Riesz's Functions In Weighted Hardy And Bergman Spaces

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Riesz's Functions In Weighted Hardy And Bergman Spaces

by

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and

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Dedicated to Professor Fumi-Yuki Maeda on his sixtieth birthday

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Abstract. Let μ be a finite positive Borel measure on the closed unit disc $\bar{D}$. For each a in $\bar{D}$, put

$$S(a) = inf \int_{\bar{D}} \mid f \mid^p d\mu$$

where f ranges over all analytic polynomials with $f(a) = 1$. This upper semicontinuous function $S(a)$ is called a Riesz's function and studied in detail. Moreover several applications are given to weighted Bergman and Hardy spaces.

§1. Introduction

Let D be the open unit disc in the complex plane $\mathbf{C}$. P denotes a set of all analytic polynomials and H denotes a set of all analytic functions on D. Suppose $0 < p < \infty$. When μ is a finite positive Borel measure on $\bar{D}$ and $a \in \bar{D}$, put

$$S(\mu, a) = S(\mu, p, a) = inf\{\int_{\bar{D}} | f |^p \, d\mu \; ; \; f \in P \text{ and } f(a) = 1\}$$

and

$$R(\mu, a) = R(\mu, p, a) = sup\{| f(a) |^p \; ; \; f \in P \text{ and } \int_{\bar{D}} | f |^p \, d\mu \leq 1\}.$$

When μ is a finite positive Borel measure on D and $a \in D$, put

$$s(\mu, a) = s(\mu, p, a) = inf\{\int_{D} | f |^p \, d\mu \; ; \; f \in H \text{ and } f(a) = 1\}$$

and

$$r(\mu, a) = r(\mu, p, a) = sup\{| f(a) |^p \; ; \; f \in H \text{ and } \int_{D} | f |^p \, d\mu \leq 1\}.$$

The four functions S, R, s and r are called Riesz's functions. In this paper we study these four Riesz's functions. M.Riesz used such functions to solve the moment problem on the real line (cf. [6, Chapter 5]). T.Kriete and T.Trent [7] also investigated the relationship between μ and $R(\mu, 2, a)$. In the investigations of Riesz's functions, the most fundamental and important result is the following theorem by G.Szegő (cf. [5, Chapter 3]). He proved it only when $p = 2$ but it can be proved for arbitrary p.

Szegő's Theorem. Suppose $0 < p < \infty$, μ is a finite positive Borel measure on $\bar{D}$ with $supp\ \mu \subseteq \partial D$ and $d\mu/(d\theta/2\pi) = w(e^{i\theta})$.

Then,

$$S(\mu, p, a) = (1- | a |^2) exp(log w)\hat{}(a) \qquad (a \in D)$$

where $(log w)\hat{}(a) = \int_0^{2\pi} log w(e^{i\theta}) \frac{1- | a |^2}{| 1 - \bar{a} e^{i\theta} |^2} d\theta/2\pi.$

It is most desirable to describe $S(\mu, p, a)$ using μ as in Szegő's Theorem, when μ is an arbitrary measure on $\bar{D}$. However such a problem is very difficult except for some special measures μ. In Section 2, we study the behaviour of $S(\mu, p, a)$ as $| a | \longrightarrow 1$ for an arbitrary measure on $\bar{D}$. Moreover we note that $S(\mu, p, a)\ R(\mu, p, a) = 1 \quad (a \in \bar{D})$. Thus we need to know only S or R. In this paper, the results and the proofs about s and r are very similar to those about S and R. Hence we concentrate on only S or R in

Sections 2, 3 and 4. Let m be the normalized area measure on D, that is, $dm = rdrd\theta/\pi$. In Section 3, we give the several lower estimates of S using $d\mu/dm$. It is more difficult to give the upper estimates of S. We do it only in very special cases. In Section 4, we show that $R(\mu, p, a)$ is not in $L^1(\mu)$ if *supp* μ is not a finite set

Suppose $0 < p < \infty$. $H^p(\mu)$ denotes the closure of P in $L^p(\mu)$ when μ is a finite positive Borel measure on $\bar{D}$. $H^p(\mu)$ is called a weighted Hardy space. If $d\mu = d\theta/2\pi, H^p(\mu) = H^p$ is the classical Hardy space. When μ is a finite positive Borel measure on D, then one defines $L^p_a(\mu) = H \cap L^p(\mu)$. $L^p_a(\mu)$ is called a weighted Bergman space. If $\mu = m, L^p_a(\mu) = L^p_a$ is the usual Bergman space. H^p can be embedded in H. $L^p_a = H^p(m)$ and hence L^p_a is closed. We are interested in the following questions : (1) When $H^p(\mu)$ can be embedded in H ? (2) When $L^p_a(\mu)$ is closed ? (3) When $H^p(\mu)$ can be embedded in $L^p_a(\mu)$? Of course it is very interesting to know when $L^p_a(\mu) = H^p(\mu)$, where μ is a measure on D. This problem is classical and important (cf. [2]). However, in this paper we are not going to consider this problem. The problem (2) was studied by M.Yamada [13]. If μ is a measure on D, the problem (1) is equivalent to (3). Note that the measure μ for (2) satisfies (3). In Section 5, we study the three problems above. For example, for some compact set K in D, if $\int_{\bar{D}\setminus K} logW dm > -\infty$ then $H^p(\mu)$ can be embedded in H where $W = d\mu/dm$. This result follows from the lower estimate of $S(\mu, p, a)$ in Section 3.

In this paper, we will use the following notations. For each $a \in D$, *let* ϕ_a be the Möbius function on D, that is,

$$\phi_a(z) = \frac{a-z}{1-\bar{a}z} \qquad (z \in D),$$

and put

$$\beta(a,z) = \frac{1}{2}\, log\frac{1+ \mid \phi_a(z) \mid}{1- \mid \phi_a(z) \mid} \qquad (a, z \in D).$$

For $0 < r \leq \infty$ and $a \in D$,

$$D_r(a) = \{z \in D \ ; \ \beta(a,z) < r\}$$

be the Bergman disc with 'center' a and 'radius' r. For $u \in L^1(m)$,

$$\tilde{u}(a) = \int_D u \circ \phi_a(z) dm(z) \qquad (a \in D).$$

Then $\tilde{u}$ may be bounded on D even if u is not bounded on D.

§2. Riesz's function

If $\mu = m$, then for $0 < p < \infty$ $S(m,p,a) = (1- \mid a \mid^2)^2$. Hence $\mu = m$ or $supp\ \mu \subseteq \partial D$, by Szegö's Theorem $\lim_{r \to 1-} S(\mu, p, re^{i\theta}) = 0$ $a.e.\theta$. In this section, we show that this is true in general. In particular, R is not bounded on D. In fact, for arbitrary μ, we show that $\lim_{r \to 1-} S(\mu, p, re^{i\theta}) = 0$ except a countable set of θ.

