CONICS ON A GENERIC HYPERSURFACE

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Abstract. In this paper, we compute the contributions from double cover maps to genus 0 degree 2 Gromov-Witten invariants of general type projective hypersurfaces. Our results correspond to a generalization of Aspinwall-Morrison formula to general type hypersurfaces in some special cases.
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1. Introduction

In this paper, we discuss a generalization of the multiple cover formula for rational Gromov-Witten invariants of Calabi-Yau manifolds [AM], [M] to double cover maps of a line L on a degree k hypersurface M_N^k in $\mathbf{P}^{N-1}$. Näively, for a given finite set of elements $\alpha_j \in H^*(M_N^k, \mathbf{Z})$, the rational Gromov-Witten invariant $\langle \mathcal{O}_{\alpha_1}\mathcal{O}_{\alpha_2}\cdots\mathcal{O}_{\alpha_n}\rangle_{0,d}$ of M_N^k counts the number of degree d (possibly singular and reducible) rational curves on M_N^k that intersect real sub-manifolds of M_N^k that are Poincaré-dual to α_j.

Recently, the mirror computation of rational Gromov-Witten invariants of M_N^k with negative first chern class $(k-N>0)$ was established in [CG], [Iri], [J]. Using the method presented in these articles, we can compute $\langle \mathcal{O}_{e^{m_1}}\mathcal{O}_{e^{m_2}}\cdots\mathcal{O}_{e^{m_n}}\rangle_{0,d}$ where e is the generator of $H^{1,1}(M_N^k, \mathbf{Z})$. Briefly, mirror computation of M_N^k $(k>N)$ in [J] goes as follows. We start from the following ODE:

$$(1)\quad \Big((\partial_x)^{N-1} - k\cdot\exp(x)\cdot(k\partial_x+k-1)(k\partial_x+k-2)\cdots(k\partial_x+1)\Big)w(x)=0,$$

and construct the virtual Gauss-Manin system associated with (1):

$$(2)\quad \partial_x\tilde{\psi}_{N-2-m}(x) = \tilde{\psi}_{N-1-m}(x)+\sum_{d=1}^{\infty}\exp(dx)\cdot\tilde{L}_m^{N,k,d}\cdot\tilde{\psi}_{N-1-m-(N-k)d}(x),$$

where m runs through all the integers and $\tilde{L}_m^{N,k,d}$ is non-zero only if $0\leq m\leq N-1+(k-N)d$. From the compatibility of (1) and (2), we can derive the recursive formulas that determine all the $\tilde{L}_m^{N,k,d}$'s:

$$\sum_{n=0}^{k-1}\tilde{L}_n^{N,k,1}w^n = k\cdot\prod_{j=1}^{k-1}(jw+(k-j)),$$

$$\sum_{m=0}^{N-1+(k-N)d}\tilde{L}_m^{N,k,d}z^m = \sum_{l=2}^{d}(-1)^l\sum_{0=i_0<\cdots<i_l=d}\times$$

$$\times\sum_{j_l=0}^{N-1+(k-N)d}\cdots\sum_{j_2=0}^{j_3}\sum_{j_1=0}^{j_2}\prod_{n=1}^{l}\Big(\big(\frac{i_{n-1}+(d-i_{n-1})z}{d}\big)^{j_n-j_{n-1}}\cdot\tilde{L}^{N,k,i_n-i_{n-1}}_{j_n+(N-k)i_{n-1}}\Big).$$

With these data, we can construct the formulas that represent rational three point Gromov-Witten invariant $\langle\mathcal{O}_e\mathcal{O}_{e^{N-2-m}}\mathcal{O}_{e^{m-1-(k-N)d}}\rangle_d$ in terms of $\tilde{L}_m^{N,k,d}$. These three point Gromov-Witten invariants are enough for reconstruction of all the rational Gromov-Witten invariants $\langle\mathcal{O}_{e^{m_1}}\mathcal{O}_{e^{m_2}}\cdots\mathcal{O}_{e^{m_n}}\rangle_{0,d}$ [KM]. In particular, we obtain the following formula in the $d=2$ case:

$$\langle\mathcal{O}_e\mathcal{O}_{e^{N-2-m}}\mathcal{O}_{e^{m-1-(k-N)2}}\rangle_2 = k\cdot\left(\tilde{L}_n^{N,k,2}-\tilde{L}_{1+2(k-N)}^{N,k,2}-2\tilde{L}_{1+(k-N)}^{N,k,1}\Big(\sum_{j=0}^{k-N}(\tilde{L}_{n-j}^{N,k,1}-\tilde{L}_{1+2(k-N)-j}^{N,k,1})\Big)\right). \tag{3}$$

According to the results of this procedure, rational three point Gromov-Witten invariants can be rational numbers with large denominator if $k>N$, in contrast to the Calabi-Yau case where rational three point Gromov-Witten invariants are always integers.

One of the reasons of this rationality (non-integrality) comes from the contributions of multiple cover maps to Gromov-Witten invariants. In the Calabi-Yau case ($N=k$), for any divisor m of d there are some contributions from degree m multiple cover maps ϕ of a rational curve $\mathbf{P}^1$ onto a degree $\frac{d}{m}$ rational curve $C\hookrightarrow M_k^k$. The contributions from the multiple cover maps are expressed in terms of the virtual fundamental class of Gromov-Witten invariants. Let C be a general degree d rational curve in M_k^k. Its normal bundle N_{C/M_k^k} is decomposed into a direct sum of line bundles as follows:

$$N_{C/M_k^k}\simeq O_C(-1)\oplus O_C(-1)\oplus O_C^{\oplus(k-5)}.$$

Let $\phi:\mathbf{P}^1\to C$ be a holomorphic map of degree m. Since the pull-back $\phi^*(N_{C/M_k^k})$ is given by

$$\phi^*(N_{C/M_k^k})\simeq O_{\mathbf{P}^1}(-m)\oplus O_{\mathbf{P}^1}(-m)\oplus O_{\mathbf{P}^1}^{\oplus k-5},$$

we obtain $h^1(\phi^*(N_{C/M_k^k}))=2m-2$. On the other hand, let $\overline{M}_{0,0}(M,d)$ be the moduli space of 0-pointed stable maps of degree d from genus 0 curve to M. Then the moduli space of ϕ is the fiber space $\pi:\overline{M}_{0,0}(C,m)\to\overline{M}_{0,0}(M_k^k,\frac{d}{m})$, whose fibre $\overline{M}_{0,0}(C,m)$ over C (fixed) has complex dimension $2m-2$. Then the push-forward of the virtual fundamental class $\pi_*(c_{top}(H^1(\phi^*N_{C/M_k^k})))$ can be computed only by intersection theory on the fiber $\overline{M}_{0,0}(C,m)$, which turns out to be equal to $\frac{1}{d^3}$. This depends on neither the structure of the base $\overline{M}_{0,0}(M_k^k,\frac{d}{m})$ nor the global structure of the fibration.

But when $k<N$, the situation is more complicated than M_k^k because of negative first Chern class. Let us concentrate on the case of $d=2, m=2$ that we discuss in this paper. In this case, C is just a line L on the hypersurface M_N^k. The moduli space $\overline{M}_{0,0}(M_N^k,1)$ is a sub-manifold of $\overline{M}_{0,0}(\mathbf{P}^{N-1},1)$, while $\overline{M}_{0,0}(\mathbf{P}^{N-1},1)$ is the Grassmannian $G(2,N)$, the moduli space of rank 2 quotients of $V=\mathbf{C}^N$. As will be shown later, for a generic line L, N_{L/M_N^k} is decomposed into

$$N_{L/M_N^k}\simeq O_L(-1)^{\oplus k-N+2}\oplus O_L^{\oplus 2N-k-5}.$$

By pulling back it by the degree 2 map $\phi : \mathbf{P}^1 \to L$, we obtain,

$$\phi^* N_{L/M_N^k} \simeq O_{\mathbf{P}^1}(-2)^{\oplus k-N+2} \oplus O_{\mathbf{P}^1}^{\oplus 2N-k-5}.$$

Therefore, $h^1(\phi^*(N_{L/M_N^k})) = k-N+2$, which is strictly grater than two, the complex dimension of the fiber $\overline{M}_{0,0}(L,2)$. Thus we need to know the global structure of the fibration π in order to compute the multiple cover contribution to degree 2 rational Gromov-Witten invariants of M_N^k.

In order to estimate the contributions from double cover maps $\phi : \mathbf{P}^1 \to L$ to $\langle \mathcal{O}_{e^a}\mathcal{O}_{e^b}\mathcal{O}_{e^c}\rangle_{0,2}$, we first computed the number of conics, that intersect cycles Poincaré dual to e^a, e^b and e^c, on M_N^k (whose normal bundle are of the same type) by using the method in [K2]. Then we found the following formula by comparing these integers with the results obtained from (3):

$$\begin{aligned}(4)\qquad &\langle \mathcal{O}_{e^a}\mathcal{O}_{e^b}\mathcal{O}_{e^c}\rangle_{0,2} = (\text{number of corresponding conics}) + \\ &\int_{G(2,N)} c_{top}(S^kQ) \wedge \Big[\frac{c(S^{k-1}Q)}{1-\frac{1}{2}c_1(Q)}\Big]_{k-N} \wedge \sigma_{a-1} \wedge \sigma_{b-1} \wedge \sigma_{c-1},\end{aligned}$$

where Q is the universal rank 2 quotient bundle of $G(2,N)$, σ_a is a Schubert cycle defined by $\sum_{a=0}^{\infty} \sigma_a := \frac{1}{c(Q^\vee)}$ and $[*]_{k-N}$ is the operation of picking up degree $2(k-N)$ part of Chern classes.

