(A_2)-Conditions and Carleson Inequalities

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(A_2)-Conditions and Carleson Inequalities

in

Bergman Spaces

by

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and

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Dedicated to Professor Mitsuru Nakai on his sixtieth birthday

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Abstract. Let ν and μ be finite positive measures on the open unit disk D. We say that ν and μ satisfy the (ν, μ)-Carleson inequality, if there is a constant $C > 0$ such that

$$\int_D |f|^2 \, d\nu \leqq C \int_D |f|^2 \, d\mu$$

for all analytic polynomials f. In this paper, we study the necessary and sufficient condition for the (ν, μ)-Carleson inequality. We establish it when ν or μ is an absolutely continuous measure with respect to the Lebesgue area measure which satisfies the (A_2)-condition. Moreover, many concrete examples of such measures are given.

§ 1. Introduction

Let D denote the open unit disk in the complex plane. For $1 \leqq p \leqq \infty$, let L^p denote the Lebesgue space on D with respect to the normalized Lebesgue area measure m, and $\|\cdot\|_p$ represents the usual L^p-norm. For $1 \leqq p < \infty$, let L^p_a be the collection of analytic functions f on D such that $\|f\|_p$ is finite, which are so called the Bergman spaces. For any z in D, let ϕ_z be the Möbius function on D, that is

$$\phi_z(w) = \frac{z-w}{1-\bar{z}w} \qquad (w \in D),$$

and put

$$\beta(z,w) = 1/2 \log(1+|\phi_z(w)|)(1-|\phi_z(w)|)^{-1} \qquad (z,w \in D).$$

For $0 < r < \infty$ and z in D, set

$$D_r(z) = \{w \in D;\ \beta(z,w) < r\}$$

be the Bergman disk with "center" z and "radius" r, and we define an average of a finite positive measure μ on $D_r(a)$ by

$$\hat{\mu}_r(a) = \frac{1}{m(D_r(a))} \int_{D_r(a)} d\mu \qquad (a \in D),$$

and if there exists a non-negative function u in L^1 such that $d\mu = u\, dm$, then we may write it $\hat{u}_r$ instead of $\hat{\mu}_r$.

Let ν and μ be finite positive measures on D, and let P be the set of all analytic polynomials. We say that ν and μ satisfy the (ν,μ)-Carleson inequality, if there is a constant $C > 0$ such that

$$\int_D |f|^2\, d\nu \leqq C \int_D |f|^2\, d\mu$$

for all f in P. Our purpose of this paper is to study conditions on ν and μ so that the (ν,μ)-Carleson inequality is satisfied. If $\nu \leqq C\mu$

on D, then the (ν, μ)-Carleson inequality is true. However it is clear that this sufficient condition for the (ν, μ)-Carleson inequality is too strong. A reasonable and natural condition is the following: there exist $r>0$ and $\gamma>0$ such that

$$(*) \qquad \hat{\nu}_r(a) \leqq \gamma \hat{\mu}_r(a) \qquad (a \in D).$$

The averages $\hat{\mu}_r(a)$ are sometimes computable. If $\mu = m$, then $\hat{\mu}_r(a)=1$ on D. If $d\mu=(1-|z|^2)^\alpha dm$ for $\alpha>-1$, then $\hat{\mu}_r(a)$ is equivalent to $(1-|a|^2)^\alpha$ on D.

When $d\mu=(1-|z|^2)^\alpha dm$ for $\alpha>-1$, Oleinik-Pavlov [7], Hastings [2], or Stegenga [8] showed that ν and μ satisfy the Carleson inequality if and only if they satisfy (*). In §3 of this paper, when $d\mu = u\,dm$ and u satisfies the $(A_2)_\partial$-condition (the definition is in § 3), we obtain that the (ν, μ)-Carleson inequality is satisfied if and only if they satisfy (*). We show that if both u and u^{-1} are in BMO_∂ (see [9;p127]), then u satisfies the $(A_2)_\partial$-condition. We give some concrete examples which satisfy the $(A_2)_\partial$-condition.

When $\nu = m$ and $d\mu = \chi_G\, dm$, where χ_G is a characteristic function of a measurable subset G of D, Luecking [4] showed the equivalence between the (ν, μ)-Carleson inequality and the condition (*). If we do not put any hypotheses on μ, the problem is very difficult. The equivalence between the (ν, μ)-Carleson inequality and the condition (*) is not known even if $\nu = m$. Luecking [5] showed the following:

(1) If there exists $\gamma>0$ such that $\hat{m}_r(a) \leqq \gamma \hat{\mu}_r(a)$ for all $r>0$ and a in D, then the (m, μ)-Carleson inequality is satisfied.

(2) Suppose the (μ, m)-Carleson inequality is valid (equivalently $\hat{\mu}_r$ is bounded on D). Then the (m, μ)-Carleson inequality implies the condition (*).