Proposition 1. Suppose $0 < p < \infty$ and μ is a finite positive Borel measure. Then the following are valid for $R(a) = R(\mu, p, a)$ and $S(a) = S(\mu, p, a)$.

(1) $R(\mu, p, a)\ S(\mu, p, a) = 1$ for $a \in \bar{D}$, assuming $\infty \times 0 = 1$.

(2) $R(\mu)$ is lower semicontinuous on $(0, \infty) \times D$, and $S(\mu)$ is upper semicontinuous on the same set. Moreover $R(\mu, p, a) \geq 1/\mu(\bar{D})$ and $S(\mu, p, a) \leq \mu(\bar{D})$.

(3) If $logR$ or R is in $L^1(m)$, then for $a \in D$

$$R(a) \leq exp(logR)^{\sim}(a) \leq \tilde{R}(a).$$

(4) If $r < \infty$, then for $a \in D$

$$logR(a) \leq \left(\frac{1 + s \mid a \mid}{1 - s \mid a \mid}\right)^2 \frac{1}{m(D_r(a))} \int_{D_r(a))} logRdm$$

where $s = \tanh r$. Hence for $a \in D$

$$logR(a) \leq \left(\frac{1 + \mid a \mid}{1 - \mid a \mid}\right)^2 \int_D logRdm.$$

These inequalities are also valid for R instead of $logR$.

(5) For $a \in D$,

$$S(\mu, p, a) \geq S(S(\mu)dm, p, a).$$

(6) R is not bounded on D and $\bar{D}$.

Proof. (1) It is easy to see that $1 \leq R(a)S(a)$ for $a \in \bar{D}$. If $1 < R(a)S(a)$, then there exists a positive constant γ such that $1 \leq \gamma\ S(a)$ and $\gamma < R(a)$. Hence $1 \leq \gamma \int \mid g \mid^p d\mu$ for any $g \in P$ with $g(a) = 1$ and so

$$\mid f(a) \mid^p \leq \gamma \int_{\bar{D}} \mid f \mid^p d\mu \text{ for any } f \in P.$$

This implies $\gamma \geq R(a)$. This contradiction shows that $1 = R(a)\ S(a)$. (2) is clear by (1). (3) If $f \in P$, then $log \mid f \mid$ is subharmonic on D and hence for any $a \in D$,

$$log \mid f(a) \mid^p \leq \int_D log \mid f(z) \mid^p \frac{(1- \mid a \mid^2)^2}{\mid 1 - \bar{a}z \mid^4} dm(z).$$

Assuming $\int | f |^p \, d\mu \leq 1$, by definition on R

$$logR(a) \leq \int_D logR(z) \, \frac{(1-|a|^2)^2}{|1-\bar{a}z|^4} dm(z).$$

This implies $R(a) \leq exp(logR)^{\sim}(a) \leq \tilde{R}(a)$. (4) If $0 < r < \infty$, for any $a \in D_r(0)$ and any $f \in P$,

$$log \, | f(a) |^p \leq \frac{1}{m(D_r(0))} \int_{D_r(a)} log \, | f(z) |^p \, \frac{(1-|a|^2)^2}{|1-\bar{a}z|^4} dm(z)$$

and hence

$$log \, | f(a) |^p \leq \frac{1}{m(D_r(a))} \left(\frac{1+s|a|}{1-s|a|}\right)^2 \int_{D_r(a)} log \, | f |^p \, dm$$

where $s = \tanh r$. This proof is same to that of [14, Proposition 4.3.8.]. Assuming $\int | f |^p \, d\mu \leq 1$, we get (4) as in (3). (5) By (1),

$$\int | f |^p \, d\mu \geq S(\mu, z) \, | f(z) |^p \qquad (z \in D).$$

and hence $\int | f |^p \, d\mu \geq \int | f |^p \, S(\mu)dm$. Assuming $f(a) = 1$ and $a \in D$, we get $S(\mu, a) \geq S(S(\mu)dm, a)$. (6) If $R(\mu, p, a)$ is bounded on $\bar{D}$, then $H^p(\mu) \subset L^\infty(\mu)$. By [11, Theorem 5.2], $H^p(\mu)$ is finitely dimensional. It is easy to see that *supp* μ is a finite set. Then trivially $R(\mu, p, a) = \infty$ except *supp* μ. The proof of the statement for D is same to that for $\bar{D}$, assuming $\mu = \mu \mid D$.

Even if v is not bounded, $\tilde{v}$ may be bounded. However (3) and (6) of Proposition 1 show that $\tilde{R}$ is also not bounded. The following theorem gives a stronger result.

Theorem 2. Suppose $0 < p < \infty$ and μ is a finite positive Borel measure. If $a \in \partial D$, then the following are valid.

(1) $\mu(\{a\}) = 0$ if and only if $S(\mu, p, a) = 0$.

(2) $\lim_{r \to 1-} S(\mu, p, ra) = 0$ except a countable set of a in ∂D.

(3) If $\mu(\{a\}) = 0$ and $\{a_n\}$ is a sequence in D with $lim \; a_n = a$, then $\lim_{n \to \infty} S(\mu, p, a_n) = 0$.

(4) If $\mu(\{a\}) > 0$, then for each n, the set $\{z \in D \; ; \; | z - a | < 1/n\} \cap \{z \in D \; ; \; S(\mu, p, z) < 1/n\}$ is a nonempty open set.

(5) If $b < c$ and $E = \{z \in D \; ; \; z = re^{i\theta}, 0 \leq r < 1 \text{ and } b \leq \theta \leq c\}$, then R is not bounded on E.

Proof. We may assume $a = 1$. (1) If $\mu(\{1\}) > 0$, then $| f(1) |^p \leq \int | f |^p \, d\mu/\mu(\{1\})$ and so $R(\mu, p, 1) \leq 1/\mu(\{1\})$. (1) of Proposition 1 implies $S(\mu, p, 1) > 0$. Conversely suppose $\mu(\{1\}) = 0$. If $z \in \bar{D}$ and $z \neq 1$, then $\lim_{t \to 1+} | (1-t)/(z-t) | = 0$ and

$$\left| \frac{z-1}{z-t} - 1 \right| = \left| \frac{1-t}{z-t} \right| < 1 \qquad (t > 1).$$

For any $t > 1$,

$$S(\mu, p, 1) \leq \int_{\bar{D}} \left|1 - \frac{z-1}{z-t}\right|^p d\mu(z) = \int_{\bar{D}\setminus\{1\}} \left|\frac{1-t}{z-t}\right|^p d\mu(z).$$

As $t \to 1$, by the Lebesgue's dominated convergence theorem, $S(\mu, p, 1) = 0$. (2) Suppose $\mu(\{1\}) = 0$. If there exist a sequence $\{r_n\}$ and a positive constant ε such that $0 < r_n < 1$ with $r_n \to 1$ and $S(\mu, p, r_n) \geq \varepsilon > 0$, then

$$| f(r_n) |^p \leq \frac{1}{\varepsilon} \int_{\bar{D}} | f |^p \, d\mu \text{ and so } | f(1) |^p \leq \frac{1}{\varepsilon} \int_{\bar{D}} | f |^p \, d\mu.$$