On the other hand, we have the following formula which directly follows from the definition of the virtual fundamental class of $\overline{M}_{0,0}(M_N^k, 2)$:

$$\begin{aligned}(5)\qquad &\langle \mathcal{O}_{e^a}\mathcal{O}_{e^b}\mathcal{O}_{e^c}\rangle_{0,2} = (\text{number of corresponding conics}) + \\ &8\int_{G(2,N)} c_{top}(S^kQ) \wedge \big[\pi_*(c_{top}(H^1(\phi^* N_{L/M_N^k})))\big]_{k-N} \wedge \sigma_{a-1} \wedge \sigma_{b-1} \wedge \sigma_{c-1}.\end{aligned}$$

where $\pi : \overline{M}_{0,0}(L,2) \to \overline{M}_{0,0}(M_N^k,1)$ is the natural projection. Here, the factor 8 comes from the divisor axiom of Gromov-Witten invariants.

In this paper, we prove the following formula

$$(6)\qquad \pi_*(c_{top}(H^1(\phi^* N_{L/M_N^k}))) = \frac{1}{8}\Big[\frac{c(S^{k-1}Q)}{1-\frac{1}{2}c_1(Q)}\Big]_{k-N}.$$

By combining (5) with (6), we can derive the formula (4) immediately.

From (4), we see that $\langle \mathcal{O}_{e^a}\mathcal{O}_{e^b}\mathcal{O}_{e^c}\rangle_{0,2}$ of M_N^k is a rational number with denominator at most 2^{k-N}. Therefore rationality (non-integrality) of the Gromov-Witten invariant $\langle \mathcal{O}_{e^a}\mathcal{O}_{e^b}\mathcal{O}_{e^c}\rangle_{0,2}$ is caused by the effect of multiple cover map in this case.

We note here that the total moduli space of double cover maps of lines is isomorphic to $\mathbf{P}(S^2Q)$ over $G := \overline{M}_{0,0}(M_N^k,1) \hookrightarrow G(2,N)$, which is an algebraic $\mathbf{Q}$-stack $\mathbf{P}(S^2Q)^{stack}$ (in the sense of Mumford). As a consequence, the union of all $H^1(\phi^* N_{C/M_N^k})$ turns out to be a coherent sheaf on $\mathbf{P}(S^2Q)^{stack}$ with fractional Chern class in (6), as was suggested in [BT]. See [V, Section 9].

We also did some numerical experiments on degree 3 Gromov-Witten invariants of M_N^k by using the results of [ES]. For $k-N>0$, there is a new contribution from multiple cover maps to nodal conics in M_N^k that did not appear in the Calabi-Yau case. Therefore, multiple cover map contributions are far more complicated than Calabi-Yau, and we leave general analysis on this problem to future works.

This paper is organized as follows. In Section 1, we analyze characteristics of moduli space of lines in M_N^k and derive $N_{L/M_N^k} \simeq O_L(-1)^{\oplus k-N+2} \oplus O_L^{\oplus 2N-k-5}$. In Section 2, we study the moduli space $\overline{M}_{0,0}(\mathbf{P}^1, 2)$ from the point of view of stability and identify it with $\mathbf{P}^2$ and show that the moduli space $\overline{M}_{0,0}(\mathbf{P}^1, 2)$ is isomorphic to $\mathbf{P}(S^2Q)$ over G. In section 4, we describe $H^1(\phi^* N_{L/M_N^k})$ as an coherent sheaf over $\mathbf{P}(S^2Q)^{stack}$. In section 5, we derive the main theorem (6) of this paper by using Segre classes. In Section 6, we mention some generalization to degree 3 Gromov-Witten invariants.

2. Lines on a hypersurface

Let M be a generic hypersurface of degree k of the projective space $\mathbf{P}^{N-1} = \mathbf{P}(V)$. We assume $2N-5 \geq k \geq N-2 \geq 2$ throughout this note. In this note we count the number of rational curves of virtual degree two, namely rational curves which doubly cover lines on M.

Let $\mathbf{P} = \mathbf{P}(V)$ be the projective space parameterizing all one-dimensional quotients of V, which is usually denoted by $\mathbf{P}(V)$ in the standard notation in algebraic geometry. In this notation let W be a subspace of V. Then $\mathbf{P}(W)$ is naturally a linear subspace of $\mathbf{P}(V)$ of dimension $\dim W - 1$.

Let $G(2,V)$ be the Grassmann variety of lines in $\mathbf{P}(V)$, the scheme parameterizing all lines of $\mathbf{P} = \mathbf{P}(V)$. This is also the universal scheme parameterizing all one-dimensional quotient linear spaces of V. Let W be a two dimensional quotient linear space, $\psi \in G(2,V)$, namely $\psi : \mathbf{P}(W) \to \mathbf{P}(V)$ the natural immersion and $i_\psi^* : V \to W$ the quotient homomorphism. The space W is denoted by $W(\psi)$ when necessary.

There exists the universal bundle $Q_{G(2,V)}$ over $G(2,V)$ and a homomorphism $i^{\mathrm{univ}\,*} : O_{G(2,V)} \otimes V \to Q_{G(2,V)}$ whose fiber $i_\psi^{\mathrm{univ}\,*} : V \to Q_{G(2,V),\psi}$ is the quotient $i_\psi^* : V \to W(\psi)$ of V corresponding to ψ.

2.1. **Existence of a line on M.** Let $L = \mathbf{P}(W)$ be a line of $\mathbf{P}$, equivalently $W \in G(2,V)$. Then the condition $L \subset M$ imposes at most $k+1$ conditions on W, while the number of moduli of lines of $\mathbf{P}$ equals $\dim G(2,V) = 2N-4$. Hence we infer

Lemma 2.2. *If $2N \geq k+5$, then there exists at least a line on M.*

See also [Katz,p.152]. Let G be the subscheme of $G(2,V)$ parameterizing all lines of $\mathbf{P}(V)$ lying on M, $Q = (Q_{G(2,V)})_{|G}$ the restriction of $Q_{G(2,V)}$ to G. By Lemma 2.2, G is nonempty. Let $i^* : O_G \otimes V \to Q$ be the restriction of $i^{\mathrm{univ}\,*}$ to G. Let $P = \mathbf{P}(Q)$ and $\pi : P \to G$ the natural projection. Then π is the universal line of M over G, to be more exact, the universal family over G of lines lying on M. In other words, the natural epimorphism $i^* : O_G \otimes V \to Q$ induces a morphism $i : P \to \mathbf{P}_G(V) := G \times \mathbf{P}(V)$, which is a closed immersion into $\mathbf{P}_G(V)$, thus P is a subscheme of $\mathbf{P}_G(V)$ such that $\pi = (p_1)_{|P}$. Let $L_\psi = \mathbf{P}(Q_\psi)$. Note that

$$L_\psi = P_\psi := \pi^{-1}(\psi) \simeq \mathbf{P}(Q_\psi) \subset \{\psi\} \times \mathbf{P}(V) \simeq \mathbf{P}(V).$$

2.3. **The normal bundle $N_{L/M}$.** The argument of this section is standard and well known. Let $\mathbf{P}=\mathbf{P}(V)$, $L=\mathbf{P}(W)$ and $i_W^*: V \twoheadrightarrow W \in G$. Let us recall the following exact sequence:

$$0 \longrightarrow O_{\mathbf{P}} \longrightarrow O_{\mathbf{P}}(1)\otimes V^\vee \xrightarrow{D} T_{\mathbf{P}} \longrightarrow 0$$

where the homomorphism D is defined by

$$D(a\otimes v^\vee) := aD_{(v^\vee)} \quad (a\in O_{\mathbf{P}}(1))$$
$$(D_{v^\vee}F)(u^\vee) := (\frac{d}{dt}F(u^\vee+tv^\vee))_{t=0}$$

for a homogeneous polynomial $F\in S(V)$ and $u^\vee, v^\vee \in V^\vee$. We note $H^0(O_{\mathbf{P}}(1))\otimes V^\vee = V\otimes V^\vee = \mathrm{End}(V,V)$ and that the image of $H^0(O_{\mathbf{P}})$ in $\mathrm{End}(V,V)$ is $\mathbf{C}\,\mathrm{id_V}$. We also have the following exact sequences:

$$0 \longrightarrow T_L \longrightarrow (T_{\mathbf{P}})_L \longrightarrow N_{L/\mathbf{P}} \longrightarrow 0$$
$$0 \longrightarrow O_L \longrightarrow O_L(1)\otimes V^\vee \xrightarrow{D_L} (T_{\mathbf{P}})_L \longrightarrow 0.$$

Lemma 2.4. *Let $L=\mathbf{P}(W)$. Then*

$$N_{L/\mathbf{P}} \simeq O_L(1)\otimes (V^\vee/W^\vee),\ H^0(N_{L/\mathbf{P}})\simeq W\otimes (V^\vee/W^\vee).$$

Proof. The assertion is clear from the following commutative diagram with exact rows and columns:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & O_L & \longrightarrow & O_L(1)\otimes W^\vee & \xrightarrow{(D_L)_{|W^\vee}} & T_L & \longrightarrow & 0\\
\downarrow & & \downarrow & & \downarrow{\scriptstyle \mathrm{id}\otimes i^\vee} & & \downarrow & & \downarrow\\
0 & \longrightarrow & O_L & \longrightarrow & O_L(1)\otimes V^\vee & \xrightarrow{D_L} & (T_{\mathbf{P}})_L & \longrightarrow & 0\\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow\\
0 & \longrightarrow & 0 & \longrightarrow & O_L(1)\otimes (V^\vee/W^\vee) & \longrightarrow & N_{L/\mathbf{P}} & \longrightarrow & 0
\end{array}$$