In §2 of this paper, we give a sufficient condition (close to that of (1)) for the (ν, μ)-Carleson inequality when ν is not necessarily m. Moreover, using the idea of Luecking's proof of (2), a generalization of (2) is given. In §4, when $d\nu = v\, dm$ and v satisfies the (A_2)-condition (the definition is in §3), we establish a more natural extension of (2) under some condition of a quantity $\varepsilon_r(\nu)$ (the definition is in §2), that is, $\varepsilon_r(\nu)\to 0$ as $r\to\infty$. The (A_2)-condition is weaker than the $(A_2)_\partial$ -condition. We give some concrete examples which satisfy the (A_2)-condition or the above condition of $\varepsilon_r(\nu)$.

§ 2. (ν,μ)-Carleson inequality

Let G be a measurable subset of D and u be a non-negative function in L^1, and put

$$(u_G^{-1})\hat{}_r(a)=\frac{1}{m(D_r(a))}\int_{D_r(a)} u^{-1}\chi_G\, dm.$$

Particularly, when $G=D$, we will omit the letter D in the above notation. The following proposition 1 gives a general sufficient condition on ν and μ which satisfy the (ν,μ)-Carleson inequality. In order to prove it we use ideas in [5] and [9;p109]. Since $(u^{-1})\hat{}_r(a)^{-1} \leq \hat{u}_r(a)$ for all a in D, proposition 1 is also related with (1) of §1 (cf.[5; Theorem 4.2]).

Proposition 1. Suppose that $d\mu=u\,dm$. Put $E_r=\{z\in D$;there is a $w\in \operatorname{supp}\nu$ such that $\beta(z,w)<r/2\}$. If there exist $r>0$ and $\gamma>0$ such that $u>0$ a.e. on $E=E_r$, and $\hat{\nu}_r(a)\times(u_E^{-1})\hat{}_r(a)\leq\gamma$ for all a in D, then there is a constant $C>0$ such that

$$\int_D |f|^2\, d\nu \leq C\int_E |f|^2\, d\mu$$

for all f in P.

Proof. Suppose that $\hat{\nu}_{2r}(a)\times(u_E^{-1})\hat{}_{2r}(a)\leq\gamma$ for all a in D, and put $E=\{z\in D$;there is a $w\in\operatorname{supp}\nu$ such that $\beta(z,w)<r\}$. By an elementary theory for Bergman disks, there is a positive integer $N=N_r$

such that there exists $\{\lambda_n\} \subset D$ satisfying that $D = \cup D_r(\lambda_n)$ and any z in D belongs to at most N of the sets $D_{2r}(\lambda_n)$ (cf.[9;p62]), therefore

$$\int_{\operatorname{supp} \nu} |f|^2 \, d\nu$$

$$\leqq \sum \int_{D_r(\lambda_n) \cap \operatorname{supp} \nu} |f|^2 \, d\nu$$

$$\leqq \sum \nu(D_r(\lambda_n)) \times \sup\{|f(z)|^2 ; z \in D_r(\lambda_n) \cap \operatorname{supp} \nu\}.$$

By proposition 4.3.8 in [9;p62], there is a constant $C = C_r > 0$ such that

$$|f(z)| \leqq \frac{C}{m(D_r(z))} \int_{D_r(z)} |f(w)| \, dm(w)$$

for all f analytic, z in D. If z in $D_r(\lambda_n) \cap \operatorname{supp} \nu$, then $D_r(z)$ is contained in $D_{2r}(\lambda_n) \cap E$, and there exists a constant $K = K_r > 0$ such that $m(D_{2r}(\lambda_n)) \leqq K m(D_r(z))$ for all $n \geqq 1$ (cf. [9;p61]). Hence the Cauchy-Schwarz's inequality implies that

$$\int_D |f|^2 \, d\nu$$

$$\leqq \sum \nu(D_r(\lambda_n)) \times \left(\frac{KC}{m(D_{2r}(\lambda_n))} \int_{D_{2r}(\lambda_n) \cap E} |f| \, dm \right)^2$$

$$\leqq \sum \nu(D_r(\lambda_n)) \times K^2 C^2$$

$$\times \left(\frac{1}{m(D_{2r}(\lambda_n))} \int_{D_{2r}(\lambda_n)} |f|^2 u \chi_E \, dm \right)$$

$$\times \left(\frac{1}{m(D_{2r}(\lambda_n))} \int_{D_{2r}(\lambda_n)} u^{-1} \chi_E \, dm \right)$$

$$\leqq K^2 C^2 \sum \hat{\nu}_{2r}(\lambda_n) \times (\widehat{u_E^{-1}})_{2r}(\lambda_n)$$

$$\times \left(\int_{D_{2r}(\lambda_n) \cap E} |f|^2 u \, dm \right).$$

By the hypothesis and a choice of disks, it follows that

$$\int_D |f|^2 \, d\nu \leqq K^2 C^2 \gamma N \int_E |f|^2 \, d\mu .$$

This completes the proof. ▮

Let μ be a finite nonzero positive measure on D. For any a in D, put

$$k_a(z) = (1-|a|^2)/(1-\bar{a}z)^2 \qquad (z \in D),$$

and a function $\tilde{\mu}$ on D is defined by

$$\tilde{\mu}(a) = \int_D |k_a|^2 \, d\mu .$$

Moreover, for any fixed $r < \infty$, put

$$\varepsilon_r(\mu) = \sup_{a \in D} \left(\int_{D \setminus D_r(a)} |k_a|^2 \, d\mu \right) \times \left(\int_D |k_a|^2 \, d\mu \right)^{-1} .$$

