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**R. Agemi and K. Yokoyama**

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# The Null Condition and Global Existence of Solutions to Systems of Wave Equations with Different Speeds

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## Abstract

In this paper, we consider the initial value problems to systems of quasilinear wave equations with different speeds in two space dimensions. Applying John-Shatah observations to our problem, we introduce the null condition for the system with different speeds. Moreover, we prove a global existence theorem for a class satisfying the null condition.

## 1 Introduction.

We shall start this paper with the description of John-Shatah observations on the null condition. We consider the scalar quasilinear wave equations with quadratic nonlinearity in three space dimensions. Introducing the space-time gradient of unknown, one can find that components of the gradient satisfy some hyperbolic system of first order. The plane wave solutions of this system satisfy hyperbolic systems of first order in *one* space dimension. Making use of the results in F. John[3], F. John and J. Shatah have proved in F. John[5] the following

remarkable fact: The requirement that no plane wave solution of this system is genuinely nonlinear leads to a class of equations which satisfy the null condition (S. Klainerman[9]).

We apply John-Shatah observations to a system of quasilinear wave equations with different speeds in two space dimensions. We consider the system with unknown vector  $u(t, x) = {}^t(u^1(t, x), \dots, u^m(t, x))$  in the form

$$(1.1) \quad \partial_t^2 u^i - c_i^2 \Delta u^i = \sum_{j=1}^m \sum_{\alpha, \beta=0}^2 C_{ij}^{\alpha\beta}(\partial u) \partial_\alpha \partial_\beta u^j \quad (i = 1, \dots, m),$$

where  $\partial u$  stands for space-time gradient of  $u$ , i.e.

$$(1.2) \quad \begin{aligned} \partial u &= {}^t(\partial u^1, \dots, \partial u^m), \\ \partial u^i &= {}^t(\partial_0 u^i, \partial_1 u^i, \partial_2 u^i), \\ \partial_0 &= \frac{\partial}{\partial t}, \quad \partial_1 = \frac{\partial}{\partial x_1}, \quad \partial_2 = \frac{\partial}{\partial x_2} \end{aligned}$$

and the  $c_i$  ( $i = 1, \dots, m$ ) are positive constants different from each other. We assume that  $C_{ij}^{\alpha\beta}$  are  $C^\infty$ -functions of their arguments which vanish at  $\partial u = 0$  of second order. Set  $v = \partial u$ . Then one can find from (1.1) and (1.2) that the vector  $v$  satisfies the system of first order which is hyperbolic near  $v = 0$ :

$$(1.3) \quad \sum_{\alpha=0}^2 a^\alpha(v) \partial_\alpha v = 0.$$

For the concrete expression of  $a^\alpha(v)$  see section 2. We next consider the plane wave solutions  $w$  of the system (1.3):

$$(1.4) \quad v(t, x) = w(t, s), \quad s = \sum_{i=1}^2 \zeta_i x_i$$

where  $\zeta = (\zeta_1, \zeta_2) \in \mathbf{R}^2$  and  $\zeta \neq 0$ . Then one can find from (1.3) and (1.4) that the vector  $w$  satisfies the system in one space dimension:

$$(1.5) \quad a^0(w) \partial_t w + \sum_{i=1}^2 \zeta_i a^i(w) \partial_s w = 0.$$

We take the initial values for the solutions  $w$  of (1.5) in the form

$$(1.6) \quad w(0, s) = \varepsilon \varphi(s),$$

where  $\varphi$  has compact support and  $\varepsilon$  is small positive constant.

Since the system (1.1) has the cubic nonlinearity, the system (1.5) is not genuinely non-linear. Thus it is natural to require that the lifespan  $T_\varepsilon$  of solutions to (1.5) and (1.6) is at least of order  $\varepsilon^{-3}$  for any  $\zeta$ . Making use of the results in Li Ta-t sien, Kong De-xing and Zhou Yi[12], we shall prove in section 2 that the requirement above is equivalent to the following fact: it holds that

$$(1.7) \quad \sum_{\alpha, \beta, \gamma, \delta=0}^2 \frac{\partial^2 C_{ii}^{\alpha\beta}}{\partial(\partial_\gamma u^i) \partial(\partial_\delta u^i)} \Big|_{\partial u=0} X_\alpha^i X_\beta^i X_\gamma^i X_\delta^i = 0 \quad (i = 1, \dots, m)$$

for any real vector  $X^i = (X_0^i, X_1^i, X_2^i)$  satisfying

$$(1.8) \quad (X_0^i)^2 - c_i^2 \sum_{j=1}^2 (X_j^i)^2 = 0.$$

Thus we can interpret this as the null condition for the system (1.1) with different speeds.

If

$$(1.9) \quad \frac{\partial^2 C_{ii}^{\alpha\beta}}{\partial(\partial_\gamma u^i) \partial(\partial_\delta u^i)} \Big|_{\partial u=0} = 0 \quad \left( \begin{array}{l} i = 1, \dots, m \\ \alpha, \beta, \gamma, \delta = 0, 1, 2 \end{array} \right)$$

then the null condition (1.7) is automatically satisfied. The main aim of this paper is to prove the global existence of solution to (1.1) with small data under the assumption (1.9). It is still open whether the null condition (1.7) guarantees the global existence of solution to (1.1).

The null condition for the nonlinear elastic wave equation is more complicated comparing with (1.7). T. Sideris has studied in [13] a class satisfying the null condition.

In section 3, we introduce some notations and in section 4 we state the main result. In section 5, using the representation of solution to (1.1) in Kovalyov[10], we estimate the first order derivatives of solution. Finally we prove the main results in section 7 using the estimates and energy inequalities in section 6.

## 2 The null condition.

In this section we introduce the null condition for the system (1.1) with different speeds stated in Introduction.

We consider the system in the form

$$(2.1) \quad \sum_{j=1}^m \sum_{\alpha, \beta=0}^2 a_{ij}^{\alpha\beta}(\partial u) \partial_\alpha \partial_\beta u^j = 0 \quad (i = 1, \dots, m),$$

where  $\partial u$  stands for space-time gradient of  $u$

$$\begin{aligned} \partial u &= {}^t(\partial u^1, \dots, \partial u^m), \\ \partial u^i &= {}^t(\partial_0 u^i, \partial_1 u^i, \partial_2 u^i), \\ \partial_0 &= \frac{\partial}{\partial t}, \quad \partial_1 = \frac{\partial}{\partial x_1}, \quad \partial_2 = \frac{\partial}{\partial x_2}. \end{aligned}$$

We assume that

$$(2.2) \quad \begin{aligned} a_{ij}^{\alpha\beta} &= a_{ij}^{\beta\alpha}, \quad a_{ii}^{00}(0) = 1, \\ a_{ij}^{\alpha\beta}(0) &= -c_i^2 \delta_{ij} \delta_{\alpha\beta} \quad \text{for } (\alpha, \beta) \neq (0, 0), \\ c_i &> 0, \quad c_i \neq c_j \quad \text{for } i \neq j \end{aligned}$$

and  $a_{ii}^{00}(\partial u) - 1, a_{ij}^{\alpha\beta}(\partial u) + c_i^2 \delta_{ij} \delta_{\alpha\beta}$  vanish at  $\partial u = 0$  at least of second order, that is,

$$(2.3) \quad \begin{aligned} a_{ii}^{00}(\partial u) &= 1 + O(|\partial u|^2) \\ a_{ij}^{\alpha\beta}(\partial u) &= -c_i^2 \delta_{ij} \delta_{\alpha\beta} + O(|\partial u|^2) \end{aligned}$$

near  $\partial u = 0$ .

Set

$$(2.4) \quad \begin{aligned} v &= {}^t(v^1, \dots, v^m) \\ v^i &= {}^t(v_0^i, v_1^i, v_2^i), \quad v_\alpha^i = \partial_\alpha u^i. \end{aligned}$$

Then we find from (2.1) and (2.4) that the vector  $v$  satisfies a system of first order which is hyperbolic near  $v = 0$ :

$$(2.5) \quad \sum_{\alpha=0}^2 a^\alpha(v) \partial_\alpha v = 0$$

Here the  $3m \times 3m$  matrices  $a^\alpha$  are defined by

$$(2.6) \quad a^\alpha = (A_{ij}^\alpha : i, j = 1, \dots, m),$$

where

$$(2.7) \quad A_{ij}^0 = \begin{pmatrix} a_{ij}^{00} & \mathbf{0} \\ \mathbf{0} & \delta_{ij} \end{pmatrix}, \quad A_{ij}^1 = \begin{pmatrix} 2a_{ij}^{10} & a_{ij}^{11} & a_{ij}^{12} \\ -\delta_{ij} & \mathbf{0} \\ 0 & \mathbf{0} \end{pmatrix},$$

$$A_{ij}^2 = \begin{pmatrix} 2a_{ij}^{20} & a_{ij}^{21} & a_{ij}^{22} \\ 0 & \mathbf{0} \\ -\delta_{ij} & \mathbf{0} \end{pmatrix}.$$

We next consider the plane wave solution  $v(t, x)$  of the equation(2.5):

$$(2.8) \quad v(t, x) = w(t, s), \quad s = \sum_{i=1}^2 \zeta_i x_i,$$

where  $\zeta = (\zeta_1, \zeta_2) \in \mathbf{R}^2$  and  $\zeta \neq 0$ . Then we find from (2.5) and (2.8) that the vector  $w$  satisfies a system in one space dimension.

$$(2.9) \quad \partial_t w + a(w) \partial_s w = 0$$

where

$$(2.10) \quad a(w) = a^0(w)^{-1} \sum_{i=1}^2 \zeta_i a^i(w).$$

We take the initial values for the solution  $w$  of (2.9) in the form

$$(2.11) \quad w(0, s) = \varepsilon \varphi(s),$$

where  $\varphi$  has compact support and  $\varepsilon > 0$ .

We shall seek the eigenvalues  $\lambda = \lambda(w)$  of the matrix  $a(w)$  and the corresponding eigenvector  $\xi = \xi(w)$ . By the definition,  $\lambda$  satisfies an equation

$$(2.12) \quad \det \left( \lambda a^0(w) - \sum_{i=1}^2 \zeta_i a^i(w) \right) = 0.$$

We can verify by induction that

$$(2.13) \quad \det \left( \lambda a^0(w) - \sum_{i=1}^2 \zeta_i a^i(w) \right) = \lambda^m \det(p_{ij} : i, j = 1, \dots, m)$$



where

$$(2.14) \quad p_{ij} = a_{ij}^{00} \lambda^2 - 2\lambda \sum_{k=1}^2 a_{ij}^{k0} \zeta_k + \sum_{k,l=1}^2 a_{ij}^{kl} \zeta_k \zeta_l.$$

Therefore we find from (2.2), (2.3), (2.12), (2.13) and (2.14) that the eigenvalues  $\lambda_i^\pm(0)$  ( $i = 1, \dots, m$ ) of the matrix  $a(0)$ , aside from the trivial multiple eigenvalue  $\lambda = 0$ , become

$$(2.15) \quad \lambda_i^\pm(0) = \pm c_i |\zeta|, \quad |\zeta| = (\zeta_1^2 + \zeta_2^2)^{1/2}.$$

According to (2.4) we arrange the components of a vector  $\xi \in \mathbf{R}^{3m}$  as follows:

$$(2.16) \quad \begin{aligned} \xi &= {}^t(\xi^1, \dots, \xi^m), \\ \xi^i &= {}^t(\xi_0^i, \xi_1^i, \xi_2^i). \end{aligned}$$

Then we find from (2.2), (2.3), (2.7) and (2.12) that the eigenvector  $\xi_i^\pm(0)$  corresponding to  $\lambda_i^\pm(0)$  becomes

$$(2.17) \quad \begin{aligned} (\xi_i^\pm(0))^j &= {}^t(0, 0, 0) \quad \text{for } j \neq i \\ (\xi_i^\pm(0))^i &= {}^t(\mp 1, \zeta_1/c_i |\zeta|, \zeta_2/c_i |\zeta|) \end{aligned}$$

and the eigenvectors  $\xi_i$  ( $i = 1, \dots, m$ ) corresponding to the trivial eigenvalue 0 become

$$\begin{aligned} (\xi_i)^j &= {}^t(0, 0, 0) \quad \text{for } j \neq i, \\ (\xi_i)^j &= {}^t(0, \zeta_2, -\zeta_1). \end{aligned}$$

Since  $\xi_i^\pm(0), \xi_i$  ( $i = 1, \dots, m$ ) are linearly independent, we see that the system (2.9) is hyperbolic near  $w = 0$ .

We now require that a solution  $w(t, s)$  to the initial value problem (2.9), (2.11) has a lifespan  $T_\varepsilon$  which is at least of order  $\varepsilon^{-3}$  for any  $\zeta \in \mathbf{R}^2$ . This requirement is equivalent to the following facts[12]:

$$(2.18) \quad \sum_{j=1}^m \sum_{\alpha=0}^2 \frac{\partial \lambda_i^\pm}{\partial w_\alpha^j} \Big|_{w=0} (\xi_i^\pm(0))_\alpha^j = 0$$

and

$$(2.19) \quad \sum_{j,k=1}^m \sum_{\alpha,\beta=0}^2 \frac{\partial^2 \lambda_i^\pm}{\partial w_\alpha^j \partial w_\beta^k} \Big|_{w=0} (\xi_i^\pm(0))_\alpha^j (\xi_i^\pm(0))_\beta^k = 0 \quad \text{for } i = 1, \dots, m \text{ and } \zeta \in \mathbf{R}^2.$$

Set

$$P(\lambda) = \det (p_{ij}(\lambda) : i, j = 1, \dots, m).$$

Differentiating the equations

$$(2.20) \quad P(\lambda_i^\pm(w)) = 0 \quad (i = 1, \dots, m)$$

in a variable  $w_\alpha^j$  and evaluating the results at  $w = 0$ , we get

$$2c_i |\zeta|^{2m-1} \prod_{l \neq i} (c_i^2 - c_l^2) \frac{\partial \lambda_i^\pm}{\partial w_\alpha^j} \Big|_{w=0} = 0$$

which implies

$$(2.21) \quad \frac{\partial \lambda_i^\pm}{\partial w_\alpha^j} \Big|_{w=0} = 0$$

for all  $i, j, \alpha$ . Therefore it follows from (2.2), (2.3), (2.14) and (2.21) that the condition (2.18) holds trivially and

$$(2.22) \quad \frac{\partial (p_{ij}(\lambda_k^\pm))}{\partial w_\alpha^l} \Big|_{w=0} = 0$$

for all  $i, j, k, l, \alpha$ . Next differentiating twice the equations (2.20) in variables  $w_\alpha^j$  and  $w_\beta^k$  and evaluating the results at  $w = 0$ , we get

$$(2.23) \quad |\zeta|^{2m-2} \prod_{l \neq i} (c_i^2 - c_l^2) \frac{\partial^2 (p_{ii}(\lambda_i^\pm))}{\partial w_\alpha^j \partial w_\beta^k} \Big|_{w=0} = 0$$

for all  $i, j, k, \alpha, \beta$ . By the definition (2.14) of  $p_{ii}(\lambda)$ , we have

$$(2.24) \quad \frac{\partial^2 (p_{ii}(\lambda_i^\pm))}{\partial w_\alpha^j \partial w_\beta^k} \Big|_{w=0} = 2\lambda_i^\pm(0) \frac{\partial^2 \lambda_i^\pm}{\partial w_\alpha^j \partial w_\beta^k} \Big|_{w=0} + \lambda_i^\pm(0)^2 \frac{\partial^2 a_{ii}^{00}}{\partial w_\alpha^j \partial w_\beta^k} \Big|_{w=0} \\ - 2\lambda_i^\pm(0) \sum_{l=1}^2 \frac{\partial^2 a_{ii}^{l0}}{\partial w_\alpha^j \partial w_\beta^k} \Big|_{w=0} \zeta_l + \sum_{h,l=1}^2 \frac{\partial a_{ii}^{hl}}{\partial w_\alpha^j \partial w_\beta^k} \Big|_{w=0} \zeta_h \zeta_l.$$

