BACKWARD SHIFT INVARIANT SUBSPACES IN THE BIDISC II

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BACKWARD SHIFT INVARIANT SUBSPACES IN THE BIDISC II

KEIJI IZUCHI, TAKAHIKO NAKAZI, and MICHIO SETO

ABSTRACT. For every invariant subspace M in the Hardy spaces $H^2(\Gamma^2)$, let V_z and V_w be mulitplication operators on M. Then it is known that the condition $V_z V_w^* = V_w^* V_z$ on M holds if and only if M is a Beurling type invariant subspace. For a backward shift invariant subspace N in $H^2(\Gamma^2)$, two operators S_z and S_w on N are defined by $S_z = P_N L_z P_N$ and $S_w = P_N L_w P_N$, where P_N is the orthogonal projection from $L^2(\Gamma^2)$ onto N. It is given a characterization of N satisfying $S_z S_w^* = S_w^* S_z$ on N.

KEYWORDS: Backward shift invariant subspaces, the Hardy space in the bidisc

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1. INTRODUCTION

Let Γ^2 be the 2-dimensional unit torus. We denote by $(z, w) = (e^{i\theta}, e^{i\phi})$ the variables in $\Gamma^2 = \Gamma_z \times \Gamma_w$. Let $L^2 = L^2(\Gamma^2)$ be the usual Lebesgue space on Γ^2 with the norm $\|f\|_2 = (\int_{\Gamma^2} |f(e^{i\theta}, e^{i\phi})|^2 d\theta d\phi/(2\pi)^2)^{1/2}$. The space L^2 is a Hilbert space with the usual inner product. For $f \in L^2$, the Fourier coefficients are given by

$$\hat{f}(n,m) = \int_{\Gamma^2} f(e^{i\theta}, e^{i\phi}) e^{-in\theta} e^{-im\phi}\, d\theta d\phi/(2\pi)^2 = \langle f, z^n w^m \rangle.$$

Let $H^2 = H^2(\Gamma^2)$ be the Hardy space on Γ^2, that is,

$$H^2 = \{f \in L^2; \hat{f}(n,m) = 0 \text{ if } n < 0 \text{ or } m < 0\}.$$

For $f \in H^2$, we can write f as

$$f = \sum_{i,j=0}^{\infty} \oplus a_{i,j} z^i w^j, \text{ where } \sum_{i,j=0}^{\infty} |a_{i,j}|^2 < \infty.$$

Let P be the orthogonal projection from L^2 onto H^2. For a closed subspace M of L^2, we denote by P_M the orthogonal projection from

L^2 onto M. For a function $\psi \in L^\infty$, let $L_\psi f = \psi f$ for $f \in L^2$. The Toeplitz operator T_ψ on H^2 is defined by $T_\psi f = PL_\psi f$ for $f \in H^2$. It is well known that $T_\psi^* = T_{\overline{\psi}}$. It holds that $T_{z^n}^* T_{w^m} = T_{w^m} T_{z^n}^*$ for $n, m \geq 1$. A function $f \in H^2$ is called to be inner if $|f| = 1$ on Γ^2 almost everywhere. A closed subspace M of H^2 is called invariant if $zM \subset M$ and $wM \subset M$. In one variable case, an invariant subspace M of $H^2(\Gamma)$ has a form $M = qH^2(\Gamma)$, where q is inner. This is the well known Beurling theorem [2]. In two variable case, the structure of invariant subspaces of H^2 is complicated, see [1, 9, 11].

Let M be an invariant subspace of H^2. Then $T_z^*(H^2 \ominus M) \subset (H^2 \ominus M)$ and $T_w^*(H^2 \ominus M) \subset (H^2 \ominus M)$. We call a closed subspace N of H^2 to be backward shift invariant if $T_z^* N \subset N$ and $T_w^* N \subset N$. If N is a backward shift invariant subspace of H^2, then $H^2 \ominus N$ is invariant. There are studies of backward shift invariant subspaces of the unit circle Γ, see [3, 12].

Let M be an invariant subspace of H^2 and $\psi \in L^\infty$. Let V_ψ be the operator on M defined by $V_\psi = P_M L_\phi|_M$. Then $V_z = T_z$ and $V_z^* = V_{\overline{z}}$ on M. In [8], Mandrekar proved that $V_z V_w^* = V_w^* V_z$ on M holds if and only if M is Beurling type, that is, $M = qH^2$ for some inner function q, see also [4, 9, 10].

In this paper, we study a similar type problem on a backward shift invariant subspace N of H^2. For $\psi \in L^\infty$, put

$$S_\psi = P_N L_\psi|_N \quad \text{on } N.$$

Then we have $S_\psi^* = S_{\overline{\psi}}$ and $S_z^* = T_z^*$ on N. Our purpose is to characterize backwad shift invariant subspaces N which satisfiy the condition $S_z S_w^* = S_w^* S_z$ on N. Recently, this problem is studied in [5, 6]. Our theorem in this paper is the following complete characterization.

THEOREM 2.1. *Let N be a backward shift invariant subspace of H^2 and $N \neq H^2$. Then $S_z S_w^* = S_w^* S_z$ on N holds if and only if N has one of the following forms;*
(i) $N = H^2 \ominus q_1(z)H^2$,
(ii) $N = H^2 \ominus q_2(w)H^2$,
(iii) $N = (H^2 \ominus q_1(z)H^2) \cap (H^2 \ominus q_2(w)H^2)$,
where $q_1(z)$ and $q_2(w)$ are one variable inner functions.

In Section 2, we prove our theorem as a continuation of the study of [6]. In Section 3, we study the above problem from another view point.

Let $H^2(\Gamma_z)$ and $H^2(\Gamma_w)$ be the Hardy spaces on the unit circle Γ in variables z and w, respectively. We think that $H^2(\Gamma_z) \subset H^2$

and $H^2(\Gamma_w) \subset H^2$. In [6], if $S_z S_w^* = S_w^* S_z$ and $N \neq H^2$, then either $(H^2 \ominus N) \cap H^2(\Gamma_z) \neq \{0\}$ or $(H^2 \ominus N) \cap H^2(\Gamma_w) \neq \{0\}$ holds. We prove the following.