If there exists a non-negative function u in L^1 such that $d\mu = u\, dm$, then making a change of variable, it is easy to see that

$$\varepsilon_r(\mu) = \sup_{a \in D} \left(\int_{D \setminus D_r(0)} u \circ \phi_a \, dm \right) \times \left(\int_D u \circ \phi_a \, dm \right)^{-1} .$$

In general $0 < \varepsilon_r(\mu) \leqq 1$. In this section and §4, this quantity ε_r is important. The following proposition 2 gives two general necessary conditions on ν and μ which satisfy the (ν, μ)-Carleson inequality. In order to prove (2) of proposition 2 we use ideas in [5; Theorem 4.3]. Since $\varepsilon_r(m) < 1$ and $\varepsilon_r(m) \to 0$ $(r \to \infty)$, (2) of proposition 2 is related with (2) of §1.

Lemma 1. Let μ be a finite positive measure on D and $0 < r < \infty$, then the following (1)~(3) are equivalent.

(1) $\varepsilon_r(\mu) < 1$.

(2) There is a $\delta = \delta_r < \infty$ such that

$$\int_{D \setminus D_r(a)} |k_a|^2 \, d\mu \leqq \delta \int_{D_r(a)} |k_a|^2 \, d\mu$$

for all a in D.

(3) There is a $\rho = \rho_r < \infty$ such that

$$\tilde{\mu}(a) \leqq \rho \, \hat{\mu}_r(a)$$

for all a in D.

Proof. The implication (1)⇒(2) is trivial. (2)⇒(3) and (3)⇒(1) follow from lemma 4.3.3 in [9;p60]. In fact, by lemma 4.3.3, there exist $L = L_r > 0$ and $M = M_r > 0$ such that

$$L \leqq m(D_r(a)) \times \inf\{|k_a(z)|^2 ; z \in D_r(a)\}$$

and

$$m(D_r(a)) \times \sup\{|k_a(z)|^2 ; z \in D_r(a)\} \leqq M$$

for all a in D. Thus remainder implications are obtained. ∎

Proposition 2. Suppose that ν and μ satisfy the (ν, μ)-Carleson inequality, then the following are true.

(1) If there exists $r < \infty$ such that $\varepsilon_r(\mu) < 1$, then there exists $\gamma > 0$ such that $\hat{\nu}_r(a) \leqq \gamma \hat{\mu}_r(a)$ for all a in D.

(2) If $d\nu = v\,dm$, $v > 0$ a.e. on D, $\varepsilon_t(\nu) \to 0$ $(t \to \infty)$, and there are $l > 0$ and $\gamma' > 0$ such that $\hat{\mu}_l(a) \times (v^{-1})\hat{}_l(a) \leqq \gamma'$ for all a in D, then there are $r > 0$ and $\gamma = \gamma_r > 0$ such that $\hat{\nu}_r(a) \leqq \gamma \hat{\mu}_r(a)$ for all a in D.

Proof. Since $k_a(z)$ is uniformly approximated by polynomials, the inequality is valid for $f = k_a$, that is

$$\int_D |k_a|^2\, d\nu \leqq C \int_D |k_a|^2\, d\mu .$$

Firstly, we show that (1) is true. The above inequality and lemma 1 imply that

$$\tilde{\nu}(a) \leqq C \tilde{\mu}(a)$$

$$\leqq C \rho \hat{\mu}_r(a)$$

for all a in D. Moreover, by lemma 4.3.3 in [9;p60], there exists a constant $L > 0$ such that

$$\hat{\nu}_r(a) \leqq L^{-1} \tilde{\nu}(a)$$

for all a in D. Hence we have that

$$\hat{\nu}_r(a) \leqq C\rho L^{-1}\hat{\mu}_r(a)$$

for all a in D.

Next, we prove that (2) is true. For any a in D and $r \geqq l$, put $d\mu_{a,r} = (1-\chi_{D_r(a)})d\mu$. By the latter half of the hypothesis in (2), we have that

$$(\mu_{a,r})\hat{}_l(\lambda)\times(v^{-1})\hat{}_l(\lambda) \leqq \gamma'$$

for all a, λ in D, and $r \geqq l$. Set $E_{a,r,l} = \{z \in D$;there is a w in $\operatorname{supp}\mu_{a,r}$ such that $\beta(z,w) < l/2\}$. By proposition 1, there exists a constant $C' > 0$ such that

$$\int_{D\backslash D_r(a)} |f|^2\, d\mu \leqq C' \int_{E_{a,r,l}} |f|^2\, d\nu$$

