On Blow up at Space Infinity for Semilinear Heat Equations

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Abstract

A nonnegative blowing up solution of the semilinear heat equation $u_t = \Delta u + u^p$ with $p > 1$ is considered when initial data u_0 satisfies

$$\lim_{|x|\to\infty} u_0 = M > 0, \quad u_0 \le M \quad \text{and} \quad u_0 \not\equiv M.$$

It is shown that the solution blows up only at space infinity and that $\lim_{|x|\to\infty} u(x,t)$ is the solution of the ordinary differential equation $v_t = v^p$ with $v(0) = M$.

1 Introduction and main theorems

We are interested in solutions of semilinear heat equations which blow up at space infinity.

We consider nonnegative solutions of the initial value problem for the equation

$$\begin{cases} u_t = \Delta u + u^p, & x \in \mathbf{R}^n, t > 0, \\ u(x,0) = u_0(x), & x \in \mathbf{R}^n, \end{cases} \tag{1}$$

where $p > 1$ and u_0 is a nonnegative continuous function in $\mathbf{R}^n$ satisfying

$$\lim_{|x|\to\infty} u_0 = M > 0, \quad u_0 \le M \quad \text{and} \quad u_0 \not\equiv M. \tag{2}$$

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Problem (1) has a unique, nonnegative and bounded solution at least locally in time. However, the solution necessarily blows up in finite time ([10, Theorem 3.2]). (The solution of (1) with initial value decaying slowly at space infinity blows up surely, let alone the solution with initial value not decaying.) For a given initial value u_0 let $T^* = T^*(u_0, p)$ be the maximal existence time of the solution. If $T^* = \infty$, the solution exists globally in time. If $T^* < \infty$, we say that the solution blows up in finite time. It is well known that

$$\limsup_{t \to T^*} \|u(t)\|_\infty = \infty, \tag{3}$$

where $\|u\|_\infty$ denotes the L^∞-norm of u in space variables.

In this paper, we are interested in behavior of a blowing up solution near space infinity as well as location of blow up points defined below. A point $x_{BU} \in \mathbf{R}^n$ is called a *blow up point* if there exists a sequence $\{(x_m, t_m)\}_{m=1}^\infty$ such that

$$t_m \uparrow T^*, \quad x_m \to x_{BU} \quad \text{and} \quad u(x_m, t_m) \to \infty \quad \text{as} \quad m \to \infty.$$

If there exists a sequence $\{(x_m, t_m)\}_{m=1}^\infty$ such that

$$t_m \uparrow T^*, \quad |x_m| \to \infty \quad \text{and} \quad u(x_m, t_m) \to \infty \quad \text{as} \quad m \to \infty.$$

we say that the solution blows up at space infinity.

We consider the solution $v(t)$ of an ordinary differential equation

$$\begin{cases} v_t = v^p, & t > 0, \\ v(0) = M. \end{cases} \tag{4}$$

An explicit form of the solution is

$$v(t) = \frac{1}{(p-1)^{1/(p-1)}(T_v - t)^{1/(p-1)}},$$

where $T_v = T^*(M, p)$ is the maximal existence time of the solution of (4) and its explicit form is

$$T_v = \frac{1}{(p-1)M^{p-1}}.$$

We are now in position to state our main results.

Theorem 1. *Assume that $p > 1$. Let u_0 be a nonnegative continuous function satisfying* (2)*. Then the solution $u(x,t)$ of* (1) *blows up at $T_v = T^*(M, p)$ and satisfies*

$$\lim_{|x| \to \infty} u(x,t) = v(t).$$

The convergence is uniform in an every compact subset of $\{t : 0 \le t < T_v\}$.

This result in particular implies that

$$\sup_{0<t<T^*} (T^* - t)^{1/(p-1)} \|u(t)\|_\infty < \infty. \tag{5}$$

Such a blow up rate estimate is known for subcritical p; see e.g. [4], [6], [7] for general bounded initial data without assuming (2). The blow up time $T^*(u_0, p)$ may be larger than T_v with $v(0) = \|u_0\|_\infty$. However, for supercritical p such a blow up rate estimate (5) may not hold in general; see e.g. [1], [8]. If one considers only radial solution of (1) for supercritical p less than $1 + 4/(n - 4 - 2(n-1)^{1/2})$ or $n \le 10$, then the estimate (5) holds [11]. We would like to emphasize that Theorem 1 requires no restriction on p.

Our second main result is on the location of blow up points.

Theorem 2. *Assume the same hypotheses of Theorem 1. Then the solution of* (1) *has no blow up points in* $\boldsymbol{R}^n$. (*It blows up only at space infinity.*)

There are huge literature on location of blow up points since the work of Weissler [15] and Friedman-McLeod [2]. (We do not intend to exhaust references in this paper.) However, most of results consider either bounded domains or solutions decaying at space infinity; such a solution does not blowup at space infinity [5].

As far as the authors know, the only paper discussing blow up at space infinity is the work of Lacey [9]. He considered the Dirichlet problem in a half line. He studied various nonlinear terms and proved that a solution blows up only at space infinity.

In particular, his result implies that the solution of

$$\begin{cases} u_t = u_{xx} + u^p, & x > 0, t > 0, \\ u(0,t) = 1, & t > 0, \\ u(x,0) = u_0(x) \ge 1, & x > 0 \end{cases}$$

blows up only at space infinity, where u_0 satisfies (2) with $M > 1$.

His method is based on construction of suitable subsolutions and supersolutions. However, the construction heavily depends on the Dirichlet condition at $x = 0$ and does not apply to the Cauchy problem even for the case $n = 1$.