THEOREM 3.1. *Let N be a backward shift invariant subspace of H^2 and $M = H^2 \ominus N$. Suppose that $M \cap H^2(\Gamma_z) \neq \{0\}$. Put $M \cap H^2(\Gamma_z) = q_1(z) H^2(\Gamma_z)$, where $q_1(z)$ is an inner function. Put $\tilde{M} = M \ominus q_1(z) H^2$. Then the following conditions are equivalent.*

(i) $S_z S_w^* = S_w^* S_z$.
(ii) $T_z^* \tilde{M} \subset \tilde{M}$.
(iii) *Either $\tilde{M} = \{0\}$ or $\tilde{M} = q_2(w)(H^2 \ominus q_1(z) H^2)$ holds for some inner function $q_2(w) \in H^2(\Gamma_w)$.*
(iv) *Either $M = q_1(z) H^2$ or $M = q_1(z) H^2 + q_2(w) H^2$ holds.*

Theorem 3.1 follows from Theorem 2.1 without so difficulty. We also give a proof of Theorem 3.1 without using Theorem 2.1. Since Theorem 2.1 follows from Theorem 3.1, this means that we give two different type proofs of Theorem 2.1. In the forthcoming paper [7], we study backward shift invariant subspaces N satisfying $S_z S_w^* \neq S_w^* S_z$ and $S_{z^2} S_w^* = S_w^* S_{z^2}^2$. In [7], both ideas will be used effectively.

2. PROOF OF THEOREM 2.1

To prove our theorem, we need some lemmas. The following two lemmas are proved in [6].

LEMMA 2.2. *Let N be a backward shift invariant subspace of H^2 and $M = H^2 \ominus N$. Then the following conditions are equivalent.*

(i) $S_z S_w^* = S_w^* S_z$.
(ii) $S_w S_z^* = S_z^* S_w$.
(iii) $(M \ominus zM) \ominus (M \cap H^2(\Gamma_w)) \subset (M \cap H^2(\Gamma_z)) \oplus wM$.
(iv) $(M \ominus wM) \ominus (M \cap H^2(\Gamma_z)) \subset (M \cap H^2(\Gamma_w)) \oplus zM$.

LEMMA 2.3. *Let N be a backward shift invariant subspace of H^2 such that $N \neq H^2$. Let $M = H^2 \ominus N$. If $S_z S_w^* = S_w^* S_z$ holds, then either $M \cap H^2(\Gamma_z) \neq \{0\}$ or $M \cap H^2(\Gamma_w) \neq \{0\}$ holds.*

LEMMA 2.4. *Let $q_1(z)$ and $q_2(w)$ be one variable inner functions. Then $M = q_1(z) H^2 + q_2(w) H^2$ is an invariant subspace of H^2.*

Proof. We need to prove that M is closed. Since

$$H^2 \ominus q_2(w)H^2 = \sum_{j=0}^{\infty} \oplus z^j \Big(H^2(\Gamma_w) \ominus q_2(w)H^2(\Gamma_w) \Big),$$

$H^2 \ominus q_2(w)H^2$ is z-invariant. Then $q_1(z)(H^2 \ominus q_2(w)H^2) \perp q_2(w)H^2$ and

$$\begin{aligned} M &= q_1(z)H^2 + q_2(w)H^2 \\ &= q_1(z)\Big((H^2 \ominus q_2(w)H^2) \oplus q_2(w)H^2 \Big) + q_2(w)H^2 \\ &= \Big(q_1(z)(H^2 \ominus q_2(w)H^2) \Big) \oplus q_2(w)H^2. \end{aligned}$$

Hence M is closed. □

Proof of Theorem 2.1. Put $M = H^2 \ominus N$. Then M is an invariant subspace. Suppose that (i) holds. Then $M = q_1(z)H^2$, so that $M \ominus wM = q_1(z)H^2(\Gamma_z)$ and $M \cap H^2(\Gamma_z) = q_1(z)H^2(\Gamma_z)$. Hence $(M \ominus wM) \ominus (M \cap H^2(\Gamma_z)) = \{0\}$. By Lemma 2.2, $S_z S_w^* = S_w^* S_z$ holds. Similarly if (ii) holds, then $S_z S_w^* = S_w^* S_z$.

Suppose that (iii) holds. By Lemma 2.4, we have $M = q_1(z)H^2 + q_2(w)H^2$. Then we have

$$q_1(z), q_2(w) \in M. \tag{2.1}$$

If either $q_1(z)$ or $q_2(z)$ is constant, then we have $M = H^2$, so that $N = \{0\}$. In this case, trivially $S_z S_w^* = S_w^* S_z$ holds. Hence we may assume that both of $q_1(z)$ and $q_2(w)$ are not constant functions. We have $M \cap H^2(\Gamma_z) = q_1(z)H^2(\Gamma_z)$, $M \cap H^2(\Gamma_w) = q_2(w)H^2(\Gamma_w)$, and

$$M \ominus zM \subset q_1(z)H^2(\Gamma_w) + q_2(w)H^2(\Gamma_w). \tag{2.2}$$

By Lemma 2.2, it is sufficient to prove

$$(M \ominus zM) \ominus q_2(w)H^2(\Gamma_w) \subset q_1(z)H^2(\Gamma_z) \oplus wM. \tag{2.3}$$

Let

$$f \in (M \ominus zM) \ominus q_2(w)H^2(\Gamma_w). \tag{2.4}$$

Then by (2.2),

$$f = q_1(z)h_1(w) + q_2(w)h_2(w), \quad h_1(w), h_2(w) \in H^2(\Gamma_w). \tag{2.5}$$

By (2.4), $f \perp zM$. Since $q_2(w)h_2(w) \perp zM$, we have

$$q_1(z)h_1(w) \perp z\Big(q_1(z)H^2 + q_2(w)H^2 \Big).$$

Since $q_1(z)h_1(w) \perp zq_1(z)H^2$, we have $q_1(z)h_1(w) \perp zq_2(w)H^2$. Since $q_1(z)$ is not constant, $q_1(z) \not\perp z^n$ for some $n \geq 1$. Since $q_1(z)h_1(w) \perp$

$z^n q_2(w)H^2(\Gamma_w)$, we get $h_1(w) \perp q_2(w)H^2(\Gamma_w)$. Hence $q_1(z)h_1(w) \perp q_2(w)H^2(\Gamma_w)$. By (2.4), $f \perp q_2(w)H^2(\Gamma_w)$. Therefore by (2.5), $q_2(w)h_2(w) \perp q_2(w)H^2(\Gamma_w)$. Thus we get $h_2(w) = 0$. Let $h_1(w) = \hat{h}_1(0) + wh_1'(w)$, where $h_1'(w) \in H^2(\Gamma_w)$. By (2.1), $q_1(z)h_1'(w) \in M$. Hence we get

$$\begin{aligned} f &= q_1(z)h_1(w) \\ &= \hat{h}_1(0)q_1(z) + q_1(z)wh_1'(w) \\ &\in q_1(z)H^2(\Gamma_z) \oplus wM. \end{aligned}$$

Thus (2.3) holds. Therefore $S_z S_w^* = S_w^* S_z$ holds.