The second assertion is clear from $H^0(L, O_L(1)) = W$. □

Since $T_L \simeq O_L(2)$, there follow exact sequences

$$0 \longrightarrow H^0(T_L) \longrightarrow H^0((T_{\mathbf{P}})_L) \longrightarrow H^0(N_{L/\mathbf{P}}) \longrightarrow 0$$
$$0 \longrightarrow H^0(O_L) \longrightarrow H^0(O_L(1))\otimes V^\vee \xrightarrow{H^0(D_L)} H^0((T_{\mathbf{P}})_L) \longrightarrow 0.$$

We also note

$$H^0(T_L) = \mathrm{Lie}\,\mathrm{Aut}^0(L) = \mathrm{End}(W,W)/\,\mathrm{center} = \mathrm{End}(W,W)/\mathbf{C}\,\mathrm{id_W}\,.$$

Since $H^0(O_L(1)) = W$, we see

$$H^0((T_{\mathbf{P}})_L) = W\otimes V^\vee/\,\mathrm{Im}\,H^0(O_L) = \mathrm{Hom}(V,W)/\mathbf{C}i_W^*.$$

Hence we again see

$$H^0((N_{L/\mathbf{P}})) = (\mathrm{Hom}(V,W)/\mathbf{C}i_W^*)/(\mathrm{Hom}(W,W)/\mathbf{C}\,\mathrm{id}_W)$$
$$= W\otimes (V^\vee/W^\vee) = \mathrm{Hom}(V/W,W).$$

For any line $L=\mathbf{P}(W)$ of $\mathbf{P}$ the following sequence is exact:

(7) $$0\to N_{L/M}\to N_{L/\mathbf{P}}\to (N_{M/\mathbf{P}})_L(\simeq O_L(k))\to 0.$$

Hence so is the following sequence as well:

$$\begin{aligned} 0 \longrightarrow H^0(N_{L/M}) \longrightarrow H^0(N_{L/\mathbf{P}}) &\xrightarrow{H^0(D_L)} H^0(O_L(k)) \\ \longrightarrow H^1(N_{L/M}) \longrightarrow \quad 0. \end{aligned}$$

Hence we have

Lemma 2.5. *The following is exact:*

$$0 \to H^0(N_{L/M}) \to W \otimes (V^\vee/W^\vee) \overset{H^0(D_L)}{\longrightarrow} S^kW \to H^1(N_{L/M}) \to 0. \tag{8}$$

Corollary 2.6. $\dim G \geq 2N - k - 5$, *equality holding if* $H^1(N_{L/M}) = 0$.

Proof. As is well-known, $\dim G \geq h^0(N_{L/M}) - h^1(N_{L/M})$. Note $\dim W \otimes (V^\vee/W^\vee) = 2(N-2)$ and $\dim S^kW = k+1$. Hence the corollary follows from Lemma 2.5. □

Lemma 2.7. *For a generic line L on a generic hypersurface M of degree k*

(i) $N_{L/M} \simeq O_L^{\oplus a} \oplus O_L(-1)^{\oplus b}$, *where* $a = 2N - k - 5$ *and* $b = k - N + 2$,

(ii) $\operatorname{Coker} H^0(D_L^-) \simeq S^{k-1}W/(V^\vee/W^\vee)$ *where* $D_L^- := D_L \otimes O_L(-1)$.

Proof. Let M be a generic hypersurface of degree k and L a generic line L on M. Without loss of generality we may assume that $W^\vee$ is generated by $e_1^\vee$ and $e_2^\vee$, in other words, $\psi : L \to \mathbf{P}$ is given by

$$\psi : [s : t] \to [x_1, \cdots, x_N] = [s, t, 0, \cdots, 0].$$

Then F, the polynomial of degre k defining M, is written as

$$F = x_3F_3 + x_4F_4 + \cdots + x_NF_N$$

for some polynomials F_j of degree $k-1$. Let $f_j = \psi^*F_j = F_j(s, t, 0, \cdots, 0)$.

Now we consider the exact sequence

$$\begin{aligned} 0 \longrightarrow H^0(N_{L/M}(-1)) \longrightarrow H^0(N_{L/\mathbf{P}}(-1)) &\xrightarrow{H^0(D_L^-)} H^0(O_L(k-1)) \\ \longrightarrow H^1(N_{L/M}(-1)) \longrightarrow \quad 0. \end{aligned}$$

where we note $H^0(N_{L/\mathbf{P}}(-1)) = V^\vee/W^\vee$. Hence the following is exact:

$$\begin{aligned} 0 \longrightarrow H^0(N_{L/M}(-1)) \longrightarrow V^\vee/W^\vee &\xrightarrow{H^0(D_L^-)} S^{k-1}W \\ \longrightarrow H^1(N_{L/M}(-1)) \longrightarrow \quad 0 \end{aligned} \tag{9}$$

where $H^0(D_L^-)$ is given by $H^0(D_L^-)(e_j^\vee) = f_j \quad (j = 3, 4, \cdots, N)$.

A generic choice of F implies a generic choice of degree $k-1$ polynomials f_j $(j = 3, 4, \cdots, N)$ in s and t. By the assumptions

$$\dim S^{k-1}W = k \geq N - 2 = \dim V^\vee/W^\vee,$$
$$\dim W \otimes V^\vee/W^\vee = 2(N-2) \geq k + 1 = \dim S^kW,$$

the generic choice of F implies that we can choose $f_j \in S^{k-1}W$ $(j = 3, 4, \cdots, N)$ (and fix once for all) such that

(iii) f_j $(j = 3, 4, \cdots, N)$ are linearly independent,

(iv) $Wf_3 + Wf_4 + \cdots + Wf_N = S^kW$.

Hence $H^0(D_L^-)$ is injective by (iii). It follows that $H^0(N_{L/M}(-1)) = 0$. Hence (ii) is clear. Next we consider $H^0(D_L)$. By (iv), we see

$$S^kW = W \cdot H^0(D_L^-)(V^\vee/W^\vee) = H^0(D_L)(W \otimes V^\vee/W^\vee),$$

whence $H^0(D_L)$ is surjective. It follows that $H^1(N_{L/M}) = 0$. Hence $N_{L/M} \simeq O_L^{\oplus a} \oplus O_L(-1)^{\oplus b}$ for some a and b. Since $a + b = \mathrm{rank}(N_{L/M}) = N - 3$ and $-b = \deg(N_{L/M}) = N - 2 - k$, we have (i). □

2.8. **Lines on a quintic hypersurface in $\mathbf{P}^4$.** See [Katz, Appendix A] for the subsequent examples. Let $N = 5$ and $k = 5$. Hence M is a hypersurface of degree 5 in $\mathbf{P}^4$, a Calabi-Yau 3-fold. Let

$$F = x_4x_1^4 + x_5x_2^4 + x_3^5 + x_4^5 + x_5^5.$$

First we note that $M = \{F = 0\}$ is nonsingular. Let $L = \{x_3 = x_4 = x_5 = 0\} = \{[s, t, 0, 0, 0]\}$. In this case $f_3 = 0$, $f_4 = s^4$ and $f_5 = t^4$. In the exact sequence (1) we see $H^0(N_{L/M}(-1)) = \mathrm{Ker}\, H^0(D_L^-) = \mathbf{C}e_3^\vee$ and $H^1(N_{L/M}(-1)) = \mathrm{Coker}\, H^0(D_L^-)$ is 3-dimensional. Hence $N_{L/M} = O_L(1) \oplus O_L(-3)$.

We summarize the above. If $\dim \mathrm{Ker}\, H^0(D_L^-) = 1$ and if M is nonsingular, then $N_{L/M} = O_L(1) \oplus O_L(-3)$. Hence $H^0(N_{L/M}) = \mathrm{Ker}\, H^0(D_L) = W \otimes \mathrm{Ker}\, H^0(D_L^-)$ is 2-dimensional. Therefore we can choose $f_3 = 0$ and a linearly independent pair f_4 and $f_5 \in S^4W$ so that $Wf_4 + Wf_5$ is 4-dimensional. The choice $f_4 = s^4$ and $f_5 = t^4$ satisfies the conditions. This enables us to find a nonsingular hypersurface M as above. However if we choose $f_3 = 0$, $f_4 = s^4$ and $f_5 = s^3t$, then $Wf_4 + Wf_5$ is 3-dimensional. Hence M is singular.

Next in the same manner we find L on a nonsingular hypersurface M with $N_{L/M} = O_L \oplus O_L(-2)$ or $N_{L/M} = O_L(-1)^{\oplus 2}$. Let

$$F = x_3x_1^4 + x_4x_1^3x_2 + x_5x_2^4 + x_3^5 + x_4^5 + x_5^5.$$

Then we have $f_3 = s^4$, $f_4 = s^3t$ and $f_5 = t^4$. Since $Wf_3 + Wf_4 + Wf_5$ is 5-dimensional, $H^0(N_{L/M}(-1)) = \mathrm{Ker}\, H^0(D_L^-) = 0$, $H^0(N_{L/M}) = \mathrm{Ker}\, H^0(D_L) = \mathbf{C}(te_3^\vee - se_4^\vee)$. We see also that $\dim H^1(N_{L/M}) = \dim \mathrm{Coker}\, H^0(D_L) = 1$ and $N_{L/M} = O_L \oplus O_L(-2)$. The hypersurface $M = \{F = 0\}$ is easily shown to be nonsingular.

If $F = x_3x_1^4 + x_4x_1^2x_2^2 + x_5x_2^4 + x_3^5 + x_4^5 + x_5^5$ and $M = \{F = 0\}$, then $N_{L/M} = O_L(-1)^{\oplus 2}$.