To prove Theorem 1 we shall estimate $\|u(t)\|_\infty$ from above. The key step is an estimate of $\liminf_{|x|\to\infty} u(x,t)$ from below, where we first assume $\Delta u_0 \to 0$ as $|x| \to \infty$. A key observation is that the effect of Δu is negligible near the space infinity. The case of general initial data can be proved by a comparison argument.

To prove Theorem 2 we shall construct a supersolution $\bar{u}$ so that $\limsup_{t\to T_v} \bar{u}(x,t)v(t) < 1$.

For subcritical p (i.e., $(n-2)p < n+2$ or $n \leq 2$) by [5] this estimate implies that x is not a blow up point. The proof for the supercritical case is more involved but can be done along the line of [5]. We reproduce some of their arguments for the reader's convenience.

This paper is organized as follows. In section 2 we prove Theorem 1. The proof of Theorem 2 is given in section 3 at least for subcritical case. In section 4 we extend removability results for blow up points developed by [5]. In appendix we give a key estimate for removability results which is essentially the same as in [5].

2 Behavior at space infinity

Our goal in this section is to prove Theorem 1. We begin by estimating $\|u(t)\|_\infty$ from above.

Lemma 2.1. *Let u be a solution of* (1) *in $\boldsymbol{R}^n \times (0, T^*)$. Then for each $t_0 \in [0, T^*)$ the estimate*

$$\|u(\cdot,t)\|_\infty \leq \|u(\cdot,t_0)\|_\infty + \int_{t_0}^t \|u(\cdot,s)\|_\infty^p ds$$

holds for all $t \in [t_0, T^*)$, where $T^* = T^*(u_0, p)$.

Proof. Since u is a solution of (1), it fulfills an integral equation of the form

$$u(x,t) = (e^{(t-s)\Delta}u(\cdot,t_0))(x) + \int_{t_0}^t (e^{(t-s)\Delta}u(\cdot,s))(x)ds, \tag{6}$$

where $e^{t\Delta}$ is the solution operator of the heat equation defined by

$$e^{t\Delta}f(x) = (4\pi t)^{-n/2} \int_{\mathbf{R}^n} e^{-|x-y|^2/4t} f(y)dy. \tag{7}$$

Since $\|e^{t\Delta}f\|_\infty \leq \|f\|_\infty$, we observe that

$$\|u(\cdot,t)\|_\infty \leq \|u(\cdot,t_0)\|_\infty + \int_{t_0}^t \|u^p(\cdot,s)\|_\infty ds.$$

Since $\|u^p\|_\infty = \|u\|_\infty^p$, we obtain the desired inequality. □

We shall discuss the estimate from below for $\liminf_{|x|\to\infty} u(x,t)$ assuming that u_0 is C^2-function and $\Delta u_0 \to 0$ as $|x| \to \infty$. For this purpose we first study behaviour at space infinity for the heat equation.

Lemma 2.2. *Let ξ be a nonnegative continuous function in $\boldsymbol{R}^n$ satisfying*

$$\lim_{|x|\to\infty} \xi(x) = 0. \tag{8}$$

Then

$$\lim_{|x|\to\infty} (e^{s\Delta}\xi)(x) = 0 \tag{9}$$

and the convergence is uniform in $s \in [0,t]$ for every finite $t > 0$.

Proof. Since $h(\sigma) = \sigma^n e^{-\sigma^2}$ is decreasing for large σ and $\lim_{\sigma\to\infty} h(\sigma) = 0$, for $\epsilon > 0$ there is a large R_0 depending on t and ϵ such that

$$h\left(\frac{R}{\sqrt{4s}}\right) \le h\left(\frac{R}{\sqrt{4t}}\right) < \frac{\epsilon\pi^{n/2}}{2^{n+1}\|\xi\|_\infty} \qquad \text{for } s \in (0,t)$$

for $R \ge R_0$. We may assume that

$$\xi(x) < \frac{\epsilon}{2} \qquad \text{for} \qquad |x| > R_0 \tag{10}$$

by taking R_0 larger.

By this choice of R_0 for $|x| > 2R_0$ we observe that

$$\begin{aligned} e^{s\Delta}\xi(x) &= (4\pi s)^{-n/2} \int_{\mathbf{R}^n} e^{-|x-y|^2/4s}\xi(y)dy \\ &= \pi^{-n/2} \int_{\mathbf{R}^n} e^{-|y|^2}\xi(x-\sqrt{4s}y)dy \\ &= \pi^{-n/2} \int_{|x-\sqrt{4s}y|<R_0} + \int_{|x-\sqrt{4s}y|>R_0} e^{-|y|^2}\xi(x-\sqrt{4s}y)dy \end{aligned}$$

Here, consider the integration inside. Then we have

$$\begin{aligned} \pi^{-n/2} \int_{|x-\sqrt{4s}y|<R_0} & e^{-|y|^2}\xi(x-\sqrt{4s}y)dy \\ &< (4\pi s)^{-n/2} \int_{|x-y|<R_0} \|\xi\|_\infty e^{-R_0^2/4s}dy \\ &\le \frac{2^n\|\xi\|_\infty}{\pi^{n/2}} h(R_0/\sqrt{4s}) < \frac{\epsilon}{2} \end{aligned}$$

for all $s \in [0,t]$. On the other hand, we consider this outside, and we take

$$\begin{aligned} \pi^{-n/2} \int_{|x-\sqrt{4s}y|>R_0} & e^{-|y|^2}\xi(x-\sqrt{4s}y)dy \\ &< \pi^{-n/2} \int_{|x-\sqrt{4s}y|>R_0} \frac{\epsilon}{2} e^{-|y|^2}dy \\ &\le \pi^{-n/2} \int_{\mathbf{R}^n} \frac{\epsilon}{2} e^{-|y|^2}dy = \frac{\epsilon}{2} \end{aligned}$$

for all $s \in [0, t]$. We thus obtain

$$e^{s\Delta}\xi(x) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

for all $s \in [0, t]$. Thus, we have proved that

$$\sup_{0 \le s \le t} (e^{s\Delta}\xi)(x) \to 0 \qquad \text{as } |x| \to \infty. \quad \square$$