Then it follows from (2.15), (2.17), (2.23) and (2.24) that

$$(2.25) \quad \frac{\partial^2 \lambda_i^\pm}{\partial w_\alpha^j \partial w_\beta^k} \Big|_{w=0} = \frac{\mp c_i |\zeta|}{2} \sum_{\gamma, \delta=0}^2 \frac{\partial^2 a_{ii}^{\gamma\delta}}{\partial w_\alpha^j \partial w_\beta^k} \Big|_{w=0} (\xi_i^\pm(0))_\gamma (\xi_i^\pm(0))_\delta$$

Therefore we find from (2.17) and (2.25) that the condition (2.19) is equivalent to

$$(2.26) \quad \sum_{\alpha, \beta, \gamma, \delta=0}^2 \frac{\partial^2 a_{ii}^{\gamma\delta}}{\partial w_\alpha^i \partial w_\beta^i} \Big|_{w=0} (\xi_i^\pm(0))_\alpha^i (\xi_i^\pm(0))_\beta^i (\xi_i^\pm(0))_\gamma^i (\xi_i^\pm(0))_\delta^i = 0$$

for  $i = 1, \dots, m$  and  $\zeta \in \mathbf{R}^2$ . By the definition (2.17) of  $(\xi_i^\pm(0))^i$ , we have

$$(2.27) \quad \left\{ (\xi_i^\pm(0))_0^i \right\}^2 - c_i^2 \sum_{j=1}^2 \left\{ (\xi_i^\pm(0))_j^i \right\}^2 = 0$$

for all  $\zeta \in \mathbf{R}^2$ .

Consequently we have proved the following

**Proposition 2.1** *The lifespan  $T_\varepsilon$  of a unique solution  $w(t, s)$  of the initial value problem (2.9), (2.11) is at least of order  $\varepsilon^{-3}$  for any  $\zeta \in \mathbf{R}^2$  if and only if it holds that*

$$(2.28) \quad \sum_{\alpha, \beta, \gamma, \delta=0}^2 \frac{\partial^2 a_{ii}^{\gamma\delta}}{\partial w_\alpha^i \partial w_\beta^i} \Big|_{w=0} X_\alpha^i X_\beta^i X_\gamma^i X_\delta^i = 0 \quad (i = 1, \dots, m)$$

for all real vector  $X^i = (X_0^i, X_1^i, X_2^i)$  satisfying

$$(2.29) \quad (X_0^i)^2 - c_i^2 \sum_{j=1}^2 (X_j^i)^2 = 0.$$

Setting

$$\begin{aligned} C_{ii}^{00}(\partial u) &= 1 - a_{ii}^{00}(\partial u), \\ C_{ij}^{\alpha\beta}(\partial u) &= -c_i^2 \delta_{ij} \delta_{\alpha\beta} - a_{ij}^{\alpha\beta}(\partial u) \quad (\alpha, \beta) \neq (0, 0), \end{aligned}$$

we see that the null condition (1.7), (1.8) follows from Proposition 2.1.

### 3 Notations.

To begin with, we introduce some notations that are used throughout the paper.

Partial derivatives are denoted by

$$\partial_0 = \partial_t = \frac{\partial}{\partial t}, \quad \partial_1 = \frac{\partial}{\partial x_1}, \quad \partial_2 = \frac{\partial}{\partial x_2}.$$

We also use the angular derivative:

$$\Omega = x_1 \partial_2 - x_2 \partial_1.$$

We set

$$\mathcal{D} = (\mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3) = (\partial_1, \partial_2, \Omega)$$

and define

$$\mathcal{D}^A = \mathcal{D}_1^{A_1} \mathcal{D}_2^{A_2} \mathcal{D}_3^{A_3}, |A| = A_1 + A_2 + A_3,$$

where  $A = (A_1, A_2, A_3)$  is a multi-index.

Let  $u = {}^t(u^1, \dots, u^m)$  be an unknown vector and set

$$(3.1) \quad w_i(t, r) = (r+1)^{1/2-\gamma} (t+r+1)^\gamma (|r-c_i t|+1)^{1/2}$$

for  $0 < \gamma < 1/2$ . Then we define, for a non-negative integer  $k$ ,

$$(3.2) \quad \begin{aligned} |\partial u(t, x)| &= \sum_{i=1}^m \sum_{\alpha=0}^2 |\partial_\alpha u^i(t, x)| \\ |\partial u(t)|_k &= \sum_{|A| \leq k} \sum_{i=1}^m \sum_{\alpha=0}^2 \sup_{x \in \mathbf{R}^2} |\mathcal{D}^A \partial_\alpha u^i(t, x)| \\ \|\partial u(t)\|_k &= \sum_{|A| \leq k} \sum_{i=1}^m \sum_{\alpha=0}^2 \left\| \mathcal{D}^A \partial_\alpha u^i(t, \cdot) \right\|_{L^2(\mathbf{R}^2)} \\ [\partial u(t)]_k &= \sum_{|A| \leq k} \sum_{i=1}^m \sum_{\alpha=0}^2 \sup_{x \in \mathbf{R}^2} |w_i(t, |x|) \mathcal{D}^A \partial_\alpha u^i(t, x)|. \end{aligned}$$

Moreover, we define

$$(3.3) \quad \begin{aligned} |\partial u|_k(t) &= \sup_{0 < s < t} |\partial u(s)|_k, \\ [\partial u]_k(t) &= \sup_{0 < s < t} [\partial u(s)]_k. \end{aligned}$$

In what follows,  $M$  denotes various constant depending on  $F_i$ ,  $f^i$ ,  $g^i$  and  $c_i$ .

## 4 Statement of the Main Result.

The initial value problem to be considered is

$$(4.1) \quad \begin{cases} \partial_t^2 u^i - c_i^2 \Delta u^i = F_i(\partial u, \partial^2 u) & \text{in } [0, \infty) \times \mathbf{R}^2 \\ u^i(0, \cdot) = \varepsilon f^i, \partial_t u^i(0, \cdot) = \varepsilon g^i & \text{in } \mathbf{R}^2 \end{cases} \quad (i = 1, \dots, m)$$

where  $c_i$  are positive constants and  $\varepsilon > 0$  is small parameter. Moreover,  $f^i$  and  $g^i$  are  $C^\infty$ -functions with compact support. We describe some assumptions on the initial value problem (4.1) and state the main theorem.

First, we assume that  $F_i$  are of first degree with respect to the second derivatives of  $u$ :

$$(4.2) \quad F_i(\partial u, \partial^2 u) = \sum_{j=1}^m \sum_{\alpha, \beta=0}^2 C_{ij}^{\alpha\beta}(\partial u) \partial_\alpha \partial_\beta u^j + E_i(\partial u).$$

Here,  $C_{ij}^{\alpha\beta}$  and  $E_i$  are  $C^\infty$ -functions of  $\partial u$  in  $\{|\partial u| < 1\}$  that satisfy

$$(4.3) \quad C_{ij}^{\alpha\beta} = C_{ij}^{\beta\alpha} = C_{ji}^{\alpha\beta},$$

$$(4.4) \quad |C_{ij}^{\alpha\beta}(\partial u)| \leq M |\partial u|^2,$$

$$(4.5) \quad |E_i(\partial u)| \leq M |\partial u|^3.$$

Assuming (4.2)-(4.5), M. Kovalyov[10] proved the almost global existence of the solution to (4.1).

Second, we assume the null condition (1.9) for different speeds introduced in Introduction:

$$(4.6) \quad c_i \neq c_j \quad \text{for } i \neq j$$

$$(4.7) \quad \left. \frac{\partial^2 C_{ii}^{\alpha\beta}}{\partial(\partial_\gamma u^i) \partial(\partial_\delta u^i)} \right|_{\partial u=0} = 0 \quad \left( \begin{array}{l} i = 1, \dots, m \\ \alpha, \beta, \gamma, \delta = 0, 1, 2 \end{array} \right).$$

The condition for  $E_i$  are

$$(4.8) \quad \left. \frac{\partial^3 E_i}{\partial(\partial_\alpha u^i) \partial(\partial_\beta u^i) \partial(\partial_\gamma u^i)} \right|_{\partial u=0} = 0 \quad \left( \begin{array}{l} i = 1, \dots, m \\ \alpha, \beta, \gamma = 0, 1, 2 \end{array} \right)$$

in accordance with (4.7).

**Theorem** *Let us assume (4.2)-(4.8). Then there exists a positive constant  $\varepsilon_0$  depending on given functions such that the initial value problem (4.1) has a unique  $C^\infty$ -solution in  $[0, \infty) \times \mathbf{R}^2$  for all  $\varepsilon$  with  $0 < \varepsilon < \varepsilon_0$ .*

M. Kovalyov showed in [11] that the theorem holds when  $C_{ij}^{\alpha\beta} = 0$  and  $E_i$  ( $i = 1, \dots, m$ ) satisfy the condition

$$\left. \frac{\partial^3 E_i}{\partial(\partial_\alpha u^j) \partial(\partial_\beta u^j) \partial(\partial_\gamma u^j)} \right|_{\partial u=0} = 0 \quad \left( \begin{array}{l} i, j = 1, \dots, m \\ \alpha, \beta, \gamma = 0, 1, 2 \end{array} \right)$$

instead of (4.8).

## 5 Estimate of the First Derivatives of the Solutions to Initial Value Problems.

The aim of this section is to estimate the first derivatives of the solution to the initial value problem:

$$(5.1) \quad \begin{cases} \partial_t^2 u - \Delta u = F(t, x) & \text{in } [0, T) \times \mathbf{R}^2 \\ u(0, \cdot) = \partial_t u(0, \cdot) = 0 & \text{in } \mathbf{R}^2 \end{cases}$$

Here,  $F$  is a  $C^\infty$  function in  $[0, T) \times \mathbf{R}^2$ . For this purpose, we use the representation formula of the solution to (5.1) which has proved by M. Kovalyov[10]:

**Proposition 5.1** *Let  $u \in C^\infty([0, T) \times \mathbf{R}^2)$  be the solution of the initial value problem (5.1).*

*Then,  $u$  has the following representation:*

$$u(t, x) = \frac{1}{2\pi} \iint_{D'} r dr ds \int_{-\varphi}^{\varphi} K_1 F(s, re^{\sqrt{-1}(\theta+\psi)}) d\psi \\ + \frac{1}{2\pi} \chi(t-a) \iint_{D''} r dr ds \int_{-\pi}^{\pi} K_1 F(s, re^{\sqrt{-1}(\theta+\psi)}) d\psi$$

where

$$x = (a \cos \theta, a \sin \theta) = ae^{\sqrt{-1}\theta} \\ \varphi = \arccos \frac{a^2 + r^2 - (t-s)^2}{2ar} \quad \text{for } (s, r) \in D' \\ K_1 = \frac{1}{\{(t-s)^2 - a^2 - r^2 + 2ar \cos \psi\}^{1/2}} \\ \chi(s) = \begin{cases} 1 & (s > 0) \\ 0 & (s \leq 0) \end{cases}$$

Moreover, the domains  $D'$  and  $D''$  are defined as follows.

$$D' = \{(s, r) \mid 0 < s < t, r_1 < r < r_2\} \\ D'' = \begin{cases} \{(s, r) \mid 0 < s < t-a, 0 < r < r_1\} & \text{for } t > a \\ \emptyset & \text{for } t \leq a \end{cases}$$

where

$$(5.2) \quad r_1 = |a - t + s|, \quad r_2 = a + t - s.$$

Next, we derive representation formulae for the first derivatives of the solution of the initial value problem (5.1) from Proposition 5.1. In order to present the formulae, we set

$$(5.3) \quad \begin{aligned} \delta &= \min\{1/2, a\} \\ \tilde{\delta} &= \min\{1/2, (t - a)/2\} \end{aligned} \quad (a = |x|)$$

and split the domains  $D'$  and  $D''$  as follows:

$$\begin{aligned} D' &= \text{blue} \cup \text{white} \\ \text{blue} &= \{(s, r) \in D' \mid r_1 < r \leq r_1 + \delta \text{ or } r_2 - \delta \leq r < r_2\} \\ \text{white} &= \begin{cases} D' \setminus \text{blue} & \text{for } \delta = 1/2 \\ \emptyset & \text{for } \delta = a \end{cases} \\ D'' &= \text{black} \cup \text{red} \\ \text{black} &= \{(s, r) \in D'' \mid r_1 - \tilde{\delta} \leq r < r_1 \text{ or } 0 < r \leq \tilde{\delta}\} \\ \text{red} &= \begin{cases} D'' \setminus \text{black} & \text{for } \tilde{\delta} = 1/2 \\ \emptyset & \text{for } \tilde{\delta} = (t - a)/2 \end{cases} \end{aligned}$$

We set

$$\begin{aligned} I_{\text{blue}}(F)(t, x) &= \iint_{\text{blue}} r dr ds \int_{-\varphi}^{\varphi} K_1 F(s, r e^{\sqrt{-1}(\theta+\psi)}) d\psi \\ I_{\text{white}}(F)(t, x) &= \iint_{\text{white}} r dr ds \int_{-\varphi}^{\varphi} K_1 F(s, r e^{\sqrt{-1}(\theta+\psi)}) d\psi \\ I_{\text{black}}(F)(t, x) &= \iint_{\text{black}} r dr ds \int_{-\pi}^{\pi} K_1 F(s, r e^{\sqrt{-1}(\theta+\psi)}) d\psi \\ I_{\text{red}}(F)(t, x) &= \iint_{\text{red}} r dr ds \int_{-\pi}^{\pi} K_1 F(s, r e^{\sqrt{-1}(\theta+\psi)}) d\psi \end{aligned}$$