for all a in D, $r \geqq l$ and f in P. Here we claim that $E_{a,r,l}$ is contained in $D\backslash D_{r/2}(a)$. In fact, since $D\backslash D_r(a)$ contains $\operatorname{supp}\mu_{a,r}$ and $r \geqq l$, if z belongs to $E_{a,r,l}$ then there exists w in D such that $\beta(w,a) \geqq r$ and $\beta(w,z) < r/2$. Therefore,

$$\begin{aligned} r &\leqq \beta(w,a) \\ &\leqq \beta(w,z) + \beta(z,a) \\ &< r/2 + \beta(z,a), \end{aligned}$$

thus we have that z is contained in $D\backslash D_{r/2}(a)$. Particularly put $f = k_a$ in the above inequality, then

$$\int_{D\backslash D_r(a)} |k_a|^2\, d\mu \leqq C' \int_{D\backslash D_{r/2}(a)} |k_a|^2\, d\nu$$

for all a in D and $r \geqq l$. It follows that

$$\begin{aligned} &\int_{D_r(a)} |k_a|^2\, d\mu \\ &= \int_D |k_a|^2\, d\mu - \int_{D\backslash D_r(a)} |k_a|^2\, d\mu \\ &\geqq C^{-1}\int_D |k_a|^2\, d\nu - C'\int_{D\backslash D_{r/2}(a)} |k_a|^2\, d\nu. \end{aligned}$$

By the definition of $\varepsilon_t(\nu)$, the above inequality implies that

$$\int_{D_r(a)} |k_a|^2 \, d\mu$$

$$\geq (C^{-1} - C' \varepsilon_{r/2}(\nu)) \int_D |k_a|^2 \, d\nu$$

for all a in D and $r \geq l$. Here let r be sufficiently large, then by the hypothesis on $\varepsilon_r(\nu)$, $C^{-1} - C' \varepsilon_{r/2}(\nu) > 0$, and by lemma 4.3.3 in [9;p60], we conclude that

$$\hat{\mu}_r(a) \geq [M^{-1}(C^{-1} - C' \varepsilon_{r/2}(\nu))L] \, \hat{\nu}_r(a)$$

for all a in D. ∎

§ 3. (A_2)-condition

For a complex measure μ on D, recall that a function $\tilde{\mu}$ on D is defined by

$$\tilde{\mu}(a) = \int_D |k_a|^2 \, d\mu .$$

Particularly, if there exists a complex valued L^1-function u such that $d\mu = u \, dm$, then we denote the function by $\tilde{u}$ instead of $\tilde{\mu}$, and say that $\tilde{u}$ is the Berezin transform of the function u.

Let v and u be non-negative functions in L^1, put $d\nu = v \, dm$ and $d\mu = u \, dm$. Suppose that there is a constant $\gamma > 0$ such that

$$\tilde{v}(a) \times (u^{-1})^{\sim}(a) \leq \gamma$$

for all a in D, then lemma 4.3.3 in [9;p60] implies that there exist $r > 0$ and $\gamma' > 0$ such that

$$\hat{v}_r(a) \times (u^{-1})^{\wedge}_r(a) \leq \gamma'$$

for all a in D, and hence by proposition 1, we obtain that the (ν, μ)-Carleson inequality is satisfied. In the above two inequalities, if we put $u = v$, then such a function u is interesting for us.

A non-negative function u in L^1 is said to satisfy a $(A_2)_\partial$-condition, if there exists a constant $A > 0$ such that

$$\tilde{u}(a) \times (u^{-1})^{\sim}(a) \leqq A$$

for all a in D. If there exist $r > 0$ and $A_r > 0$ such that

$$\hat{u}_r(a) \times (u^{-1})^{\wedge}_r(a) \leqq A_r$$

for all a in D, then we say that u satisfies a (A_2)-condition. In [6], the (A_2)-condition is called Condition C_2. It is known that u satisfies the (A_2)-condition for some $0 < r < \infty$ if and only if u satisfies the (A_2)-condition for all $0 < r < \infty$ [6]. Hence it shows that the definition of the (A_2)-condition is independent of r. In general, lemma 4.3.3 in [9;p60] and the familiar inequality between the harmonic and arithmetic means imply that for any $0 < r < \infty$ there exists a constant $M = M_r > 0$ such that $M^{-1}(u^{-1})^{\sim\,-1} \leqq (u^{-1})^{\wedge\,-1}_r \leqq \hat{u}_r \leqq M\tilde{u}$. Therefore, if u satisfies the $(A_2)_\partial$-condition, then $(u^{-1})^{\sim\,-1}$, $(u^{-1})^{\wedge\,-1}_r$, $\hat{u}_r$, and $\tilde{u}$ are equivalent. Similarly, if u satisfies the (A_2)-condition, then $(u^{-1})^{\wedge\,-1}_r$, and $\hat{u}_r$ are equivalent. When u is in $L^1(\partial D)$ (L^1 is a usual Lebesgue space on the unit circle and $k_a(z)$ is a normalized reproducing kernel of a Hardy space), the $(A_2)_\partial$-condition has been studied in [3; (c) of Theorem2].

