



Title	Backward shift invariant subspaces in the bidisc
Author(s)	Izuchi, K.; Nakazi, T.
Citation	Hokkaido University Preprint Series in Mathematics, 572, 1-8
Issue Date	2002-11
DOI	10.14943/83717
Doc URL	<a href="http://hdl.handle.net/2115/69321">http://hdl.handle.net/2115/69321</a>
Type	bulletin (article)
File Information	pre572.pdf



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Series #572. November 2002

HOKKAIDO UNIVERSITY  
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# Backward Shift Invariant Subspaces in the Bidisc

by

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\*This research was partially supported by Grant-in-Aid for Scientific Research, Ministry of Education

2000 Mathematics Subject Classification : Primary 47 A 15, 46 J 15, Secondary 47 A 20.

Key words and phrases : bidisc, Hardy space, backward shift, invariant subspace, double commuting

**Abstract.** Suppose that  $T_\phi$  is a Toeplitz operator with a symbol  $\phi$  on the Hardy space  $H^2$  on the bidisc. Let  $N$  be a backward shift invariant subspace of  $H^2$ , that is,  $N$  is an invariant subspace under  $T_z^*$  and  $T_w^*$ . Let  $P$  be the orthogonal projection from  $H^2$  onto  $N$ . For  $\phi$  in  $H^\infty$ , put  $S_\phi = PT_\phi|N$ . In this paper, we give a characterization of a backward shift invariant subspace which satisfies  $S_z S_w^* = S_w^* S_z$ .

## §1. Introduction

Let  $T^2$  be the torus that is the Cartesian product of two unit circles  $T$  in  $\mathbf{C}$ . Let  $p = 1$  or  $p = \infty$ . The usual Lebesgue spaces, with respect to the Haar measure  $m$  on  $T^2$ , are denoted by  $L^p = L^p(T^2)$ , and  $H^p = H^p(T^2)$  is the space of all  $f$  in  $L^p$  whose Fourier coefficients

$$\hat{f}(j, \ell) = \int_{T^2} f(z, w) \bar{z}^j \bar{w}^\ell dm(z, w)$$

are 0 as soon as at least one component of  $(j, \ell)$  is negative. Then  $H^p$  is called the Hardy space. As  $T^2 = (z, T) \times (w, T)$ ,  $H^p(z, T)$  and  $H^p(w, T)$  denote the one variable Hardy spaces.

Let  $P_{H^2}$  be the orthogonal projection from  $L^2$  onto  $H^2$ . For  $\phi$  in  $L^\infty$ , the Toeplitz operator  $T_\phi$  is defined by

$$T_\phi f = P_{H^2}(\phi f) \quad (f \in H^2).$$

A closed subspace  $N$  of  $H^2$  is said to be backward shift invariant if

$$T_z^* N \subset N \quad \text{and} \quad T_w^* N \subset N.$$

A closed subspace  $M$  of  $H^2$  is said to be shift invariant if  $T_z M \subset M$  and  $T_w M \subset M$ . The orthogonal complement of  $N$  is shift invariant. Let  $P_N$  and  $P_M$  be the orthogonal projections from  $H^2$  onto  $N$  and  $M$ , respectively. For  $\phi$  in  $H^\infty$ , put

$$S_\phi = P_N T_\phi P_N|N \quad \text{and} \quad V_\phi = P_M T_\phi P_M|M.$$

It is known in [2] that  $V_z V_w^* = V_w^* V_z$  if and only if  $M = qH^2$  for some inner function  $q$  in  $H^\infty$ . In this paper, we are interested in backward shift invariant subspaces  $N$  which satisfy  $S_z S_w^* = S_w^* S_z$ . Let  $M = H^2 \ominus N$ . We will write  $P = P_N$  and  $Q = I - P_N$ , where  $I$  is the identity operator on  $H^2$ . In this paper, we also study two operators

$$A = QT_z P \quad \text{and} \quad B = PT_w^* Q.$$

In §2, we show that  $AB|M = V_w^* V_z - V_z V_w^*$  and  $BA|N = S_z S_w^* - S_w^* S_z$ . Then  $AB = 0$  is equivalent to  $V_z V_w^* = V_w^* V_z$ , and  $BA = 0$  is equivalent to  $S_z S_w^* = S_w^* S_z$ . Moreover we determine backward shift invariant subspaces satisfying  $A = 0$  or  $B = 0$ . In §3, we give a characterization of backward shift invariant subspaces satisfying  $BA = 0$ , equivalently

$S_z S_w^* = S_w^* S_z$ . And we give simple sufficient conditions to be  $S_z S_w^* = S_w^* S_z$ . In §4, we give a conjecture, that is, the sufficient condition is also necessary one.