2.9. **Lines on a generic hypersurface M_7^8 of $\mathbf{P}^6$.** Let $N = 7$ and $k = 8$. In view of Lemma 2.2 there exists a line L on any generic hypersurface M of degree 8 in $\mathbf{P}(V) = \mathbf{P}^6$. In view of Lemma 2.7, $a = 1$, $b = 3$ and $N_{L/M} \simeq O_L \oplus O_L(-1)^{\oplus 3}$. For example let $L : x_j = 0$ $(j \geq 3)$ and we take

$$F_3 = 8x_1^7, F_4 = 8x_1^6x_2, F_5 = 8x_1^4x_2^3, F_6 = 8x_1^2x_2^5, F_7 = 8x_2^7,$$
$$F = x_3F_3 + x_4F_4 + x_5F_5 + x_6F_6 + x_7F_7 + x_3^8 + x_4^8 + x_5^8 + x_6^8 + x_7^8.$$

and let $M = M_7^8 : F = 0$. We see that M is nonsingular near L and has at most isolated singularities. However it is still unclear to us whether $M = M_7^8$ is nonsingular everywhere. The space $H^0(N_{L/M})$ is spanned by $te_3^\vee - se_4^\vee$, hence an infinitesimal deformation L_ε of L is given by

$$[s,t] \mapsto [s,t,\varepsilon t, -\varepsilon s, 0,0,0]$$

which yields $F_{|L_\varepsilon} = \varepsilon^8(s^8+t^8) \equiv 0 \mod \varepsilon^8$. Since $H^1(N_{L/M}) = 0$, this infinitesimal deformation is integrable and G (:= the moduli of lines of $\mathbf{P}^6$ contained in M) is nonsingular and one dimensional at the point $[L]$.

We note that M also contains 8 lines

$$L' := L'_{\varepsilon_8} : \varepsilon_8 x_1 - x_2 = x_3 + \varepsilon_8 x_4 = x_j = 0 \ (j \geq 5),$$

with $N_{L'/M} = O_{L'}(1)^{\oplus 3} \oplus O_{L'}(-6)$ where $\varepsilon_8^8 = -1$.

3. Stability

Definition 3.1. Suppose that a reductive algebraic group G acts on a vector space V. Let $v \in V$, $v \neq 0$.

(1) the vector v is said to be *semistable* if there exists a G-invariant homogeneous polynomial F on V such that $F(v) \neq 0$,
(2) the vector v is said to be *stable* if p has a closed G-orbit in X_{ss} and the stabilizer subgroup of v in G is finite.

Let $\pi : V \setminus \{0\} \to \mathbf{P}(V^\vee)$ be the natural surjection. Then $v \in V$ is semistable (resp. stable) if and only if $\pi(v)$ is semistable (resp. stable).

3.2. **Grassmann variety.** Let V be an N-dimensional vector space, and $G(r,N)$ the Grassmann variety parameterizing all r-dimensional quotient spaces of V. Here is a natural way of understanding $G(r,N)$ via GIT-stability. Let U be an r-dimensional vector space, $X = \mathrm{Hom}(V,U)$ and $\pi : X \setminus \{0\} \to \mathbf{P}(X^\vee)$ the natural map. Then $\mathrm{SL}(U)$ acts on X from the left by:

$$(g \cdot \phi^*)(v) = g \cdot (\phi^*(v)) \quad \text{for } \phi^* \in X, \ v \in V.$$

We see that for $\phi^* \in X$

$$\phi^* \text{ is } \mathrm{SL}(U)\text{-stable} \iff \operatorname{rank} \phi^* = r,$$
$$\phi^* \text{ is } \mathrm{SL}(U)\text{-semistable} \iff \phi^* \text{ is } \mathrm{SL}(U)\text{-stable}.$$

In fact, if $\operatorname{rank}\phi^* = r-1$, then there is a one-parameter torus T of $\mathrm{SL}(U)$ such that the closure of the orbit $T \cdot \phi$ contains the zero vector as the following simple example ($r=2$) shows

$$\lim_{t\to 0} \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1N} \\ 0 & 0 & \cdots & 0 \end{pmatrix} = \lim_{t\to 0} \begin{pmatrix} ta_{11} & ta_{12} & \cdots & ta_{1N} \\ 0 & 0 & \cdots & 0 \end{pmatrix}.$$

Let X_{s} be the set of all (semi)stable points and $\mathbf{P}_s$ the image of X_{s} by π. It is, as we saw above, just the set of all $\phi \in X$ with $\operatorname{rank}\phi^* = r$. Therefore the GIT-orbit space $\mathbf{P}_s /\!/ \mathrm{SL}(U)$ is the orbit space $\mathbf{P}_s / \mathrm{SL}(U)$ by the free action, the Grassmann variety $G(r,N)$.

3.3. **Moduli of double coverings of $\mathbf{P}^1$ (1).** Let W and U be a pair of two dimensional vector spaces, $X = \mathrm{Hom}(W, S^2U)$, and $\pi : X \setminus \{0\} \to \mathbf{P}(X^\vee)$ the natural morphism. Note that $\mathrm{SL}(U)$ acts on S^2U from the left via the natural action: $\sigma(u_1u_2) = \sigma(u_1)\sigma(u_2)$ for $\forall u_1, u_2 \in U$. Thus $\mathrm{SL}(U)$ acts on X from the left in the same manner in the subsection 3.2.

Lemma 3.4. *Let $\phi^* \in X$.*

(i) *ϕ^* is unstable iff $\phi^*(w)$ has a double root for any $w \in W$,*
(ii) *ϕ^* is semistable iff $\phi^*(w)$ has no double roots for some nonzero $w \in W$,*
(iii) *ϕ^* is stable iff $\phi^*(W)$ is a base-point free linear subsystem of S^2U on $\mathbf{P}(U)$.*

Proof. We note that ϕ^* is unstable iff there is a suitable basis s and t of U such that $\phi^*(w) = a(w)s^2$ for any $w \in W$ since a torus orbit $T \cdot \phi^*$ contains the zero vector. This proves (i). This also proves (ii). Next we prove (iii). If $\phi^*(W)$ has a base point, then it is clear that ϕ^* is not stable. If ϕ^* is semistable and it is not stable, then we choose a basis s, t of U and a basis w_1, w_2 of W such that $\phi^*(w_1) = st$. If $\phi(w_2) = as^2 + bst$, then ϕ^* is not stable. This proves the lemma. □

Theorem 3.5. *Let X_{ss} be the Zariski open subset of X consisting of all semistable points of X, $\pi(X_{ss})$ the image of X_{ss} by π, and $Y := \pi(X_{ss})/\!/\,\mathrm{SL}(U)$. Then $Y \simeq \mathbf{P}^2$.*

Proof. First consider a simplest case. We choose a basis s, t of U. Let w_1 and w_2 be a basis of W, T the subgroup of $\mathrm{SL}(U)$ of diagonal matrices and $X' = \{\phi^* \in X; \phi^*(w_1) = 2st\}$. Let $Z' = \mathrm{SL}(U) \cdot X'$.

We note that Z' is an $\mathrm{SL}(U)$-invariant subset of X_{ss}. We prove $\pi(Z')/\!/\,\mathrm{SL}(U) \simeq \mathbf{C}^2$. Let ϕ^* and ψ^* be points of X'. Let $\phi^*(w_2) = As^2 + 2Bst + Ct^2$ and $\psi^*(w_2) = as^2 + 2bst + ct^2$. Then it is easy to check

$$\begin{aligned} g \cdot \phi^* = \psi^* \ \text{ for } \exists\, g \in \mathrm{SL}(U) &\Longleftrightarrow g \cdot \phi^* = \psi^* \ \text{ for } \exists\, g \in T \\ &\Longleftrightarrow A = au^2,\ B = b,\ C = u^{-2}c \ \text{ for } \exists\, u \neq 0. \end{aligned}$$

Therefore each equivalence class of $\pi(Z)/\!/\,\mathrm{SL}(U)$ is represented by the pair (AC, B), which proves $\pi(Z)/\!/\,\mathrm{SL}(U) \simeq \mathbf{C}^2$.

Now we prove the lemma. Let $\phi^* \in X_{ss}$, $\phi_j = \phi^*(w_j)$ and $\phi_0 = -(\phi_1 + \phi_2)$. Let

$$\begin{aligned} \phi_0 &= r_1s^2 + 2r_2st + r_3t^2, \\ \phi_1 &= p_1s^2 + 2p_2st + p_3t^2, \\ \phi_2 &= q_1s^2 + 2q_2st + q_3t^2, \end{aligned}$$

and we define

$$\begin{gathered} D_1 = p_2^2 - p_1p_3, \quad D_2 = q_2^2 - q_1q_3, \\ D_0 = r_2^2 - r_1r_3 = D_1 + D_2 + 2p_2q_2 - (p_1q_3 + p_3q_1). \end{gathered}$$

To show the lemma, we prove the more precise isomorphism

$$\pi(X_{ss})/\!/\,\mathrm{SL}(U) = \mathrm{Proj}\,\mathbf{C}[D_0, D_1, D_2]$$

For this purpose we define $Y_j = \pi(\{\phi^* \in X_{ss}; \phi_j \text{ has no double roots}\})/\!/\,\mathrm{SL}(U)$. It suffices to prove $Y_1 = \mathrm{Spec}\,\mathbf{C}[\frac{D_0}{D_1}, \frac{D_2}{D_1}]$ by reducing it to the first simplest case.