Next, we prove the converse assertion. We may assume that $N \neq \{0\}$. Suppose that $S_z S_w^* = S_w^* S_z$. By Lemma 2.3, we may further assume that $M \cap H^2(\Gamma_w) \neq \{0\}$ holds. In this case, we shall prove that N has the form either (ii) or (iii). Similarly, if $M \cap H^2(\Gamma_z) \neq \{0\}$ holds, then we can prove that N has the form either (i) or (iii).

By the Beurling theorem [2],

$$M \cap H^2(\Gamma_w) = q_2(w)H^2(\Gamma_w), \tag{2.6}$$

where $q_2(w)$ is an inner function. By Lemma 2.2,

$$(M \ominus zM) \ominus q_2(w)H^2(\Gamma_w) \subset (M \cap H^2(\Gamma_z)) \oplus wM.$$

Put

$$K_0 = (M \ominus zM) \ominus q_2(w)H^2(\Gamma_w). \tag{2.7}$$

Then

$$K_0 \subset (M \cap H^2(\Gamma_z)) \oplus wM \tag{2.8}$$

and

$$K_0 \perp \Big(zM \oplus q_2(w)H^2(\Gamma_w)\Big). \tag{2.9}$$

We have

$$q_2(w)H^2 = q_2(w)H^2(\Gamma_w) \oplus zq_2(w)H^2. \tag{2.10}$$

By (2.6), we have $q_2(w) \in M$. Then $zq_2(w)H^2 \subset zM$. Hence by (2.9) and (2.10),

$$K_0 \perp q_2(w)H^2. \tag{2.11}$$

We also have

$$q_2(w)H^2 = \sum_{j=0}^{\infty} \oplus z^j q_2(w)H^2(\Gamma_w). \tag{2.12}$$

Then

$$\begin{aligned} M &= \sum_{j=0}^{\infty} \oplus z^j (M \ominus zM) \\ &= \sum_{j=0}^{\infty} \oplus z^j \Big(K_0 \oplus q_2(w) H^2(\Gamma_w) \Big) \qquad \text{by (2.7)} \\ &= \Big(\sum_{j=0}^{\infty} \oplus z^j q_2(w) H^2(\Gamma_w) \Big) \oplus \Big(\sum_{j=0}^{\infty} \oplus z^j K_0 \Big) \\ &= q_2(w) H^2 \oplus \Big(\sum_{j=0}^{\infty} \oplus z^j K_0 \Big) \qquad \text{by (2.12).} \end{aligned}$$

Hence

$$M = q_2(w) H^2 \oplus \Big(\sum_{j=0}^{\infty} \oplus z^j K_0 \Big). \tag{2.13}$$

Since (2.8) holds, it occurs one of the following three cases;

$$K_0 = \{0\}, \quad K_0 \subset wM, \quad \text{and} \quad K_0 \not\subset wM.$$

Case 1. $K_0 = \{0\}$.

In this case, by (2.13) it holds that $M = q_2(w)H^2$. Therefore $N = H^2 \ominus M = H^2 \ominus q_2(w)H^2$. Hence (ii) holds.

Case 2. $K_0 \subset wM$.

In this case, we shall prove that $K_0 = \{0\}$. Let $F \in K_0$. By our assumption of Case 2,

$$F = wf, \quad f \in M. \tag{2.14}$$

We shall prove that

$$f \in K_0. \tag{2.15}$$

We have

$$\begin{aligned} \Big\langle f, q_2(w)H^2 \oplus \sum_{j=1}^{\infty} \oplus z^j K_0 \Big\rangle &= \Big\langle wf, w\Big(q_2(w)H^2 \oplus \sum_{j=1}^{\infty} \oplus z^j K_0 \Big) \Big\rangle \\ &= \Big\langle F, z\Big(\sum_{j=1}^{\infty} \oplus z^{j-1} w K_0 \Big) \Big\rangle \qquad \text{by (2.11) and (2.14)} \\ &= 0. \end{aligned}$$

The last equation follows from the facts

$$z\Big(\sum_{j=1}^{\infty} \oplus z^{j-1} w K_0\Big) \subset zM, \;\; F \in K_0\,, \;\; \text{and} \;\; K_0 \perp zM.$$

Then by (2.13), we have (2.15). Hence $F \in \bigcap_{n=1}^{\infty} w^n K_0$ holds, so that $F = 0$.

Case 3. $K_0 \not\subset wM$.

In this case, by (2.8) it holds that $M \cap H^2(\Gamma_z) \neq \{0\}$. By the Beurling theorem,

(2.16) $$M \cap H^2(\Gamma_z) = q_1(z) H^2(\Gamma_z), \;\; \text{where } q_1(z) \text{ is inner.}$$

By (2.8) again, $K_0 \subset q_1(z) H^2(\Gamma_z) \oplus wM$ holds. Let $G \in K_0$. Then $G = q_1(z) h_0(z) \oplus w h_1$, where $h_0(z) \in H^2(\Gamma_z)$ and $h_1 \in M$. We have

(2.17) $$G = \hat{h}_0(0) q_1(z) \oplus z q_1(z) h_2(z) \oplus w h_1 \;\; \text{for some } h_2(z) \in H^2(\Gamma_z).$$

By (2.16), we have $q_1(z) \in M$. Hence $z q_1(z) h_2(z) \in zM$. Then by (2.9), $G \perp z q_1(z) h_2(z)$ holds. Therefore by (2.17), $z q_1(z) h_2(z) = 0$, so that $G = \hat{h}_0(0) q_1(z) \oplus w h_1$ holds. Thus we get