The following theorem 3 gives a necessary and sufficient condition in order to satisfy the (ν, μ)-Carleson inequality when $d\mu = u\,dm$ and u satisfies the $(A_2)_\partial$-condition.

Theorem 3. Suppose that u satisfies the $(A_2)_\partial$-condition, then the following are equivalent.

(1) There is a constant $C > 0$ such that

$$\int_D |f|^2\, d\nu \leqq C \int_D |f|^2\, u\, dm$$

for all f in P.

(2) There exist $r > 0$ and $\gamma > 0$ such that

$$\hat{\nu}_r(a) \leqq \gamma\, \hat{u}_r(a)$$

for all a in D.

(3) For any $r > 0$, there exists $\gamma = \gamma_r > 0$ such that

$$\hat{\nu}_r(a) \leqq \gamma\, \hat{u}_r(a)$$

for all a in D.

Proof. Suppose that (1) holds. Since u satisfies the $(A_2)_\partial$-condition, by (1) of proposition 8, u satisfies a relation in (3) of lemma 1 for all $r>0$. Therefore, (3) follows from (1) of proposition 2. The implication (3)⇒(2) is obvious. We will show that (2)⇒(1). Since u satisfies the $(A_2)_\partial$-condition, u^{-1} is integrable, hence $u>0$ a.e. on D. Moreover, by (5) of proposition 4, u satisfies the (A_2)-condition for all $r>0$ and therefore (2) implies that

$$\hat{\nu}_r(a)\times(u^{-1})\hat{}_r(a) \leqq A_r\gamma$$

for all a in D. In the statement of proposition 1, put $E=D$, then the above fact shows that the inequality in (1) is satisfied. This completes the proof. ▮

For any u in L^2, a in D, we put

$$MO(u)(a)=\{\,|u|^2\,\tilde{}\,(a)-|\tilde{u}(a)|^2\,\}^{1/2},$$

and let BMO_∂ be the space of functions u such that $MO(u)(a)$ is bounded on D (cf.[9;p127]). We give several simple sufficient conditions.

Proposition 4. Let u be a non-negative function in L^1, then the following are true.

(1) If both $\tilde{u}$ and $(u^{-1})\tilde{}$ are in L^∞, then u satisfies the $(A_2)_\partial$-condition.

(2) If both u and u^{-1} are in BMO_∂, then u satisfies the $(A_2)_\partial$-condition.

(3) Let $1<p,\ q<\infty$ and $1/p+1/q=1$. If u_1^p and u_2^q satisfy the $(A_2)_\partial$-condition, then $u=u_1u_2$ satisfies the $(A_2)_\partial$-condition.

(4) Suppose that f is a complex valued function in L^1 such that $f\neq 0$ on D, f^{-1} is in L^1, $\tilde{f}\times(f^{-1})\tilde{}$ is in L^∞, and $|\arg f|\leqq\pi/2-\varepsilon$ for some $\varepsilon>0$. If $u=|f|$, then u satisfies the $(A_2)_\partial$-condition.

(5) If u satisfies the $(A_2)_\partial$-condition, then u satisfies the (A_2)-condition.

Proof. (1) is trivial. By proposition 6.1.7 in [9;p108], we have that

$$\tilde{u}(a)\times(u^{-1})\tilde{}(a) \leqq MO(u)(a)\times MO(u^{-1})(a)+1.$$

This implies that (2) is true. The Hölder's inequality implies that (3) is true. (5) follows from lemma 4.3.3 in [9;p60].

We show that (4) is true. Suppose that $u=|f|$ and there exists $\varepsilon>0$ such that $|\arg f|\leqq \pi/2-\varepsilon$ on D. Since $|\arg f|\leqq \pi/2-\varepsilon$ on D, there exists $\delta>0$ such that $\cos(\arg f)\geqq \delta$ on D. Therefore, we have that

$$\begin{aligned}\mathrm{Re} f &= |f|\times\cos(\arg f) \geqq |f|\cdot\delta \\ &= \delta u.\end{aligned}$$

For any a in D, it follows that

$$\delta\tilde{u}(a) \leqq \int \mathrm{Re} f\cdot|k_a|^2\, dm \leqq |\tilde{f}(a)|.$$

Similarly, we have that

$$\delta(u^{-1})^{\sim}(a) \leqq |(f^{-1})^{\sim}(a)|.$$

Thus,

$$\tilde{u}(a)\times(u^{-1})^{\sim}(a) \leqq \delta^{-2}\times|\tilde{f}(a)|\times|(f^{-1})^{\sim}(a)|$$

for all a in D, and hence (4) follows. ∎

We exhibit some concrete examples which satisfy the $(A_2)_{\bar\partial}$-condition.

Proposition 5. If u is a function that is given by (1), (2), or (3), then u satisfies the $(A_2)_{\bar\partial}$-condition.

(1) For any $-1<\alpha<1$, put $u(z)=(1-|z|^2)^\alpha$.