This implies $S(\mu, p, 1) > 0$ and contradicts (1). Hence if $\mu(\{1\}) = 0$, then $\lim_{r \to 1-} S(\mu, p, r) = 0$. This implies (2) because $\{a \in \partial D \; ; \; \mu(\{a\}) > 0\}$ is a countable set. (3) is clear by the proof of (2). (4) Suppose $\mu(\{1\}) > 0$ and for each n, put

$$G_n = \{z \in \bar{D} \; ; \; | z - 1 | < \frac{1}{n}\} \cap \{z \in \bar{D} \; ; \; S(\mu, p, z) < \frac{1}{n}\}.$$

Since $\{z \in \partial D \; ; \; \mu(\{z\}) > 0\}$ is a countable set, for each n there exists $b_n \in \{z \in \partial D \; ; \; | z - 1 | < \frac{1}{n}\}$ with $\mu(\{b_n\}) = 0$. Then $S(\mu, p, b_n) = 0$ by (1) and hence G_n is not empty. G_n is a relatively open set in $\bar{D}$ by (2) of Proposition 1 and so $G_n \cap D$ is a nonempty open set. (5) follows from (2).

If $R(\mu, 2, a) < \infty$, then there exists k_a in $H^2(\mu)$ such that $f(a) = \int f(z)\overline{k_a(z)} d\mu(z)$ for any f in $H^2(\mu)$ and hence $R(\mu, 2, a) = \int | k_a(z) |^2 \, d\mu(z)$. Thus the results in this section give the informations about the reproducing kernel k_a.

§3. Estimate of Riesz's function

In this section we give upper and lower estimates of S. The lower ones will be used later. The following proposition is a generalization of Szegő's Theorem in Introduction. In fact, if $\mu \mid D$ is a zero measure, then it gives Szegő's Theorem.

Proposition 3. Suppose $0 < p < \infty$ and μ is a finite positive Borel measure such that $(d\mu \mid \partial D)/(d\theta/2\pi) = w(e^{i\theta}), \mu \mid D = \sum a_j \delta_{z_j}$ and $\sum(1- \mid z_j \mid) < \infty$. Let b be a Blaschke product of $\{z_\ell\}$ and b_j a Blaschke product of $\{z_\ell\}_{\ell \neq j}$. Then for all $a \in D, (1- \mid a \mid^2) exp(log w)^\wedge(a) \leq S(\mu, p, a)$. If $a \in D \setminus \{z_\ell\}$, then

$$S(\mu, p, a) \leq | b(a) |^{-p} (1- | a |^2) exp(logw)^\wedge(a).$$

If $a = z_j$, then

$$S(\mu, p, a) \leq | b_j(a) |^{-p} (1- | a |^2) exp(logw)^\wedge(a) + a_j.$$

In particular, $S(\mu, p, a) > 0$ if and only if $logw \in L^1(d\theta)$.

Proof. Since $S(\mu, p, a) \geq S(wd\theta/2\pi, p, a)$ for all $a \in D$, by Szegő's Theorem $(1- | a |^2) exp(logw)^\wedge(a) \leq S(\mu, p, a)$ for all $a \in D$. Let B_n be a finite Blaschke product of $\{z_1, z_2, \cdots, z_n\}$. If $a \in D \setminus \{z_\ell\}$, then

$$\begin{aligned}
& S(\mu, p, a) \\
\leq \quad & inf\{\int | \frac{B_n}{B_n(a)} g |^p \, d\mu \mid \partial D + \sum_{j=1}^{\infty} a_j | \frac{B_n(z_j)}{B_n(a)} g(z_j) |^p \; ; \; g \in P \text{ and } g(a) = 1\} \\
= \quad & \frac{1}{| B_n(a) |^p} inf\{\int | B_n g |^p \, d\mu \mid \partial D + \sum_{j=n+1}^{\infty} a_j | B_n(z_j) |^p | g(z_j) |^p \; ; \; g \in P \\
\text{and} \quad & g(a) = 1\}.
\end{aligned}$$

As $n \to \infty$,

$$S(\mu, p, a) \leq \frac{1}{| b(a) |^p} inf\{\int | g |^p \, d\mu \mid \partial D \; ; \; g \in P \text{ and } g(a) = 1\}.$$

Now by Szegő's Theorem, for each $a \in D$ $S(\mu, p, a) \leq | b(a) |^{-p} (1- | a |^2) exp(logw)^\wedge(a)$. Let $B_{j,n}$ be a finite Blaschke product of $\{z_1, z_2, \cdots, z_n\} \setminus \{z_j\}$. If $a = z_j$ and $n > j$, then

$$\begin{aligned}
& S(\mu, p, a) \\
\leq \quad & inf\{\int | \frac{B_{j,n}}{B_{j,n}(a)} g |^p \, d\mu \; ; \; g \in P \text{ and } g(a) = 1\} \\
= \quad & \frac{1}{| B_{j,n}(a) |^p} inf\{\int | B_{j,n} g |^p \, d\mu \mid \partial D + a_j | B_{j,n}(a) |^p \\
+ \quad & \sum_{\ell \geq n+1} a_\ell | B_{j,n}(z_\ell) |^p | g(z_\ell) |^p \; ; \; g \in P \text{ and } g(a) = 1\}.
\end{aligned}$$

As $n \to \infty$, by Szegő's Theorem, for $a = z_j$,

$$S(\mu, p, a) \leq | b_j(a) |^{-p} (1- | a |^2) exp(logw)^\wedge(a) + a_j.$$

The following proposition is related to Theorem 2 in this paper and Theorem in [7]. In fact, if $\tilde{W}$ is bounded on D, then $(1- | a |^2)^{-2} S(Wdm, p, a)$ is bounded on D. Moreover if W is continuous on $\bar{D}$, then for all $e^{i\theta}$

$$\lim_{a\to e^{i\theta}}(1-|a|^2)^2R(Wdm,p,a)=1/W(e^{i\theta}).$$

Proposition 4. Suppose $0<p<\infty$ and μ is a finite positive Borel measure on $\bar{D}$.

(1) $\tilde{\mu}(a)\ge(S(\mu))^{\sim}(a) \quad (a\in D)$.

(2) If $d\mu=Wdm$ and $a\in D$, then

$$(1-|a|^2)^2exp(logW)^{\sim}(a)\le S(\mu,p,a)\le(1-|a|^2)^2\tilde{W}(a).$$

(3) $S(Wdm,a)=(1-|a|^2)^2S(W\circ\phi_a dm,0)$ for $a\in D$.