(2.18) $$G = a_0 q_1(z) \oplus w h_1, \quad h_1 \in M.$$

Here we shall prove that

(2.19) $$h_1 \in K_0.$$

Since $q_2(w) \in M$, $M = q_2(w) H^2 \oplus (M \ominus q_2(w) H^2)$. Put $h_1 = h_1' \oplus h_2'' \in q_2(w) H^2 \oplus (M \ominus q_2(w) H^2)$. Then we have $G = a_0 q_1(z) \oplus w h_1' \oplus w h_2''$. Since $w h_1' \in q_2(w) H^2$, by (2.11) $w h_1' \perp K_0$ holds. Since $G \in K_0$, we have $h_1' = 0$. Thus we get

(2.20) $$h_1 \perp q_2(w) H^2.$$

We have

(2.21) $$q_1(z) \perp w\Big(\sum_{j=1}^{\infty} \oplus z^j K_0\Big).$$

Since $w(\sum_{j=1}^{\infty} \oplus z^j K_0) \subset zM$, $G \in K_0$, and $K_0 \perp zM$, we have

(2.22) $$G \perp w\Big(\sum_{j=1}^{\infty} \oplus z^j K_0\Big).$$

Then we have

$$\begin{aligned} \Big\langle h_1, \sum_{j=1}^{\infty} \oplus z^j K_0 \Big\rangle &= \Big\langle wh_1, w\Big(\sum_{j=1}^{\infty} \oplus z^j K_0\Big)\Big\rangle \\ &= \Big\langle G - a_0 q_1(z), w\Big(\sum_{j=1}^{\infty} \oplus z^j K_0\Big)\Big\rangle \qquad \text{by (2.18)} \\ &= 0 \qquad \text{by (2.21) and (2.22).} \end{aligned}$$

Hence $h_1 \perp \sum_{j=1}^{\infty} \oplus z^j K_0$. Therefore by (2.13) and (2.20), we get (2.19).

Applying (2.18) and (2.19) infinitely many times, we have

$$G = \sum_{j=0}^{\infty} \oplus a_j q_1(z) w^j = q_1(z)\Big(\sum_{j=0}^{\infty} \oplus a_j w^j\Big) \in q_1(z) H^2(\Gamma_w).$$

Hence $K_0 \subset q_1(z)H^2(\Gamma_w)$, so that

$$\sum_{j=0}^{\infty} \oplus z^j K_0 \subset q_1(z) H^2.$$

Therefore by (2.13), $M \subset q_1(z)H^2 + q_2(w)H^2$. By (2.6) and (2.16), we have $q_1(z), q_2(w) \in M$. Then $q_1(z)H^2 + q_2(w)H^2 \subset M$. Thus we get $M = q_1(z)H^2 + q_2(w)H^2$. Hence $N = (H^2 \ominus q_1(z)H^2) \cap (H^2 \ominus q_2(w)H^2)$. □

Corollary 2.5. *Let N be a backward shift invariant subspace of H^2 and $N \neq H^2$. Let $M = H^2 \ominus N$. Then $S_z S_w^* = S_w^* S_z$ holds if and only if M has one of the following forms;*

(i) $M = q_1(z)H^2$,
(ii) $M = q_2(w)H^2$,
(iii) $M = q_1(z)H^2 + q_2(w)H^2$,

where $q_1(z)$ and $q_2(w)$ are one variable inner functions.

3. Another proof of Theorem 2.1

Let N be a backward shift invariant subspace of H^2 and $M = H^2 \ominus N$. Then M is an invariant subspace. Let $q_1(z)$ be an inner function in $H^2(\Gamma_z)$. In this section, we assume that

$$q_1(z)H^2 \subset M \quad \text{and} \quad M \cap H^2(\Gamma_z) = q_1(z)H^2(\Gamma_z). \tag{3.1}$$

Then $q_1(z)H^2 \subset M$. Put

$$\tilde{M} = M \ominus q_1(z)H^2. \tag{3.2}$$

Then

$$H^2 \ominus q_1(z)H^2 = \tilde{M} \oplus N \tag{3.3}$$

and $\tilde{M}$ is w-invariant. The following is the main theorem in this section.

THEOREM 3.1. *Let N be a backward shift invariant subspace of H^2 and $M = H^2 \ominus N$. Suppose that $M \cap H^2(\Gamma_z) \neq \{0\}$. Put $M \cap H^2(\Gamma_z) = q_1(z)H^2(\Gamma_z)$, where $q_1(z)$ is an inner function. Put $\tilde{M} = M \ominus q_1(z)H^2$. Then the following conditions are equivalent.*

(i) $S_z S_w^* = S_w^* S_z$.
(ii) $T_z^* \tilde{M} \subset \tilde{M}$.
(iii) *Either $\tilde{M} = \{0\}$ or $\tilde{M} = q_2(w)(H^2 \ominus q_1(z)H^2)$ holds for some inner function $q_2(w) \in H^2(\Gamma_w)$.*
(iv) *Either $M = q_1(z)H^2$ or $M = q_1(z)H^2 + q_2(w)H^2$ holds.*

To prove our theorem, we need to study the properties of $\tilde{M}$.

LEMMA 3.2. *Let $f \in \tilde{M}$. Then we have the following.*

(i) *$T_w^* f \in \tilde{M}$ if and only if $f \in w\tilde{M}$.*
(ii) *$T_w^* f \perp \tilde{M}$ if and only if $f \in \tilde{M} \ominus w\tilde{M}$.*

Proof. (i) Suppose that $T_w^* f \in \tilde{M}$. Put

$$f = \sum_{j=0}^{\infty} \oplus w^j f_j(z), \quad f_j(z) \in H^2(\Gamma_z). \tag{3.4}$$

Then

$$\sum_{j=1}^{\infty} \oplus w^{j-1} f_j(z) \in \tilde{M}. \tag{3.5}$$

Since $w\tilde{M} \subset \tilde{M}$, it holds that $\sum_{j=1}^{\infty} \oplus w^j f_j(z) \in \tilde{M}$. By (3.4), we have $f_0(z) \in \tilde{M}$. Then by (3.1),

$$f_0(z) \in \tilde{M} \cap H^2(\Gamma_z) \subset M \cap H^2(\Gamma_z) = q_1(z)H^2(\Gamma_z).$$

Then by (3.2), $f_0(z) \perp \tilde{M}$. Thus we get $f_0(z) = 0$. Hence, by (3.4) and (3.5), $f \in w\tilde{M}$ holds. The converse is trivial.