(2) Let $\{b_j\}$ be a finite sequence of complex numbers in $D\cup\partial D$ with $b_i\neq b_j (i\neq j)$, and let $0\leqq \alpha(j)<2$ for all j or $-2<\alpha(j)\leqq 0$ for all j. Put $u=\prod p_j{}^{\alpha(j)}$ where $p_j(z)=|z-b_j|$.

(3) Let $\{b_j\}$, $\{p_j\}$ as in (2) and $-1<\alpha(j)<1$ for all j. Put $u=\prod p_j{}^{\alpha(j)}$.

Proof. We suppose that u has the form of (1). For any a in D, making a change of variable, we have that

$$\tilde{u}(a)\times(u^{-1})^{\sim}(a)=\int (1-|a|^2)^\alpha(1-|z|^2)^\alpha|1-\bar{a}z|^{-2\alpha}\, dm(z)$$

$$\times \int (1-|a|^2)^{-\alpha}(1-|z|^2)^{-\alpha}|1-\bar{a}z|^{2\alpha}\, dm(z)$$

$$= \int (1-|z|^2)^{\alpha}|1-\bar{a}z|^{-2\alpha}\, dm(z)$$

$$\times \int (1-|z|^2)^{-\alpha}|1-\bar{a}z|^{2\alpha}\, dm(z).$$

Since $-1<\alpha<1$, Rudin's lemma(cf.[9;p53]) implies that both factors of the right hand side in the above equality are bounded. Hence u satisfies the $(A_2)_\partial$-condition.

We show that u satisfies the $(A_2)_\partial$-condition when u has the form of (2). Let α be a real number such that $0<\alpha<2$. For any fixed b in D, put $p(z)=|z-b|$. Firstly, we show that the Berezin transform of $p^{-\alpha}$ is bounded. In fact, making a change of variable, elementary calculations show that

$$(p^{-\alpha})^{\sim}(a) \leqq |1-\bar{a}b|^{-\alpha} \|1-\bar{a}z\|_\infty^{\alpha} \times \int |\phi_a(b)-z|^{-\alpha}\, dm(z).$$

Since $\phi_a(b)-z$ lies in $2D=\{2z\,;z\in D\}$ for any a, z in D and an area measure is translation invariant, we have that

$$(p^{-\alpha})^{\sim}(a) \leqq (1-|b|)^{-\alpha} \|1-\bar{a}z\|_\infty^{\alpha} \times \int_{2D} |w|^{-\alpha}\, dm(w)$$

for all a in D. Hence we obtain that the Berezin transform of $p^{-\alpha}$ is bounded. Next, let b be in ∂D and put $p(z)=|z-b|$. Then, as in the proof of the above case, we have that

$$(p^{\alpha})^{\sim}(a) \leqq |a-b|^{\alpha} \cdot \|\phi_a(b)-z\|_\infty^{\alpha} \times \int |1-\bar{a}z|^{-\alpha}\, dm(z),$$

and

$$(p^{-\alpha})^{\sim}(a) \leqq |a-b|^{-\alpha} \|1-\bar{a}z\|_\infty^{\alpha} \times \int_{2D} |w|^{-\alpha}\, dm(w).$$

Therefore, Rudin's lemma implies that p^{α} satisfies the $(A_2)_\partial$-condition. For any b_1 in D and b_2 in ∂D, put $p_1(z)=|z-b_1|$ and $p_2(z)=|z-b_2|$. Fix $0<\alpha(j)<2$ for $j=1,\ 2$ and $\varepsilon>0$. Because

$b_1 \neq b_2$, there exist measurable subsets B_j of D such that $B_1 \cap B_2 = \phi$ and $p_j \geqq \varepsilon$ on $B_j{}^c$ for $j = 1, 2$. Set $B_0 = D \backslash B_1 \cup B_2$, then

$$(p_1{}^{\alpha(1)} \cdot p_1{}^{\alpha(2)})^{\widetilde{}}(a) \times (p_1{}^{-\alpha(1)} \cdot p_2{}^{-\alpha(2)})^{\widetilde{}}(a)$$

$$\leqq (p_1{}^{\alpha(1)} \cdot p_2{}^{\alpha(2)})^{\widetilde{}}(a) \times \left(\varepsilon^{-\alpha(1)-\alpha(2)} \int_{B_0} |k_a|^2 \, dm \right.$$

$$\left. + \ \varepsilon^{-\alpha(2)} \int_{B_1} p_1{}^{-\alpha(1)} |k_a|^2 dm \ + \ \varepsilon^{-\alpha(1)} \int_{B_2} p_2{}^{-\alpha(2)} |k_a|^2 dm \right)$$

$$\leqq M_0 \times \varepsilon^{-\alpha(1)-\alpha(2)} + M_0 \times \varepsilon^{-\alpha(2)} \cdot (p_1{}^{-\alpha(1)})^{\widetilde{}}(a)$$

$$+ \ M_1 \times \varepsilon^{-\alpha(1)} \cdot (p_2{}^{\alpha(2)})^{\widetilde{}}(a) \cdot (p_2{}^{-\alpha(2)})^{\widetilde{}}(a),$$

where $M_0 = \| p_1{}^{\alpha(1)} \cdot p_2{}^{\alpha(2)} \|_\infty$ and $M_1 = \| p_1{}^{\alpha(1)} \|_\infty$. Hence we have that $p_1{}^{\alpha(1)} \cdot p_2{}^{\alpha(2)}$ satisfies the $(A_2)_\partial$ -condition. If u has the form of (2), then applying the same argument for finitely many factors of u and u^{-1}, we obtain that u satisfies $(A_2)_\partial$-condition.