Proof. (1) For all $z\in D$

$$\int|f|^p\,d\mu\ge|f(z)|^p\,S(z) \text{ and so } \int|f|^p\,d\mu\ge\int|f|^p\,Sdm.$$

Assuming $f(z)=\{(1-|a|^2)/(1-\bar{a}z)^2\}^{2/p}$ for $a\in D,\tilde{\mu}(a)\ge\tilde{S}(a)$. (2) If $logW\in L^1(m)$, then

$$S(Wdm,p,a)=inf\{\int|f|^p\,Wdm\ ;\ f\in P \text{ and } f(a)=1\}$$

$$=inf\{\int|g|^p\,W\circ\phi_a\frac{(1-|a|^2)^2}{|1-\bar{a}z|^4}dm\ ;\ g\in H^p(W\circ\phi_a dm) \text{ and } g(0)=1\}$$

$$=(1-|a|^2)^2inf\{\int|k|^p\,W\circ\phi_a dm\ ;\ k\in H^p(W\circ\phi_a dm) \text{ and } k(0)=1\}$$

$$\ge(1-|a|^2)^2exp\int(logW)\circ\phi_a dm=(1-|a|^2)^2exp(logW)^{\sim}(a).$$

The inequality above is proved by two Jensen's inequalities. (3) is clear by the proof of (2).

In (2) of Proposition 4, we can get estimates of $S(\mu,p,a)$ as in Proposition 3 when $d\mu=Wdm+\sum_{j=1}^{\infty}a_j\delta_{z_j}$, $\{z_j\}\subset D$ and $\sum(1-|z_j|)<\infty$. The following theorem is important in this paper and the following lemma is used to prove it.

Lemma 1. Let $\Delta_s(a)$ be the set $\{z\in D\ ;\ |(a-z)/(1-\bar{a}z)|<s\}$ where $a\in D$ and $s\in(0,1)$. If $t\in(0,1)$ and $1-s^2=(1-|a|^2)(1-t^2)/5$, then $\overline{\Delta_t(0)}\subset\Delta_s(a)$.

Proof. The Euclidean center and radius of $\Delta_s(a)$ are

$$C=\frac{1-s^2}{1-s^2\mid a\mid^2}a\ ,\ R=\frac{1-\mid a\mid^2}{1-s^2\mid a\mid^2}s$$

respectively. Hence to prove $\overline{\Delta_t(0)}\subset\Delta_s(a)$, it is sufficient to show that

$$t+\frac{1-s^2}{1-s^2\mid a\mid^2}\mid a\mid\leq\frac{1-\mid a\mid^2}{1-s^2\mid a\mid^2}s.$$

If $1-s^2=(1-\mid a\mid^2)(1-t^2)/5$, then

$$1-s^2\leq\frac{(1-\mid a\mid^2)(1-t^2)}{5-\mid a\mid^2}$$

and hence $s^2\geq\{4+(1-\mid a\mid^2)t^2\}/(5-\mid a\mid^2)$. The last inequality is equivalent to

$$1-s^2\leq\frac{(1-\mid a\mid^2)(s^2-t^2)}{4}.$$

Then

$$1-s^2\leq\frac{(1-\mid a\mid^2)(s-t)}{2}\ \frac{s+t}{2}\leq\frac{(1-\mid a\mid^2)(s-t)}{\mid a\mid(t\mid a\mid+1)}$$

because $s+t\leq 2$ and $\mid a\mid(t\mid a\mid+1)\leq 2$. This implies that

$$t+\frac{1-s^2}{1-s^2\mid a\mid^2}\mid a\mid\leq\frac{1-\mid a\mid^2}{1-s^2\mid a\mid^2}s.$$

Theorem 5. Suppose $0<p<\infty$ and μ is a finite positive Borel measure on $\bar{D}$. Let $d\mu/dm=Wdm, K$ an arbitrary compact set in D and $t=max\{\mid z\mid\ ;\ z\in K\}$. Then, for $a\in D$

$$S(\mu,p,a)\geq\frac{(1-\mid a\mid^2)^3(1-t^2)}{5}exp\frac{2^4\cdot 5}{(1-\mid a\mid^2)^3(1-t^2)}\int_{K^c}log(W\wedge 1)dm.$$

If $1\leq p<\infty$ and $a\in D$, then

$$S(\mu,p,a)\geq\frac{(1-\mid a\mid^2)^{3(2-\frac{1}{p})}(1-t^2)^{2-\frac{1}{p}}}{2^{4(1-\frac{1}{p})}\cdot 5^{2-\frac{1}{p}}}\left(\int_{K^c}W^{-\frac{1}{p-1}}dm\right)^{\frac{1}{p}-1}.$$

Proof. By two Jensen's inequalities, for $a\in D$

$$\begin{aligned}&S(\mu,p,a)\geq S(Wdm,p,a)\\&=\ inf\{\int\mid g\mid^p W\circ\phi_a\frac{(1-\mid a\mid^2)^2}{\mid 1-\bar{a}z\mid^4}dm\ ;\ g(0)=1\}\\&=\ (1-\mid a\mid^2)^2 inf\{\int\mid k\mid^p W\circ\phi_a dm\ ;\ k(0)=1\}\end{aligned}$$

$$\begin{aligned}
&\geq (1-|a|^2)^2\int_0^1 2rdr\ exp\int_0^{2\pi} logW\circ\phi_a d\theta/2\pi \\
&\geq (1-|a|^2)^2(1-s^2)\int_s^1 \frac{2r}{1-s^2}dr\ exp\int_0^{2\pi} logW\circ\phi_a d\theta/2\pi \\
&\geq (1-|a|^2)^2(1-s^2)\ exp\frac{1}{1-s^2}\int_s^1 2dr\int_0^{2\pi} logW\circ\phi_a d\theta/2\pi \\
&= (1-|a|^2)^2(1-s^2)\ exp\frac{1}{1-s^2}\int_{D\backslash\Delta_s(0)} logW\circ\phi_a dm \\
&= (1-|a|^2)^2(1-s^2)\ exp\frac{1}{1-s^2}\int_{D\backslash\Delta_s(a)} logW\frac{(1-|a|^2)^2}{|1-\bar{a}z|^4}dm \\
&\geq (1-|a|^2)^2(1-s^2)\ exp\frac{(1-|a|^2)^2}{(1-|a|)^4}\ \frac{1}{1-s^2}\int_{D\backslash\Delta_s(a)} log(W\wedge 1)dm
\end{aligned}$$

where $s \in (0,1)$ and $\Delta_s(a) = \{z \in D\ ;\ |(a-z)/(1-\bar{a}z)| < s\}$. For each compact set $K \subset D$, if $t = max\{|z|\ ;\ z \in K\}$ and $1-s^2 = (1-|a|^2)(1-t^2)/5$, then by Lemma 1 $\overline{\Delta_t(0)} \subset \Delta_s(a)$. Hence $K \subset \Delta_s(a)$ and so $K^c \supset D\backslash\Delta_s(a)$. Thus, if $1-s^2 = (1-|a|^2)(1-t^2)/5$, then

$$\frac{(1-|a|^2)^2}{(1-|a|)^4}\ \frac{1}{1-s^2} = \frac{(1+|a|)^4}{(1-|a|^2)^2(1-s^2)} \leq \frac{2^4\cdot 5}{(1-|a|^2)^3(1-t^2)}$$

and hence for all $a \in D$

$$S(\mu,p,a) \geq \frac{(1-|a|^2)^3(1-t^2)}{5}\ exp\frac{2^4\cdot 5}{(1-|a|^2)^3(1-t^2)}\int_{K^c} log(W\wedge 1)dm.$$