Lemma 2.3. *Let u be a solution of* (1) *in $\boldsymbol{R}^n \times (0, T^*)$. If u_0 is C^2-function and $\lim_{|x|\to\infty} \Delta u_0 = 0$, then*

$$\left(\liminf_{|x|\to\infty} u(x,t)\right) - \left(\liminf_{|x|\to\infty} u(x,t_0)\right) \ge \int_{t_0}^{t} \liminf_{|x|\to\infty} u^p(x,s)ds$$

for $t > t_0$ satisfying t, $t_0 \in (0, T^*)$.

Proof. Differentiate (1) with respect to x twice to get

$$\begin{aligned}(\Delta u)_t &= \Delta(\Delta u) + pu^{p-1}\Delta u + p(p-1)u^{p-2}|\nabla u|^2 \\ &\ge \Delta(\Delta u) + pu^{p-1}\Delta u.\end{aligned}$$

Thus, we observe that $(-\Delta u)$ satisfies

$$(-\Delta u)_t \le \Delta(-\Delta u) + pu^{p-1}(-\Delta u).$$

We consider the solution of

$$\begin{cases} f_t(x,t) = \Delta f(x,t) + g(t)f(x,t), & x \in \mathbf{R}^n, t > 0, \\ f(x,0) = f_0(x), & x \in \mathbf{R}^n. \end{cases} \tag{11}$$

The solution of (11) is of the form

$$f(x,t) = e^{t\Delta} f_0(x) \exp\left(\int_0^t g(s)ds\right). \tag{12}$$

We consider the solution of

$$\begin{cases} \eta_t(x,t) = \Delta\eta(x,t) + pu^{p-1}\eta(x,t), & x \in \mathbf{R}^n, t > 0, \\ \eta(x,0) = (-\Delta u_0)_+(x), & x \in \mathbf{R}^n. \end{cases} \tag{13}$$

where $(u)_+ = \min\{u, 0\}$. Comparing with (12), we see that the solution of (13) is estimated by

$$0 \le \eta(x,t) \le \left(e^{t\Delta}(-\Delta u_0)_+\right)(x) \exp\left(\int_0^t p\|u(\cdot,s)\|_\infty^{p-1} ds\right).$$

From this estimate and comparison theorem it follows that

$$(-\Delta u)_+(x,t) \le \left(e^{t\Delta}(-\Delta u_0)_+\right)(x) \exp\left(\int_0^t p\|u(\cdot,s)\|_\infty^{p-1} ds\right). \tag{14}$$

Since $\lim_{|x|\to\infty} \Delta u_0 = 0$, applying Lemma 2.2 to (14) yields

$$\limsup_{|x|\to\infty} (-\Delta u)_+(x,t) = 0 \tag{15}$$

for $t \in [0, T^*)$. This implies

$$\liminf_{|x|\to\infty} \Delta u(x,t) \ge 0. \tag{16}$$

Integrating (1) with respect to t, we see that the solution $u(x,t)$ of (1) satisfies

$$u(x,t) - u(x,t_0) = \int_{t_0}^t \{\Delta u(x,s) + u^p(x,s)\}\, ds \tag{17}$$

for $t \in (t_0, T^*)$. Then from (16) and (17), it follows that

$$\liminf_{|x|\to\infty} \{u(x,t) - u(x,t_0)\} \ge \liminf_{|x|\to\infty} \int_{t_0}^t u^p(x,s) ds.$$

For the function $a(x,t)$, $b(x,t)$ satisfying $|\liminf_{|x|\to\infty} a(x,t)| < \infty$ and $|\liminf_{|x|\to\infty} b(x,t)| < \infty$, it is clear that

$$\liminf_{|x|\to\infty} a(x,t) + \liminf_{|x|\to\infty} b(x,t) \le \liminf_{|x|\to\infty} (a(x,t) + b(x,t)).$$

Thus, if we set $a(x,t) = u(x,t) - u(x,t_0)$ and $b(x,t) = u(x,t_0)$, then we have

$$\left(\liminf_{|x|\to\infty} u(x,t)\right) - \left(\liminf_{|x|\to\infty} u(x,t_0)\right) \ge \liminf_{|x|\to\infty} \{u(x,t) - u(x,t_0)\}.$$

By this observation and Fatou's lemma or

$$\liminf_{|x|\to\infty} \int_{t_0}^t u^p(x,s) ds \ge \int_{t_0}^t \liminf_{|x|\to\infty} u^p(x,s) ds,$$

we now have

$$\left(\liminf_{|x|\to\infty} u(x,t)\right) - \left(\liminf_{|x|\to\infty} u(x,t_0)\right) \ge \int_{t_0}^t \liminf_{|x|\to\infty} u^p(x,s) ds. \quad \square$$

Lemma 2.3 yields the estimate

$$\liminf_{|x|\to\infty} u(x,t) \geq v(t) \quad \text{for} \quad t \in [0, \min\{T^*, T_v\})$$

if we admit the next elementary lemma.