Then, by Proposition 5.1,  $\partial_\mu u$  ( $\mu = 0, 1, 2$ ) is represented as

$$(5.4) \quad \begin{aligned} \partial_\mu u &= \frac{1}{2\pi} \{ I_{\text{blue}}(\partial_\mu F) + \chi(a - 1/2) I_{\text{white}}(\partial_\mu F) \\ &\quad + \chi(t - a) I_{\text{black}}(\partial_\mu F) + \chi(t - a - 1) I_{\text{red}}(\partial_\mu F) \}. \end{aligned}$$

Following [10], we change the variable of integration from  $\psi$  to  $\tau$  by the map  $\psi = \Psi$ , where

$$\begin{aligned}\Psi &= \arccos[1 + P\tau - \tau], \\ P &= \frac{a^2 + r^2 - (t - s)^2}{2ar}.\end{aligned}$$

Then we have the following

**Proposition 5.2**

$$\begin{aligned}I_{white}(\partial_\mu F)(t, x) &= \sum_{j, \alpha=0}^1 \left\{ \iint_{\partial(white)} d\sigma \int_0^1 r K_2 a_\mu^\alpha(\theta + \Psi_j) n_\alpha F(s, re^{\sqrt{-1}(\theta + \Psi_j)}) d\tau \right. \\ &\quad - \iint_{white} dr ds \int_0^1 \nabla_\alpha \{r K_2 a_\mu^\alpha(\theta + \Psi_j)\} F(s, re^{\sqrt{-1}(\theta + \Psi_j)}) d\tau \\ &\quad - \iint_{white} r dr ds \int_0^1 K_2 a_\mu^\alpha(\theta + \Psi_j) (\Omega F)(s, re^{\sqrt{-1}(\theta + \Psi_j)}) \nabla_\alpha \Psi_j d\tau \left. \right\} \\ &\quad + \iint_{white} dr ds \int_{-\varphi}^\varphi K_1 a_\mu^2(\theta + \psi) (\Omega F)(s, re^{\sqrt{-1}(\theta + \psi)}) d\psi \\ I_{red}(\partial_\mu F)(t, x) &= \sum_{\alpha=0}^1 \iint_{\partial(red)} r d\sigma \int_{-\pi}^\pi K_1 a_\mu^\alpha(\theta + \psi) n_\alpha F(s, re^{\sqrt{-1}(\theta + \psi)}) d\psi \\ &\quad - \sum_{\alpha=0}^1 \iint_{red} dr ds \int_{-\pi}^\pi \nabla_\alpha \{r K_1\} a_\mu^\alpha(\theta + \psi) F(s, re^{\sqrt{-1}(\theta + \psi)}) d\psi \\ &\quad + \iint_{red} dr ds \int_{-\pi}^\pi K_1 a_\mu^2(\theta + \psi) (\Omega F)(s, re^{\sqrt{-1}(\theta + \psi)}) d\psi\end{aligned}$$

where

$$\begin{aligned}K_2 &= \frac{1}{\{2ar\tau(1 - \tau)(2 + P\tau - \tau)\}^{1/2}}, \\ \Psi_j &= (-1)^j \Psi,\end{aligned}$$

$$a_\mu^0(\theta) = \delta_\mu^0, \quad a_\mu^1(\theta) = \begin{cases} 0 & (\mu = 0) \\ \cos \theta & (\mu = 1) \\ \sin \theta & (\mu = 2) \end{cases}, \quad a_\mu^2(\theta) = \begin{cases} 0 & (\mu = 0) \\ -\sin \theta & (\mu = 1) \\ \cos \theta & (\mu = 2) \end{cases},$$

$$\nabla_\alpha = \begin{cases} \partial_s & (\alpha = 0) \\ \partial_r & (\alpha = 1) \end{cases},$$



$\vec{n} = {}^t(n_0, n_1)$  is the unit outer normal vector field on  $\partial(\text{white}) \cup \partial(\text{red})$ , and  $d\sigma$  is the line element on  $\partial(\text{white}) \cup \partial(\text{red})$ .

M. Kovalyov used these formulae in his work[10], but he has omitted the terms containing the first derivatives of  $\Psi_j$  in the above formulae. So we show the proof for completeness.

Proof. We denote  $F(s, re^{\sqrt{-1}\theta}) = G(s, r, \theta)$ . Then,

$$(\partial_\mu F)(s, re^{\sqrt{-1}\theta}) = \sum_{\alpha=0}^1 a_\mu^\alpha(\theta)(\nabla_\alpha G)(s, r, \theta) + \frac{a_\mu^2(\theta)}{r}(\partial_\theta G)(s, r, \theta).$$

Therefore,

$$\begin{aligned} I_{\text{white}}(\partial_\mu F) &= \sum_{\alpha=0}^1 \iint_{\text{white}} r dr ds \int_{-\varphi}^{\varphi} K_1 a_\mu^\alpha(\theta + \psi)(\nabla_\alpha G)(s, r, \theta + \psi) d\psi \\ &\quad + \iint_{\text{white}} dr ds \int_{-\varphi}^{\varphi} K_1 a_\mu^2(\theta + \psi)(\partial_\theta G)(s, r, \theta + \psi) d\psi. \end{aligned}$$

Changing variable from  $\psi$  to  $\tau$  by the map  $\psi = \Psi$ , we have

$$\begin{aligned} (5.5) \quad &\iint_{\text{white}} r dr ds \int_{-\varphi}^{\varphi} K_1 a_\mu^\alpha(\theta + \psi)(\nabla_\alpha G)(s, r, \theta + \psi) d\psi \\ &= \sum_{j=0}^1 \iint_{\text{white}} r dr ds \int_0^1 K_2 a_\mu^\alpha(\theta + \Psi_j)(\nabla_\alpha G)(s, r, \theta + \Psi_j) d\tau. \end{aligned}$$

Notice that

$$(5.6) \quad (\nabla_\alpha G)(s, r, \theta + \Psi_j) = \nabla_\alpha \{G(s, r, \theta + \Psi_j)\} - (\partial_\theta G)(s, r, \theta + \Psi_j) \nabla_\alpha \Psi_j.$$

Substituting (5.6) into (5.5) and integrating by parts give

$$\begin{aligned} (5.7) \quad I_{\text{white}}(\partial_\mu F) &= \sum_{j,\alpha=0}^1 \left\{ \iint_{\partial(\text{white})} d\sigma \int_0^1 r K_2 a_\mu^\alpha(\theta + \Psi_j) n_\alpha G(s, r, \theta + \Psi_j) d\tau \right. \\ &\quad - \iint_{\text{white}} dr ds \int_0^1 \nabla_\alpha \{r K_2 a_\mu^\alpha(\theta + \Psi_j)\} G(s, r, \theta + \Psi_j) d\tau \\ &\quad - \iint_{\text{white}} r dr ds \int_0^1 K_2 a_\mu^\alpha(\theta + \Psi_j) (\partial_\theta G)(s, r, \theta + \Psi_j) \nabla_\alpha \Psi_j d\tau \left. \right\} \\ &\quad + \iint_{\text{white}} dr ds \int_{-\varphi}^{\varphi} K_1 a_\mu^2(\theta + \psi) (\partial_\theta G)(s, r, \theta + \psi) d\psi. \end{aligned}$$

Similarly,

$$\begin{aligned}
I_{red}(\partial_\mu F) &= \sum_{\alpha=0}^1 \iint_{red} r dr ds \int_{-\pi}^{\pi} K_1 a_\mu^\alpha(\theta + \psi) (\nabla_\alpha G)(s, r, \theta + \psi) d\psi \\
&\quad + \iint_{red} dr ds \int_{-\pi}^{\pi} K_1 a_\mu^2(\theta + \psi) (\partial_\theta G)(s, r, \theta + \psi) d\psi \\
(5.8) \quad &= \sum_{\alpha=0}^1 \iint_{\partial(red)} r d\sigma \int_{-\pi}^{\pi} K_1 a_\mu^\alpha(\theta + \psi) n_\alpha G(s, r, \theta + \psi) d\psi \\
&\quad - \sum_{\alpha=0}^1 \iint_{red} dr ds \int_{-\pi}^{\pi} \nabla_\alpha \{r K_1\} a_\mu^\alpha(\theta + \psi) G(s, r, \theta + \psi) d\psi \\
&\quad + \iint_{red} dr ds \int_{-\pi}^{\pi} K_1 a_\mu^2(\theta + \psi) (\partial_\theta G)(s, r, \theta + \psi) d\psi.
\end{aligned}$$

Thus we get the representation formula from (5.7) and (5.8). ■

The following proposition is used to estimate the terms appearing in Proposition 5.2. This was shown in M. Kovalyov[10], except the estimates containing the derivatives of  $\Psi$ . For the sake of completeness, we give the proof of all.

### Proposition 5.3

I. Let  $(s, r) \in D'$ . Then the following estimates hold:

$$\begin{aligned}
(i) \quad &\int_{-\varphi}^{\varphi} K_1 d\psi = 2 \int_0^1 K_2 d\tau \leq \frac{M}{(ar)^{1/2}} \log \left[ 2 + \frac{ar}{(r-r_1)(r+r_2)} \chi(t-s-a) \right] \\
(ii) \quad &\int_0^1 \{|\partial_s K_2| + |\partial_r K_2|\} d\tau \leq \frac{M}{(ar)^{1/2}(r+s+a-t)} \\
(iii) \quad &\int_0^1 K_2 \{|\partial_s \Psi| + |\partial_r \Psi|\} d\tau \leq \frac{M(a+r)}{\{ar(r^2-r_1^2)(r_2^2-r^2)\}^{1/2}}
\end{aligned}$$

II. Let  $(s, r) \in D''$ . Then the following estimates hold:

$$\begin{aligned}
(i) \quad &\int_{-\pi}^{\pi} K_1 d\psi \leq \frac{M}{\{(r+r_1)(r_2-r)\}^{1/2}} \log \left[ 2 + \frac{ar}{(r_1-r)(r_2+r)} \right] \\
(ii) \quad &\int_{-\pi}^{\pi} \{|\partial_s K_1| + |\partial_r K_1|\} d\psi \leq \frac{M}{(r_1-r)\{(r+r_1)(r_2-r)\}^{1/2}}
\end{aligned}$$

Proof. The following identity can be easily verified by simple computation.

$$(5.9) \quad 1 + P = \frac{(r+r_2)(r+a-t+s)}{2ar}, \quad 1 - P = \frac{(r_2-r)(t-s-a+r)}{2ar}$$

I-(i). Changing variable by the map  $\psi = \Psi$ , we have

$$(5.10) \quad \begin{aligned} \int_{-\varphi}^{\varphi} K_1 d\psi &= 2 \int_0^1 K_2 d\tau \\ &= \frac{2^{1/2}}{(ar)^{1/2}} \int_0^1 \{\tau(1-\tau)(2+P\tau-\tau)\}^{-1/2} d\tau. \end{aligned}$$

First, we notice that in the domain  $D'$ ,

$$|P| < 1$$

$$2 + P\tau - \tau = (P+1)\tau + 2(1-\tau) \geq 2(1-\tau) \quad \text{for } \tau > 0.$$

Thus, splitting the interval of integration into two pieces, we have

$$(5.11) \quad \int_0^{1/2} \{\tau(1-\tau)(2+P\tau-\tau)\}^{-1/2} d\tau \leq 2^{1/2} \int_0^{1/2} \tau^{-1/2} d\tau = 2,$$

$$(5.12) \quad \begin{aligned} &\int_{1/2}^1 \{\tau(1-\tau)(2+P\tau-\tau)\}^{-1/2} d\tau \\ &\leq 2^{1/2} \int_{1/2}^1 \{(1-\tau)(2+P\tau-\tau)\}^{-1/2} d\tau \\ &\leq 2^{1/2} \int_0^1 \partial_\tau \{-2(1-\tau)^{1/2}\} (2+P\tau-\tau)^{-1/2} d\tau \\ &= 2 + 2^{1/2}(1-P) \int_0^1 (1-\tau)^{1/2} (2+P\tau-\tau)^{-3/2} d\tau \\ &\leq 2 + (1-P) \int_0^1 (2+P\tau-\tau)^{-1} d\tau \\ &\leq M \log \left[ 2 + \frac{1}{1+P} \right]. \end{aligned}$$

Since  $P+1 > 1/2$  for  $t-s \leq a$ , the estimate I-(i) follows from (5.10)-(5.12) and (5.9).

I-(ii). Since

$$\begin{aligned} \partial_s K_2 &= \frac{-(t-s)\tau^{1/2}}{(2ar)^{3/2}(1-\tau)^{1/2}(2+P\tau-\tau)^{3/2}} \\ \partial_\tau K_2 &= -\frac{1}{2r} K_2 - \frac{\tau^{1/2}}{2^{3/2}(ar)^{1/2}(1-\tau)^{1/2}(2+P\tau-\tau)^{3/2}} \left( \frac{1}{a} - \frac{P}{r} \right), \end{aligned}$$

then we have

$$(5.13) \quad \begin{aligned} &\int_0^1 \{|\partial_s K_2| + |\partial_\tau K_2|\} d\tau \\ &\leq \frac{1}{2r} \int_0^1 K_2 d\tau + \frac{t-s+a+r}{(2ar)^{3/2}} \int_0^1 \frac{\tau^{1/2}}{(1-\tau)^{1/2}(2+P\tau-\tau)^{3/2}} d\tau. \end{aligned}$$

By I-(i), we have

$$(5.14) \quad \begin{aligned} \frac{1}{2r} \int_0^1 K_2 d\tau &\leq Ma(ar)^{-3/2} \log[2 + (1 + P)^{-1}] \\ &\leq Ma(ar)^{-3/2} (1 + P)^{-1}. \end{aligned}$$

On the other hand, since

$$\int_0^1 \frac{\tau^{1/2}}{(1 - \tau)^{1/2} (2 + P\tau - \tau)^{3/2}} d\tau \leq \int_0^1 (1 - \tau)^{-1/2} (2 + P\tau - \tau)^{-3/2} d\tau,$$

we have by the method from which (5.12) was derived,

$$(5.15) \quad \int_0^1 \frac{\tau^{1/2}}{(1 - \tau)^{1/2} (2 + P\tau - \tau)^{3/2}} d\tau \leq \frac{3}{2^{1/2}} \frac{1}{1 + P}.$$

Therefore it follows from (5.13), (5.14) and (5.15) that

$$\begin{aligned} \int_0^1 \{|\partial_s K_2| + |\partial_r K_2|\} d\tau &\leq \frac{M(t - s + a + r)}{(ar)^{3/2}} \frac{1}{1 + P} \\ &\leq \frac{M}{(ar)^{1/2} (r + s + a - t)}. \end{aligned}$$

I-(iii). We can easily verify that

$$\begin{aligned} \partial_r \Psi &= \left( \frac{P}{r} - \frac{1}{a} \right) \frac{\tau^{1/2}}{\{(1 - P)(2 + P\tau - \tau)\}^{1/2}} \\ \partial_s \Psi &= -\frac{t - s}{ar} \frac{\tau^{1/2}}{\{(1 - P)(2 + P\tau - \tau)\}^{1/2}}. \end{aligned}$$

We use the same method as we used in I-(i) and obtain

$$\begin{aligned} &\int_0^1 K_2 \{|\partial_r \Psi| + |\partial_s \Psi|\} d\tau \\ &= \frac{1}{(2ar)^{1/2}} \left\{ \left| \frac{1}{a} - \frac{P}{r} \right| + \frac{t - s}{ar} \right\} \frac{1}{(1 - P)^{1/2}} \int_0^1 \frac{d\tau}{(1 - \tau)^{1/2} (2 + P\tau - \tau)} \\ &\leq \frac{M}{(ar)^{1/2}} \frac{a + r}{ar} \frac{1}{\{(1 - P)(1 + P)\}^{1/2}}. \end{aligned}$$

Thus we get the estimate I-(iii).