(ii) follows from the fact that, $T_w^* f \perp \tilde{M}$ if and only if $f \perp w\tilde{M}$. □

We denote by $P_\perp$ the orthogonal projection from H^2 onto $H^2 \ominus q_1(z)H^2$. Then we have a Toeplitz type operator Q_z on $H^2 \ominus q_1(z)H^2$

such that

$$Q_z : H^2 \ominus q_1(z)H^2 \ni f \to P_\perp(T_z f) \in H^2 \ominus q_1(z)H^2. \tag{3.6}$$

Since $zM \subset M$, by (3.2), it holds that $Q_z\tilde{M} \subset \tilde{M}$. By (3.3), Q_z has the following matrix form;

$$Q_z = \begin{pmatrix} * & P_{\tilde{M}}T_z|_N \\ 0 & S_z \end{pmatrix} \quad \text{on} \quad H^2 \ominus q_1(z)H^2 = \begin{pmatrix} \tilde{M} \\ \oplus \\ N \end{pmatrix}. \tag{3.7}$$

Since $H^2 \ominus q_1(z)H^2$ is backward shift invariant, it holds that $T_w^*(H^2 \ominus q_1(z)H^2) \subset H^2 \ominus q_1(z)H^2$. Since $T_w^*N \subset N$, the operator T_w^* on $H^2 \ominus q_1(z)H^2$ has the following matrix form;

$$T_w^* = \begin{pmatrix} * & 0 \\ P_N T_w^*|_{\tilde{M}} & S_w^* \end{pmatrix} \quad \text{on} \quad H^2 \ominus q_1(z)H^2 = \begin{pmatrix} \tilde{M} \\ \oplus \\ N \end{pmatrix}. \tag{3.8}$$

Put

$$A = P_{\tilde{M}}T_z|_N \quad \text{and} \quad B = P_N T_w^*|_{\tilde{M}}. \tag{3.9}$$

LEMMA 3.3. *We have the following.*
(i) $T_w^*Q_z = Q_zT_w^*$ *on* $H^2 \ominus q_1(z)H^2$.
(ii) $T_wQ_z = Q_zT_w$ *on* $H^2 \ominus q_1(z)H^2$.

Proof. Let $f \in H^2 \ominus q_1(z)H^2$. Put

$$zf = f_1 \oplus f_2 \in (H^2 \ominus q_1(z)H^2) \oplus q_1(z)H^2. \tag{3.10}$$

Then $Q_zf = f_1$. Hence $T_w^*Q_zf = T_w^*f_1$. On the other hand, by (3.10) we have

$$zT_w^*f = T_w^*zf = T_w^*f_1 + T_w^*f_2.$$

Since $T_w^*q_1(z)H^2 \subset q_1(z)H^2$, it holds that $T_w^*f_2 \in q_1(z)H^2$. Since $T_w^*f_1 \in H^2 \ominus q_1(z)H^2$, by the above we have $Q_zT_w^*f = T_w^*f_1$. Thus we get $T_w^*Q_z = Q_zT_w^*$.

Since $T_w(H^2 \ominus q_1(z)H^2) \subset H^2 \ominus q_1(z)H^2$, similarly we have $T_wQ_z = Q_zT_w$ on $H^2 \ominus q_1(z)H^2$. □

LEMMA 3.4. $S_zS_w^* = S_w^*S_z$ *holds if and only if* $BA = 0$.

Proof. By Lemma 3.3(i), $T_w^*Q_z = Q_zT_w^*$ on $H^2 \ominus q_1(z)H^2$. Then by (3.7) and (3.8), we have $BA + S_w^*S_z = S_zS_w^*$. Then $S_zS_w^* = S_w^*S_z$ if and only if $BA = 0$. □

THEOREM 3.5. *Let N be a backward shift invariant subspace of H^2 and $M = H^2 \ominus N$. Suppose that $M \cap H^2(\Gamma_z) \neq \{0\}$. Put $M \cap H^2(\Gamma_z) = q_1(z)H^2(\Gamma_z)$, where $q_1(z)$ is a one variable inner function. Put $\tilde{M} = M \ominus q_1(z)H^2$. Then the following conditions are equivalent.*

(i) $S_z S_w^* = S_w^* S_z$.

(ii) $\tilde{M} \ominus \{f \in \tilde{M}; T_z^* f \in \tilde{M}\} \subset w\tilde{M}$.

(iii) $T_z^* \tilde{M} \subset \tilde{M}$.

Proof. (i) $\Leftrightarrow$ (ii) By Lemma 3.4, condition (i) is equivalent to $BA = 0$. By (3.3), (3.9), and Lemma 3.2(i), we have that

$$\ker B = \{f \in \tilde{M}; T_w^* f \in \tilde{M}\} = w\tilde{M}.$$

We denote by [ran A] the closed range of A. Let $A_1 = P_{\tilde{M}} T_z P_N$ on $\tilde{M} \oplus N$. Then we have [ran A] = [ran A_1]. Since $A_1^* = P_N T_z^* P_{\tilde{M}}$, we have

$$\ker A_1^* = N \oplus \{f \in \tilde{M}; T_z^* f \in \tilde{M}\}.$$

Then

$$[\operatorname{ran} A] = [\operatorname{ran} A_1] = (\tilde{M} \oplus N) \ominus \ker A_1^* = \tilde{M} \ominus \{f \in \tilde{M}; T_z^* f \in \tilde{M}\}.$$

Therefore it holds that $BA = 0$ if and only if

$$\tilde{M} \ominus \{f \in \tilde{M}; T_z^* f \in \tilde{M}\} \subset w\tilde{M}.$$

Thus we get (i) $\Leftrightarrow$ (ii).