Now we will prove the second inequality. Instead of two Jensen's inequalities, we will use Kolmogoroff's inequality (cf.[12, Theorem 4.3.1]). For $a \in D$, if $1 \leq p < \infty$ and $1/p + 1/q = 1$,

$$\begin{aligned}
&S(\mu,p,a) \\
&\geq (1-|a|^2)^2\int_0^1 2rdr\left(\int_0^{2\pi}(W\circ\phi_a)^{-\frac{1}{p-1}}d\theta/2\pi\right)^{-\frac{1}{q}} \\
&\geq (1-|a|^2)^2(1-s^2)\int_s^1 \frac{2r}{1-s^2}dr\left(\int_0^{2\pi}(W\circ\phi_a)^{-\frac{1}{p-1}}d\theta/2\pi\right)^{-\frac{1}{q}} \\
&\geq (1-|a|^2)^2(1-s^2)\left(\frac{1}{1-s^2}\int_s^1 2rdr\int_0^{2\pi}(W\circ\phi_a)^{-\frac{1}{p-1}}d\theta/2\pi\right)^{-\frac{1}{q}} \\
&= (1-|a|^2)^2(1-s^2)^{1+\frac{1}{q}}\left(\int_{D\backslash\Delta_s(0)}(W\circ\phi_a)^{-\frac{1}{p-1}}dm\right)^{-\frac{1}{q}} \\
&= (1-|a|^2)^2(1-s^2)^{1+\frac{1}{q}}\left(\int_{D\backslash\Delta_s(a)} W^{-\frac{1}{p-1}}\frac{(1-|a|^2)^2}{|1-\bar{a}z|^4}dm\right)^{-\frac{1}{q}} \\
&\geq (1-|a|^2)^2(1-s^2)^{1+\frac{1}{q}}\left\{\frac{(1-|a|^2)^2}{(1-|a|)^4}\int_{D\backslash\Delta_s(a)} W^{-\frac{1}{p-1}}dm\right\}^{-\frac{1}{q}}
\end{aligned}$$

$$\geq \frac{(1-|a|^2)^{2(1+\frac{1}{q})}(1-s^2)^{1+\frac{1}{q}}}{2^{\frac{4}{q}}}\left(\int_{D\backslash\Delta_s(a)} W^{-\frac{1}{p-1}}dm\right)^{-\frac{1}{q}}$$

where $s \in (0,1)$. As in the proof of the first inequality, for each compact set $K \subset D$, if $t = max\{|z| \; ; \; z \in K\}$ and $1-s^2 = (1-|a|^2)(1-t^2)/5$, then $K^c \supset D\backslash\Delta_s(a)$. Thus, if $1-s^2 = (1-|a|^2)(1-t^2)/5$, then for all $a \in D$

$$S(\mu,p,a) \geq \frac{(1-|a|^2)^{3(1+\frac{1}{q})}(1-t^2)^{1+\frac{1}{q}}}{2^{\frac{4}{q}}\cdot 5^{1+\frac{1}{q}}}\left(\int_{K^c} W^{-\frac{1}{p-1}}dm\right)^{-\frac{1}{q}}.$$

The second inequality of Theorem 5 implies $S(\mu,1,a) \geq (1-|a|^2)^3 \times (1-t^2)(1/5)ess.inf\{W(x) \; ; \; x \in K^c\}$. Let σ be a finite positive Borel measure on [0,1]. $\mu(re^{i\theta}) = \sigma(r) \times W(re^{i\theta})d\theta/2\pi$ is more general than $Wdm = 2rdr \times W(re^{i\theta})d\theta/2\pi$. If $\sigma(r)$ is singular to the Lebesgue measure on [0, 1], then μ is singular to m. However we can give an interesting lower estimate. It is different from that of Theorem 5 in case of $\mu = Wdm$.

Theorem 6. Suppose $0 < p < \infty$ and $d\mu = \sigma(r) \times W(re^{i\theta})d\theta/2\pi$ where $\sigma(r)$ is a finite positive Borel measure on [0, 1]. If $\mathbf{W}(e^{i\theta}) = \sup_r W(re^{i\theta})$ and $W_r(e^{i\theta}) = W(re^{i\theta})$, then for $a \in D$

$$\begin{aligned}
&(1-|a|^2)\int_{|a|}^1 exp(logW_r)^\wedge(a)d\sigma(r) \leq S(\mu,p,a) \\
&\leq \sigma([0,1])\, inf\left\{\sup_r \int_0^{2\pi} |f(re^{i\theta})|^p\, W(re^{i\theta})d\theta/2\pi \; ; \; f(a)=1\right\} \\
&\leq \sigma([0,1])\, inf\left\{\sup_r \int_0^{2\pi} |f(re^{i\theta})|^p\, \mathbf{W}(e^{i\theta})d\theta/2\pi \; ; \; f(a)=1\right\}.
\end{aligned}$$

Proof. For $a \in D$,

$$\begin{aligned}
S(\mu,p,a) &= inf\left\{\int_0^1 d\sigma(r)\int_0^{2\pi} |f(re^{i\theta})|^p\, W(re^{i\theta})d\theta/2\pi \; ; \; f(a)=1\right\} \\
&\geq \int_0^1 d\sigma(r)\, inf\left\{\int_0^{2\pi} |f(re^{i\theta})|^p\, W(re^{i\theta})d\theta/2\pi \; ; \; f(a)=1\right\} \\
&= \int_{|a|}^1 d\sigma(r)\, inf\left\{\int_0^{2\pi} |f(re^{i\theta})|^p\, W(re^{i\theta})d\theta/2\pi \; ; \; f(a)=1\right\} \\
&= \int_{|a|}^1 (1-|a|^2)\; exp(logW_r)^\wedge(a)d\sigma(r).
\end{aligned}$$

We used Szegő's Theorem in the last equality. The upper estimates are trivial.

Corollary 1. Let $d\mu = \sigma(r) \times W(re^{i\theta})d\theta/2\pi$ as in Theorem 6 and $0 < p < \infty$.
(1) If $W(re^{i\theta}) \equiv 1$, then for $a \in D$

$$(1-|a|^2)\,\sigma([|a|,1]) \le S(\mu,p,a) \le (1-|a|^2)\sigma([0,1]).$$

In particular, $S(\mu,p,0)=\sigma([0,1])$.