Apparently, (3) follows from (2) of this proposition and (3) of proposition 4. In fact, we let $-1 < \alpha(j) < 1$ for all j, and set $j(+) = \{j ;\ \alpha(j) \geqq 0\}$, $j(-) = \{j ;\ \alpha(j) < 0\}$. Put $u_1 = \prod_{j(+)} p_j{}^{\alpha(j)}$ and $u_2 = \prod_{j(-)} p_j{}^{\alpha(j)}$, then $u_1{}^2$ and $u_2{}^2$ satisfy the $(A_2)_\partial$ -condition. Hence, (3) of proposition 4 implies that $u = u_1 \times u_2$ satisfies the $(A_2)_\partial$-condition. ∎

Corollary 1 is a partial result of [2], [7] and [8].

Corollary 1. (Oleinik-Pavlov-Hastings-Stegenga) Let ν be a finite positive measure on D. For any $-1 < \alpha < 1$, there is a constant $C > 0$ such that

$$\int_D |f|^2 \, d\nu \ \leqq \ C \int_D |f|^2 \, (1 - |z|^2)^\alpha \, dm$$

for all f in P if and only if there exist $r > 0$ and $\gamma > 0$ such that

$$\hat{\nu}_r(a) \ \leqq \ \gamma (1 - |a|^2)^\alpha$$

for all a in D.

Proof. Since $[(1-|z|^2)^\alpha]^\frown(a)$ is comparable to $(1-|a|^2)^\alpha$, by theorem 3 and (1) of proposition 5 the corollary follows. ∎

Lemma 2. Let $\{b_j\}$ be a finite sequence of complex numbers in $D\cup\partial D$ with $b_i \neq b_j (i\neq j)$, and let $\{\alpha(j)\}$ be a finite sequence of real numbers such that $-2<\alpha(j)$ when j is in Λ^c(the definition of Λ is below). Put $p_j(z)=|z-b_j|$ and $u=\prod p_j{}^{\alpha(j)}$, and let $0<r<\infty$, then there are constants $\gamma_1>0$ and $\gamma_2>0$ such that

$$\gamma_1 \hat{u}_r(a) \leqq \prod_{j\in\Lambda} |a-b_j|^{\alpha(j)} \leqq \gamma_2 \hat{u}_r(a)$$

for all a in D, here $\Lambda=\{j\,;b_j \text{ is in } \partial D\}$.

Proof. For any fixed $0<r<\infty$, in general, lemma 4.3.3 in [9;p60] implies that there are constants $L>0$ and $M>0$ such that

$$L\hat{u}_r(a) \leqq \int_{D_r(0)} u\circ\phi_a \, dm \leqq M\hat{u}_r(a)$$

for all a in D, where u is a non-negative integrable function on D. Let $u=\prod|z-b_j|^{\alpha(j)}$, $\{b_j\}\subset D\cup\partial D$, $b_i\neq b_j(i\neq j)$, and $\alpha(j)$ be real numbers. Then, by the same calculations in the proof of (2) of proposition 5, we have that

$$\int_{D_r(0)} u\circ\phi_a \, dm$$

$$= \prod|1-\bar{a}b_j|^{\alpha(j)} \int_{D_r(0)} \prod|\phi_a(b_j)-z|^{\alpha(j)}\cdot|1-\bar{a}z|^{-\Sigma\alpha(j)} \, dm(z).$$

Put

$$I(a) = \int_{D_r(0)} \prod|\phi_a(b_j)-z|^{\alpha(j)} \, dm(z),$$

then it is easy to see that $\int_{D_r(0)} u\circ\phi_a \, dm$ is equivalent to $I(a)\times \prod_{j\in\Lambda}|a-b_j|^{\alpha(j)}$.

Firstly, we show that the lemma is true when $0\leqq\alpha(j)$ for all j. By the above facts, it is enough to prove that the integration

$$I(a) = \int_{D_r(0)} \prod|\phi_a(b_j)-z|^{\alpha(j)} \, dm(z)$$

is bounded below for all a in D, because $0 \leqq \alpha(j)$. Conversely, suppose that there exists $\{a_n\} \subset D$ such that $I(a_n) < 1/n$. Here we can choose a subsequence $\{a_k\} \subset \{a_n\}$ such that $a_k \to a'(k \to \infty)$, where a' may be in $D \cup \partial D$. Therefore, Fatou's lemma implies that $I(a') = 0$, thus it follows that $\prod |\phi_{a'}(b_j) - z|^{\alpha(j)} = 0$ on $D_r(0)$. This contradiction implies that the assertion is true when $0 \leqq \alpha(j)$ for all j.