Throughout this paper, for a subset  $H$  of  $H^2$ ,  $[H]_2$  denotes the closed linear span of  $H$  and  $[H]$  the linear span of  $H$ .

## §2. Invariant subspace with $A = 0$ or $B = 0$

Let  $N$  be a backward shift invariant subspace and  $M$  be the orthogonal complement of  $N$  in  $H^2$ . Put  $P = P_N$  and  $Q = I - P_N$ , then  $Q$  is the orthogonal projection from  $H^2$  onto  $M$ .

### Lemma 2.1.

- (1)  $AB = QT_w^* QT_z Q - QT_z QT_w^* Q$  and so  $AB|M = V_w^* V_z - V_z V_w^*$ .
- (2)  $BA = PT_z PT_w^* P - PT_w^* PT_z P$  and so  $BA|N = S_z S_w^* - S_w^* S_z$
- (3)  $\ker A = \{f \in N : T_z f \in N\} \oplus M$ .
- (4)  $\ker B = \{f \in M : T_w^* f \in M\} \oplus N$ .

Proof. (1) Since  $T_z Q = QT_z Q$  and  $T_z T_w^* = T_w^* T_z$ ,

$$\begin{aligned} AB &= QT_z PT_w^* Q \\ &= QT_z T_w^* Q - QT_z QT_w^* Q \\ &= QT_w^* QT_z Q - QT_z QT_w^* Q. \end{aligned}$$

(2) Since  $T_w^* P = PT_w^* P$  and  $T_w^* T_z = T_z T_w^*$ ,

$$\begin{aligned} BA &= PT_w^* QT_z P \\ &= PT_w^* T_z P - PT_w^* PT_z P \\ &= PT_z PT_w^* P - PT_w^* PT_z P. \end{aligned}$$

The properties (3) and (4) are clear.

### Theorem 2.2.

- (1)  $A = 0$  if and only if  $N = H^2$  or  $N = H^2 \ominus qH^2$  where  $q$  is a one variable inner function with  $q = q(w)$ .
- (2)  $B = 0$  if and only if  $M = [0]$  or  $M = qH^2$  where  $q$  is a one variable inner function with  $q = q(z)$ .
- (3)  $A = B = 0$  if and only if  $N = [0]$  or  $N = H^2$ .

Proof. (2) follows from (1). We will show (1). We have  $H^2 = N \oplus M$  and  $T_z M \subset M$ . Suppose  $A = 0$ . By Lemma 2.1 (3),  $T_z N \subset N$ . Put  $N_0 = N \ominus T_z N$  and

$M_0 = M \ominus T_z M$ . Then

$$H^2 = \sum_{n=0}^{\infty} \oplus (N_0 \oplus M_0) z^n = \sum_{n=0}^{\infty} \oplus H^2(w, T) z^n$$

because  $zH^2 = zN \oplus zM$  and so  $N_0 \oplus M_0 = H^2(w, T)$ . By Lemma 2.1 (1),  $V_w^* V_z = V_z V_w^*$  and so  $V_z^* V_w = V_w V_z^*$  because  $AB = 0$ . Hence  $V_w(\ker V_z^*) \subseteq \ker V_z^*$  and  $\ker V_z^* = M_0$ . Therefore by a theorem of Beurling [1], if  $M_0 \neq [0]$ ,  $M_0 = qH^2(w, T)$  and  $q$  is a one variable inner function with  $q = q(w)$ . Hence  $M = qH^2$  and so  $N = H^2 \ominus qH^2$ . If  $M_0 = [0]$ , then  $M = [0]$ , and so  $N = H^2$ .