II-(i). In the domain  $D''$ ,  $P < -1$  and  $t - s > a + r$ . Therefore,

$$(5.16) \quad \int_{-\pi}^{\pi} K_1 d\psi = \frac{2}{\{(t-s)^2 - a^2 - r^2\}^{1/2}} \int_0^{\pi} \frac{d\psi}{(1 - P^{-1} \cos \psi)^{1/2}} \\ \leq \frac{2}{\{(t-s)^2 - a^2 - r^2\}^{1/2}} \left\{ \int_0^{3\pi/4} \frac{d\psi}{(1 - 2^{-1/2})^{1/2}} + \int_{3\pi/4}^{\pi} \frac{d\psi}{(1 - P^{-1} \cos \psi)^{1/2}} \right\}$$

Further,

$$(5.17) \quad \int_{3\pi/4}^{\pi} \frac{d\psi}{(1 - P^{-1} \cos \psi)^{1/2}} = \int_0^{\pi/4} \frac{d\psi}{(1 + P^{-1} \cos \psi)^{1/2}} \\ = \int_0^{\pi/4} \frac{d\psi}{\{1 - \cos \psi + (1 + P^{-1}) \cos \psi\}^{1/2}} \\ \leq \int_0^{\pi/4} \frac{d\psi}{\{2\pi^{-2}\psi^2 + (1 + P^{-1})2^{-1/2}\}^{1/2}} \\ \leq M\{1 - \log(1 + P^{-1})\} \\ = M\left\{1 + \log\left(1 - \frac{1}{1 + P}\right)\right\}$$

From (5.17), (5.9) and

$$(t-s)^2 - a^2 - r^2 = -2arP \geq ar(1 - P) = \frac{1}{2}(r + r_1)(r_2 - r),$$

we get II-(i).

II-(ii). We can easily see that

$$\partial_s K_1 = \frac{t-s}{((t-s)^2 - a^2 - r^2 + 2ar \cos \psi)^{3/2}} \\ \partial_r K_1 = \frac{r - a \cos \psi}{((t-s)^2 - a^2 - r^2 + 2ar \cos \psi)^{3/2}}$$

Thus,

$$(5.18) \quad \int_{-\pi}^{\pi} \{|\partial_s K_1| + |\partial_r K_1|\} d\psi \leq 2(t-s) \int_{-\pi}^{\pi} \frac{d\psi}{((t-s)^2 - a^2 - r^2 + 2ar \cos \psi)^{3/2}} \\ = \frac{4(t-s)}{(-2arP)^{3/2}} \int_0^{\pi} \frac{d\psi}{(1 - P^{-1} \cos \psi)^{3/2}}.$$

We get by the same way as the proof of II-(i) that

$$(5.19) \quad \int_0^{\pi} \frac{d\psi}{(1 - P^{-1} \cos \psi)^{3/2}} \leq \int_0^{4\pi/3} \frac{d\psi}{(1 - 2^{-1/2})^{3/2}} + \int_0^{\pi/4} \frac{d\psi}{\{2\pi^{-2}\psi^2 + (1 + P^{-1})2^{-1/2}\}^{3/2}} \\ \leq M\left(1 + \frac{P}{1 + P}\right) \\ \leq M\frac{P}{1 + P}.$$

Therefore it follows from (5.18) and (5.19) that

$$\begin{aligned}
\int_{-\pi}^{\pi} \{|\partial_s K_1| + |\partial_r K_1|\} d\psi &\leq \frac{M(t-s)}{(-2arP)^{3/2}} \frac{P}{1+P} \\
&\leq \frac{M(t-s)}{(ar)^{3/2}(1-P)^{1/2}} \frac{-1}{1+P} \\
&\leq \frac{M}{(r_1-r)(ar)^{1/2}(1-P)^{1/2}}
\end{aligned}$$

and we get the estimate II-(ii). ■

Now we can show the estimates for the first derivatives of the solution to the initial value problem (5.1).

**Proposition 5.4** *Let  $u \in C^\infty([0, T] \times \mathbf{R}^2)$  be the solution of the initial value problem (5.1).*

*And let  $w(s, r)$  be a positive function that satisfies*

$$(5.20) \quad (i) \quad k-1 \leq r \leq k+1 \Rightarrow \frac{1}{M} w(s, k) \leq w(s, r) \leq M w(s, k)$$

$(k = 1, 2, \dots; M \text{ is independent of } s, k, r)$

$$\begin{aligned}
(5.21) \quad (ii) \quad \frac{1}{w(s, r)} &\leq M \left\{ \sum_{i=1}^L \frac{1}{(r+s+1)(|r-c_i s|+1)} + \frac{1}{(r+s+1)^{1+2\gamma}(r+1)^{1-2\gamma}} \right. \\
&\quad \left. + \frac{1}{(r+s+1)^{1+\epsilon}(|r-s|+1)^{1-\epsilon}} \right\} \\
&\quad (c_i \neq 1 \ (i = 1, 2, \dots, L), \ 0 < \gamma < 1/2, \ 0 < \epsilon < 1)
\end{aligned}$$

Then the following estimate holds:

$$\begin{aligned}
|\partial u(t, x)| &\leq \frac{M}{(|x|+1)^{1/2-\gamma}(|x|+t+1)^\gamma(|x|-t+1)^{1/2}} \left\{ \sum_{|A| \leq 3} \sup_{0 < s < t} \|w(s, |\cdot|) \mathcal{D}^A F(s, \cdot)\|_0 \right. \\
(5.22) \quad &\quad \left. + \sum_{|A| \leq 2} \sup_{0 < s < t} \|w(s, |\cdot|) \mathcal{D}^A \partial_t F(s, \cdot)\|_0 \right\}.
\end{aligned}$$

Proof. By (5.4) and Proposition 5.2, we have

$$\begin{aligned}
|\partial u(t, x)| &\leq \left\{ \sup_{0 < s < t} \sup_{x \in \mathbf{R}^2} |x|^{1/2} w(t, |x|) |F(t, x)| + \sup_{0 < s < t} \sup_{x \in \mathbf{R}^2} |x|^{1/2} w(t, |x|) |\partial F(t, x)| \right. \\
(5.23) \quad &\quad \left. + \sup_{0 < s < t} \sup_{x \in \mathbf{R}^2} |x|^{1/2} w(t, |x|) |\Omega F(t, x)| \right\} \{I'_1 + \dots + I'_5 + I''_1 + \dots + I''_4\}
\end{aligned}$$

where  $I'_i$  ( $i = 1, \dots, 5$ ) and  $I''_i$  ( $i = 1, \dots, 4$ ) are defined as follows.

$$\begin{aligned}
I'_1 &= \iint_{\text{blue}} \frac{r^{1/2}}{w(s, r)} dr ds \int_{-\varphi}^{\varphi} K_1 d\psi \\
I'_2 &= \iint_{\partial(\text{white})} \frac{r^{1/2}}{w(s, r)} d\sigma \int_0^1 K_2 d\tau \\
I'_3 &= \iint_{\text{white}} \frac{1}{r^{1/2} w(s, r)} dr ds \int_0^1 K_2 d\tau \\
I'_4 &= \iint_{\text{white}} \frac{r^{1/2}}{w(s, r)} dr ds \int_0^1 \{|\partial_s K_2| + |\partial_r K_2|\} d\tau \\
I'_5 &= \iint_{\text{white}} \frac{r^{1/2}}{w(s, r)} dr ds \int_0^1 K_2 \{|\partial_s \Psi| + |\partial_r \Psi|\} d\tau \\
I''_1 &= \iint_{\text{black}} \frac{r^{1/2}}{w(s, r)} dr ds \int_{-\pi}^{\pi} K_1 d\psi \\
I''_2 &= \iint_{\partial(\text{red})} \frac{r^{1/2}}{w(s, r)} d\sigma \int_{-\pi}^{\pi} K_1 d\psi \\
I''_3 &= \iint_{\text{red}} \frac{1}{r^{1/2} w(s, r)} dr ds \int_{-\pi}^{\pi} K_1 d\psi \\
I''_4 &= \iint_{\text{red}} \frac{r^{1/2}}{w(s, r)} dr ds \int_{-\pi}^{\pi} \{|\partial_s K_1| + |\partial_r K_1|\} d\psi
\end{aligned}$$

Here,  $I'_i$  ( $i = 1, \dots, 5$ ) are integrals that are related to the domain  $D'$ , and  $I''_i$  ( $i = 1, \dots, 4$ ) to the domain  $D''$ . We show in the following that

$$(5.24) \quad I'_i \leq \frac{M}{(a+1)^{1/2-\gamma}(a+t+1)^\gamma(|a-t|+1)^{1/2}} \quad (i = 1, \dots, 5),$$

$$(5.25) \quad I''_i \leq \frac{M}{(a+1)^{1/2-\gamma}(a+t+1)^\gamma(|a-t|+1)^{1/2}} \quad (i = 1, \dots, 4),$$

where  $a = |x|$ . By Lemma 6 in [10], it holds that

$$(5.26) \quad |x| |f(x)| w(s, |x|)^2 \leq M \sum_{|A| \leq 2} \|w(s, |\cdot|) \mathcal{D}^A f\|_0^2$$

for  $f \in C_0^\infty(\mathbf{R}^2)$ . Then we get the estimate (5.22) from (5.23), (5.24), (5.25) and (5.26).

First, we prove (5.24). To prove this, we introduce some notations. Set

$$\begin{aligned}
\xi(s, r) &= \xi_1(s, r) + \xi_2(s, r), \\
\xi_1(s, r) &= \sum_{i=1}^L \frac{1}{(r+s+1)(|r-c_i s|+1)} + \frac{1}{(r+s+1)^{1+2\gamma}(r+1)^{1-2\gamma}}, \\
\xi_2(s, r) &= \frac{1}{(r+s+1)^{1+\epsilon}(|r-s|+1)^{1-\epsilon}}.
\end{aligned}$$

Then by the assumption (5.21) on  $w(s, r)$ ,

$$(5.27) \quad \frac{1}{w(s, r)} \leq M\xi(s, r).$$

Moreover, set

$$\begin{aligned} \eta(s, r) &= \eta_1(s, r) + \eta_2(s, r), \\ \eta_1(s, r) &= \sum_{i=1}^L \frac{1}{(r+s+1)^{\tilde{\lambda}}(|r-c_i s|+1)} + \frac{1}{(r+s+1)^{\tilde{\lambda}+2\gamma}(r+1)^{1-2\gamma}}, \\ \eta_2(s, r) &= \frac{1}{(r+s+1)^{\tilde{\lambda}+\epsilon}(|r-s|+1)^{1-\epsilon}}, \end{aligned}$$

where

$$(5.28) \quad 0 < \lambda < \min\{\gamma, 1/2 - \gamma\}, \quad \tilde{\lambda} = 1/2 - \gamma - \lambda.$$

Since  $r+s \geq |a-t|$  for  $(s, r) \in D'$ , we have

$$(5.29) \quad \xi_i(s, r) \leq \frac{M}{(|a-t|+1)^{1/2+\gamma+\lambda}} \eta_i(s, r) \quad (i=1, 2)$$

$$(5.30) \quad \frac{1}{w(s, r)} \leq \frac{M}{(|a-t|+1)^{1/2+\gamma+\lambda}} \eta(s, r)$$

for  $(s, r) \in D'$ . But in the estimate of  $I'_1$  and  $I'_5$ ,  $\xi_2(s, r)$  is treated in another way.

(i) *Estimate of  $I'_1$*

By Proposition 5.3.I.(i),

$$(5.31) \quad I'_1 \leq \frac{M}{a^{1/2}} \iint_{blue} \frac{1}{w(s, r)} \log \left[ 2 + \frac{ar}{(r-r_1)(r+r_2)} \chi(t-s-a) \right] dr ds.$$

Therefore it follows from (5.31) and (5.20) that

$$(5.32) \quad I'_1 \leq \frac{M}{a^{1/2}} \left\{ \int_0^t \frac{ds}{w(s, r_1)} \int_{r_1}^{r_1+\delta} \log \left[ 2 + \frac{ar}{(r-r_1)(r+r_2)} \chi(t-s-a) \right] dr \right. \\ \left. + \int_0^t \frac{ds}{w(s, r_2)} \int_{r_2-\delta}^{r_2} \log \left[ 2 + \frac{ar}{(r-r_1)(r+r_2)} \chi(t-s-a) \right] dr \right\}.$$

Let us consider the integrals of  $\log[2 + ar/(r-r_1)(r+r_2) \cdot \chi(t-s-a)]$ . For  $0 < s < t-a$  and  $r_2 - \delta < r < r_2$ , it follows from (5.2) and (5.3) that

$$r - r_1 > r_2 - \delta - r_1 = 2a - \delta > a.$$



Then we have

$$(5.33) \quad \log \left[ 2 + \frac{ar}{(r-r_1)(r+r_2)} \right] \leq \log \left[ 2 + \frac{a}{r-r_1} \right] \leq \log 3.$$

For  $0 < s < t - a$  and  $r_1 < r < r_1 + \delta$ , we have

$$(5.34) \quad \begin{aligned} \int_{r_1}^{r_1+\delta} \log \left[ 2 + \frac{ar}{(r-r_1)(r+r_2)} \right] dr &\leq \int_{r_1}^{r_1+\delta} \log \left[ 2 + \frac{r}{r-r_1} \right] dr \\ &= \delta \left[ \log(3\delta + r_1) - \log \delta \right] + \frac{r_1}{3\delta} \log(1 + 3\delta/r_1) \\ &\leq \delta \log(3/2 + t - a) + \delta^{1/2} 2e^{-1} + \delta \\ &\leq M\delta^{1/2} \log[2 + |a - t|]. \end{aligned}$$

