(f \in N)$ where P is the orthogonal projection from H^2 to N. For a symbol ϕ in L^∞, T_ϕ denotes the usual Toeplitz operator on H^2.

Proposition 2. *Suppose $\Phi = q\bar{\phi}$ belongs to H^∞. Then*
(1) $\| S_\phi^* S_\phi - S_\phi S_\phi^* \| \le Area(\overline{\Phi(D)})/\pi$;
(2) $\| S_\phi^* S_\phi^n - S_\phi^n S_\phi^* \| \le \{Area(\overline{\Phi(D)})/\pi\}^{n+1}$ *for* $n \ge 0$.

Proof. By a well known theorem of Sarason [8],

$$\| S_\phi \| = \| \phi + qH^\infty \| = \| \bar{q}\phi + H^\infty \| = \| \bar{\Phi} + H^\infty \| .$$

By Nehari's theorem [6], $\| \bar{\Phi} + H^\infty \| = \| H_{\bar{\Phi}} \|$ where $H_{\bar{\Phi}}$ denotes a Hankel operator from H^2 to $\bar{z}\bar{H}^2$. Since $\| H_{\bar{\Phi}} \|^2 = \| T_\Phi^* T_\Phi - T_\Phi T_\Phi^* \|$ where T_Φ denotes a Toeplitz operator on H^2, by the Putnam inequality

$$\| T_\Phi^* T_\Phi - T_\Phi T_\Phi^* \| \le Area(\sigma(T_\Phi))/\pi = Area(\overline{\Phi(D)})/\pi.$$

Now since $\| S_\phi^* S_\phi - S_\phi S_\phi^* \| \le \| S_\phi \|^2$, (1) follows. (2) is also clear by the proof above and Corollary 1.

Proposition 3. *Suppose f and g are in H^∞. Then*

$$\|T_f^* T_g - T_g T_f^*\| \le \{Area(\overline{f(D)})/\pi\}^{1/2}\{Area(\overline{g(D)})/\pi\}^{1/2}$$

Proof. It is easy to see that $T_f^* T_g - T_g T_f^* = H_{\bar{g}}^* H_{\bar{f}}$. Hence

$$\|T_f^* T_g - T_g T_f^*\| \le \|H_{\bar{g}}\| \cdot \|H_{\bar{f}}\|.$$

Since $H_{\bar{f}}^* H_{\bar{f}} = T_f^* T_f - T_f T_f^*$, by the Putnam inequalty

$$\|T_f^* T_g - T_g T_f^*\| \leq \{Area(\overline{f(D)})/\pi\}^{1/2} \{Area(\overline{g(D)})/\pi\}^{1/2}.$$

References

1. H. Alexander, Projections of polynomial hulls, J. Funct. Anal. 13(1973), 13-19.
2. W. Arveson, Interpolation problems in nest algebras, J. Funct. Amal. 20(1975), 208-233.
3. M. Chō and M. Itoh, Putnam inequality for p-hyponormal operators, Proc. Amer. Math. Soc. 123(1995), 2435-2440.
4. F. Hansen, An operator inequality, Math. Ann. 246(1980), 249-250.
5. T. Nakazi, Complete spectral area estimates and self-commutators, Michigan Math. J. 35(1988), 435-441.
6. Z. Nehari, On bounded bilinear forms, Ann. of Math. (2) 65(1957), 153-162.
7. C. R. Putnam, An inequality for the area of hyponormal spectra, Math. Z. 116(1970), 323-330.
8. D. Sarason, Generalized interpolation in H^∞, Trans. Amer. Math. Soc. 127(1967), 179-203.
9. B. Sz.-Nagy and C. Foias, Harmonic Analysis Of Operators On Hilbert Space. American Elsevier, New York, 1970.
10. A. Uchiyama, Berger-Shaw's theorem for p-hyponormal operators, Integr. Equ. Oper. Theory 33(1999), 221-230.
11. A. Uchiyama, Inequalities of Putnam and Berger-Shaw for p-quasihyponormal operators, Integr. Equ. Oper. Theory 34(1999), 91-106.
12. T. Yoshino, Subnormal operators with a cyclic vector, Tohoku Math. 21(1969), 47-55.

M. Chō
Department of Mathematics
Kanagawa University
Japan
chiyom01@kanagawa-u.ac.jp

T. Nakazi
Department of Mathematics
Faculty of Science
Hokkaido University
Sapporo 060-0810, Japan
nakazi@math.sci.hokudai.ac.jp