(2) If $W(re^{i\theta}) = |h(re^{i\theta})|$ for some outer function h in $H^1(d\theta)$, then for $a \in D$

$$(1-|a|^2)\int_{|a|}^{1} W(ra)d\sigma(r) \le S(\mu,p,a) \le (1-|a|^2)W(a)\sigma([0,1]).$$

(3) If $1<p<\infty$ and $\mathbf{W}(e^{i\theta}) = \sup_r W(re^{i\theta})$ satisfies the A_p condition, then there exists a positive constant γ such that for $a \in D$

$$S(\mu,p,a) \le \gamma(1-|a|^2)\, exp(log\mathbf{W})^\wedge(a)\sigma([0,1]).$$

Proof. (1) is a special case of (2). (2) Since h is an outer function in H^1, for $a \in D$

$$exp(logW_r)^\wedge(a) = exp(log\,|h_r|)^\wedge(a) = |h(ra)| = W(ra)$$

and

$$\begin{aligned} &\inf_f \left\{\sup_r \int_0^{2\pi} |f(re^{i\theta})|^p\, W(re^{i\theta})d\theta/2\pi\right\} \\ &= \inf_f \int_0^{2\pi} |f(e^{i\theta})|^p |h(e^{i\theta})|\, d\theta/2\pi = (1-|a|^2)\,|h(a)| = (1-|a|^2)W(a). \end{aligned}$$

Now Theorem 6 implies (2). (3) By a theorem of M.Rosenblum (cf. [10] and [9, Theorem 2.2]), there exists a positive constant γ such that for any $f \in P$

$$\sup_r \int_0^{2\pi} |f(re^{i\theta})|^p\, \mathbf{W}(e^{i\theta})d\theta/2\pi \le \gamma \int_0^{2\pi} |f(e^{i\theta})|^p\, \mathbf{W}(e^{i\theta})d\theta/2\pi$$

because $\mathbf{W} \in A_p$. By Theorem 6 and Szegő's Theorem, for $a \in D$

$$\begin{aligned} &\inf_f \left\{\sup_r \int_0^{2\pi} |f(re^{i\theta})|^p\, \mathbf{W}(e^{i\theta})d\theta/2\pi\right\} \le \gamma \inf_f \int_0^{2\pi} |f(e^{i\theta})|^p\, \mathbf{W}(e^{i\theta})d\theta/2\pi \\ &= \gamma(1-|a|^2)exp(log\mathbf{W})^\wedge(a). \end{aligned}$$

This implies (3).

§4. Carleson inequality and Riesz's function

Let ν and μ be finite positive Borel measures on $\bar{D}$ and $1 \le p < \infty$. We say that ν and μ satisfy the (ν,μ,p) - Carleson inequality, if there exists a constant $\gamma > 0$ such that

$$\int_{\bar{D}} |f|^p \, d\nu \leq \gamma \int_{\bar{D}} |f|^p \, d\mu$$

for all $f \in P$ (see [8]). ν and μ satisfy the (ν, μ, p) - Carleson inequality if and only if $H^p(\mu) \subset H^p(\nu)$ and the inclusion mapping $i_p : H^p(\mu) \to H^p(\nu)$ is bounded. We say that for $p > 1, \nu$ and μ satisfy the (ν, μ, p)-vanishing Carleson inequality if $H^p(\mu) \subset H^p(\nu)$ and $i_p : H^p(\mu) \to H^p(\nu)$ is compact. We say that for $p = 1$, ν and μ satisfy the (ν, μ, p)-vanishing Carleson inequality if i_p is star-compact. We could not prove Theorem 7 for $p = 1$ because we do not know anything about the predual of $H^1(\mu)$. Using Riesz's functions, we will show vanishing Carleson inequalities. As a result, we show that $R(\mu, p) \notin L^1(\mu)$ if *supp* μ is not a finite set. Moreover, from given a measure μ, we will show how to construct a measure ν such that the (ν, μ, p) -vanishing Carleson inequality is valid.

Theorem 7. Suppose $1 < p < \infty$, and ν and μ are finite positive Borel measures on $\bar{D}$.

(1) If $\int R(\mu, p) d\nu < \infty$, then ν and μ satisfy the (ν, μ, p)-vanishing Carleson inequality and

$$R(\mu, p, a) \leq \left(\int R(\mu, p) d\nu \right) R(\nu, p, a) \qquad (a \in \bar{D}).$$

(2) If V is a Borel function such that $0 \leq V \leq S$ on $\bar{D}$, then $V \mid g \mid^p$ is bounded on $\bar{D}$ for each g in $H^p(\mu)$, and Vdm and μ satisfy the (Vdm, μ, p)-vanishing Carleson inequality.

Proof. (1) By definition of $R(\mu, p, a)$, for $a \in \bar{D}$,

$$|f(a)|^p \leq R(\mu, p, a) \int |f|^p \, d\mu \qquad (f \in P).$$

Hence if $\gamma = \int R(\mu, p) d\nu < \infty$, then $\int |f|^p \, d\nu \leq \gamma \int |f|^p \, d\mu$ $(f \in P)$ and so $i_p : H^p(\mu) \to H^p(\nu)$ is bounded. We will show that i_p is compact. If $f_n \to f$ weakly in $H^p(\mu)$, then there exists a finite positive constant γ' such that

$$\int |f_n - f|^p \, d\mu \leq \gamma' \quad \text{for all } n.$$

By the hypothesis, $R(\mu, p, a) < \infty \quad \nu - a.e.$ on $\bar{D}$ and so $f_n \to f \quad \nu - a.e.$ on $\bar{D}$ because $f_n \to f$ weakly. Moreover by definition of $R(\mu, p, a), |f_n(a) - f(a)|^p \leq \gamma' R(\mu, p, a)$ and by the hypothesis, $R(\mu, p, a) \in L^1(\nu)$. Thus

$$\int |f_n - f|^p \, d\nu \to 0 \quad \text{as } n \to \infty.$$

by the Lebesgue's dominated convergence theorem. This implies i_p is compact. Since $\int |f|^p d\nu \leq \gamma \int |f|^p d\mu$ and $\gamma = \int R(\mu,p)d\nu$, assuming $f(a)=1$, we get $S(\nu,p,a) \leq \gamma\, S(\mu,p,a)$. Now by (1) of Proposition 1, we get the inequality of (1). (2) If $0 \leq V \leq S$, then $VR \leq 1$ and hence $V(a) \mid f(a) \mid^p$ is bounded on $\bar{D}$ by $\int |f|^p d\mu$, for each $f \in H^p(\mu)$. Moreover if $\nu = Vdm$ and $0 \leq V \leq S$, then $\int R(\mu,p)d\nu \leq \int dm = 1$ and hence by (1) ν and μ satisfy the (ν,μ,p)-vanishing Carleson inequality.

Corollary 2. If $0 < p < \infty$ and *supp* μ is not a finite set, then $R(\mu,p) \notin L^1(\mu)$.