Therefore it follows from (5.32), (5.33) and (5.34) that

$$(5.35) \quad I'_1 \leq \frac{M\delta^{1/2}}{a^{1/2}} \log[2 + |a - t|] \left\{ \int_0^t \frac{ds}{w(s, r_1)} + \int_0^t \frac{ds}{w(s, r_2)} \right\}.$$

We next show

$$(5.36) \quad \int_0^t \frac{ds}{w(s, r_i)} \leq \frac{M}{(|a - t| + 1)^{1/2 + \gamma + \lambda}} \quad (i = 1, 2).$$

We use (5.30) for  $1/w(s, r_2)$  and obtain

$$(5.37) \quad \int_0^t \frac{ds}{w(s, r_2)} \leq \frac{M}{(|a - t| + 1)^{1/2 + \gamma + \lambda}} \int_0^t \eta(s, r_2) ds.$$

Moreover,

$$(5.38) \quad \begin{aligned} &\int_0^t \eta(s, r_2) ds \\ &\leq M \int_0^t \left\{ \sum_{i=1}^L \frac{1}{(|r_2 - c_i s| + 1)^{1+\bar{\lambda}}} + \frac{1}{(|r_2 - s| + 1)^{1+\bar{\lambda}}} + \frac{1}{(r_2 + 1)^{1+\bar{\lambda}}} \right\} ds \\ &\leq M \int_{-\infty}^{\infty} \left\{ \sum_{i=1}^L \frac{1}{(|r_2 - c_i s| + 1)^{1+\bar{\lambda}}} + \frac{1}{(|r_2 - s| + 1)^{1+\bar{\lambda}}} + \frac{1}{(|r_2| + 1)^{1+\bar{\lambda}}} \right\} ds \leq M. \end{aligned}$$

Therefore, from (5.37) and (5.38) we have (5.36) for  $i = 2$ . The treatment for  $i = 1$  is slightly different. We remark that  $|r_1 - s| = |a - t|$  for  $(t - a)_+ < s < t$  by the definition (5.2), where  $x_+ = \max\{0, x\}$ . Then we see from (5.28) that

$$(5.39) \quad \xi_2(s, r_1) \leq \frac{1}{(|a - t| + 1)^{1/2 + \gamma + \lambda} (r_1 + s + 1)^{1 + \min\{\epsilon, \bar{\lambda}\}}}$$

for  $(t-a)_+ < s < t$ . Therefore it follows (5.27), (5.29) and (5.39) that

$$\begin{aligned} \int_0^t \frac{ds}{w(s, r_1)} &\leq \frac{M}{(|a-t|+1)^{1/2+\gamma+\lambda}} \left\{ \int_0^t \eta_1(s, r_1) ds + \int_0^{(t-a)_+} \eta_2(s, r_1) ds \right. \\ &\quad \left. + \int_{(t-a)_+}^t \frac{ds}{(r_1+s+1)^{1+\min\{\epsilon, \lambda\}}} \right\} \\ &\leq \frac{M}{(|a-t|+1)^{1/2+\gamma+\lambda}}. \end{aligned}$$

Combining (5.35) and (5.36), we have

$$\begin{aligned} I'_1 &\leq \frac{M\delta^{1/2}}{a^{1/2}(|a-t|+1)^{1/2+\gamma+\lambda}} \log[2+|a-t|] \\ &\leq \frac{M}{(a+1)^{1/2}(|a-t|+1)^{1/2+\gamma}} \\ (5.40) \quad &\leq \frac{M}{(a+1)^{1/2-\gamma}(a+t+1)^\gamma(|a-t|+1)^{1/2}}. \end{aligned}$$

Here we use the fact that

$$(5.41) \quad \frac{a+t+1}{(a+1)(|a-t|+1)} \leq 4 \quad \text{for } a, t \geq 0.$$

(ii) *Estimate of  $I'_2$*

By Proposition 5.3.I.(i),

$$(5.42) \quad I'_2 \leq \frac{M}{a^{1/2}} \iint_{\partial(\overline{white})} \frac{1}{w(s, r)} \log \left[ 2 + \frac{ar}{(r-r_1)(r+r_2)} \chi(t-s-a) \right] d\sigma.$$

Let  $0 \leq s \leq t-a$  and  $(s, r) \in \overline{white}$ . Then

$$\frac{ar}{(r-r_1)(r+r_2)} \leq \frac{r}{r-r_1} \leq \frac{r_1+1/2}{1/2} \leq 2(t-a+1/2).$$

So we have

$$(5.43) \quad \frac{ar}{(r-r_1)(r_2+r)} \chi(t-s-a) \leq 2(2+|a-t|)$$

for  $(s, r) \in \overline{white}$ . Hence from (5.42) and (5.43) we get

$$(5.44) \quad I'_2 \leq \frac{M}{a^{1/2}} \log[2+|a-t|] \iint_{\partial(\overline{white})} \frac{d\sigma}{w(s, r)}.$$

We have already computed the integral of  $1/w(s, r)$  in the estimate of  $I'_1$ , except the one on  $\{0\} \times (|a - t| + \delta, a + t - \delta)$ . Applying (5.30) for  $s = 0$ , we have

$$(5.45) \quad \int_{|a-t|+\delta}^{a+t-\delta} \frac{dr}{w(0, r)} \leq \frac{M}{(|a-t|+1)^{1/2+\gamma+\lambda}}.$$

Therefore it follows from (5.44), (5.36) and (5.45) that

$$(5.46) \quad I'_2 \leq \frac{M}{a^{1/2}(|a-t|+1)^{1/2+\gamma+\lambda}} \log[2 + |a-t|].$$

Since  $a \geq \delta = 1/2$  when the domain *white* is not empty, we have (5.24) for  $i = 2$  by the way from which (5.40) was derived.

(iii) *Estimate of  $I'_3$*

By Proposition 5.3.I.(i),

$$(5.47) \quad I'_3 \leq \frac{M}{a^{1/2}} \iint_{white} \frac{1}{rw(s, r)} \log \left[ 2 + \frac{ar}{(r-r_1)(r+r_2)} \chi(t-s-a) \right] dr ds.$$

Further, by (5.30) and (5.43),

$$(5.48) \quad I'_3 \leq \frac{M}{a^{1/2}(|a-t|+1)^{1/2+\gamma+\lambda}} \log[2 + |a-t|] \iint_{white} \frac{\eta(s, r)}{r} dr ds.$$

Since  $r \geq \delta = 1/2$  in the domain *white*, we have

$$(5.49) \quad \frac{\eta(s, r)}{r} \leq M \left\{ \sum_{i=1}^L \frac{1}{(r+1)^{1+\bar{\lambda}/2}(|r-c_i s|+1)^{1+\bar{\lambda}/2}} + \frac{1}{(r+1)^{2+\bar{\lambda}}} + \frac{1}{(r+1)^{1+\bar{\lambda}/2}(|r-s|+1)^{1+\bar{\lambda}/2}} \right\}$$

for  $(s, r) \in white$ . Concerning the right-hand side of (5.49), the integral of the first and the third term are shown to be bounded by a constant  $M$  in the same way as (5.38). As for the second term, we see that

$$(5.50) \quad \begin{aligned} \iint_{white} \frac{dr ds}{(r+1)^{2+\bar{\lambda}}} &\leq \int_0^t ds \int_{r_1}^{r_2} \frac{dr}{(r+1)^{2+\bar{\lambda}}} \\ &\leq \frac{1}{1+\bar{\lambda}} \int_0^t \frac{ds}{(r_1+1)^{1+\bar{\lambda}}} \leq M. \end{aligned}$$

Therefore it follows from (5.49) and (5.50) that

$$(5.51) \quad \iint_{white} \frac{\eta(s, r)}{r} dr ds \leq M.$$

Hence from (5.48) and (5.51) we have (5.24) for  $i = 3$ .

(iv) *Estimate of  $I'_4$*

By Proposition 5.3.I.(ii),

$$(5.52) \quad I'_4 \leq \frac{M}{a^{1/2}} \iint_{white} \frac{1}{w(s, r)(r + s + a - t)} dr ds.$$

Applying (5.30) to the right-hand side of (5.52), we obtain

$$(5.53) \quad I'_4 \leq \frac{M}{a^{1/2}(|a - t| + 1)^{1/2 + \gamma + \lambda}} \iint_{white} \frac{\eta(s, r)}{r + s + a - t} dr ds.$$

Since  $r + s + a - t \geq \delta = 1/2$  in the domain *white*, we have

$$\begin{aligned} \frac{\eta(s, r)}{r + s + a - t} &\leq \frac{M}{r + s + a - t + 1} \left\{ \sum_{i=1}^L \frac{1}{(r + s + 1)^{\bar{\lambda}/2} (|r - c_i s| + 1)^{1 + \bar{\lambda}/2}} \right. \\ &\quad \left. + \frac{1}{(r + s + 1)^{\bar{\lambda}/2} (r + 1)^{1 + \bar{\lambda}/2}} + \frac{1}{(r + s + 1)^{\bar{\lambda}/2} (|r - s| + 1)^{1 + \bar{\lambda}/2}} \right\} \end{aligned}$$

for  $(s, r) \in white$ . Hence we find by the change of variables  $(\alpha, \beta) = (s + r, s - r)$  that

$$(5.54) \quad \iint_{white} \frac{\eta(s, r)}{r + s + a - t} dr ds \leq M.$$

Therefore from (5.53), (5.54) and (5.41) we obtain (5.24) for  $i = 4$ .

(v) *Estimate of  $I'_5$*

By Proposition 5.3.I.(iii),

$$(5.55) \quad I'_5 \leq \frac{M}{a^{1/2}} \iint_{white} \frac{1}{w(s, r)} \frac{a + r}{\{(r^2 - r_1^2)(r_2^2 - r^2)\}^{1/2}} dr ds.$$

We notice that for  $(s, r) \in white$ ,

$$r_2 + r \geq a, \quad r_2 + r \geq r;$$

$$r + r_1 \geq a, \quad r + r_1 \geq r \quad \text{for } r \geq (r_2 - r_1)/2;$$

$$r_2 - r \geq a, \quad r_2 - r \geq r \quad \text{for } r \leq (r_2 - r_1)/2.$$

Hence we have

$$(5.56) \quad \frac{a + r}{\{(r^2 - r_1^2)(r_2^2 - r^2)\}^{1/2}} \leq 2 \left\{ \frac{1}{(r^2 - r_1^2)^{1/2}} + \frac{1}{\{(r - r_1)(r_2 - r)\}^{1/2}} \right\}$$

for  $(s, r) \in \text{white}$ . Therefore it follows from (5.55), (5.27) and (5.56) that

$$(5.57) \quad I'_5 \leq \frac{M}{a^{1/2}} \iint_{\text{white}} \xi(s, r) \left\{ \frac{1}{(r^2 - r_1^2)^{1/2}} + \frac{1}{\{(r - r_1)(r_2 - r)\}^{1/2}} \right\} dr ds.$$

We show in the following that

$$(5.58) \quad \iint_{\text{white}} \xi(s, r) \frac{1}{(r^2 - r_1^2)^{1/2}} dr ds \leq \frac{M}{(|a - t| + 1)^{1/2 + \gamma + \lambda}}.$$

We use (5.29) for  $\xi_1(s, r)$  and obtain

$$(5.59) \quad \begin{aligned} \iint_{\text{white}} \xi_1(s, r) \frac{1}{(r^2 - r_1^2)^{1/2}} dr ds \\ \leq \frac{M}{(|a - t| + 1)^{1/2 + \gamma + \lambda}} \iint_{\text{white}} \eta_1(s, r) \frac{1}{(r^2 - r_1^2)^{1/2}} dr ds. \end{aligned}$$

Since  $r \pm (s + a - t) \geq \delta = 1/2$  for  $(s, r) \in \text{white}$ , we have

$$\begin{aligned} \frac{\eta_1(s, r)}{(r^2 - r_1^2)^{1/2}} &\leq \frac{M}{(r + s + a - t + 1)^{1/2} (r - s - a + t + 1)^{1/2}} \eta_1(s, r) \\ &\leq \frac{M}{r + s + a - t + 1} \left\{ \sum_{i=1}^L \frac{1}{(r + s + 1)^{\bar{\lambda}/2} (|r - c_i s| + 1)^{1 + \bar{\lambda}/2}} \right. \\ &\quad \left. + \frac{1}{(r + s + 1)^{\bar{\lambda}/2} (r + 1)^{1 + \bar{\lambda}/2}} \right\} \\ &\quad + \frac{M}{r - s - a + t + 1} \left\{ \sum_{i=1}^L \frac{1}{(|r - s| + 1)^{\bar{\lambda}/2} (|r - c_i s| + 1)^{1 + \bar{\lambda}/2}} \right. \\ &\quad \left. + \frac{1}{(|r - s| + 1)^{\bar{\lambda}/2} (r + 1)^{1 + \bar{\lambda}/2}} \right\} \end{aligned}$$

for  $(s, r) \in \text{white}$ . Hence by the change of variables  $(\alpha, \beta) = (s + r, s - r)$  we have

$$(5.60) \quad \iint_{\text{white}} \frac{\eta_1(s, r)}{(r^2 - r_1^2)^{1/2}} dr ds \leq M.$$

Therefore it follows from (5.59) and (5.60) that

$$(5.61) \quad \iint_{\text{white}} \frac{\xi_1(s, r)}{(r^2 - r_1^2)^{1/2}} dr ds \leq \frac{M}{(|a - t| + 1)^{1/2 + \gamma + \lambda}}.$$

On the other hand, we see that, for  $(s, r) \in \text{white}$ ,

$$\frac{\xi_2(s, r)}{(r^2 - r_1^2)^{1/2}}$$

$$\begin{aligned}
&\leq \frac{M}{(r+s+a-t+1)^{1/2}(r-s-a+t+1)^{1/2}} \left[ \{1 - \chi(|r-s| - |a-t|/2)\} \xi_2(s, r) \right. \\
&\quad \left. + \chi(|r-s| - |a-t|/2) \xi_2(s, r) \right] \\
&\leq \frac{M}{(|a-t|+1)^{1/2+\gamma+\lambda}} \frac{1}{(r+s+a-t+1)^{1/2}} \cdot \\
&\quad \cdot \left\{ \frac{1}{(r+s+1)^{1/2+\bar{\lambda}+\epsilon}(|r-s|+1)^{1-\epsilon}} + \frac{1}{(r-s-a+t+1)^{1/2}(r+s+1)^{1+\min\{\epsilon, \bar{\lambda}\}}} \right\} \\
&\leq \frac{M}{(|a-t|+1)^{1/2+\gamma+\lambda}} \frac{1}{(r+s+a-t+1)^{1/2}} \cdot \\
&\quad \cdot \left\{ \frac{1}{(r+s+1)^{1/2+\bar{\lambda}/2}(|r-s|+1)^{1+\bar{\lambda}/2}} \right. \\
&\quad \left. + \frac{1}{(r-s-a+t+1)^{1/2}(|r-s|+1)^{1/2+\min\{\epsilon, \bar{\lambda}\}/2}(r+s+1)^{1/2+\min\{\epsilon, \bar{\lambda}\}/2}} \right\}.
\end{aligned}$$

Here, we have used that  $t - a > 0$  on the support of  $1 - \chi(|r - s| - |a - t|/2)$ . Therefore it follows that

$$(5.62) \quad \iint_{white} \frac{\xi_2(s, r)}{(r^2 - r_1^2)^{1/2}} dr ds \leq \frac{M}{(|a-t|+1)^{1/2+\gamma+\lambda}}.$$

Combining (5.61) and (5.62), we obtain (5.58). To estimate the second term in the right hand side of (5.57), we note that

$$(5.63) \quad \frac{1}{\{(r-r_1)(r_2-r)\}^{1/2}} \leq \frac{1}{\{(a+t-s-r+1)(r+s+a-t+1)\}^{1/2}} + \frac{1}{\{(a+t-s-r+1)(r-s-a+t+1)\}^{1/2}}.$$

Using (5.29) for the first term of (5.63) and the method above for the second term of (5.63), we get

$$(5.64) \quad \iint_{white} \frac{\xi(s, r)}{\{(r-r_1)(r_2-r)\}^{1/2}} dr ds \leq \frac{M}{(|a-t|+1)^{1/2+\gamma+\lambda}}.$$

Therefore from (5.57), (5.58) and (5.64) we obtain (5.24) for  $i = 5$ . Consequently, we have proved (5.24).