Lemma 2.4. *Let v be the solution of* (4) *in* $[0, T_v)$. *Let $\tilde{v}$ be a nonnegative measurable function on* $[0, T_0)$ *with some* $T_0 \in (0, T_v)$. *Assume that $\tilde{v}$ satisfies*

$$\tilde{v}(t) - \tilde{v}(t_0) \geq (\leq) \int_{t_0}^{t} \tilde{v}^p(s)ds \quad \text{for} \quad t_0, t \in [0, T_0) \quad \text{with} \quad t_0 \leq t. \tag{18}$$

Assume that $\tilde{v}(0) = M$. *Then*

$$\tilde{v}(t) \geq (\leq) v(t) \quad \text{for} \quad t \in [0, T_0).$$

Proof. We shall only prove the case $\tilde{v}(t) - \tilde{v}(t_0) \geq \int_{t_0}^{t} \tilde{v}^p(s)ds$ since the proof of the other case is parallel. Integrating the first formula of (4) from t_0 to t, we have

$$v(t) - v(t_0) = \int_{t_0}^{t} v^p(s)ds \tag{19}$$

for $t_0 \in [0, t]$. Since $\tilde{v}(0) = v(0) = M$, the estimate (18) together with (19) yields

$$\tilde{v}(t) - v(t) \geq \int_0^t (\tilde{v}^p(s) - v^p(s))ds.$$

By the mean value theorem we observe that

$$\tilde{v}(t) - v(t) \geq \int_0^t c(s)\left(\tilde{v}(s) - v(s)\right)ds,$$

where

$$c(s) = \int_0^1 p\left(\theta v(s) + (1-\theta)\tilde{v}(s)\right)^{p-1} d\theta.$$

We set $\psi_\epsilon(t) = \tilde{v}(t) - v(t) + \epsilon$ with $\epsilon > 0$, and observe that $\psi_\epsilon(t)$ satisfies

$$\psi_\epsilon \geq \int_0^t c(s)\psi_\epsilon(s)ds + \epsilon\left(1 - \int_0^t c(s)ds\right).$$

We set

$$t_0 = \sup\left\{t > 0; \int_0^t c(s)ds < \frac{1}{2}\right\}.$$

Then, for $t \in [0, t_0]$ we have

$$\psi_\epsilon(t) \geq \int_0^t c(s)\psi_\epsilon(s)ds + \frac{\epsilon}{2}. \tag{20}$$

We shall argue by contradiction to prove $\psi_\epsilon(t) \geq 0$. Suppose that $\psi_\epsilon(t) < 0$ for some $t \in [0, t_0]$. Then $\psi_\epsilon(\tau) = 0$ for

$$\tau = \inf \{t \in [0, t_0]; \psi_\epsilon < 0\}. \tag{21}$$

This τ must be positive. Indeed, since $\tilde{v}$ is nondecreasing by (18) and v is continuous, $\psi_\epsilon(0) > \epsilon$ implies $\tau > 0$.

Since $\int_0^\tau c(s)\psi_\epsilon(s)ds \geq 0$ and (21) imply $\psi_\epsilon(\tau) \leq 0$, we get a contradiction. We thus proved that

$$\psi_\epsilon(t) \geq 0.$$

Since this holds for all $\epsilon > 0$, we get $\tilde{v}(t) \geq v(t)$ for $t \in [0, t_0]$. (If $\tilde{v}(t) < v(t)$ for some t, there exist $\epsilon > 0$ such that $\psi_\epsilon < 0$ for such t.)

Next, since $\tilde{v}(t) \geq v(t)$ for $t \in [0, t_0]$, we observe that

$$\psi_\epsilon \geq \int_{t_1}^t c(s)\psi_\epsilon(s)ds + \epsilon\left(1 - \int_{t_1}^t c(s)ds\right).$$

We set

$$t_1 = \sup\left\{t > t_0; \int_{t_0}^t c(s)ds < \frac{1}{2}\right\}$$

and observe that

$$\psi_\epsilon \geq \int_{t_0}^t c(s)\psi_\epsilon(s)ds + \frac{\epsilon}{2}.$$

for $t \in [t_0, t_1]$. By the same argument one can prove $\psi_\epsilon \geq 0$ for all $\epsilon > 0$, and $\tilde{v}(t) \geq v(t)$ for $t \in [t_0, t_1]$.

We repeat this argument and conclude that

$$\tilde{v}(t) \geq v(t)$$

for all $t \in [0, T_v)$. By the same argument, we find if

$$\tilde{v}(t) - \tilde{v}(t_0) \leq \int_{t_0}^t \tilde{v}^p(s)ds \quad \text{for} \quad t_0, t \in [0, T_0) \quad \text{with} \quad t_0 \leq t,$$

then

$$\tilde{v}(t) \le v(t) \quad \text{for} \quad t \in [0, T_0). \quad \Box$$

Remark 2.5. In the proof of Lemma 2.4, we take ϵ so that ψ_ϵ is strictly large than $\tilde{v}(t) - v(t)$. If $\epsilon = 0$, then τ may be zero; in this case the above argument does not yield a contradiction.