Proof. Suppose $1 < p < \infty$. If $R(\mu,p) \in L^1(\mu)$, then the inclusion map $i_p : H^p(\mu) \to H^p(\mu)$ is compact. It is easy to see that i_p is an identity operator. Hence the unit ball of $H^p(\mu)$ is compact with respect to the norm. Therefore $H^p(\mu)$ is finitely dimensional. This contradicts that *supp* μ is not a finite set. This implies that $R(\mu,p) \notin L^1(\mu)$. For $0 < p \leq 1$, the proof is due to the referee. Choose n sufficiently large that $np > 1$. If $g(a) = 1$ then $g^n(a) = 1$ as well, and g^n is a polynomial if g is a polynomial. Thus,

$$\begin{aligned} S(\mu,p,a) &= inf\{\int_{\bar{D}} |f|^p d\mu \ ;\ f \in P, f(a) = 1\} \\ &\leq inf\{\int_{\bar{D}} |g^n|^p d\mu \ ;\ g \in P,\ g(a) = 1\} = S(\mu,np,a). \end{aligned}$$

This implies that $R(\mu,p) \notin L^1(\mu)$ for $0 < p \leq 1$.

Corollary 3. Suppose $1 < p < \infty$ and $d\mu/dm = W$

(1) If $logW \in L^1(m)$ and $d\nu = (1-|z|^2)^2 exp(logW)^{\sim} dm$, then ν and μ satisfy the (ν,μ,p) - vanishing Carleson inequality.

(2) If $\chi_{K^c}\log(W \wedge 1) \in L^1(m)$ for some compact set K in D, there exist positive constant a and nonpositive constant b such that $d\nu = a(1-|z|^2)^3\{\exp b(1-|z|^2)^{-3}\}dm$ and μ satisfy the (ν,μ,p)-vanishing Carleson inequality.

(3) Suppose $\chi_{K^c}W^{-\frac{1}{p-1}} \in L^1(m)$ for some compact set K in D. If $d\nu = c(1-|z|^2)^{3(2-\frac{1}{p})}dm$, then ν and μ satisfy the (ν,μ,p)-vanishing Carleson inequality.

Suppose $1 < p < \infty$ and $d\mu/dm = W$. If $\chi_{K^c} logW \in L^1(m)$ for some compact set K in D, there exist positive constant a and nonpositive constant b such that

$$\{a(1-|z|^2)^3 \exp\ b(1-|z|^2)^{-3}\} \mid f(z) \mid^p$$

is bounded on D for each $f \in H^p(\mu)$. Here a and b do not depend on f. This is a corollary of (2) in Theorem 7.

§5. $H^p(\mu)$ and $L^p_a(\mu)$

The following is a result of Theorem 5. If $d\mu/dm = W$ and $logW$ is integrable on the complement K^c of a compact set in D, then $H^p(\mu) \subseteq L^p_a(\mu)$. In this section, we show that if $logW$ is locally integrable on K^c, then the same result is true. We give a necessary and sufficient condition for $H^p(\mu) \subset L^p_a(\mu)$ using Riesz's function. Theorem 8 is a joint work with K.Takahashi. A subset E of D is a uniqueness set if E satisfies the following : If f in H is zero on E, then $f \equiv 0$ on D.

Lemma 2. Suppose $0 < p < \infty$ and μ is a finite positive Borel measure on D. Then, the following (1) $\sim$ (3) are equivalent.

(1) $\sup_{a\in K} R(\mu,p,a) < \infty$ for all compact set K in D.

(2) $\int_K R(\mu,p)dm < \infty$ for all compact set K in D.

(3) $\int_K log\ R(\mu,p)dm < \infty$ for all compact set K in D.

Proof. Both (1) $\Rightarrow$ (2) and (2) $\Rightarrow$ (3) are trivial. We will show (3) $\Rightarrow$ (1). We may assume that $\mu(D) = 1$. For any $f \in P$,

$$log \mid f(0) \mid^p \leq \frac{1}{m(D_r(0))}\int_{D_r(0)} log \mid f \mid^p dm.$$

If $a \in D_r(0)$, then for all $f \in P$

$$log \mid f(a) \mid^p \leq \frac{1}{m(D_r(0))}\int_{D_r(a)} log \mid f \mid^p \frac{(1-\mid a \mid^2)^2}{\mid 1 - \bar{a}z \mid^4} dm.$$

Assuming $\int \mid f \mid^p d\mu \leq 1$, we get

$$log\ R(\mu,p,a) \leq \frac{1}{m(D_r(0))} \frac{(1+\mid a \mid)^2}{(1-\mid a \mid)^2}\int_{D_r(a)} log\ R(\mu,p)dm.$$

Since $D_r(a) \subset D_{2r}(0)$ and $R(\mu,p,a) \geq 1$, for each $a \in D_r(0)$ there exists a finite positive constant γ_r such that

$$log\ R(\mu,p,a) \leq \gamma_r \int_{D_{2r}(0)} log\ R(\mu,p)dm.$$

This implies (1).

Lemma 3. Let X be a Banach space which consists of analytic functions on D and contains 1. Suppose there exists a dense subspace Y of X such that if f in Y, then $(f-f(a))/(z-a)$ belongs to Y for some a in D. If $(z-a)X$ is not dense in X, then the functial $f \to f(a)$ is bounded.

Proof. By the hypothesis, if $f \in Y$, then $f = f(a) + (z-a)g$ for some $g \in Y$. Since $(z-a)X$ is not dense in X, there exists $\phi \in X^*$ such that $<(z-a)h, \phi>= 0$ for all $h \in X$ but $<1, \phi> \neq 0$. Hence, $<f, \phi>= f(a) <1, \phi>$. Thus $| f(a) | \leq \gamma \|f\|$ for all $f \in X$ where $\gamma = |<1, \phi>|^{-1} \|\phi\|_*$.

Theorem 8. Suppose $1 \leq p < \infty$ and μ is a finite positive Borel measure on D such that $(supp\ \mu) \cap D$ is a uniqueness set for H.