Next, we prove (5.25).  $\xi(s, r)$  and  $\eta(s, r)$  are used again, but we do not consider  $\xi_1(s, r)$  and  $\xi_2(s, r)$  separately. Since  $t > a$  when  $D''$  is not empty,  $r_1 = t - a - s$ .

(vi) *Estimate of  $I_1''$*

By Proposition 5.3.II.(i),

$$(5.65) \quad I_1'' \leq M \iint_{black} \frac{r^{1/2}}{w(s,r)\{(r+r_1)(r_2-r)\}^{1/2}} \log \left[ 2 + \frac{ar}{(r_1-r)(r+r_2)} \right] dr ds.$$

In the domain  $D''$ , we use the following facts:

$$(5.66) \quad \frac{1}{\{(r+r_1)(r_2-r)\}^{1/2}} \leq \frac{M}{\{(t-a)(a+t)\}^{1/2}} \quad \text{for } r+s \leq (t-a)/2,$$

$$(5.67) \quad \frac{1}{w(s,r)} \leq \frac{M}{(|a-t|+1)^{1/2+\gamma+\lambda}} \eta(s,r) \quad \text{for } r+s \geq (t-a)/2,$$

$$(5.68) \quad r_2 - r \geq 2a + \tilde{\delta} \quad \text{for } r \leq r_1 - \tilde{\delta},$$

$$(5.69) \quad \frac{ar}{(r_1-r)(r+r_2)} \leq \frac{r}{r_1-r} \leq M\{(t-a)\chi(2r-r_1)+1\} \quad \text{for } r \leq r_1 - \tilde{\delta}.$$

It follows from (5.66)-(5.69) and (5.3) that

$$(5.70) \quad \begin{aligned} I_1'' &\leq \frac{M}{\{(t-a)(a+t)\}^{1/2}} \iint_{r+s \leq (t-a)/2}^{black} \frac{r^{1/2}}{w(s,r)} dr ds \\ &+ \frac{M}{(2a+\tilde{\delta})^{1/2}(|a-t|+1)^{1/2+\gamma+\lambda}} \iint_{(t-a)/2 \leq r+s \leq t-a-\tilde{\delta}}^{black} \eta(s,r) \frac{r^{1/2}}{(r+r_1)^{1/2}} dr ds \\ &+ \frac{M}{(|a-t|+1)^{1/2+\gamma+\lambda}} \cdot \\ &\cdot \iint_{t-a-\tilde{\delta} \leq r+s}^{black} \eta(s,r) \frac{r^{1/2}}{\{(r+r_1)(r_2-r)\}^{1/2}} \log \left[ 2 + \frac{ar}{(r_1-r)(r+r_2)} \right] dr ds. \end{aligned}$$

Moreover,

$$(5.71) \quad \iint_{r+s \leq (t-a)/2}^{black} \frac{r^{1/2}}{w(s,r)} dr ds \leq M \int_0^{t-a} \xi(s,0) ds \int_0^{\tilde{\delta}} r^{1/2} dr \leq M \tilde{\delta}^{3/2},$$

$$(5.72) \quad \begin{aligned} &\iint_{(t-a)/2 \leq r+s \leq t-a-1/2}^{black} \eta(s,r) \frac{r^{1/2}}{(r+r_1)^{1/2}} dr ds \\ &\leq M \int_0^{t-a} \frac{\eta(s,0)}{(1+r_1)^{1/2}} ds \int_0^{1/2} dr \\ &\leq M \int_0^{t-a} \frac{1}{(1+t-a-s)^{1/2}} \left\{ \sum_{i=1}^L \frac{1}{(s+1)^{\tilde{\lambda}}(c_i s+1)} + \frac{1}{(s+1)^{\tilde{\lambda}+2\gamma}} + \frac{1}{(s+1)^{1+\tilde{\lambda}}} \right\} ds \\ &\leq M \end{aligned}$$

because  $\tilde{\lambda} + 2\gamma > 1/2$  by (5.28).

It remains to estimate the third term of the right-hand side of (5.70). We show that

$$(5.73) \quad \int_{(r_1-\delta)_+}^{r_1} \frac{r^{1/2}}{\{(r+r_1)(r_2-r)\}^{1/2}} \log \left[ 2 + \frac{ar}{(r_1-r)(r+r_2)} \right] dr \\ \leq \frac{M}{(a+1)^{1/2}} \log[2+t-a],$$

from which we obtain

$$(5.74) \quad \iint_{\substack{\text{black} \\ t-a-\delta \leq r+s}} \eta(s,r) \frac{r^{1/2}}{\{(r+r_1)(r_2-r)\}^{1/2}} \log \left[ 2 + \frac{ar}{(r_1-r)(r+r_2)} \right] dr ds \\ \leq M \int_0^{t-a} \eta(s,r_1) ds \int_{(r_1-\delta)_+}^{r_1} \frac{r^{1/2}}{\{(r+r_1)(r_2-r)\}^{1/2}} \log \left[ 2 + \frac{ar}{(r_1-r)(r+r_2)} \right] dr \\ \leq \frac{M}{(a+1)^{1/2}} \log[2+t-a].$$

To prove (5.73), we consider the following two cases separately: (a)  $1 \leq a$  and (b)  $0 < a < 1$ .

(a)  $1 \leq a$

Since  $r_2 - r \geq r_2 - r_1 \geq a + 1$ , we have

$$\int_{(r_1-\delta)_+}^{r_1} \frac{r^{1/2}}{\{(r+r_1)(r_2-r)\}^{1/2}} \log \left[ 2 + \frac{ar}{(r_1-r)(r+r_2)} \right] dr \\ \leq \frac{1}{(a+1)^{1/2}} \int_{r_1-\delta}^{r_1} \log \left[ 2 + \frac{r}{r_1-r} \right] dr.$$

Hence by the way from which we derive (5.34), we have (5.73) for  $a \geq 1$ .

(b)  $0 < a < 1$

Since  $\log[2 + ar/(r_1-r)(r+r_2)] \leq 1 + a^{1/2}/(r_1-r)^{1/2}$  and  $r_2 - r \geq 2a$ , we have

$$\int_{(r_1-\delta)_+}^{r_1} \frac{r^{1/2}}{\{(r+r_1)(r_2-r)\}^{1/2}} \log \left[ 2 + \frac{ar}{(r_1-r)(r+r_2)} \right] dr \\ \leq \int_{r_1-\delta}^{r_1} \left\{ \frac{1}{(r_2-r)^{1/2}} + \frac{1}{(r_1-r)^{1/2}} \right\} dr \\ \leq 4\tilde{\delta}^{1/2} \leq 2\sqrt{2}.$$

Thus we obtain (5.73) for  $0 < a < 1$ .

Therefore it follows from (5.70)-(5.72) and (5.74) that

$$I_1'' \leq \frac{M\tilde{\delta}}{\{(t-a)(a+t)\}^{1/2}} + \frac{M}{(2a+1/2)^{1/2}(|a-t|+1)^{1/2+\gamma+\lambda}}$$



$$\begin{aligned}
& + \frac{M}{(a+1)^{1/2}(|a-t|+1)^{1/2+\gamma+\lambda}} \log[2+t-a] \\
(5.75) \quad & \leq \frac{M}{(t-a+1)^{1/2}(t+a+1)^{1/2}} + \frac{M}{(a+1)^{1/2}(|a-t|+1)^{1/2+\gamma}} \\
& \leq \frac{M}{(a+1)^{1/2-\gamma}(a+t+1)^\gamma(|a-t|+1)^{1/2}}.
\end{aligned}$$

(vii) *Estimate of  $I_2''$*

By Proposition 5.3.II.(i),

$$I_2'' \leq M \iint_{\partial(\text{red})} \frac{r^{1/2}}{w(s,r)\{(r+r_1)(r_2-r)\}^{1/2}} \log \left[ 2 + \frac{ar}{(r_1-r)(r+r_2)} \right] d\sigma.$$

Moreover, by (5.66)-(5.69), we have

$$\begin{aligned}
(5.76) \quad I_2'' & \leq \frac{M}{\{(a+t+1)(t-a+1)\}^{1/2}} \iint_{\substack{\partial(\text{red}) \\ s+r \leq (t-a)/2}} \frac{r^{1/2}}{w(s,r)} d\sigma \\
& + \frac{M}{(a+1)^{1/2}(|a-t|+1)^{1/2+\gamma+\lambda}} \log[2+t-a] \iint_{\substack{\partial(\text{red}) \\ s+r \geq (t-a)/2}} \eta(s,r) \frac{r^{1/2}}{(r+r_1)^{1/2}} d\sigma.
\end{aligned}$$

Here we notice that  $(t-a)/2 \geq \tilde{\delta} = 1/2$  when the domain *red* is not empty. We further see that

$$(5.77) \quad \iint_{\substack{\partial(\text{red}) \\ s+r \leq (t-a)/2}} \frac{r^{1/2}}{w(s,r)} d\sigma \leq M,$$

$$(5.78) \quad \iint_{\substack{\partial(\text{red}) \\ s+r \geq (t-a)/2}} \eta(s,r) \frac{r^{1/2}}{(r+r_1)^{1/2}} d\sigma \leq M.$$

Hence from (5.76), (5.77) and (5.78) we obtain (5.25) for  $i = 2$ .

(viii) *Estimate of  $I_3''$*

By Proposition 5.3.II.(i),

$$I_3'' \leq M \iint_{\text{red}} \frac{1}{r^{1/2}w(s,r)\{(r+r_1)(r_2-r)\}^{1/2}} \log \left[ 2 + \frac{ar}{(r_1-r)(r+r_2)} \right] dr ds.$$

Further, by (5.66)-(5.69), we have

$$\begin{aligned}
(5.79) \quad I_3'' & \leq \frac{M}{\{(t-a+1)(a+t+1)\}^{1/2}} \iint_{\substack{\text{red} \\ s+r \leq (t-a)/2}} \frac{1}{r^{1/2}w(s,r)} dr ds \\
& + \frac{M}{(a+1)^{1/2}(|a-t|+1)^{1/2+\gamma+\lambda}} \log[2+t-a] \iint_{\substack{\text{red} \\ s+r \geq (t-a)/2}} \frac{\eta(s,r)}{r^{1/2}(r+r_1)^{1/2}} dr ds.
\end{aligned}$$

Since  $r \geq \tilde{\delta} = 1/2$  in the domain  $red$ , we have

$$\begin{aligned} \frac{1}{r^{1/2}w(s,r)} &\leq \frac{M\xi(s,r)}{(r+1)^{1/2}} \\ &\leq M \left\{ \sum_{i=1}^L \frac{1}{(r+1)^{5/4}(|r-c_i s|+1)^{5/4}} + \frac{1}{(r+s+1)^{1+\gamma}(r+1)^{1+1/2-\gamma}} \right. \\ &\quad \left. + \frac{1}{(r+1)^{5/4}(|r-s|+1)^{5/4}} \right\} \end{aligned}$$

for  $(s,r) \in red$ . Therefore it follows that

$$(5.80) \quad \iint_{\substack{red \\ s+r \leq (t-a)/2}} \frac{1}{r^{1/2}w(s,r)} dr ds \leq M.$$

Moreover,

$$\begin{aligned} \frac{\eta(s,r)}{r^{1/2}(r+r_1)^{1/2}} &\leq \frac{\eta(s,r)}{(r+1)^{1/2}(r+r_1+1)^{1/2}} \\ &\leq M \left\{ \sum_{i=1}^L \frac{1}{(r+1)^{1+\tilde{\lambda}/2}(|r-c_i s|+1)^{1+\tilde{\lambda}/2}} \right. \\ &\quad + \frac{1}{(r-s+t-a+1)^{1/2}(|r-s|+1)^{1/2+\min\{\gamma-\lambda,\tilde{\lambda}/2\}}(r+1)^{1+\tilde{\lambda}/2}} \\ &\quad \left. + \frac{1}{(r+1)^{1+\tilde{\lambda}/2}(|r-s|+1)^{1+\tilde{\lambda}/2}} \right\} \end{aligned}$$

for  $(s,r) \in red$ . Therefore it follows that

$$(5.81) \quad \iint_{\substack{red \\ s+r \geq (t-a)/2}} \frac{\eta(s,r)}{r^{1/2}(r+r_1)^{1/2}} dr ds \leq M.$$

Hence from (5.79), (5.80) and (5.81) we obtain (5.25) for  $i = 3$ .