### §3. Invariant subspace with $AB = 0$ or $BA = 0$

Suppose that  $N$  is a backward shift invariant subspace and  $M = H^2 \ominus N$ . By Lemma 2.1,  $AB = 0$  if and only if  $V_w^* V_z = V_z V_w^*$ , and  $BA = 0$  if and only if  $S_z S_w^* = S_w^* S_z$ . Hence we know (see [2],[3],[4]) that  $AB = 0$  if and only if  $M = qH^2$  for some inner function  $q$ . In this section, we study  $N$  when  $BA = 0$ , that is,  $S_z S_w^* = S_w^* S_z$ .

#### Lemma 3.1.

$$[\text{ran } A]_2 = \{M \ominus zM\} \ominus \{H^2(w, T) \cap M\}$$

and

$$\ker B = \{H^2(z, T) \cap M\} \oplus wM \oplus N.$$

Proof. Since  $(T_w^* f, g) = (f, wg)$  if  $f, g \in H^2$ ,

$$\{f \in M ; T_w^* f \in M\} = M \cap \{H^2 \ominus wN\} = \{H^2(z, T) \cap M\} \oplus wM,$$

because  $H^2 \ominus wN = (H^2 \ominus wH^2) \oplus w(H^2 \ominus N)$  and  $N = H^2 \ominus M$ . Hence by Lemma 2.1 (4),  $\ker B = \{f \in M ; T_w^* f \in M\} \oplus N = \{H^2(z, T) \cap M\} \oplus wM \oplus N$ . By the same argument,  $\ker A^* = \{H^2(w, T) \cap M\} \oplus zM \oplus N$  and so

$$\begin{aligned} [\text{ran } A]_2 &= H^2 \ominus \ker A^* \\ &= \{M \ominus zM\} \ominus \{H^2(w, T) \cap M\}. \end{aligned}$$

#### Lemma 3.2.

- (1)  $A = 0$  if and only if  $M = \{H^2(w, T) \cap M\} \oplus zM$ .
- (2)  $B = 0$  if and only if  $M = \{H^2(z, T) \cap M\} \oplus wM$ .
- (3)  $BA = 0$  if and only if  $\{H^2(z, T) \cap M\} \oplus wM \supseteq \{M \ominus zM\} \ominus \{H^2(w, T) \cap M\}$ .

Proof. These follow from Lemma 3.1.

For a subset  $H$  of  $H^2$ , let  $H_k = \sum_{i+j=k} z^i w^j H$  for  $k \geq 0$ .

**Theorem 3.3.** *Let  $N$  be a backward shift invariant subspace of  $H^2$  and  $M$  its orthogonal complement. Suppose  $N \neq H^2$ .*

(1)  $S_z S_w^* = S_w^* S_z$  if and only if  $M = H + M_1$  and if and only if  $M = \sum_{j=0}^{k-1} H_j + M_k$  for any  $k \geq 1$ , where  $H = H_0 = H^2(z, T) \cap M + H^2(w, T) \cap M$ . If  $S_z S_w^* = S_w^* S_z$ , then  $H \neq [0]$ .

(2) When  $M \cap H^2(z, T) = [0]$  or  $M \cap H^2(w, T) = [0]$ ,  $S_z S_w^* = S_w^* S_z$  if and only if  $M = qH^2 + M_k$  for any  $k \geq 1$  where  $q$  is a one variable inner function such that  $M \cap H^2(z, T) = qH^2(z, T)$  or  $M \cap H^2(w, T) = qH^2(w, T)$ .

(3) When  $M \cap H^2(z, T) \neq [0]$  and  $M \cap H^2(w, T) \neq [0]$ ,  $S_z S_w^* = S_w^* S_z$  if and only if  $M = q_1 H^2 + q_2 H^2 + M_k$  for any  $k \geq 1$  where  $q_1 = q_1(z)$  and  $q_2 = q_2(w)$  are one variable inner functions such that  $M \cap H^2(z, T) = q_1 H^2(z, T)$  and  $M \cap H^2(w, T) = q_2 H^2(w, T)$ .