Proof of Theorem 1. (*The case* $\lim_{|x|\to\infty} \Delta u_0 = 0$) Assume that u_0 is C^2-function and $\lim_{|x|\to\infty} \Delta u_0(x) = 0$. By (4) we see that the solution $v(t)$ of (4) satisfies

$$v(t) - v(t_0) = \int_{t_0}^{t} \{v^p(s)\}\, ds \tag{22}$$

for $t \in (t_0, T_v)$ and $t_0 \in [0, T_v)$. This together with Lemmas 2.1 and 2.4 yields

$$\|u(\cdot, t)\|_\infty \le v(t) \text{ for } t \in (0, T_v) \text{ and } T_v \le T^*.$$

Similarly, from Lemmas 2.3 and 2.4 it follows that

$$\liminf_{|x|\to\infty} u(x,t) \ge v(t) \text{ for } t \in (0, T^*) \text{ and } T_v \ge T^*.$$

Since $\|u(\cdot,t)\|_\infty \ge \liminf_{|x|\to\infty} u(x,t)$, we have $T_v = T^*$ and

$$\|u(\cdot, t)\|_\infty = v(t) = \lim_{|x|\to\infty} u(x,t) \text{ for } t \in (0, T_v). \quad \Box$$

We have proved Theorem 1 in the case $\lim_{|x|\to\infty} \Delta u_0(x) = 0$. It remains to prove Theorem 1 for general initial data.

Lemma 2.6. *Let $u_0(x)$ satisfy* (2)*. Then, there exist C^2-functions $\overline{u}_0(x)$ and $\underline{u}_0(x)$ satisfying*

$$\begin{cases} 0 \le \underline{u}_0(x) \le u_0(x) \le \overline{u}_0(x) \le M, \\ \lim\limits_{|x|\to\infty} \overline{u}_0(x) = \lim\limits_{|x|\to\infty} \underline{u}_0 = M, \\ \overline{u}_0 \not\equiv M \textit{ and } \lim\limits_{|x|\to\infty} \Delta \underline{u}_0(x) = \lim\limits_{|x|\to\infty} \Delta \overline{u}_0(x) = 0, \\ \underline{u}_0 \textit{ is radial with respect to the origin.} \end{cases} \tag{23}$$

Proof. Since $u_0(x) \not\equiv M$, we can easily find $\overline{u}_0(x)$ satisfying above conditions. It remains to construct $\underline{u}_0(x)$.

We set $\eta_1(r) = \inf_{r\ge|x|} u_0(x)$. Then $\eta_1(r)$ is an increasing function with respect to r. We then set

$$\eta_2(r) = \begin{cases} (r - [r])\eta_1([r]-1) + (1 - r + [r])\eta_1([r]-2), & r \ge 2, \\ \eta_1(0), & 0 \le r < 2, \end{cases}$$

where $[r]$ is the greatest integer not greater than r. If we set $\underline{u}_0(x) = \eta_2(|x|)$, then it satisfies all desired properties except the last one. The function $\eta_2(r)$ is a piecewise linear function of r and is not C^2 so we shall modify its corners.

Let $\mu_n(r)$ be a polynomial of degree at most five satisfying

$$\mu_n\left(n \pm \frac{1}{4}\right) = \eta_2\left(n \pm \frac{1}{4}\right),$$
$$\frac{d\mu_n}{dr}\left(n \pm \frac{1}{4}\right) = \frac{d\eta_2}{dr}\left(n \pm \frac{1}{4}\right) \text{ and } \frac{d^2\mu_n}{dr^2}\left(n \pm \frac{1}{4}\right) = 0.$$

We set

$$\eta_3(r) = \begin{cases} \mu_n(r), & n - \frac{1}{4} \leq r \leq n + \frac{1}{4} \qquad (n \in \mathbf{N}), \\ \eta_2(r), & 0 \leq r < \frac{3}{4}, n + \frac{1}{4} < r < n + \frac{3}{4} \quad (n \in \mathbf{N}). \end{cases}$$

Then $\eta_3(r)$ is C^2-function with respect to r for $r \geq 0$.

We now set $\underline{u}_0(x) = \eta_3(|x|)$ and conclude that $\underline{u}_0(x)$ is a C^2 radial function satisfying

$$\lim_{|x| \to \infty} \Delta \underline{u}_0(x) = 0$$

as well as other desired properties. $\square$

Completion of the proof of Theorem 1. For general u_0 satisfying (2) we apply Lemma 2.6 and construct $\underline{u}$ and $\overline{u}$ satisfying both (2) and (23) with $\underline{u}_0 \leq u_0 \leq \overline{u}_0$. Let $\underline{u}$ and $\overline{u}$ be solutions of (1) with initial value $\underline{u}_0$ and $\overline{u}_0$, respectively. Then, by comparison theorem we have $\underline{u}(x,t) \leq u(x,t) \leq \overline{u}(x,t)$ for $(x,t) \in \mathbf{R}^n \times [0, T_v)$ and $\lim_{|x|\to\infty} \underline{u}(x,t) = \lim_{|x|\to\infty} \overline{u}(x,t) = v(t)$. Thus, we have proved $\lim_{|x|\to\infty} u(x,t) = v(t)$ for general initial data u_0 satisfying (2). $\square$

Remark 2.7. (Generalization) Conclusion of Theorem 1 is still valid even if one replaces u^p by a general term $f(u)$ provided that

$$f' > 0 \quad \text{and} \quad \int_1^\infty \frac{ds}{f(s)} < \infty.$$

3 No blow up point in $\mathbf{R}^n$

We first show Theorem 2 for subcritical p, i.e., $1 < p < (n+2)/(n-2)$ or $n \leq 2$ since the supercritical case is more involved.

Proof of Theorem 2. Let v be the solution of (4). By Theorem 1 the solution u of (1) blows up at $T_v = T^*(M,p)$ which is the blow up time of v.

We shall construct a supersolution of (1) which blows up only at space infinity.

Let w be the solution of the heat equation

$$\begin{cases} w_t = \Delta w, & x \in \mathbf{R}^n, t > 0, \\ w(x,0) = u_0(x)/M, & x \in \mathbf{R}^n. \end{cases}$$

Since $u_0(x)/M < 1$ and $u_0 \not\equiv M$, by the strong maximum principle (see [13]), we see that $w(x,t) < 1$ for all $x \in \mathbf{R}^n$, $t \geq 0$.