Let $\phi^* \in Y_1$. Let α and β be the roots of $\phi_1 = 0$. By the assumption ϕ_1 has no double roots, hence $\alpha \neq \beta$. Let

$$u = \frac{1}{\gamma}(s - \alpha t), \quad v = \frac{1}{\gamma}(s - \beta t), \quad g = \frac{1}{\gamma}\begin{pmatrix} 1 & -\alpha \\ 1 & -\beta \end{pmatrix}$$

where $\gamma = \sqrt{\alpha - \beta}$. Note that $g \in \mathrm{SL}(U)$. Hence we see

$$(\phi_1(s,t), \phi_2(s,t)) \equiv (p_1\gamma^4 uv, A_1u^2 + 2B_1uv + C_1v^2)$$

where

$$\begin{aligned} A_1 &= q_1^2\beta^2 + 2q_2\beta + q_3, \\ -B_1 &= q_1\alpha\beta + q_2(\alpha + \beta) + q_3, \\ C_1 &= q_1^2\alpha^2 + 2q_2\alpha + q_3. \end{aligned}$$

Thus we see

$$(\phi_1(s,t), \phi_2(s,t)) \equiv (2st, As^2 + 2Bst + Ct^2)$$

where

$$\begin{aligned} A = \frac{2A_1}{p_1\gamma^4},\ B = \frac{2B_1}{p_1\gamma^4},\ C = \frac{2C_1}{p_1\gamma^4},\ p_1^2\gamma^4 = 4D_1, \\ AC = B^2 - \frac{D_2}{D_1},\ B = \frac{D_0 - D_1 - D_2}{2D_1}. \end{aligned}$$

Therefore by the first half of the proof

$$Y_1 \simeq \mathrm{Spec}\ \mathbf{C}[AC, B] = \mathrm{Spec}\ \mathbf{C}[\frac{D_0}{D_1}, \frac{D_2}{D_1}].$$

This completes the proof of the lemma. □

Corollary 3.6. *Let $Y^s = \pi(X_s)/\!/ \mathrm{SL}(U)$. Then $Y \setminus Y^s$ is a conic of Y defined by*

$$Y \setminus Y^s : D_0^2 + D_1^2 + D_2^2 - 2D_0D_1 - 2D_1D_2 - 2D_2D_0 = 0.$$

Proof. In view of Theorem 3.5, $Y_1 \simeq \mathrm{Spec}\ \mathbf{C}[AC, B]$. The complement of Y_s in Y_1 is then the curve defined by $AC = 0$, which is easily identified with the above conic. □

Corollary 3.7. *Let X^0 be the Zariski open subset of X consisting of all semistable points ϕ^* of X with* $\mathrm{rank}\,\phi^* = 2$, *and let $Y^0 := \pi(X^0)/\!/ \mathrm{SL}(U)$. Then $Y^0 \simeq \pi(X^0)/\mathrm{SL}(U) \simeq Y \simeq \mathbf{P}^2$.*

Proof. It suffices to compare Y_1 and $Y^0 \cap Y_1$. As in the proof of Theorem 3.5 we let $X' = \{\phi^* \in X; \phi^*(w_1) = 2st\}$. Let $Z = \mathrm{SL}(U) \cdot X'$ and $Z^0 = \mathrm{SL}(U) \cdot (X' \cap X^0)$.

Then with the notation in Theorem 3.5, we recall $X' = \{\phi^* \in X; \phi^*(w_1) = 2st, \phi^*(w_2) = As^2 + 2Bst + Ct^2\}$, $\pi(Z)/\!/ \mathrm{SL}(U) \simeq \mathrm{Spec}\ \mathbf{C}[AC, B]$ where

$$X' \cap X^0 = \{\phi^* \in X'; A \neq 0 \text{ or } C \neq 0\}.$$

In the same manner as before we see $\pi(Z^0)/\!/ \mathrm{SL}(U) \simeq \mathrm{Spec}\ \mathbf{C}[AC, B]$, whence $\pi(Z^0)/\!/ \mathrm{SL}(U) = \pi(Z)/\!/ \mathrm{SL}(U)$. This proves $Y^0 \cap Y_1 = Y_1$. This completes the proof of the corollary. □

3.8. **Moduli of double coverings of $\mathbf{P}(W)$ (2).** There is an alternative way of understanding $\pi(X_{ss})/\!/\operatorname{SL}(U) \simeq \mathbf{P}^2$ by using the isomorphism $S^2\mathbf{P}^1 \simeq \mathbf{P}^2$. We use the following convention to denote a point of $\mathbf{P}(U) = U^\vee \setminus \{0\}/\mathbf{G}_m$: $(u : v) = us^\vee + vt^\vee \in U^\vee$ where $s^\vee$ and $t^\vee$ are a basis dual to s and t. In what follows we fix a basis w_1 and w_2 of W. Let $P := (a_1 : a_2)$ and $Q := (b_1 : b_2)$ be a pair of points of $\mathbf{P}(W) \simeq \mathbf{P}^1$. If $P \neq Q$, there is a double covering $\phi : \mathbf{P}(U) \to \mathbf{P}(W)$ ramifying at P and Q, unique up to isomorphism once we fix the base w_1 and w_2:

$$\frac{b_2w_1 - b_1w_2}{a_2w_1 - a_1w_2} = (\frac{t}{s})^2.$$

Thus ϕ is given explicitly by

$$\phi_1 := \phi^*(w_1) = b_1s^2 - a_1t^2,\ \phi_2 := \phi^*(w_2) = b_2s^2 - a_2t^2,\ \phi_0 = -(\phi_1 + \phi_2)$$

for which we have

$$D_1 = a_1b_1,\ D_2 = a_2b_2,\ D_0 = (a_1 + a_2)(b_1 + b_2).$$

The isomorphism $S^2\mathbf{P}^1 \simeq \mathbf{P}^2$ is given by $(P, Q) \mapsto (D_0, D_1, D_2)$. This shows

Corollary 3.9. *We have a natural isomorphism:* $Y \simeq \mathbf{P}(S^2W)$.

4. The virtual normal bundle of a double covering

4.1. **The case $N = 7$ and $k = 8$ revisited.** We revisit the example in the subsection 2.9. Let $N = 7$ and $k = 8$. Let $L : x_j = 0$ $(j \geq 3)$ and we take

$$F_3 = 8x_1^7, F_4 = 8x_1^6x_2, F_5 = 8x_1^4x_2^3, F_6 = 8x_1^2x_2^5, F_7 = 8x_2^7,$$
$$F = x_3F_3 + x_4F_4 + x_5F_5 + x_6F_6 + x_7F_7 + x_3^8 + x_4^8 + x_5^8 + x_6^8 + x_7^8.$$

and let $M = M_8^5 : F = 0$. We often denote L also by $\mathbf{P}(W)$ with W a two dimensional vector space for later convenience. Since $H^0(D_L^-)$ is injective and $H^0(D_L)$ is surjective, we have $N_{L/M} \simeq O_L \oplus O_L(-1)^{\oplus 3}$. Hence $H^1(N_{L/M}(-1)) = H^1(O_L(-2)^{\oplus 3})$ is 3-dimensional. As we see easily, this follows also from the fact that $\operatorname{Coker} H^0(D_L^-)$ is freely generated by $x_1^5x_2^2$, $x_1^3x_2^4$ and $x_1x_2^6$.

Let $\phi^* = (\phi_1, \phi_2) \in X^0$. Then $\operatorname{Ker} H^0(\phi^*D_L)$ is generated by a single element $\phi_2e_3^\vee - \phi_1e_4^\vee$, while $\operatorname{Coker} H^0(\phi^*D_L)$ is generated by $S^2U \cdot \phi_1^5\phi_2^2$, $S^2U \cdot \phi_1^3\phi_2^4$ and $S^2U \cdot \phi_1\phi_2^6$. To be more precise, we see

$$\operatorname{Coker} H^0(\phi^*D_L) = \{\phi_1^5\phi_2^2, \phi_1^3\phi_2^4, \phi_1\phi_2^6\} \otimes S^2U/\{\phi_1, \phi_2\}.$$

In fact, this is proved as follows: first we consider the case where ϕ_1 and ϕ_2 has no common zeroes. In this case ϕ^* gives rise to a double covering $\phi : \mathbf{P}(U) \to \mathbf{P}(W)$ $(= L)$, which we denote by L_ϕ for brevity. By pulling back by ϕ^* the normal sequence $0 \to N_{L/M} \to N_{L/\mathbf{P}} \to O_L(k) \to 0$ $(k = 8)$ for the line L we infer an exact sequence

$$0 \to \phi^*N_{L/M} \to \phi^*N_{L/\mathbf{P}} \xrightarrow{\phi^*D_L} \phi^*O_L(k) \to 0,$$

which yields an exact sequence

$$\begin{aligned} 0 \longrightarrow H^0(\phi^*N_{L/M}) \longrightarrow S^2U \otimes (V^\vee/W^\vee) \xrightarrow{H^0(\phi^*D_L)} H^0(O_{L_\phi}(2k)) \\ \longrightarrow H^1(\phi^*N_{L/M}) \longrightarrow 0. \end{aligned}$$

Let $\eta = q_3 e_3^\vee + \cdots + q_7 e_7^\vee \in \operatorname{Ker} H^0(\phi^* D_L)$, $q_j \in S^2U$. Then we have

$$\phi_1^2(q_3\phi_1^5 + q_4\phi_1^4\phi_2 + q_5\phi_1^2\phi_2^3 + q_6\phi_2^5) = -q_7\phi_2^7.$$

Since ϕ_1 and ϕ_2 are mutually prime and q_j is of degree two, we have $q_7 = 0$ and

$$\phi_1^2(q_3\phi_1^3 + q_4\phi_1^2\phi_2 + q_5\phi_2^3) = -q_6\phi_2^5,$$

Hence $q_6 = 0$ and similarly we infer also $q_5 = 0$. Thus we have $q_3\phi_1 + q_4\phi_2 = 0$. This proves that $\operatorname{Ker} H^0(\phi^* D_L)$ is generated by $\phi_2 e_3^\vee - \phi_1 e_4^\vee$.