(ii) $\Rightarrow$ (iii) Suppose that

$$\tilde{M} \ominus \{f \in \tilde{M}; T_z^* f \in \tilde{M}\} \subset w\tilde{M}. \tag{3.11}$$

Since $\{f \in \tilde{M}; T_z^* f \in \tilde{M}\}$ is a closed subspace, by (3.11) we have

$$\tilde{M} \ominus w\tilde{M} \subset \{f \in \tilde{M}; T_z^* f \in \tilde{M}\}. \tag{3.12}$$

Since $w\tilde{M} \subset \tilde{M}$, we have

$$\tilde{M} = \sum_{j=0}^{\infty} \oplus w^j (\tilde{M} \ominus w\tilde{M}). \tag{3.13}$$

To prove (iii), let $f \in \tilde{M}$. Then by (3.13),

$$f = \sum_{j=0}^{\infty} w^j g_j, \quad \text{where } g_j \in \tilde{M} \ominus w\tilde{M}.$$

Since $T_z^* T_w = T_w T_z^*$ on H^2, by (3.12) we have

$$T_z^* f = \sum_{j=0}^{\infty} w^j T_z^* g_j \in \tilde{M}.$$

(iii) $\Rightarrow$ (ii) is trivial. $\square$

For a one variable inner function $q(z)$, put $q^*(z) = \overline{z}(q(z) - \hat{q}(0))$.

LEMMA 3.6. *Let $q_1(z)$ and $q_2(z)$ be inner functions. Then we have the following.*

(i) $T_z^* q_1(z) = q_1^*(z)$ *and* $q_1^*(z) \perp q_1(z)H^2(\Gamma_z)$.

(ii) *If* $q_1(z)H^2(\Gamma_z) \subsetneqq q_2(z)H^2(\Gamma_z)$, *then the smallest closed* T_z^**-invariant subspace of* $H^2(\Gamma_z)$ *containing* $q_2(z)H^2(\Gamma_z) \ominus q_1(z)H^2(\Gamma_z)$ *equals to* $H^2(\Gamma_z) \ominus q_1(z)H^2(\Gamma_z)$.

(iii) *The closed subspace generated by* $T_z^{*n} q_1^*(z), n = 0, 1, 2, \ldots,$ *equals to* $H^2(\Gamma_z) \ominus q_1(z)H^2(\Gamma_z)$.

Proof. (i) Trivially $T_z^* q_1(z) = q_1^*(z)$ holds. For $h \in H^2(\Gamma_z)$, we have

$$\langle q_1^*(z), q_1(z)h \rangle = \langle T_z^* q_1(z), q_1(z)h \rangle = \langle q_1(z), z q_1(z)h \rangle = \langle 1, zh \rangle = 0.$$

Thus we get (i).

(ii) Let L be the smallest backward shift invariant subspace of $H^2(\Gamma_z)$ containing $q_2(z)H^2(\Gamma_z) \ominus q_1(z)H^2(\Gamma_z)$. Then $L \subset H^2(\Gamma_z) \ominus q_1(z)H^2(\Gamma_z)$. Let $f \in H^2(\Gamma_z) \ominus q_1(z)H^2(\Gamma_z)$ such that $f \perp L$. Since $H^2(\Gamma_z) \ominus L$ is invariant, $z^k f \perp L$ for $k \geq 0$. Hence

$$z^k f \perp q_2(z)H^2(\Gamma_z) \ominus q_1(z)H^2(\Gamma_z) \quad \text{for every } k \geq 0.$$

Since $q_2(z)H^2(\Gamma_z) \subset L \oplus q_1(z)H^2(\Gamma_z)$, we have $f \perp q_2(z)H^2(\Gamma_z)$. Hence

$$f \perp z^n \Big(q_2(z)H^2(\Gamma_z) \ominus q_1(z)H^2(\Gamma_z) \Big) \quad \text{for every } k \geq 0.$$

Since $q_2(z)H^2(\Gamma_z) \ominus q_1(z)H^2(\Gamma_z) \neq \{0\}$, we have $f = 0$. Thus we get $L = H^2(\Gamma_z) \ominus q_1(z)H^2(\Gamma_z)$.

(iii) Let E be the closed subspace generated by $T_z^{*n} q_1^*(z), n \geq 0$. By (i), $E \subset H^2(\Gamma_z) \ominus q_1(z)H^2(\Gamma_z)$ and E is a backward shift invariant subspace of $H^2(\Gamma_z)$. Then $H^2(\Gamma_z) \ominus E = q_3(z)H^2(\Gamma_z)$ for some inner function $q_3(z)$ and $q_1(z)H^2(\Gamma_z) \subset q_3(z)H^2(\Gamma_z)$. When $q_1(z)H^2(\Gamma_z) = q_3(z)H^2(\Gamma_z)$, our assertion holds.

Suppose that $q_1(z)H^2(\Gamma_z) \subsetneqq q_3(z)H^2(\Gamma_z)$. Put $q_4(z) = q_1(z)/q_3(z)$. Then $q_4(z)$ is a nonconstant inner function, and $q_1^*(z) = q_3(z)q_4^*(z) + \hat{q}_4(0)q_3^*(z)$. We have $q_4^*(z) \neq 0$, so that $q_3(z)q_4^*(z) \not\perp q_3(z)H^2(\Gamma_z)$. By (i), $q_3^*(z) \perp q_3(z)H^2(\Gamma_z)$. Hence $q_1^*(z) \not\perp q_3(z)H^2(\Gamma_z)$. Since $q_1^*(z) \in E$, $E \not\perp q_3(z)H^2(\Gamma_z)$. This is a contradiction. Hence we get our assertion. $\square$

Proof of Theorem 3.1. First, we shall prove our theorem using Corollary 2.5 and Theorem 3.5.

(i) $\Leftrightarrow$ (ii) follows from Theorem 3.5.

(i) $\Rightarrow$ (iv) follows from Corollary 2.5.

(iv) $\Leftrightarrow$ (iii) If $M = q_1(z)H^2$, then $\tilde{M} = \{0\}$. Suppose that $M = q_1(z)H^2 + q_2(w)H^2$. Then

$$\begin{aligned} M &= q_1(z)H^2 + q_2(w)\Big(q_1(z)H^2 \oplus (H^2 \ominus q_1(z)H^2)\Big) \\ &= q_1(z)H^2 + q_2(w)(H^2 \ominus q_1(z)H^2). \end{aligned}$$

Since $H^2 \ominus q_1(z)H^2$ is w-invariant, we have

$$M = q_1(z)H^2 \oplus q_2(w)(H^2 \ominus q_1(z)H^2).$$

Thus we get $\tilde{M} = q_2(w)(H^2 \ominus q_1(z)H^2)$.