We set $\bar{u} = vw$ to observe that

$$\begin{cases} \bar{u}_t = \Delta \bar{u} + v^{p-1}\bar{u}, & x \in \mathbf{R}^n, t > 0, \\ \bar{u}(x,0) = u_0(x), & x \in \mathbf{R}^n. \end{cases}$$

Since $w < 1$ so that $v^{p-1}\bar{u} > v^{p-1}w^{p-1}\bar{u} = \bar{u}^p$, we conclude that $\bar{u}$ is a supersolution of (1). By comparison we see that $u \leq \bar{u}$.

Since $w < 1$, we conclude that

$$\limsup_{t \to T_v} u(x,t)v^{-1} \leq w(x,T_v) < 1 \text{ for all } x \in \mathbf{R}^n. \tag{24}$$

For subcritical p, we just apply a criterion for a blow up point established by Giga and Kohn [3, Corollary 4] to (24) and conclude that $x \in \mathbf{R}^n$ is not a blow up point.

We shall discuss the case $p \geq (n+2)/(n-2)$ and $n \geq 3$. By the strong maximum principle [13], we see that $u(x,t) < v(t)$ instantaneously i.e., $u(x,t_0) < v(t_0)$ for $x \in \mathbf{R}^n$ for any $t_0 \in (0,T_v)$.

For $a \in \mathbf{R}^n$ there exists a radially symmetric function $\tilde{u}_0$ (with respect to a) such that $u(x,t_0) \leq \tilde{u}_0(x) < v(t_0)$ and $\lim_{|x|\to\infty} \Delta \tilde{u}_0 \equiv 0$. We may assume $t_0 = 0$ by translation of time. It suffices to prove that the solution $\tilde{u}$ of (1) starting from $\tilde{u}_0$ does not blow up in $\mathbf{R}^n$ since $u \leq \tilde{u}$ and the blow up time of $\tilde{u}$ equals that of v. We start from $\tilde{u}_0$ with $M = v(t_0)$ and construct a supersolution $\bar{u}$ as before.

Thus, we may assume that u_0 is radially symmetric with respect to a. We construct a (radially symmetric) supersolution $\bar{u} = vw$ as before. Fortunately, we have a following criterion for blow up points even for supercritical case at least for radial functions.

Proposition 3.1. *Assume the same hypothesis of Theorem 1. Assume that u is radially symmetric with respect to $a \in \boldsymbol{R}^n$. If $\lim_{t\to T_v} \sup_{|x-a|\leq\delta} u(x,t)v^{-1}(t) < 1$ for $\delta > 0$, then a is not a blow up point.*

The proof of Theorem 2 is now complete by using Proposition 3.1, whose proof is postponed in the next section.

4 A criterion for non blow up point

To prove Proposition 3.1 we recall similarity variables in [5]. We use a fundamental tool that the change of both dependent and independent variables defined by

$$w_a(y,s) = (T^* - t)^\alpha u(a + y\sqrt{T^* - t}, t), \tag{25}$$

where

$$\alpha = \frac{1}{p-1}, \qquad s = -\log(T^* - t), \tag{26}$$

and a is a given point in $\mathbf{R}^n$. One computes that $w = w_a$ solves a rescaled parabolic equation in (y,s)

$$w_s - \Delta w + \frac{y \cdot \nabla w}{2} + \frac{w}{p-1} - w^p = 0 \tag{27}$$

and the blow up time T^* corresponds to $s = \infty$.

The solution

$$v(t) = \alpha^\alpha \frac{1}{(T^* - t)^\alpha},$$

is a supersolution of (4). By comparison with v we have

$$w(y,s) \le \alpha^\alpha. \tag{28}$$

By Theorem 1 we have

$$\lim_{|y|\to\infty} w(y,s) = \alpha^\alpha. \tag{29}$$

By [3, Proposition 1] the bound (28) implies that ∇w and Δw are bounded in $\mathbf{R}^n \times (s_0 + 1, \infty)$, where $s_0 = -\log T_v$.

Lemma 4.1. *Let w_a be defined by* (25). *If w_a is bounded in $\boldsymbol{R}^n \times (-\log T_v, \infty)$, and*

$$\limsup_{\substack{s\to\infty \\ |y|<C}} w_a(y,s) = 0 \tag{30}$$

for each $C > 0$, then a is not a blowup point.

For subcritical p this Lemma 4.1 has been proved in [5, Theorem 4.2] based on [5, Proposition 3.3 and Theorem 3.5] without assuming that w_a is bounded. We are able to remove the restriction $p < (n+2)/(n-2)$ or

$n \leq 2$ by the assumption that w_a is bounded so that ∇w_a is bounded by [3, Proposition 1]. (We shall show this Lemma in detail in Appendix.)

We next prove that all nonnegative stationary solution of (27) satisfying (28) must be a constant at least when w is radial. In other words all nonnegative selfsimilar solution bounded by spatially homogeneous blow up solution must be spatially homogeneous when it is radial.