Next we prove that $\operatorname{Coker} H^0(\phi^* D_L)$ is generated by $\phi^* \operatorname{Coker} H^0(D_L^-)$ over S^2U, in fact over $S^2U/\phi^*(W)$. Without loss of generality we may assume that $\phi_1 = 2st$ and $\phi_2 = \lambda s^2 + 2\nu st + t^2$ for some $\lambda \neq 0$ and $\nu \in \mathbf{C}$. Let $\phi^* W = \{\phi_1, \phi_2\}$. Then one checks $U \cdot \phi^* W = S^3U$, and hence $S^2U \cdot \phi^* W = S^4U$, $S^{2m-2}U \cdot \phi^* W = S^{2m}U$ for $m \geq 2$. It follows $S^2U \cdot \phi^*(S^{m-1}W) = S^{2m}U$ for $m \geq 1$. In fact, by the induction on m

$$\begin{aligned} S^2U \cdot \phi^*(S^m W) &= S^2U \cdot \phi^*(W) \cdot \phi^*(S^{m-1}W) \\ &= S^4U \cdot \phi^*(S^{m-1}W) \\ &= S^2U \cdot (S^2U \cdot \phi^*(S^{m-1}W)) \\ &= S^2U \cdot S^{2m}U = S^{2m+2}U. \end{aligned}$$

Therefore $H^0(O_{L_\phi}(2k)) = S^{16}U = S^2U \cdot \phi^*(S^7W)$. Hence

$$\begin{aligned} \operatorname{Coker} H^0(\phi^* D_L) &= S^{16}U / \operatorname{Im} H^0(\phi^* D_L) \\ &= S^2U \cdot \phi^*(S^7W) / S^2U \cdot \phi^*(\operatorname{Im} H^0(D_L^-)) \\ &= (S^2U/\phi^*(W)) \cdot \phi^*(S^7W/\operatorname{Im} H^0(D_L^-)). \end{aligned}$$

because $\operatorname{Coker} H^0(D_L^-) = S^7W/\operatorname{Im} H^0(D_L^-)$ and $W \cdot S^7W \subset W \cdot \operatorname{Im} H^0(D_L^-)) = S^8W$ by the choice of L. This proves that $\operatorname{Coker} H^0(\phi^* D_L)$ is generated by $\phi^* \operatorname{Coker} H^0(D_L^-)$ over $S^2U/\phi^*(W)$. It follows $\operatorname{Coker} H^0(\phi^* D_L) = (\phi^* \operatorname{Coker} H^0(D_L^-)) \otimes (S^2U/\phi^* W)$.

Finally we consider the case where ϕ_1 and ϕ_2 has a common zero. In this case we may assume $\phi_1 = 2st$ and $\phi_2 = 2\nu st + t^2$. In this case L_ϕ is a chain of two rational curves C'_ϕ and C''_ϕ where C_ϕ is the proper transform of $\mathbf{P}(U)$, where the double covering map from L_ϕ to $\mathbf{P}(W)$ is the union of the isomorphisms ϕ' and ϕ'', say, $\phi = \phi' \cup \phi''$. Let $\psi_1 = 2s$ and $\psi_2 = 2\nu s + t$. Then ϕ' is induced by the homomorphism $(\phi')^* \in \operatorname{Hom}(W, U)$ such that $(\phi')^*(w_j) = \psi_j$. On the other hand let $U''_\phi = \mathbf{C}\lambda + \mathbf{C}t$, $\psi''_1 = 2t$ and $\psi''_2 = \lambda + 2\nu t$ where we note ψ''_j is the linear part of ϕ_j in t with $s = 1$. Then $C''_\phi = \mathbf{P}(U''_\phi)$ and ϕ'' is induced by the homomorphism $(\phi')^* \in \operatorname{Hom}(W, U''_\phi)$ such that $(\phi')^*(w_j) = \psi''_j$. Furthermore the pull back by ϕ^* of the normal sequence for L

$$0 \to \phi^* N_{L/M} \to \phi^* N_{L/\mathbf{P}} \xrightarrow{\phi^* D_L} \phi^* O_L(k) \to 0,$$

yields exact sequences with natural vertical homomorphisms:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \phi^* N_{L/M} & \longrightarrow & (\phi')^* N_{L/M} \oplus (\phi'')^* N_{L/M} & \longrightarrow & \mathbf{C} & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \phi^* N_{L/\mathbf{P}} & \longrightarrow & (\phi')^* N_{L/\mathbf{P}} \oplus (\phi'')^* N_{L/\mathbf{P}} & \longrightarrow & V^\vee/W^\vee & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \phi^* O_L(k) & \longrightarrow & O_{C'_\phi}(k) \oplus O_{C''_\phi}(k) & \longrightarrow & \mathbf{C} & \longrightarrow & 0.
\end{array}$$

This yields the following long exact sequences:

$$\begin{array}{cccccccc}
0 & \longrightarrow & H^0((\phi')^* N_{L/M}) & \longrightarrow & U \otimes V^\vee/W^\vee & \xrightarrow{H^0((\phi')^* D_L)} & S^k U \\
 & \longrightarrow & H^1((\phi')^* N_{L/M}) & \longrightarrow & 0 & & \\
0 & \longrightarrow & H^0((\phi'')^* N_{L/M}) & \longrightarrow & U''_\phi \otimes V^\vee/W^\vee & \xrightarrow{H^0((\phi'')^* D_L)} & S^k U''_\phi \\
 & \longrightarrow & H^1((\phi'')^* N_{L/M}) & \longrightarrow & 0 & &
\end{array}$$

whence $H^1((\phi')^* N_{L/M}) = H^1((\phi'')^* N_{L/M}) = 0$, and both $H^0((\phi')^* N_{L/M})$ and $H^0((\phi'')^* N_{L/M})$ are one-dimensional. Let U' be the subspace of U consisting of elements vanishing at $C'_\phi \cap C''_\phi$, namely the subspace spanned by t. Then the restriction of $H^0((\phi')^* D_L)$ to $U' \otimes V^\vee/W^\vee$ equals $t \cdot H^0((\phi')^* D_L^-)$. Hence

$$\begin{aligned}
\operatorname{Coker} H^0(\phi^* D_L) &\simeq t \cdot S^7 U / t \cdot \operatorname{Im} H^0((\phi')^* D_L^-) \oplus \operatorname{Coker} H^0((\phi'')^* D_L) \\
&\simeq S^7 U / t \cdot \operatorname{Im} H^0((\phi')^* D_L^-) \simeq \operatorname{Coker} H^0((\phi')^* D_L^-).
\end{aligned}$$

One could understand the above isomorphism as

$$\operatorname{Coker} H^0(\phi^* D_L) = \operatorname{Coker}(\phi)^* H^0(D_L^-) \otimes (S^2 U / \phi^* W).$$

Thus $H^0(\phi^* N_{L/M})$ is one-dimensional, while $H^1(\phi^* N_{L/M})$ is 3-dimensional.

This is immediately generalized into the following

Lemma 4.2. *For any $\phi^* \in X^0$ we have*

$$\begin{aligned}
\operatorname{Ker} H^0(\phi^* D_L) &= \phi^* \operatorname{Ker} H^0(D_L), \\
\operatorname{Coker} H^0(\phi^* D_L) &= (\phi^* \operatorname{Coker} H^0(D_L^-)) \otimes (S^2 U / \phi^* W).
\end{aligned}$$

Lemma 4.3. *We define a line bundle $\mathbf{L}_0$ (resp. $\mathbf{L}_1$) on Y ($\simeq \mathbf{P}(S^2 W)$) by the assignment:*

$$X^0 \ni \phi^* \mapsto \phi^* \operatorname{Ker} H^0(D_L) \ (resp. \phi^* \operatorname{Coker} H^0(D_L^-)).$$

Then $\mathbf{L}_k \simeq O_{\mathbf{P}(S^2 W)}$.

Proof. We know that $\phi^* \operatorname{Ker} H^0(D_L)$ is generated by $\phi_2 e_3^\vee - \phi_1 e_4^\vee$. By the SL(2)-variable change of s and t, ϕ_j is transformed into a new quadratic polynomial, which is however the same as the first ϕ_j. This shows the generator is unchanged, whence $\mathbf{L}_0 \simeq O_{\mathbf{P}(S^2 W)}$. The proof for $\mathbf{L}_1$ is the same. □

Lemma 4.4. *We define a coherent sheaf* $\mathbf{L}$ *on the stack* Y $(\simeq \mathbf{P}(S^2W))$ *(See Remark below) by the assignment:*

$$X^0 \ni \phi^* \mapsto S^2U/\phi^*W.$$

Then $\mathbf{L}^2 \simeq O_{\mathbf{P}(S^2W)}(-1)$.