(1) $L_a^p(\mu)$ is closed if and only if for all compact set K in D

$$\int_K log\ r(\mu, p) dm < \infty \text{ or } \int_K log\ s(\mu, p) dm > -\infty.$$

(2) $H^p(\mu) \subset L_a^p(\mu)$ if and only if for all compact set K in D

$$\int_K log\ R(\mu, p) dm < \infty \text{ or } \int_K log\ S(\mu, p) dm > -\infty.$$

Proof. (1) If $f \in L_a^p(\mu)$, then $(f - f(0))/z$ belongs to H. Since $(f - f(0))/z$ is bounded on $| z | \leq t < 1$ and $1/z$ is bounded on $| z | \geq t, (f - f(0))/z$ belongs to $L_a^p(\mu)$. This implies that $\{f \in L_a^p(\mu) \ ; \ f(0) = 0\} = zL_a^p(\mu)$ and hence $L_a^p(\mu) = \mathbf{C} \oplus zL_a^p(\mu)$. If $Af = zf$ for $f \in L_a^p(\mu)$, then A is a bounded operator on $L_a^p(\mu)$ and the range of A is complemented in $L_a^p(\mu)$ by what was just proved. By [4, Part III, Corollary 2.3], the range of A is closed and hence $zL_a^p(\mu)$ is not dense in $L_a^p(\mu)$. Applying Lemma 3 with $X = Y = L_a^p(\mu)$ and $a = 0, r(\mu, p, 0) < \infty$ follows. The same argument is true for all $a \in D \backslash \{0\}$ and hence $r(\mu, p, a) < \infty$ for all $a \in D$. By the boundedness of holomorphic functions on compact sets and the uniform boundedness principle, $\sup_{a \in K} r(\mu, p, a) < \infty$ for all compact set K in D. As Lemma 2 is also for $r(\mu, p, a)$,

$$\int_K log\ r(\mu, p) dm < \infty \quad \text{or} \quad \int_K log\ s(\mu, p) dm > -\infty.$$

Conversely, suppose $\int_K log\ r(\mu, p) dm < \infty$ for any compact set K. Then by the above lemma, $sup_k\ r(\mu, p) < \infty$ for any compact set K. If f is in the $L^p(\mu)$-norm closure of $L_a^p(\mu)$, then there exists a sequence $\{f_n\}$ in $L_a^p(\mu)$ such that $\int | f - f_n |^p d\mu \to 0$. Then by hypothesis on $r(\mu, p), sup\{| f_n(z) | \ ; \ z \in D_r(0)\} < \infty$ for each $r < \infty$, Hence, for each $r < \infty$ there exists a subsequence $\{f_{n_j}\}$ in $L_a^p(\mu)$ and an analytic function g_r on $D_r(0)$ such that $f_{n_j} \to g_r$ uniformly on $D_r(0)$. This implies that $f = g_r\ \mu - a.e.$ on $D_r(0)$ for all $r < \infty$. Thus $g = lim_r\ g_r$ is analytic on D and $f = g\ \mu - a.e.$ on D.

(2) The 'if' part is same to (1) and hence we will show the 'only if' part. Put $M = \{f \in L^p(\mu) \ ; \ zf \in H^p(\mu)\}$, then M is a closed subspace of $L^p(\mu)$ such that

$$M \supseteq H^p(\mu) \supseteq zM \supseteq H^p(\mu)_0$$

where $H^p(\mu)_0 = \{f \in H^p(\mu) \ ; \ f(0) = 0\}$. $H^p(\mu)_0$ is well defined because $H^p(\mu) \subset L^p_a(\mu)$. Suppose $H^p(\mu) \neq zM$. Then $H^p(\mu) = \mathbf{C} + H^p(\mu)_0 = \mathbf{C} + zM$ and $\mathbf{C} \cap zM = \{0\}$. As in the proof of (1), by [4, Part III, Corollary 2.3], zM is closed and hence $zH^p(\mu)$ is not dense in $H^p(\mu)$. Applying Lemma 3 with $X = H^p(\mu), Y = P$ and $a = 0, R(\mu,p,0) < \infty$ follows. Suppose $H^p(\mu) = zM$. Then $z^{-1} \in L^p(\mu)$ and hence $\mu(\{0\}) = 0$. If $Af = zf$ for $f \in M$, then A is a one-one bounded operator from M onto $H^p(\mu)$. Therefore A is invertible and hence $A(zM) = zH^p(\mu)$ is closed. Since $H^p(\mu) \subset L^p_a(\mu), zH^p(\mu) \neq H^p(\mu)$ and hence by Lemma 3, $R(\mu,p,0) < \infty$ follows. The same argument implies that $R(\mu,p,a) < \infty$ for all $a \in D$. Now, as in the proof of (1), Lemma 2 implies the 'only if' part of (2).

Corollary 4. Suppose $1 \le p < \infty$ and $d\mu/dm = W$. If $logW$ is locally integrable on K_0^c for some compact set K_0 in D, then $L^p_a(\mu)$ is closed and $H^p(\mu) \subseteq L^p_a(\mu)$.

Proof. By (1) of Theorem 8, it is sufficient to prove that for any compact set K in $D, \inf_K \ s(\mu,p) > -\infty$. If $logW$ is integrable on K_0^c, then by the proof of Theorem 5 $inf_K \ s(\mu,p) > -\infty$. For a more general W in this corollary, we have to proceed as the following. Suppose $a \in D$ and $0 < \varepsilon < \delta < 1$. As in the proof of Theorem 5,

$$\begin{aligned} & s(\mu,p,a) \\ \ge \ & (1-|a|^2)^2 \int_\varepsilon^\delta exp\left(\int_0^{2\pi} logW \circ \phi_a d\theta/2\pi\right) 2rdr \\ \ge \ & (1-|a|^2)^2(\delta^2-\varepsilon^2) exp\left(\frac{1}{\delta^2-\varepsilon^2}\int_{\Delta_\delta(0)\setminus\Delta_\varepsilon(0)} logW \circ \phi_a dm\right) \\ \ge \ & (1-|a|^2)^2(\delta^2-\varepsilon^2) exp\left(\frac{2^2}{(1-|a|)^2(\delta^2-\varepsilon^2)}\int_{\Delta_\delta(a)\setminus\Delta_\varepsilon(a)} log(W \wedge 1) dm\right). \end{aligned}$$

Suppose K is an arbitrary compact set in D. Put $t = max\{|z| \ ; \ z \in K_0\}$ and $k = max\{|z| \ ; \ z \in K\}$. The Euclidean center and radius of $\Delta_\gamma(k)$ $(0 < \gamma < 1)$ are

$$C(\gamma) = \frac{1-\gamma^2}{1-\gamma^2 k^2}k, \quad R(\gamma)\frac{1-k^2}{1-\gamma^2 k^2}\gamma$$

respectively. Put $\ell = R(\delta) + C(\delta)$ and $s = R(\varepsilon) - C(\varepsilon)$. There exist δ and ε such that

$$\Delta_\ell(0)\setminus\Delta_s(0) \subset D\setminus\Delta_t(0).$$

Then for all $a \in K$

$$\Delta_\delta(a)\setminus\Delta_\varepsilon(a) \subset \Delta_\ell(0)\setminus\Delta_s(0).$$

Hence for all $a \in K$

$$\Delta_\delta(a)\setminus\Delta_\varepsilon(a) \subset K_0^c$$

and so for all $a \in K$

$$s(\mu, p, a) \geq (1-\mid a \mid^2)^2(\delta^2 - \varepsilon^2)exp\left(\frac{2^2}{(1-\mid a \mid^2)^2(\delta^2-\varepsilon^2)}\int_{K_0^c} log(W \wedge 1)dm\right).$$

This shows the corollary.

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