The converse assertion is not difficult to prove.

(iii) $\Rightarrow$ (ii) is not difficult to prove.

Here we give another proof of (ii) $\Rightarrow$ (iii) without using Corollary 2.5. We may assume that $\tilde{M} \neq \{0\}$. By condition (ii), we have $T_z^*\tilde{M} \subset \tilde{M}$. Then $T_z^*\tilde{M} \perp N$, so that $\tilde{M} \perp zN$. Hence by (3.3) and (3.6),

$$Q_z N \subset N. \tag{3.14}$$

Since $\tilde{M} \neq \{0\}$ and $w\tilde{M} \subset \tilde{M}$, $\tilde{M} \ominus w\tilde{M} \neq \{0\}$ holds. Let $f \in \tilde{M} \ominus w\tilde{M}$. Then by (3.3) and Lemma 3.2(ii), we have $T_w^* f \in N$. Hence $T_w^* T_z^* f = T_z^* T_w^* f \in N$. Since $T_z^*\tilde{M} \subset \tilde{M}$, $T_z^* f \in \tilde{M}$ holds. Hence by Lemma 3.2(ii) again, $T_z^* f \in \tilde{M} \ominus w\tilde{M}$ holds. Thus we get

$$T_z^*(\tilde{M} \ominus w\tilde{M}) \subset \tilde{M} \ominus w\tilde{M}. \tag{3.15}$$

By (3.2), we have $f \in M$ and $zf = f_1 + f_2 \in \tilde{M} \oplus q_1(z)H^2$. Then by (3.6), we have $Q_z f = f_1 \in \tilde{M}$. Since $T_w^* f \in N$, by (3.14) and Lemma 3.3(i) we have $T_w^* Q_z f = Q_z T_w^* f \in N$. Then by (3.3) and Lemma 3.2(ii), $Q_z f \in \tilde{M} \ominus w\tilde{M}$ holds. Thus we get

$$Q_z(\tilde{M} \ominus w\tilde{M}) \subset \tilde{M} \ominus w\tilde{M}. \tag{3.16}$$

We define the operator W_z on $\tilde{M}$ to $q_1(z)H^2$ by

$$W_z = P_{q_1(z)H^2} T_z = T_z - Q_z. \tag{3.17}$$

Then by Lemma 3.3(ii),

$$W_z T_w = T_w W_z \quad \text{on } \tilde{M}. \tag{3.18}$$

Then $wW_z\tilde{M} = W_z(w\tilde{M}) \subset W_z\tilde{M}$. Hence we get

$$w\overline{W_z\tilde{M}} \subset \overline{W_z\tilde{M}}, \tag{3.19}$$

where $\overline{W_z\tilde{M}}$ is the norm closure of the space $W_z\tilde{M}$. Since $\tilde{M} \perp q_1(z)H^2$, $z\tilde{M} \perp zq_1(z)H^2$ holds. Then by (3.17), we obtain

$$W_z\tilde{M} \subset q_1(z)H^2 \ominus zq_1(z)H^2 = q_1(z)H^2(\Gamma_w).$$

Hence $\overline{q}_1(z)\overline{W_z\tilde{M}} \subset H^2(\Gamma_w)$, so that by (3.19) and the Beurling theorem,

$$\overline{q}_1(z)\overline{W_z\tilde{M}} = q_2(w)H^2(\Gamma_w) \tag{3.20}$$

for some inner function $q_2(w)$.

Let $f \in \tilde{M} \ominus w\tilde{M}$ and $g \in \tilde{M}$. Since $Q_z\tilde{M} \subset \tilde{M}$, by Lemma 3.3(ii) we have $Q_zw\tilde{M} \subset w\tilde{M}$. Then by (3.16), $Q_zf \perp Q_zwg$ holds. Since $zf \perp zwg$, by (3.17) we have

$$0 = \langle zf, zwg\rangle = \langle Q_zf \oplus W_zf, Q_zwg \oplus W_zwg\rangle = \langle W_zf, W_zwg\rangle.$$

Then $W_z(\tilde{M} \ominus w\tilde{M}) \perp W_z(w\tilde{M})$. Hence by (3.18), we get

$$W_z(\tilde{M} \ominus w\tilde{M}) \perp w\overline{W_z\tilde{M}}.$$

Therefore by (3.20), we obtain

$$W_z(\tilde{M} \ominus w\tilde{M}) \subset \overline{W_z\tilde{M}} \ominus w\overline{W_z\tilde{M}} = [q_1(z)q_2(w)],$$

where $[q_1(z)q_2(w)]$ is the linear span of a function $q_1(z)q_2(w)$. If $W_z(\tilde{M} \ominus w\tilde{M}) = \{0\}$, by (3.16) and (3.17) it holds that $z(\tilde{M} \ominus w\tilde{M}) \subset \tilde{M} \ominus w\tilde{M}$. Then $z^n(\tilde{M} \ominus w\tilde{M}) \subset \tilde{M} \ominus w\tilde{M}$ for every positive integer n. Since $\tilde{M} \ominus w\tilde{M} \neq \{0\}$, we have that $z^n(\tilde{M} \ominus w\tilde{M}) \not\perp q_1(z)H^2$ for some n. These contradict with (3.2). Thus there exists f_0 in $\tilde{M} \ominus w\tilde{M}$ such that

$$W_zf_0 = aq_1(z)q_2(w) \quad \text{and} \quad a \neq 0. \tag{3.21}$$

Since $zf_0 = Q_zf_0 + W_zf_0$, we have

$$\begin{aligned} f_0 &= T_z^*Q_zf_0 + T_z^*W_zf_0 \\ &= T_z^*Q_zf_0 + aq_1^*(z)q_2(w) \qquad \text{by (3.21) and Lemma 3.6(i).} \end{aligned}$$

Hence by (3.15) and (3.16), it holds that $q_1^*(z)q_2(w) \in \tilde{M} \ominus w\tilde{M}, n \geq 0$. By Lemma 3.6(iii), we obtain

$$q_2(w)\Big(H^2(\Gamma_z) \ominus q_1(z)H^2(\Gamma_z)\Big) \subset \tilde{M} \ominus w\tilde{M}. \tag{3.22}$$

We shall prove that

$$\tilde{M} \ominus w\tilde{M} = q_2(w)\Big(H^2(\Gamma_z) \ominus q_1(z)H^2(\Gamma_z)\Big). \tag{3.23}$$