Proof. The GIT-quotient Y^0 is covered with the images of X'_j:

$$X'_1 = \{(\phi_1, \phi_2) \in X^0; \phi_1 = 2st,\ \phi_2 = \lambda s^2 + 2\nu st + t^2,\ \lambda, \nu \in \mathbf{C}\},$$
$$X'_2 = \{(\phi_1, \phi_2) \in X^0; \phi_1 = ps^2 + 2qst + t^2,\ \phi_2 = 2st,\ p, q \in \mathbf{C}\}.$$

It is clear that the natural image of X'_j in Y is Y_j. The map ϕ given by $\phi^* = (\phi_1, \phi_2) \in Y_1$ has natural $\mathbf{Z}_2$ involution generated by,

$$r:\ (\sqrt{\lambda}s + t, \sqrt{\lambda}s - t) \to (\sqrt{\lambda}s + t, -(\sqrt{\lambda}s - t)).$$

Since

$$\begin{aligned} 2st &= \frac{1}{2\sqrt{\lambda}}((\sqrt{\lambda}s + t)^2 - (\sqrt{\lambda}s - t)^2), \\ \lambda s^2 + 2\nu st + t^2 &= \frac{\nu}{2\sqrt{\lambda}}((\sqrt{\lambda}s + t)^2 - (\sqrt{\lambda}s - t)^2) + \frac{1}{2}((\sqrt{\lambda}s + t)^2 + (\sqrt{\lambda}s - t)^2), \end{aligned}$$

it is clear that,

$$r^*(\phi_1) = \phi_1,\ \ r^*(\phi_2) = \phi_2,\ \ r^*(\lambda s^2 - t^2) = -(\lambda s^2 - t^2).$$

Therefore, we can decompose S^2U into $\langle \lambda s^2 - t^2 \rangle_{\mathbf{C}} \oplus \langle \phi_1, \phi_2 \rangle_{\mathbf{C}}$ with respect to eigenvalue of r^* and take $\lambda s^2 - t^2$ as canonical generator of S^2U/ϕ^*W. Similarly S^2U/ϕ^*W is generated by $ps^2 - t^2$ on Y_2. The problem is therefore to write $\lambda s^2 - t^2$ as an $\Gamma(O_{Y_1 \cap Y_2})$-multiple of $pu^2 - v^2$ when we write $\phi_2 = 2uv$ by a variable change in GL(2). The following variable change $(s, t) \mapsto (u, v)$ is in GL(2):

$$s = \frac{\sqrt{2\alpha}}{(\beta - \alpha)^2}(2u - \frac{(\beta - \alpha)^2}{2\alpha}v),\ \ t = \frac{\sqrt{2\alpha}}{(\beta - \alpha)^2}(2\beta u - \frac{(\beta - \alpha)^2}{2}v),$$

where α, β are roots of the equation $\lambda s^2 + 2\nu st + t^2 = 0$. Under this coordinate change, ϕ_1 and ϕ_2 is rewritten as follows:

$$\phi_1 = \frac{\lambda}{(\nu^2 - \lambda)^2}u^2 + 2\frac{\nu}{\nu^2 - \lambda}uv + v^2 =: pu^2 + 2quv + v^2,\ \ \phi_2 = 2uv.$$

Then we have

$$pu^2 - v^2 = -\frac{2}{\beta - \alpha}(\lambda s^2 - t^2) = -\frac{1}{\sqrt{\nu^2 - \lambda}}(\lambda s^2 - t^2) = -\sqrt{\frac{D_1}{D_2}}(\lambda s^2 - t^2).$$

Similarly by computing the effect on S^2U/ϕ^*W by the variable change from X'_1 into X'_0, we see that $\mathbf{L}^2$ is isomorphic to $O_{\mathbf{P}(S^2W)}(-1)$. This completes the proof. □

Remark 4.5. We remark that the space X must be regared as a $\mathbf{Q}$-stack Y^{stack} as follows: First we define $\phi_0 = -\phi_1 - \phi_2$. For each atlas X'_α we define an atlas Y^{stack}_α

$(\alpha = 0,1,2)$ by

$$
\begin{aligned}
Y_0^{stack} &= \{(\phi_0,\phi_1,\phi_2,\pm\psi_0) \in X^0 \times S^2U; \phi_0 = 2st,\ \phi_1 = as^2+2bst+t^2,\\
&\quad \psi_0 = as^2 - t^2\ a,b \in \mathbf{C}\},\\
Y_1^{stack} &= \{(\phi_0,\phi_1,\phi_2,\pm\psi_1) \in X^0 \times S^2U; \phi_1 = 2st,\ \phi_2 = \lambda s^2+2\nu st+t^2,\\
&\quad \psi_1 = \lambda s^2 - t^2\ \lambda,\nu \in \mathbf{C}\},\\
Y_2^{stack} &= \{(\phi_0,\phi_1,\phi_2,\pm\psi_2)) \in X^0 \times S^2U; \phi_1 = ps^2+2qst+t^2,\ \phi_2 = 2st,\\
&\quad \psi_2 = ps^2 - t^2, p,q \in \mathbf{C}\}.
\end{aligned}
$$

Since $\mathbf{L}^2 \simeq O_{\mathbf{P}(S^2W)}(-1)$ we have $c_1(\mathbf{L}) = -\frac{1}{2}c_1(O_{\mathbf{P}(S^2W)}(1))$ in the Chow ring $A(Y^{stack})_{\mathbf{Q}} = A(X)_{\mathbf{Q}} = A(\mathbf{P}(S^2W))_{\mathbf{Q}}$.

5. Proof of the main theorem

Theorem 5.1.

$$\pi_*(c_{top}(H^1)) = \frac{1}{8}\left[\frac{c(S^{k-1}Q)}{1-\frac{1}{2}c_1(Q)}\right]_{k-N},$$

where π is the natural projection from $\bar{M}_{0,0}(L,2)$ to G and $[*]_{k-N}$ is the operation of picking up the degree $2(k-N)$ part of Chern classes.

Proof. From now on we denote the coherent sheaf $\mathbf{L}$ in Lemma 4.4 by $O_{\mathbf{P}}(-\frac{1}{2})$. In view of the results from the previous section, what remains is to evaluate the top chern class of $\big(S^{k-1}Q/((V^\vee \otimes O_G)/Q^\vee)\big) \otimes O_{\mathbf{P}}(-\frac{1}{2})$ on $\mathbf{P}(S^2Q)$. Since double cover maps parametrized by $\mathbf{P}(S^2Q)$ have natural $\mathbf{Z}_2$ involution r given in the previous section, we have to multiply the result of integration on $\mathbf{P}(S^2Q)$ by the factor $\frac{1}{2}$ [BT], [FP]. With this set-up, let $\pi' : \mathbf{P}(S^2Q) \to G$ be the natural projection. Then what we have to compute is $\pi_*(c_{top}(H^1)) = \frac{1}{2}\pi'_*(c_{top}(H^1)) = \frac{1}{2}\pi'_*(c_{top}(\big(S^{k-1}Q/((V^\vee \otimes O_G)/Q^\vee)\big) \otimes O_{\mathbf{P}}(-\frac{1}{2})))$. Let z be $c_1(O_{\mathbf{P}}(1))$. Then we obtain,

$$
\begin{aligned}
&\frac{1}{2}\pi'_*(c_{top}(\big(S^{k-1}Q/((V^\vee \otimes O_G)/Q^\vee)\big) \otimes O_{\mathbf{P}}(-\frac{1}{2})))\\
&= \frac{1}{2}\sum_{j=0}^{k-N+2} c_{k-N+2-j}(S^{k-1}Q \oplus Q^\vee)\cdot \pi_*(z^j)\cdot(-\frac{1}{2})^j\\
&= \frac{1}{8}\sum_{j=0}^{k-N} c_{k-N-j}(S^{k-1}Q \oplus Q^\vee)\cdot s_j(S^2Q)\cdot(-\frac{1}{2})^j\\
&= \frac{1}{8}\left[\frac{c(S^{k-1}Q)\cdot c(Q^\vee)}{1-\frac{1}{2}c_1(S^2Q)+\frac{1}{4}c_2(S^2Q)-\frac{1}{8}c_3(S^2Q)}\right]_{k-N},
\end{aligned}
$$

where $s_j(S^2Q)$ is the j-th Segre class of S^2Q. But if we decompose $c(Q)$ into $(1+\alpha)(1+\beta)$, we can easily see,

$$
\begin{aligned}
\frac{c(Q^\vee)}{1-\frac{1}{2}c_1(S^2Q)+\frac{1}{4}c_2(S^2Q)-\frac{1}{8}c_3(S^2Q)} &= \frac{(1-\alpha)(1-\beta)}{(1-\alpha)(1-\frac{1}{2}(\alpha+\beta))(1-\beta)}\\
&= \frac{1}{1-\frac{1}{2}c_1(Q)}.
\end{aligned}
$$

□

Finally, by combining the above theorem with the divisor axiom of Gromov-Witten invariants, we can prove the decomposition formula of degree 2 rational Gromov-Witten invariants of M_N^k found from numerical experiments.

Corollary 5.2.

$$\langle \mathcal{O}_{e^a}\mathcal{O}_{e^b}\mathcal{O}_{e^c}\rangle_{0,2} = \langle \mathcal{O}_{e^a}\mathcal{O}_{e^b}\mathcal{O}_{e^c}\rangle_{0,2\to 2} + 8\langle \pi_*(c_{top}(H^1))\mathcal{O}_{e^a}\mathcal{O}_{e^b}\mathcal{O}_{e^c}\rangle_{0,1},$$

where $\langle \mathcal{O}_{e^a}\mathcal{O}_{e^b}\mathcal{O}_{e^c}\rangle_{0,2\to 2}$ *is the number of conics that intersect cycles Poincaré dual to* e^a, e^b *and* e^c. *We also denote by* $\langle \pi_*(c_{top}(H^1))\mathcal{O}_{e^a}\mathcal{O}_{e^b}\mathcal{O}_{e^c}\rangle_{0,1}$ *the integral:*

$$\int_{G(2,V)} c_{top}(S^kQ)\wedge \pi_*(c_{top}(H^1))\wedge \sigma_{a-1}\wedge\sigma_{b-1}\wedge\sigma_{c-1}.$$

6. Generalization to twisted cubics

In this section, we present a decomposition formula of degree 3 rational Gromov-Witten invariants found from numerical experiments using the results of [ES].