Proof. (1) Since  $S_z S_w^* = S_w^* S_z$  is equivalent to  $BA = 0$ ,  $S_z S_w^* = S_w^* S_z$  if and only if  $M = H + M_1$  by Lemma 3.2 (3). It is easy to see that  $M = H + M_1 = \sum_{j=0}^{k-1} H_j + M_k$  for any  $k \geq 1$ . If  $H = [0]$ , then  $M = M_k$  and hence  $M = [0]$ . This contradicts  $N \neq H^2$ .

(2) We may assume that  $M \cap H^2(z, T) = [0]$  and  $M \cap H^2(w, T) \neq [0]$ . By a theorem of Beurling [1],  $M \cap H^2(w, T) = qH^2(w, T)$  for some one variable inner function  $q = q(w)$ . By (1),  $S_z S_w^* = S_w^* S_z$  if and only if  $M = qH^2(w, T) + M_1$  if and only if  $M = q \sum_{j=0}^{k-1} \oplus H^2(w, T) z^j + M_k$  for any  $k \geq 1$ . This is equivalent to  $M = qH^2 + M_k$  for any  $k \geq 1$ . For,  $M_k \supseteq qz^k H^2$ .

(3) By a theorem of Beurling,  $M \cap H^2(z, T) = q_1 H^2(z, T)$  and  $M \cap H^2(w, T) = q_2 H^2(w, T)$  where  $q_1 = q_1(z)$  and  $q_2 = q_2(w)$  are one variable inner functions. By (1),  $S_z S_w^* = S_w^* S_z$  if and only if  $M = q_1 H^2(z, T) + q_2 H^2(w, T) + M_1$  if and only if

$$M = q_1 \sum_{j=0}^{k-1} \oplus H^2(z, T) w^j + q_2 \sum_{j=0}^{k-1} \oplus H^2(w, T) z^j + M_k$$

for any  $k \geq 1$ . This is equivalent to  $M = q_1 H^2 + q_2 H^2 + M_k$  for any  $k \geq 1$ . For,  $M_k \supseteq q_1 w^k H^2 + q_2 z^k H^2$ .

**Corollary 3.4.**

(1)  $AB = BA = 0$  if and only if  $A = 0$  or  $B = 0$ .

(2) If  $N = H^2 \ominus qH^2$  and  $q$  is an inner function and  $S_z S_w^* = S_w^* S_z$ , then  $q$  is a one variable.

Proof. (1) If  $AB = BA = 0$ , then by Lemma 2.1 (1)  $V_w^* V_z = V_z V_w^*$  and so  $M = qH^2$  for some inner function  $q$  (see [2], [4]). On the other hand, by Theorem 3.3 (1),  $M \cap H^2(z, T) \neq [0]$  or  $M \cap H^2(w, T) \neq [0]$  because  $S_z S_w^* = S_w^* S_z$ . Hence  $q$  is one variable. By Theorem 2.2,  $A = 0$  or  $B = 0$ .



(2) is clear by (1).

**Corollary 3.5.** *Let  $N$  be a backward shift invariant subspace and  $N \neq H^2$ .*

(1) *If  $S_z S_w^* = S_w^* S_z$ , then  $N \subseteq H^2 \ominus qH^2$  for some one variable inner function  $q$ .*

(2) *If  $N = H^2 \ominus qH^2$  for some one variable inner function  $q$ , then  $S_z S_w^* = S_w^* S_z$ .*

Proof. (1) By Theorem 3.3, if  $S_z S_w^* = S_w^* S_z$  then  $M \supseteq qH^2$  for some one variable inner function  $q$ . Hence  $N \subseteq H^2 \ominus qH^2$ . (2) is clear by Theorem 3.3 (3).

**Corollary 3.6.** *Suppose that  $A \neq 0$  and  $B \neq 0$ .*

(1) *If  $S_z S_w^* = S_w^* S_z$ , then  $N \subseteq (H^2 \ominus q_1 H^2) \cap (H^2 \ominus q_2 H^2)$  where  $q_1 = q_1(z)$  and  $q_2 = q_2(w)$  are one variable inner functions.*

(2) *If  $N = (H^2 \ominus q_1 H^2) \cap (H^2 \ominus q_2 H^2)$  where  $q_1 = q_1(z)$  and  $q_2 = q_2(w)$  are one variable inner functions, then  $S_z S_w^* = S_w^* S_z$ .*