Let

$$F \in (\tilde{M} \ominus w\tilde{M}) \ominus q_2(w)\Big(H^2(\Gamma_z) \ominus q_1(z)H^2(\Gamma_z)\Big). \tag{3.24}$$

Let i, j be nonnegative integers. Since $q_2(w)(H^2(\Gamma_z) \ominus q_1(z)H^2(\Gamma_z))$ is invariant for the operator T_z^*,

$$T_z^{*i}\Big(q_2(w)\Big(H^2(\Gamma_z) \ominus q_1(z)H^2(\Gamma_z)\Big)\Big) \in q_2(w)\Big(H^2(\Gamma_z) \ominus q_1(z)H^2(\Gamma_z)\Big).$$

Since $\tilde{M} \ominus w\tilde{M} \perp w^n(\tilde{M} \ominus w\tilde{M})$ for every positive integer n, by (3.22) and (3.24) we have

$$w^j F \perp T_z^{*i}\Big(q_2(w)\Big(H^2(\Gamma_z) \ominus q_1(z)H^2(\Gamma_z)\Big)\Big)$$

and

$$F \perp w^j T_z^{*i}\Big(q_2(w)\Big(H^2(\Gamma_z) \ominus q_1(z)H^2(\Gamma_z)\Big)\Big).$$

Hence

$$w^j F \perp \overline{z}^i q_2(w)\Big(H^2(\Gamma_z) \ominus q_1(z)H^2(\Gamma_z)\Big) \tag{3.25}$$

and

$$F \perp \overline{z}^i w^j q_2(w)\Big(H^2(\Gamma_z) \ominus q_1(z)H^2(\Gamma_z)\Big). \tag{3.26}$$

Since $q_2(w)(H^2(\Gamma_z) \ominus q_1(z)H^2(\Gamma_z))$ is invariant for the operator Q_z, similarly we have

$$w^j F \perp Q_z^i\Big(q_2(w)\Big(H^2(\Gamma_z) \ominus q_1(z)H^2(\Gamma_z)\Big)\Big) \tag{3.27}$$

and

$$F \perp w^j Q_z^i\Big(q_2(w)\Big(H^2(\Gamma_z) \ominus q_1(z)H^2(\Gamma_z)\Big)\Big). \tag{3.28}$$

By (3.6),

$$Q_z^i\Big(q_2(w)\Big(H^2(\Gamma_z)\ominus q_1(z)H^2(\Gamma_z)\Big)\Big) = P_\perp\Big(z^i q_2(w)\Big(H^2(\Gamma_z)\ominus q_1(z)H^2(\Gamma_z)\Big)\Big).$$

Since $\tilde{M} \perp q_1(z)H^2$ and $w^j F \in \tilde{M}$, by (3.27) and (3.28) we have

$$w^j F \perp z^i q_2(w)\Big(H^2(\Gamma_z) \ominus q_1(z)H^2(\Gamma_z)\Big) \tag{3.29}$$

and

$$F \perp z^i w^j q_2(w)\Big(H^2(\Gamma_z) \ominus q_1(z)H^2(\Gamma_z)\Big). \tag{3.30}$$

Since $\tilde{M} \neq \{0\}$, by (3.2) $q_1(z)$ is not constant. Hence $H^2(\Gamma_z) \ominus q_1(z)H^2(\Gamma_z) \neq \{0\}$. Therefore by (3.25), (3.26), (3.29), and (3.30), we get $F = 0$. Thus we get (3.23).

By (3.23), we obtain

$$\tilde{M} = \sum_{j=0}^{\infty} \oplus w^j(\tilde{M} \ominus w\tilde{M}) = q_2(w)(H^2 \ominus q_1(z)H^2).$$

□

The following is interesting enough in its own right.

COROLLARY 3.7. *Let $q_1(z)$ be a nonconstant inner function. Let L be a closed subspace of $H^2 \ominus q_1(z)H^2$ and $L \neq \{0\}$. Suppose that $wL \subset L$, $Q_zL \subset L$, and $Q_z^*L \subset L$. Then there exists an inner function $q_2(w)$ such that $L = q_2(w)(H^2 \ominus q_1(z)H^2)$.*

Proof. We note that $Q_z^* = T_z^*$ on $H^2 \ominus q_1(z)H^2$. Put $M = L \oplus q_1(z)H^2$. Then by our assumption, M is an invariant subspace and $q_1(z)H^2(\Gamma_z) \subset M \cap H^2(\Gamma_z)$. Put $M \cap H^2(\Gamma_z) = q_3(z)H^2(\Gamma_z)$, where $q_3(z)$ is inner. Then $q_1(z)H^2(\Gamma_z) \subset q_3(z)H^2(\Gamma_z)$.

Suppose that $q_1(z)H^2(\Gamma_z) \neq q_3(z)H^2(\Gamma_z)$. Let L_1 be the smallest closed subspace of $H^2(\Gamma_z) \ominus q_1(z)H^2(\Gamma_z)$ containing $q_3(z)H^2(\Gamma_z) \ominus q_1(z)H^2(\Gamma_z)$ such that $T_z^*L_1 \subset L_1$. By Lemma 3.6(ii), $L_1 = H^2(\Gamma_z) \ominus q_1(z)H^2(\Gamma_z)$. Since $M \cap H^2(\Gamma_z) = q_3(z)H^2(\Gamma_z)$,

$$q_3(z)H^2(\Gamma_z) \ominus q_1(z)H^2(\Gamma_z) \subset L.$$

Since $T_z^*L = Q_z^*L \subset L$, we have $L_1 \subset L$. Hence we have

$$H^2 \ominus q_1(z)H^2 = \sum_{j=0}^{\infty} \oplus w^j L_1 \subset L \subset H^2 \ominus q_1(z)H^2.$$

Therefore $L = H^2 \ominus q_1(z)H^2$. Thus we get our assertion.

Suppose that $q_1(z)H^2(\Gamma_z) = q_3(z)H^2(\Gamma_z)$. We have $L = M \ominus q_1(z)H^2$. By our assumption, $T_z^*L = Q_z^*L \subset L$. Then by Theorem 3.1, we have $L = q_2(w)(H^2 \ominus q_1(z)H^2)$ for an inner function $q_2(w)$. □

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