Conjecture 6.1. *If* $k-N=1$, *we have the following equality:*

$$\pi_*(c_{top}(H^1)) =$$
$$\frac{1}{27}\Big(\big(\frac{1}{24}(27k^2-55k+26)k(k-1)+\frac{2}{9}\big)c_1(Q)^2+\big(\frac{7}{6}(k+1)k(k-1)+\frac{1}{9}\big)c_2(Q)\Big).$$

where $\pi : \overline{M}_{0,0}(L,3)\to \overline{M}_{0,0}(M_N^k,1)$ *is the natural projection.*

In the $k-N>1$ case, we have not found the explicit formula, because in the $d=3$ case, we have another contribution from multiple cover maps of type $(2{+}1)\to(1{+}1)$. Here multiple cover map of type $(2{+}1)\to(1{+}1)$ is the map from nodal curve $\mathbf{P}^1\vee\mathbf{P}^1$ to nodal conic $L_1\vee L_2\subset M_N^k$, that maps the first (resp. the second) $\mathbf{P}^1$ to L_1 (resp. L_2) by two to one (resp. one to one). In the $k-N=1$ case, we have also determined the contributions from multiple cover maps of $(2+1)\to(1+1)$ to nodal conics.

Corollary 6.2. *If* $k-N=1$, $\langle \mathcal{O}_{e^a}\mathcal{O}_{e^b}\mathcal{O}_{e^c}\rangle_{0,3}$ *is decomposed into the following contributions:*

$$\begin{aligned}
&\langle \mathcal{O}_{e^a}\mathcal{O}_{e^b}\mathcal{O}_{e^c}\rangle_{0,3} = \langle \mathcal{O}_{e^a}\mathcal{O}_{e^b}\mathcal{O}_{e^c}\rangle_{0,3\to 3} + \frac{1}{k}\big(\frac{9}{4}\langle \mathcal{O}_{e^a}\mathcal{O}_{e^b}\mathcal{O}_{e^c}\mathcal{O}_{e^3}\rangle_{0,1}\langle \mathcal{O}_{e^{N-5}}\rangle_{0,1}\\
&+\frac{3}{2}\langle \mathcal{O}_{e^a}\mathcal{O}_{e^b}\mathcal{O}_{e^{c+2}}\rangle_{0,1}\langle \mathcal{O}_{e^{N-c-4}}\mathcal{O}_{e^c}\rangle_{0,1}+\frac{3}{2}\langle \mathcal{O}_{e^b}\mathcal{O}_{e^c}\mathcal{O}_{e^{a+2}}\rangle_{0,1}\langle \mathcal{O}_{e^{N-a-4}}\mathcal{O}_{e^a}\rangle_{0,1}\\
&+\frac{3}{2}\langle \mathcal{O}_{e^c}\mathcal{O}_{e^a}\mathcal{O}_{e^{b+2}}\rangle_{0,1}\langle \mathcal{O}_{e^{N-b-4}}\mathcal{O}_{e^b}\rangle_{0,1}\big)\\
&+27\langle \pi_*(c_{top}(H^1))\mathcal{O}_{e^a}\mathcal{O}_{e^b}\mathcal{O}_{e^c}\rangle_{0,1},
\end{aligned}$$

where $\langle \mathcal{O}_{e^a}\mathcal{O}_{e^b}\mathcal{O}_{e^c}\rangle_{0,3\to 3}$ *is the number of twisted cubics that intersect cycles Poincaré dual to* e^a, e^b *and* e^c.

Proof. In the $k-N=1$ case, dimension of moduli space of multiple cover maps of $(2+1)\to(1+1)$ to nodal conics is given by $N-6+N-6-(N-4)+2=N-6$, hence the rank of H^1 is given by $N-6-(N-5-3)=2$. On the other hand, dimension of moduli space of $d=2$ multiple cover maps of $\mathbf{P}^1\to\mathbf{P}^1$ is 2, the degree of the form of $\tilde{\pi}_*(c_{top}(H^1))$ equals to $2-2=0$, where $\tilde{\pi}$ is the projection map that projects out the fiber locally isomorphic to the moduli space of $d=2$ multiple cover maps. This situation is exactly the same as the Calabi-Yau case. Therefore, we can use the well-known result by Aspinwall and Morrison, that says for n-point rational Gromov-Witten invariants for Calabi-Yau manifold, $\tilde{\pi}_*(c_{top}(H^1))$ for degree d multiple cover map is given by,

$$\tilde{\pi}_*(c_{top}(H^1)) = \frac{1}{d^{3-n}}.$$

With this formula, we add up all the combinatorial possibility of insertion of external operator $\mathcal{O}_{e^a}$, $\mathcal{O}_{e^b}$ and $\mathcal{O}_{e^c}$,

$$\begin{aligned}
&\frac{1}{k}\big(\langle\tilde{\pi}_*(c_{top}(H^1))\mathcal{O}_{e^a}\mathcal{O}_{e^b}\mathcal{O}_{e^c}\mathcal{O}_{e^3}\rangle_{0,1}\langle\mathcal{O}_{e^{N-5}}\rangle_{0,1}\\
&+\langle\tilde{\pi}_*(c_{top}(H^1))\mathcal{O}_{e^a}\mathcal{O}_{e^b}\mathcal{O}_{e^{c+2}}\rangle_{0,1}\langle\mathcal{O}_{e^{N-c-4}}\mathcal{O}_{e^c}\rangle_{0,1}\\
&+\langle\tilde{\pi}_*(c_{top}(H^1))\mathcal{O}_{e^b}\mathcal{O}_{e^c}\mathcal{O}_{e^{a+2}}\rangle_{0,1}\langle\mathcal{O}_{e^{N-a-4}}\mathcal{O}_{e^a}\rangle_{0,1}\\
&+\langle\tilde{\pi}_*(c_{top}(H^1))\mathcal{O}_{e^c}\mathcal{O}_{e^a}\mathcal{O}_{e^{b+2}}\rangle_{0,1}\langle\mathcal{O}_{e^{N-b-4}}\mathcal{O}_{e^b}\rangle_{0,1}\\
&+\langle\mathcal{O}_{e^a}\mathcal{O}_{e^b}\mathcal{O}_{e^{c+2}}\rangle_{0,1}\langle\mathcal{O}_{e^{N-c-4}}\tilde{\pi}_*(c_{top}(H^1))\mathcal{O}_{e^c}\rangle_{0,1}\\
&+\langle\mathcal{O}_{e^b}\mathcal{O}_{e^c}\mathcal{O}_{e^{a+2}}\rangle_{0,1}\langle\mathcal{O}_{e^{N-a-4}}\tilde{\pi}_*(c_{top}(H^1))\mathcal{O}_{e^a}\rangle_{0,1}\\
&+\langle\mathcal{O}_{e^c}\mathcal{O}_{e^a}\mathcal{O}_{e^{b+2}}\rangle_{0,1}\langle\mathcal{O}_{e^{N-b-4}}\tilde{\pi}_*(c_{top}(H^1))\mathcal{O}_{e^b}\rangle_{0,1}\\
&+\langle\mathcal{O}_{e^a}\mathcal{O}_{e^b}\mathcal{O}_{e^c}\mathcal{O}_{e^3}\rangle_{0,1}\langle\mathcal{O}_{e^{N-5}}\tilde{\pi}_*(c_{top}(H^1))\rangle_{0,1}\big)\\
&=\frac{1}{k}\big(2\langle\mathcal{O}_{e^a}\mathcal{O}_{e^b}\mathcal{O}_{e^c}\mathcal{O}_{e^3}\rangle_{0,1}\langle\mathcal{O}_{e^{N-5}}\rangle_{0,1}\\
&+\langle\mathcal{O}_{e^a}\mathcal{O}_{e^b}\mathcal{O}_{e^{c+2}}\rangle_{0,1}\langle\mathcal{O}_{e^{N-c-4}}\mathcal{O}_{e^c}\rangle_{0,1}+\langle\mathcal{O}_{e^b}\mathcal{O}_{e^c}\mathcal{O}_{e^{a+2}}\rangle_{0,1}\langle\mathcal{O}_{e^{N-a-4}}\mathcal{O}_{e^a}\rangle_{0,1}\\
&+\langle\mathcal{O}_{e^c}\mathcal{O}_{e^a}\mathcal{O}_{e^{b+2}}\rangle_{0,1}\langle\mathcal{O}_{e^{N-b-4}}\mathcal{O}_{e^b}\rangle_{0,1}\\
&+\frac{1}{2}\langle\mathcal{O}_{e^a}\mathcal{O}_{e^b}\mathcal{O}_{e^{c+2}}\rangle_{0,1}\langle\mathcal{O}_{e^{N-c-4}}\mathcal{O}_{e^c}\rangle_{0,1}+\frac{1}{2}\langle\mathcal{O}_{e^b}\mathcal{O}_{e^c}\mathcal{O}_{e^{a+2}}\rangle_{0,1}\langle\mathcal{O}_{e^{N-a-4}}\pi_*\mathcal{O}_{e^a}\rangle_{0,1}\\
&+\frac{1}{2}\langle\mathcal{O}_{e^c}\mathcal{O}_{e^a}\mathcal{O}_{e^{b+2}}\rangle_{0,1}\langle\mathcal{O}_{e^{N-b-4}}\mathcal{O}_{e^b}\rangle_{0,1}\\
&+\frac{1}{4}\langle\mathcal{O}_{e^a}\mathcal{O}_{e^b}\mathcal{O}_{e^c}\mathcal{O}_{e^3}\rangle_{0,1}\langle\mathcal{O}_{e^{N-5}}\rangle_{0,1}\big).
\end{aligned}$$

The last expression is nothing but the formula we want. □

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