Proof. By Theorem 2.2, we can prove (1) as in the proof of Corollary 3.5 (1). (2) Since  $q_1 H^2 + q_2 H^2 = [q_1, q_2] + (q_1 H_1^2 + q_2 H_1^2)$ ,  $M = [q_1, q_2] + (zM + wM) = q_1 H^2 + q_2 H^2 + M_k$  for any  $k \geq 1$ . It is easy to see that  $M \cap H^2(z, T) = q_1 H^2(z, T)$  and  $M \cap H^2(w, T) = q_2 H^2(w, T)$ . Hence by Theorem 3.3 (3)  $S_z S_w^* = S_w^* S_z$ .

#### §4. Conjecture

By Corollary 3.5 (2) and Corollary 3.6, if  $N = H^2$ ,  $N = H^2 \ominus qH^2$  for some one variable inner function  $q$  or  $N = (H^2 \ominus q_1 H^2) \cap (H^2 \ominus q_2 H^2)$  for some one variable inner functions  $q_1 = q_1(z)$  and  $q_2 = q_2(w)$ , then  $S_z S_w^* = S_w^* S_z$ . Because of Theorem 3.3, we have the following conjecture. In this section, we study this conjecture.

**Conjecture.** *If  $S_z S_w^* = S_w^* S_z$ , then  $N = H^2$ ,  $N = H^2 \ominus qH^2$  for some one variable inner function  $q$  or  $N = (H^2 \ominus q_1 H^2) \cap (H^2 \ominus q_2 H^2)$ , where  $q_1 = q_1(z)$  and  $q_2 = q_2(w)$  are one variable inner functions.*

**Proposition 4.1.** *If  $M = q_1 H^2 + q_2 H^2 + M_k$  for any  $k \geq 1$ , where  $M \cap H^2(z, T) = q_1 H^2(z, T)$  and  $M \cap H^2(w, T) = q_2 H^2(w, T)$ , then  $M = q_1 (H^2 \ominus w^k H^2) + q_2 (H^2 \ominus w^k H^2) + w^k M$  for any  $k \geq 1$ . The converse is also true.*

Proof. Since  $wM \supseteq q_2 w H^2(w, T)$ , by Lemma 3.2 (3) and Theorem 3.3 (3),

$$q_1 H^2(z, T) \oplus wM \supseteq K_2 \oplus q_2 w H^2(w, T),$$

where  $M \ominus zM = K_2 \oplus q_2 H^2(w, T)$ . Thus

$$q_1 H^2(z, T) \oplus wM \supseteq \sum_{j=0}^{\infty} \oplus \{K_2 \oplus q_2 w H^2(w, T)\} z^j.$$

Since

$$M = \sum_{j=0}^{\infty} \oplus (M \ominus zM)z^j = \sum_{j=0}^{\infty} \oplus \{K_2 \oplus q_2H^2(w, T)\}z^j,$$

we have

$$q_1H^2(z, T) + q_2H^2(z, T) + wM \supseteq M.$$

Hence  $M = q_1H^2(z, T) + q_2H^2(z, T) + wM$ . This leads our assertion.

**Corollary 4.2.** *If  $M = qH^2 + M_k$  for any  $k \geq 1$  where  $M \cap H^2(z, T) = qH^2(z, T)$  and  $M \cap H^2(w, T) = [0]$ , then  $M = qH^2$ .*

*Proof.* By Proposition 4.1 and its proof,  $M = q(H^2 \ominus w^kH^2) + w^kM$  for any  $k \geq 1$ . This implies that  $M = qH^2$ , because  $q(H^2 \ominus w^kH^2)$  is orthogonal to  $w^kM$ .

It is not difficult to prove that  $q_1H^2 + q_2H^2$  is closed when  $q_1 = q_1(z)$  and  $q_2 = q_2(w)$  are one variable. Hence our conjecture is equivalent to the following one. If  $S_zS_w^* = S_w^*S_z$ , then  $M = [0]$ ,  $M = qH^2$  or  $M = q_1H^2 + q_2H^2$ . Even if  $N$  is of finite dimension,  $S_zS_w^* \neq S_w^*S_z$  may happen. In fact,  $N = \{1, z, w\}$  is such an example.

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