Lemma 4.2. *Let ψ be a nonnegative solution of*

$$\Delta\psi - \frac{y \cdot \nabla\psi}{2} - \frac{\psi}{p-1} + |\psi|^{p-1}\psi = 0. \tag{31}$$

If ψ is radial with respect to the origin and

$$0 \leq \psi \leq \left(\frac{1}{p-1}\right)^{\frac{1}{p-1}} \tag{32}$$

for every $x \in \boldsymbol{R}^n$, then

$$\psi(y) \equiv 0 \text{ or } \left(\frac{1}{p-1}\right)^{\frac{1}{p-1}}.$$

Proof. Let ψ be a radial solution of (31) so that it depends only on $r = |y|$. We shall denote ψ as a function of r. Evidently, ψ satisfies

$$\psi_{rr} + \left(\frac{n-1}{r} - \frac{r}{2}\right)\psi_r = b(r) \tag{33}$$

with

$$b(r) = \frac{\psi}{p-1} - \psi^p.$$

By assumption (32) we obtain that

$$b(r) \geq 0. \tag{34}$$

If we set

$$\phi(r) = \psi_r(r) \exp\left(\int_{r_0}^{r} \left(\frac{n-1}{s} - \frac{s}{2}\right) ds\right).$$

for $r \geq r_0 > 0$, then

$$\phi_r(r) = b(r) \exp\left(\int_{r_0}^{r} \left(\frac{n-1}{s} - \frac{s}{2}\right) ds\right).$$

Thus, we have

$$\psi_r(r) - \psi_r(r_0) = \exp\left(\int_{r_0}^{r}\left(\frac{n-1}{s} - \frac{s}{2}\right) ds\right) \times \int_{r_0}^{r} b(s) \exp\left(\int_{r_0}^{s}\left(\frac{n-1}{t} - \frac{t}{2}\right) dt\right) ds \tag{35}$$

for $r \geq r_0$. But, since ψ is bounded by (32), the estimates (34) and (35) imply that $b(r) \equiv 0$ for $r \in (0, \infty)$. This yields a contradiction.We thus conclude that

$$\psi_r(r) \equiv 0 \text{ for } r \in [0, \infty)$$

i.e., ψ is constant. Thus we obtain

$$\psi = 0 \text{ or } \left(\frac{1}{p-1}\right)^{\frac{1}{p-1}}. \square$$

Remark 4.3. It is known that all bounded solution of (32) must be a constant if $p \leq (n+2)/(n-2)$ or $n \leq 2$ (See [3]). For supercritical p there may be non constant bounded solution as proved by Troy[14].

Proof of Proposition 3.1. Since $u(x,t) \leq v(x,t)$, we have $w_a \leq \alpha^{\alpha}$. As in [3, Proposition 4] we observe that $w_a(y, s + s_k) \to \phi(y)$ locally uniformly with some ϕ by taking a subsequence $s_k \to \infty$ and that this ϕ must be a radial solution of (31) satisfying (32). From Lemma 4.2 it follows that $\phi \equiv 0$ so that

$$\lim_{s\to\infty} \sup_{|y|<C} w_a(y,s) = 0$$

for each $C > 0$. By Lemma 4.1 we see that a is not a blow up point for u. $\square$

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5 Appendix

In this section we give a detailed proof of Lemma 4.1 for the reader's convenience. Let E be an energy of the form

$$E[w] = \int_{\mathbf{R}^n} \left\{ \frac{|\nabla w|^2}{2} + \frac{|w|^2}{2(p-1)} - \frac{|w|^{p+1}}{p+1} \right\} \rho dy, \tag{36}$$

where

$$\rho(y) = \exp(-|y|^2/4).$$

The next statement is well known [5, Proposition 3.3] if p is subcritical without assuming a bound for w and ∇w. Let $B_R = B_R(0)$ be a closed ball of radius R centered at zero.

Proposition 5.1. *Suppose that w solves (27) on $B_R(0) \times (0,1)$ and w and ∇w are bounded. For any $\eta > 0$, there exist $\delta = \delta(R, n, p, M', \eta) > 0$ such that if*

$$\int_0^1 \int_{B_R} (|\nabla w|^2 + |w_s|^2) dyds + \sup_{0<s<1} \int_{B_R} |w|^2 dy \leq \delta, \tag{37}$$

then

$$|w| \leq \eta \textit{ uniformly} \text{ on } B_R \times (0,1),$$

where M' denotes a bound for $|\nabla w|$.

Proof. The proof is essentially the same as for [5, Proposition 3.3] but we give it for convenience. By the assumption we have

$$|w| \leq M \quad \text{and} \quad |\nabla w| \leq M'. \tag{38}$$

with some $M > 0$. Note that the L^2 norm of w is small uniformly in time by (37). We recall the interpolation inequality

$$\|f\|_{C^\beta(B)} \leq C \left\{ \left(\int_B |\nabla f|^q \right)^{\theta/q} \left(\int_B f^2 \right)^{(1-\theta)/2} + \left(\int_B f^2 \right)^{1/2} \right\}, \tag{39}$$

which holds for $q > n$ and $0 < \beta < 1-(n/q)$ when $\theta \in (0,1)$ is chosen so that $-\beta = (n-q)\theta/q + \frac{1}{2}n(1-\theta)$; see for example [12]. Applying this inequality (39) to $f = w(\cdot, s)$ on $B = B_R$ together with (38), we conclude that

$$|w| \leq C \left\{ (M')^\theta \delta^{(1-\theta)/2} + \delta^{1/2} \right\} \text{ in } B_R \times (0,1).$$

Since this bound tends to zero as δ tends to zero, the proof is complete. $\square$

We now recall a criterion for non blow up points.

Lemma 5.2. ([*5, Theorem 2.1*].) *There is a constant* $\epsilon > 0$ *depending only on* p *and* n *with the following property: for a point* $a \in \boldsymbol{R}^n$, $t_1 \in \boldsymbol{R}_+$ *and* $0 < r \leq 1$,*if* u *solves* (1) *on* $Q_r = B_r(a) \times [t_1 - r^2, t_1]$, *and if*

$$|u(x,t)| \leq \epsilon (t_1 - t)^{-1/(p-1)}$$

for all $(x,t) \in Q_r$, *then* v *does not blow up at* (a, t_1).