(ix) *Estimate of  $I_4''$*

By Proposition 5.3.II.(ii),

$$I_4'' \leq M \iint_{red} \frac{r^{1/2}}{w(s,r)(r_1-r)\{(r+r_1)(r_2-r)\}^{1/2}} dr ds.$$

Further, by (5.66), (5.67) and (5.68) we have

$$(5.82) \quad \begin{aligned} I_4'' &\leq \frac{M}{\{(t-a+1)(a+t+1)\}^{1/2}} \iint_{\substack{red \\ s+r \leq (t-a)/2}} \frac{r^{1/2}}{w(s,r)(r_1-r)} dr ds \\ &\quad + \frac{M}{(a+1)^{1/2}(|a-t|+1)^{1/2+\gamma+\lambda}} \iint_{\substack{red \\ s+r \geq (t-a)/2}} \frac{\eta(s,r) r^{1/2}}{(r_1-r)(r+r_1)^{1/2}} dr ds. \end{aligned}$$

Both  $r^{1/2}/w(s, r)(r_1 - r)$  and  $\eta(s, r)r^{1/2}/(r_1 - r)(r + r_1)^{1/2}$  are bounded by  $\eta(s, r)/(r_1 - r)$ . And since  $r_1 - r \geq \tilde{\delta} = 1/2$  in the domain  $red$ , we have

$$\begin{aligned}
& \iint_{red} \frac{\eta(s, r)}{r_1 - r} dr ds \\
& \leq M \iint_{red} \frac{1}{r_1 - r + 1} \left\{ \sum_{i=1}^L \frac{1}{(r + s + 1)^{\tilde{\lambda}/2} (|r - c_i s| + 1)^{1 + \tilde{\lambda}/2}} \right. \\
& \quad + \frac{1}{(r + s + 1)^{\tilde{\lambda}/2} (r + 1)^{1 + \tilde{\lambda}/2}} \\
& \quad \left. + \frac{1}{(r + s + 1)^{\tilde{\lambda}/2} (|r - s| + 1)^{1 + \tilde{\lambda}/2}} \right\} dr ds \\
(5.83) \quad & \leq M.
\end{aligned}$$

Therefore from (5.82) and (5.83) we have (5.25) for  $i = 4$ . Consequently we have proved the estimate. ■

## 6 Energy Estimates.

In this section we prove

**Proposition 6.1** *Let  $u = (u^1, \dots, u^m) \in C^\infty([0, T] \times \mathbf{R}^2; \mathbf{R}^m)$  be a solution of the following system of wave equations with  $u(0, \cdot) \in C_0^\infty(\mathbf{R}^2; \mathbf{R}^m)$ .*

$$(6.1) \quad \partial_t^2 u^i - c_i^2 \Delta u^i = \sum_{j=1}^m \sum_{\alpha, \beta=0}^2 C_{ij}^{\alpha\beta} (\partial u) \partial_\alpha \partial_\beta u^j + E_i (\partial u)$$

Here,  $C_{ij}^{\alpha\beta}, E_i$  ( $i, j = 1, 2, \dots, m; \alpha, \beta = 0, 1, 2$ ) are  $C^\infty$ -functions in  $\{|\partial u| < 1\}$ , which satisfy the conditions (4.3)-(4.8).

Moreover we assume that

$$(6.2) \quad |C_{ij}^{\alpha\beta} (\partial u)| \leq \frac{1}{4m} \min\{1, c_i^2, c_j^2\} \quad \text{for } |\partial u| < \delta_1$$

and that there exists a positive number  $T_1$  such that

$$(6.3) \quad |\partial u|_0(T_1) < 1 \quad \text{and} \quad |\partial u|_0(T_1) < \delta_1.$$

Then, we have the following energy estimates for  $0 < t < T_1$ :

$$(6.4) \quad \|\partial u(t)\|_N^2 \leq M_N \left\{ \|\partial u(0)\|_N^2 + \int_0^t (s+1)^{-1-\min\{1/3, 2\gamma\}} [\partial u(s)]_0^2 \|\partial u(s)\|_{(N+1)(N+5)}^2 ds \right\},$$

$$(6.5) \quad \|\partial u(t)\|_N^2 \leq M_N \|\partial u(0)\|_N^2 (t+1)^{M_N([\partial u]_1(t))^2}.$$

Proof. Since  $\Omega$  commutes  $\partial_t^2 - c_i^2 \Delta$ ,

$$(6.6) \quad \partial_t^2 \mathcal{D}^A u^i - c_i^2 \Delta \mathcal{D}^A u^i = \sum_{j=1}^m \sum_{\alpha, \beta=0}^2 \mathcal{D}^A \{C_{ij}^{\alpha\beta}(\partial u) \partial_\alpha \partial_\beta u^j\} + \mathcal{D}^A E_i(\partial u).$$

We set

$$(6.7) \quad a_{ij}^{\alpha\beta} = \begin{cases} 1 - C_{ii}^{00} & (\alpha = \beta = 0, j = i) \\ -c_i c_j \delta_{\alpha\beta} \delta_{ij} - C_{ij}^{\alpha\beta} & (\text{otherwise}) \end{cases}$$

$$(6.8) \quad w_A^i = \sum_{j=1}^m \sum_{\alpha, \beta=0}^2 \left[ \mathcal{D}^A \{C_{ij}^{\alpha\beta}(\partial u) \partial_\alpha \partial_\beta u^j\} - C_{ij}^{\alpha\beta}(\partial u) \partial_\alpha \partial_\beta \mathcal{D}^A u^j \right] + \mathcal{D}^A E_i(\partial u).$$

Then from (6.6), (6.7) and (6.8) we have

$$(6.9) \quad \sum_{j=1}^m \sum_{\alpha, \beta=0}^2 a_{ij}^{\alpha\beta}(\partial u) \partial_\alpha \partial_\beta \mathcal{D}^A u^j = w_A^i$$

Multiplying both side of (6.9) by  $\partial_t \mathcal{D}^A u^i$  and using (4.3), we get

$$(6.10) \quad \begin{aligned} & \sum_{i,j=1}^m \sum_{\alpha, \beta=0}^2 \left[ \partial_\alpha \{2a_{ij}^{\alpha\beta}(\partial u) \partial_t \mathcal{D}^A u^i \cdot \partial_\beta \mathcal{D}^A u^j\} - \partial_t \{a_{ij}^{\alpha\beta}(\partial u) \partial_\alpha \mathcal{D}^A u^i \cdot \partial_\beta \mathcal{D}^A u^j\} \right] \\ & = \sum_{i=1}^m 2\partial_t \mathcal{D}^A u^i \cdot w_A^i \\ & \quad + \sum_{i,j=1}^m \sum_{\alpha, \beta=0}^2 \left\{ \partial_t C_{ij}^{\alpha\beta}(\partial u) \partial_\alpha \mathcal{D}^A u^i \cdot \partial_\beta \mathcal{D}^A u^j - 2\partial_\alpha C_{ij}^{\alpha\beta}(\partial u) \partial_t \mathcal{D}^A u^i \cdot \partial_\beta \mathcal{D}^A u^j \right\} \end{aligned}$$

Integrating (6.10) over  $[0, t] \times \mathbf{R}^2$ , we have

$$(6.11) \quad \begin{aligned} & \|\partial \mathcal{D}^A u(t)\|_E^2 - \|\partial \mathcal{D}^A u(0)\|_E^2 \\ & = \int_0^t ds \iint_{\mathbf{R}^2} \left[ \sum_{i=1}^m 2\partial_t \mathcal{D}^A u^i \cdot w_A^i + \sum_{i,j=1}^m \sum_{\alpha, \beta=0}^2 \left\{ \partial_t C_{ij}^{\alpha\beta}(\partial u) \partial_\alpha \mathcal{D}^A u^i \cdot \partial_\beta \mathcal{D}^A u^j \right. \right. \\ & \quad \left. \left. - 2\partial_\alpha C_{ij}^{\alpha\beta}(\partial u) \partial_t \mathcal{D}^A u^i \cdot \partial_\beta \mathcal{D}^A u^j \right\} \right] dx, \end{aligned}$$

where

$$(6.12) \quad \begin{aligned} \|\partial \mathcal{D}^A u(t)\|_E^2 &= \iint_{\mathbf{R}^2} \sum_{i,j=1}^m \left\{ a_{ij}^{00}(\partial u) \partial_t \mathcal{D}^A u^i \cdot \partial_t \mathcal{D}^A u^j(t, x) \right. \\ &\quad \left. - \sum_{k,l=1}^2 a_{ij}^{kl}(\partial u) \partial_k \mathcal{D}^A u^i \cdot \partial_l \mathcal{D}^A u^j(t, x) \right\} dx. \end{aligned}$$

Notice that by (6.2) and (6.7) we have

$$\frac{1}{M} \|\partial \mathcal{D}^A u(t)\|_0 \leq \|\partial \mathcal{D}^A u(t)\|_E \leq M \|\partial \mathcal{D}^A u(t)\|_0.$$

Therefore it follows that

$$(6.13) \quad \|\partial \mathcal{D}^A u(t)\|_0^2 \leq M \left( \|\partial \mathcal{D}^A u(0)\|_0^2 + J_A^{(1)} + J_A^{(2)} \right),$$

where

$$\begin{aligned} J_A^{(1)} &= \sum_{i=1}^m \int_0^t ds \iint_{\mathbf{R}^2} |\partial_t \mathcal{D}^A u^i \cdot w_A^i| dx, \\ J_A^{(2)} &= \sum_{i,j=1}^2 \sum_{\alpha,\beta=0}^2 \int_0^t ds \iint_{\mathbf{R}^2} |\partial C_{ij}^{\alpha\beta}(\partial u)| |\partial \mathcal{D}^A u^i| |\partial \mathcal{D}^A u^j| dx. \end{aligned}$$

We first prove the estimate (6.4). Since  $|\partial u(s)|_0 < 1$ , then by the assumption (4.4)-(4.8), we get

$$(6.14) \quad |\mathcal{D}^B C_{ij}^{\alpha\beta}(\partial u)| \leq M_B \sum_{l=2}^{|B|+3} \sum_{j_1, \dots, j_l=1}^m \sum_{|B_1|, \dots, |B_l| \leq |B|} \delta_{i; j_1 \dots j_l} \prod_{k=1}^l |\mathcal{D}^{B_k} \partial u^{j_k}|,$$

$$(6.15) \quad |\mathcal{D}^A E_i(\partial u)| \leq M_A \sum_{l=3}^{|A|+4} \sum_{j_1, \dots, j_l=1}^m \sum_{|A_1|, \dots, |A_l| \leq |A|} \delta_{i; j_1 \dots j_l} \prod_{k=1}^l |\mathcal{D}^{A_k} \partial u^{j_k}|.$$

Here we set

$$\delta_{i; j_1 \dots j_l} = \begin{cases} 1 - \delta_{ij_1} \delta_{ij_2} \delta_{ij_3} & (l = 3) \\ 1 & (l \geq 4). \end{cases}$$

Since  $\left| \det \left( a_{ij}^{00}(\partial u) \right)_{i,j=1}^m \right| \geq 1/2^m$  from (6.2), we can solve the following simultaneous linear equation with respect to  $\partial_t^2 u^i$ :

$$\sum_{j=1}^m a_{ij}^{00}(\partial u) \partial_t^2 u^j = E_i(\partial u) - \sum_{j=1}^m \sum_{(\alpha,\beta) \neq (0,0)} a_{ij}^{\alpha\beta}(\partial u) \partial_\alpha \partial_\beta u^j \quad (i = 1, 2, \dots, m).$$

Moreover, by (4.4), (4.5), (6.2) and Cramer's formula, we see

$$(6.16) \quad \partial_t^2 u^l = - \left\{ \det \left( a_{ij}^{00}(\partial u) \right)_{i,j=1}^m \right\}^{-1} c_l^2 \Delta u^l + \text{higher order terms.}$$

Therefore it follows from (6.8), (6.14), (6.15) and (6.16) that

$$(6.17) \quad J_A^{(1)} \leq M_A \sum_{i=1}^m \sum_{l=3}^{|A|+5} \sum_{j_1, \dots, j_l=1}^m \sum_{|A_1|, \dots, |A_l| \leq |A|} \delta_{i; j_1 \dots j_l} \int_0^t ds \iint_{\mathbf{R}^2} \prod_{k=1}^l |\mathcal{D}^{A_k} \partial u^{j_k}| |\mathcal{D}^A \partial u^i| dx.$$

Next, we consider  $J_A^{(2)}$ . By (4.4) and (4.7),

$$|\partial C_{ij}^{\alpha\beta}(\partial u)| \leq M \sum_{k,l=1}^m \delta_{i;jkl} |\partial u^k| |\partial^2 u^l|.$$

Therefore it follows from (6.16) that

$$(6.18) \quad J_A^{(2)} \leq M \left\{ \sum_{i,j,k,l=1}^m \sum_{h=1}^2 \int_0^t ds \iint_{\mathbf{R}^2} \delta_{i;jkl} |\partial u^k| |\partial_h \partial u^l| |\partial \mathcal{D}^A u^i| |\partial \mathcal{D}^A u^j| dx + \int_0^t ds \iint_{\mathbf{R}^2} |\partial u|^3 |\partial \mathcal{D}^A u|^2 dx \right\}.$$

Hence we find from (6.17) and (6.18) that

$$(6.19) \quad J_A^{(1)} + J_A^{(2)} \leq M_A \sum_{l=3}^{|A|+5} \sum_{j_0, \dots, j_l=1}^m \sum_{|A_0|, \dots, |A_l| \leq |A|+1} \delta_{j_0; j_1 \dots j_l} \int_0^t ds \iint_{\mathbf{R}^2} \prod_{k=0}^l |\mathcal{D}^{A_k} \partial u^{j_k}| dx.$$

By Hölder's inequality,

$$(6.20) \quad \begin{aligned} & \iint_{\mathbf{R}^2} \prod_{k=0}^l |\mathcal{D}^{A_k} \partial u^{j_k}(s, x)| dx \\ & \leq \left\| \prod_{k=0}^{l-1} \mathcal{D}^{A_k} \partial u^{j_k}(s, \cdot) \right\|_{L^2(\mathbf{R}^2)} \left\| \mathcal{D}^{A_l} \partial u^{j_l}(s, \cdot) \right\|_{L^2(\mathbf{R}^2)} \\ & \leq \left\| \prod_{k=0}^{l-1} w_{j_k}^{-(l-1)/l}(s, |\cdot|) \right\|_{L^\infty(\mathbf{R}^2)} \\ & \quad \cdot \left\| \prod_{k=0}^{l-1} w_{j_k}^{(l-1)/l}(s, |\cdot|) \mathcal{D}^{A_k} \partial u^{j_k}(s, \cdot) \right\|_{L^2(\mathbf{R}^2)} \left\| \mathcal{D}^{A_l} \partial u^{j_l}(s, \cdot) \right\|_{L^2(\mathbf{R}^2)} \\ & \leq \left\| \prod_{k=0}^{l-1} w_{j_k}^{-(l-1)/l}(s, |\cdot|) \right\|_{L^\infty(\mathbf{R}^2)} \\ & \quad \cdot \prod_{k=0}^{l-1} \left\| w_{j_k}^{(l-1)/l}(s, |\cdot|) \mathcal{D}^{A_k} \partial u^{j_k}(s, \cdot) \right\|_{L^{2l}(\mathbf{R}^2)} \left\| \mathcal{D}^{A_l} \partial u^{j_l}(s, \cdot) \right\|_{L^2(\mathbf{R}^2)}. \end{aligned}$$

Without loss of generality we may suppose that  $j_0 = j_1 = j_2$  does not hold for  $l = 3$ . Therefore, it follows from (3.1) that

$$(6.21) \quad \left\| \prod_{k=0}^{l-1} w_{j_k}^{-(l-1)/l}(s, |\cdot|) \right\|_{L^\infty(\mathbf{R}^2)} \leq M(s+1)^{-1-\min\{1/3, 2\gamma\}}.$$

In order to estimate (6.20), we need Gagliardo-Nirenberg inequality:

**Lemma 6.1** *Let  $f \in C_0^\infty(\mathbf{R}^2)$ ,  $|A| = i \leq k$ . Then,*

$$\|\mathcal{D}^A f\|_{L^{r k/i}(X)} \leq M_A \|f\|_{L^\infty(X)}^{1-i/k} \left( \sum_{|B| \leq k} \|\mathcal{D}^B f\|_{L^r(X)} \right)^{i/k},$$

where

$$X = \mathbf{R}^2 \text{ or } \{x \mid x \in \mathbf{R}^2, n \leq |x| \leq n+1\} \quad (n = 0, 1, 2, \dots).$$

For the proof, see [6], Appendix.