The next statement is essentially well known [5, Proposition 3.5] for a sufficient condition yielding (37). In [5, Proposition 3.5] p is assumed to be subcritical, while in the next lemma we do not impose any restriction on p; however, we impose bounds for w and ∇w.

Lemma 5.3. *Suppose that* w *solves* (27) *and* w *and* ∇w *are bounded. There is a constant* $\sigma = \sigma(n,p)$ *such that if* $E[w_a](s_1) < \sigma$ *for some* $s_1 > s_0$, *then* a *is not a blow up point of* u. *The value of* σ *depends only on* n *and* p, *not on* a.

Proof. The proof is essentially the same as in [5, Proposition 3.5] but we give it for convenience. Fix $a \in \mathbf{R}^n$ and suppose that $E[w_a](s_1) < \sigma \leq 1$. Since the weighted energy E depends continuously on a (see [4, Lemma 2.3]), there is a neighborhood N of a such that

$$E[w_b](s_1) < \sigma \text{ for all } b \in N. \tag{40}$$

Applying [5, Proposition 4.1], we have

$$\int_s^{s+1} \int_{B_1(0)} (|w_{bs}|^2 + |\nabla w_b|^2) + \sup_{s \leq r \leq s+1} \int_{B_1(0)} |w_b|^2 \leq C(n,p)\sigma^{1/p} \tag{41}$$

for every $s > s_1$, where $C(n,p)$ depends only on n and p, not on w or b. By Proposition 5.1, for any $\eta > 0$, there exist $\sigma_1 = \sigma_1(n,p,\eta)$ such that if (41) with σ satisfying $\sigma < \sigma_1$ holds, then

$$|w_b(y,s)| \leq \eta \text{ when } b \in N, |y| < 1/4, s \geq s_1. \tag{42}$$

Taking $y = 0$ in (42) and rewriting the result as a statement on u, we have

$$|u(b,t)| \leq \eta (T^* - t)^{-\alpha} \text{ for } b \in N, t_1 < t < T^* \tag{43}$$

with $t_1 = T^* - \exp\{-s_1\}$. Now Lemma 5.2 (or [5, Theorem 2.1]) provides a choice of $\eta = \eta_1(n,p)$ for which (43) rules out blowup points in N. Thus the assertion of the theorem holds for any a provided that $\sigma \leq \sigma_1(n,p,\eta_1(n,p))$. $\square$

Proof of Lemma 4.1. We shall prove that $w = w_a$ satisfies $E[w](s) \to 0$ as $s \to \infty$. Since w, ∇w and Δw are bounded by [3, Proposition 1], we have

$$|w_s(y,s)| \le C(|y|+1) \text{ in } W = \mathbf{R}^n \times (-\log T_v + 1, \infty) \tag{44}$$

with some $C > 0$. Since ∇w is bounded, the estimate (44) yields

$$\iint_W \left(|w_s|^2 + |\nabla w|^2\right)\left(1+|y|^2\right)\rho dy ds < \infty. \tag{45}$$

Since

$$2E[w](s) \le \int_{\mathbf{R}^n} \left(|\nabla w|^2 + \alpha |w|^2\right)\rho dy$$

by (36), and $E[w](s) > 0$ and $(d/ds)E[w](s) \le 0$ by [5, Proposition 2.1] and [4, (2.25)], it suffices to show that

$$\liminf_{s\to\infty} \int_{\mathbf{R}^n} \left(|\nabla w(x,s)|^2 + \alpha|w(x,s)|^2\right)\rho dy = 0. \tag{46}$$

Our hypothesis (30) is equivalent to the statement that

$$w(y,s) \to 0 \text{ as } s \to \infty \text{ uniformly for } |y| \le C. \tag{47}$$

for every $C > 0$. By parabolic regularity theory it follows from (45) and (47) that

$$|\nabla w(y,s)| \to 0 \text{ as } s \to \infty \text{ uniformly for } |y| \le C. \tag{48}$$

(see [4, Lemma 3.3].) By the dominant convergence theorem (47) and (48) yield

$$\lim_{s\to\infty} \int_{|y|\le C} \left(|\nabla w|^2 + |w|^2\right)\rho dy = 0 \tag{49}$$

for any $C > 0$. Since $w(\cdot, s) \in H^1_{\mathrm{loc}}(\mathbf{R}^n)$ for any given s, [5, Lemma 4.1] yields

$$\int_{|y|>1} \rho |w|^2 dy \le C\left\{\int_{|y|>1} \rho |y|^2 |\nabla w|^2 dy + \left(\int_{|y|=1} w d\sigma\right)^2\right\}.$$

The integral over $|y| = 1$ tend to zero as $s \to \infty$, by (47), so we have

$$\begin{aligned}&\liminf_{s\to\infty} \int_{|y|>1} \left(|w|^2 + |\nabla w|^2\right)\rho dy \\ &\quad \le C \liminf_{s\to\infty} \int_{|y|>1} \rho |y|^2 |\nabla w|^2 dy = 0,\end{aligned} \tag{50}$$

by (45). Combining (49) and (50) yields (46). This now implies that $E[w](s) \to 0$ as $s \to \infty$.

We shall apply Lemma 5.3. Let σ be as in Lemma 5.3, and choose s_1 for which $E[w](s_1) < \sigma$ holds. If $s_1 = \log(T - t_1)$, then s_1 is the "initial time in similarity variables" of $u_1(x,t) = u(x, t - t_1)$, which blows up at time $T^* - t_1$. By Lemma 5.3, u_1 does not blow up at a, and so neither does u. $\square$

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