Since

$$(6.22) \quad \frac{1}{M} w_j(s, n) \leq w_j(s, r) \leq M w_j(s, n)$$

for  $n \leq r \leq n+1$ , we find from Lemma 6.1 that

$$(6.23) \quad \begin{aligned} & \left\| w_{j_k}(s, |\cdot|)^{(l-1)/l} \mathcal{D}^{A_k} \partial u^{j_k}(s, \cdot) \right\|_{L^{2l}(\mathbf{R}^2)}^{2l} \\ & \leq M \sum_{n=0}^{\infty} w_{j_k}(s, n)^{2(l-1)} \left\| \mathcal{D}^{A_k} \partial u^{j_k} \right\|_{L^{2l}(\{n \leq |x| \leq n+1\})}^{2l} \\ & \leq M_A \sum_{n=0}^{\infty} w_{j_k}(s, n)^{2(l-1)} \left\| \partial u^{j_k}(s, \cdot) \right\|_{L^\infty(\{n \leq |x| \leq n+1\})}^{2(l-1)} \\ & \quad \cdot \left( \sum_{|B| \leq |A_k|} \left\| \mathcal{D}^B \partial u^{j_k}(s, \cdot) \right\|_{L^2(\{n \leq |x| \leq n+1\})} \right)^2 \\ & \leq M_A [\partial u(s)]_0^{2(l-1)} \|\partial u(s)\|_{l|A_k|}^2 \end{aligned}$$

Therefore, from (6.3), (6.19)-(6.23) and (6.13), we get (6.4).

Next, we prove (6.5). For the proof, we use the following two lemmas:

**Lemma 6.2** *Let  $f, g \in C_0^\infty(\mathbf{R}^2)$ . Then,*

$$\left\| \mathcal{D}^A(fg) - f \mathcal{D}^A g \right\|_0 \leq M(|f|_1 \|g\|_{|A|-1} + |g|_0 \|f\|_{|A|}).$$

**Lemma 6.3** Let  $f = (f_1, \dots, f_r) \in C_0^\infty(\mathbf{R}^2; \mathbf{R}^r)$  and let  $\omega = \omega(f)$  be a  $C^\infty$ -function that satisfies

$$|\omega(f)| \leq M|f|^q.$$

Then

$$(6.24) \quad \|\mathcal{D}^A(\omega \circ f)\|_{L^p(\mathbf{R}^2)} \leq M|f|_{L^\infty(\mathbf{R}^n)}^{q-1} \sum_{|B| \leq |A|} \|\mathcal{D}^B f\|_{L^p(\mathbf{R}^2)}.$$

See [6], Appendix for the proof of Lemma 6.2, and [10], Proposition A2 for Lemma 6.3.

By these lemmas,

$$(6.25) \quad \|w_A^i\|_{L^2(\mathbf{R}^2)} \leq M_A |\partial u(s)|_0 |\partial u(s)|_1 \|\partial u(s)\|_{|A|}.$$

Therefore from (6.18) and (6.25) we get

$$(6.26) \quad J_A^{(1)} + J_A^{(2)} \leq M_A \int_0^t |\partial u(s)|_0 |\partial u(s)|_1 \|\partial u(s)\|_{|A|}^2 ds.$$

Further, it follows from (6.13) and (6.26) that

$$\|\partial u(t)\|_N^2 \leq M_N \left\{ \|\partial u(0)\|_N^2 + \int_0^t |\partial u(s)|_0 |\partial u(s)|_1 \|\partial u(s)\|_N^2 ds \right\}.$$

Hence by Gronwall's lemma we find

$$(6.27) \quad \|\partial u(t)\|_N^2 \leq M \|\partial u(0)\|_N^2 \exp \left( M_N \int_0^t |\partial u(s)|_0 |\partial u(s)|_1 ds \right).$$

Since

$$|\partial u(s)|_0 |\partial u(s)|_1 \leq M_N (s+1)^{-1} [\partial u(s)]_1^2,$$

we obtain (6.5) from (6.27). ■

## 7 Proof of the Theorem.

Making use of the method in [1] and [5], we find that a solution  $u(t, x)$  to (4.1) is unique and  $u(t, \cdot)$  ( $t \geq 0$ ) has compact support. The local existence theorem of a solution to (4.1) has proved in [4] and [7].



Let  $u(t, x)$  be a  $C^\infty$ -solution to (4.1) in  $[0, T] \times \mathbf{R}^2$ . We write  $u$  as

$$(7.1) \quad u = u_0 + u_1,$$

where  $u_0$  is the solution of the initial value problem

$$(7.2) \quad \begin{cases} \partial_t^2 u_0^i - c_i^2 \Delta u_0^i = 0 \\ u_0^i(0, \cdot) = \varepsilon f_i, \partial_t u_0^i(0, \cdot) = \varepsilon g_i \quad (i = 1, 2, \dots, m), \end{cases}$$

and  $u_1$  is the solution of the initial value problem

$$(7.3) \quad \begin{cases} \partial_t^2 u_1^i - c_i^2 \Delta u_1^i = F_i(\partial u, \partial^2 u) \\ u_1^i(0, \cdot) = \partial_t u_1^i(0, \cdot) = 0 \quad (i = 1, 2, \dots, m). \end{cases}$$

In [2], R. Glassey has proved by the method in W. von Wahl [14] that

$$(7.4) \quad |u_0^i(t, x)| \leq \frac{M\varepsilon}{\{(|x| + c_i t + 1)(||x| - c_i t| + 1)\}^{1/2}}.$$

Here  $M$  depends on  $L^1$ -norm of  $f_i$ ,  $\partial f_i$  and  $g_i$ .

We set

$$\frac{1}{\tilde{w}_i(s, r)} = \sum_{j \neq i} \frac{1}{(r + s + 1)(|r - c_j s| + 1)} + \frac{1}{(r + s + 1)^{1+2\gamma}(r + 1)^{1-2\gamma}} + \frac{1}{(r + s + 1)^{4/3}(|r - c_i s| + 1)^{2/3}}.$$

Then  $\tilde{w}_i$  satisfies (5.20) and (5.21). By Proposition 5.4, we get

$$(7.5) \quad [\partial u_1(t)]_N \leq M_N \left\{ \sum_{i=1}^m \sum_{|A| \leq N+3} \sup_{0 < s < t} \left\| \tilde{w}_i(s, |\cdot|) \mathcal{D}^A F_i(\partial u, \partial^2 u)(s, \cdot) \right\|_{L^2(\mathbf{R}^2)} + \sum_{i=1}^m \sum_{|A| \leq N+2} \sup_{0 < s < t} \left\| \tilde{w}_i(s, |\cdot|) \partial_t \mathcal{D}^A F_i(\partial u, \partial^2 u)(s, \cdot) \right\|_{L^2(\mathbf{R}^2)} \right\}.$$

Since  $[\partial u(t)]_0$  is continuous, we can take for  $0 < \varepsilon < \varepsilon_1$  a positive number  $T_1$  such that the condition (6.3) holds, provided  $\varepsilon_1$  is small enough. We suppose  $0 < \varepsilon < \varepsilon_1$  and set

$$\tilde{T}_1 = \sup\{T_1 \mid (6.3) \text{ holds.}\}.$$

Let  $0 < t < \tilde{T}_1$  in the following. In particular,  $|\partial u|_0(t) < 1$ . Then by (6.16), we have

$$(7.6) \quad \left| \mathcal{D}^A F_i(\partial u, \partial^2 u) \right| \leq M_A \sum_{l=3}^{|A|+6} \sum_{j_1, \dots, j_l=1}^m \sum_{\substack{|A_h| \leq |A|+1 \\ (h=1, 2, \dots, l)}} \delta_{i; j_1 \dots j_l} \prod_{k=1}^l \left| \mathcal{D}^{A_k} \partial u^{j_k} \right|,$$

$$(7.7) \quad \left| \partial_t \mathcal{D}^A F_i(\partial u, \partial^2 u) \right| \leq M_A \sum_{l=3}^{|A|+8} \sum_{j_1, \dots, j_l=1}^m \sum_{\substack{|A_h| \leq |A|+2 \\ (h=1, 2, \dots, l)}} \delta_{i; j_1 \dots j_l} \prod_{k=1}^l \left| \mathcal{D}^{A_k} \partial u^{j_k} \right|.$$

Hence from (7.5), (7.6) and (7.7) we get

$$(7.8) \quad \begin{aligned} & [\partial u_1(t)]_N \\ & \leq M_N \sum_{l=3}^{N+10} \sum_{j_1, \dots, j_l=1}^m \sum_{\substack{|A_h| \leq N+4 \\ (h=1, 2, \dots, l)}} \delta_{i; j_1 \dots j_l} \sup_{0 < s < t} \left\| \tilde{w}_i(s, |\cdot|) \prod_{k=1}^l \mathcal{D}^{A_k} \partial u^{j_k}(s, \cdot) \right\|_{L^2(\mathbf{R}^2)}. \end{aligned}$$

We notice that

$$\frac{1}{(w_{j_1} w_{j_2} w_{j_3})^{2/3}(s, r)} \leq \frac{M}{\tilde{w}_i(s, r)},$$

provided  $j_1 = j_2 = j_3 = i$  does not hold. Thus we get

$$(7.9) \quad \delta_{i; j_1 \dots j_l} \tilde{w}_i \leq M(w_{j_1} \dots w_{j_l})^{(l-1)/l}.$$

By (7.9) and Hölder's inequality,

$$(7.10) \quad \begin{aligned} \delta_{i; j_1 \dots j_l} \left\| \tilde{w}_i(s, |\cdot|) \prod_{k=1}^l \mathcal{D}^{A_k} \partial u^{j_k}(s, \cdot) \right\|_{L^2(\mathbf{R}^2)} & \leq M \left\| \prod_{k=1}^l w_{j_k}^{(l-1)/l}(s, |\cdot|) \mathcal{D}^{A_k} \partial u^{j_k}(s, \cdot) \right\|_{L^2(\mathbf{R}^2)} \\ & \leq M \prod_{k=1}^l \left\| w_{j_k}^{(l-1)/l}(s, |\cdot|) \mathcal{D}^{A_k} \partial u^{j_k}(s, \cdot) \right\|_{L^{2l}(\mathbf{R}^2)} \end{aligned}$$

Hence by (7.8), (7.10) and (6.23) we have

$$(7.11) \quad [\partial u_1(t)]_N \leq M_N \sup_{0 < s < t} [\partial u(s)]_0^2 \|\partial u(s)\|_{N_1},$$

where  $N_1 = (N + 10)(N + 4)$ . Therefore it follows from (7.1), (7.4) and (7.11) that

$$(7.12) \quad [\partial u(t)]_N \leq M_N \left\{ \varepsilon + \sup_{0 < s < t} [\partial u(s)]_0^2 \|\partial u(s)\|_{N_1} \right\}.$$

By Proposition 6.1,

$$(7.13) \quad \|\partial u(s)\|_{N_1}^2 \leq M_N \left\{ 1 + [\partial u]_0^2(s) \int_0^s (\tau + 1)^{-1 - \min\{1/3, 2\gamma\}} \|\partial u(\tau)\|_{(N_1+1)(N_1+5)}^2 d\tau \right\},$$

$$(7.14) \quad \|\partial u(\tau)\|_{(N_1+1)(N_1+5)} \leq M_N (\tau + 1)^{M_N [\partial u]_1^2(\tau)}.$$

We fix the constant  $M_N$  in (7.12), (7.13) and (7.14) so that

$$M_N \geq \max\{8, 2/\min\{1/3, 2\gamma\}, (2/\delta_1)^{1/2}\}.$$

We take  $\varepsilon_0$  to be

$$0 < \varepsilon_0 < \min\{1/4M_N^3, \varepsilon_1\}.$$

Moreover, we suppose that  $\varepsilon_0$  is small enough to define the following  $T_0$  for  $0 < \varepsilon < \varepsilon_0$ :

$$T_0 = \sup \{t \mid [\partial u]_N(t) \leq 4\varepsilon M_N\}.$$

Suppose that  $0 < t < T_0$ . Then,

$$[\partial u(t)]_0 \leq 4\varepsilon M_N \leq 1/M_N^2 \leq 1/2$$

$$|\partial u(t)|_0 \leq 1/M_N^2 \leq \delta_1/2.$$

Therefore  $T_0 \leq \tilde{T}_1$ . Then for  $0 < \varepsilon < \varepsilon_0$  and  $0 < t < T_0$ ,

$$[\partial u]_1^2(t) \leq (4\varepsilon M_N)^2 \leq 1/M_N^2,$$

$$(7.15) \quad -\min\{1/3, 2\gamma\} + M_N [\partial u]_1^2(\tau) \leq -\min\{1/3, 2\gamma\}/2.$$

Here from (7.13), (7.14) and (7.15) we get

$$(7.16) \quad \begin{aligned} \|\partial u(s)\|_{N_1}^2 &\leq M_N \left\{ 1 + \frac{1}{M_N^2} \cdot M_N \int_0^s (\tau + 1)^{-1 - \min\{1/3, 2\gamma\}/2} d\tau \right\} \\ &\leq 2M_N. \end{aligned}$$

Therefore it follows from (7.12) and (7.16) that

$$\begin{aligned} [\partial u(t)]_N &\leq M_N \left\{ \varepsilon + \sup_{0 < s < t} [\partial u(s)]_0 \cdot 4\varepsilon M_N \cdot (2M_N)^{1/2} \right\} \\ &\leq M_N \varepsilon + \frac{1}{2} \sup_{0 < s < t} [\partial u(s)]_0 \\ &\leq M_N \varepsilon + \frac{1}{2} \sup_{0 < s < T_0} [\partial u(s)]_N, \end{aligned}$$

which implies

$$\sup_{0 < t < T_0} [\partial u(t)]_N \leq 2M_N \varepsilon.$$

Therefore  $T_0$  cannot be finite, and we complete the proof of the theorem. ■

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