Next, we prove that the lemma is true when $-2 < \alpha(j) < 0$ for all j in Λ^c and $-\infty < \alpha(j) < 0$ for all j in Λ. In fact, we claim that $I(a)$ is bounded for all a in D. If j is in Λ, then $|\phi_a(b_j)| = 1$ for all a in D, therefore $|\phi_a(b_j) - z|^{-1}$ is bounded, because z belongs to $D_r(0)$. Analogously, if j is in Λ^c, then $|\phi_a(b_j)| \to 1$ $(|a| \to 1)$, therefore $|\phi_a(b_j) - z|^{-1}$ is bounded when a is nearby ∂D, because z belongs to $D_r(0)$. Thus, it is sufficient to prove that

$$J(a) = \int_{D_r(0)} \prod_{j \in \Lambda^c} |\phi_a(b_j) - z|^{\alpha(j)} \, dm(z)$$

is bounded for all a in $U_\eta(0) = \{a \in D; |a| \leqq \eta\}$, where $0 < \eta < 1$ is a constant which is close to 1. Put

$$\Phi_{i,j}(a) = |\phi_a(b_i) - \phi_a(b_j)| \quad (i,\ j \in \Lambda^c,\ a \in U_\eta(0)).$$

For any fixed i, $j \in \Lambda^c$, since $\Phi_{i,j}$ is a continuous function on $U_\eta(0)$ and Möbius functions are one-to-one correspondence on D, there exists $\varepsilon(i,j) > 0$ such that $\Phi_{i,j}(a) \geqq \varepsilon(i,j)$ for all a in $U_\eta(0)$ when $i \neq j$. Put $\varepsilon = \min\{\varepsilon(i,j)/2\ ; i,\ j \in \Lambda^c$ such that $i \neq j\}$, $B_j(a) = \{z \in D_r(0); |\phi_a(b_j) - z| < \varepsilon\}$ and $B_0(a) = D_r(0) \backslash \cup B_j(a)$. For any j in $\Lambda^c \cup \{0\}$, since $|\phi_a(b_i) - z| \geqq \varepsilon$ when z belongs to $B_j(a)$ and i belongs to Λ^c such that $i \neq j$, therefore we have that

$$J(a) \leqq \sum_{j \in \Lambda^c} \varepsilon^{\alpha - \alpha(j)} \int_{B_j(a)} |\phi_a(b_j) - z|^{\alpha(j)} \quad dm(z)$$

$$+ \ \varepsilon^{\alpha} \int_{B_0(a)} dm(z)$$

$$\leqq \sum_{j \in \Lambda^c} \varepsilon^{a-\alpha(j)} \int_{2D} |w|^{\alpha(j)}\, dm(w) + \varepsilon^{a},$$

where

$$a = \sum_{j \in \Lambda^c} \alpha(j).$$

Therefore, J is bounded on $D_\eta(0)$, and hence we obtain that I is bounded on D.

Using the above facts, we can show that the assertion is true when u has the general form of the statement of this lemma. Let $\{\alpha(j)\}$ be a finite sequence of real numbers such that $-2<\alpha(j)<\infty$ when j is in Λ^c and $-\infty<\alpha(j)<\infty$ when j is in Λ. As in the proof of proposition 5, set $j(+)=\{j\,;\alpha(j)\geqq 0\}$ and $j(-)=\{j\,;\alpha(j)<0\}$, then we have that

$$I(a) \leqq 2^{\Sigma_{j(+)}\alpha(j)} \int_{D_r(0)} \Pi_{j(-)} |\phi_a(b_j)-z|^{\alpha(j)}\, dm(z)$$

and

$$I(a) \geqq 2^{\Sigma_{j(-)}\alpha(j)} \int_{D_r(0)} \Pi_{j(+)} |\phi_a(b_j)-z|^{\alpha(j)}\, dm(z).$$

Therefore, we obtain that I is bounded and bounded below on D. Hence, this completes the proof. ∎

Corollary 2. Let u be a non-negative function in L^1 that is given by (2), or (3) of proposition 5 and ν be a finite positive measure on D, then there is a constant $C>0$ such that

$$\int_D |f|^2\, d\nu \leqq C \int_D |f|^2\, u\, dm$$

for all f in P if and only if there exist $r>0$ and $\gamma=\gamma_r>0$ such that

$$\hat{\nu}_r(a) \leqq \gamma \prod_{j \in \Lambda} |a-b_j|^{\alpha(j)}$$

for all a in D, here $\Lambda=\{j\,;b_j$ is in $\partial D\}$.

Proof. The corollary follows from theorem 3, proposition 5 and lemma 2. ∎

We give a characterization of u which satisfies the (A_2)-condition or the $(A_2)_\partial$-condition when u is a modulus of a rational function or a modulus of a polynomial, respectively. Let u be a non-negative integrable function on D, then it is easy to see that if u satisfies the $(A_2)_\partial$-condition then u^{-1} is integrable on D. But, we claim that the converse is true, when u is a modulus of a polynomial. As the result, we show that the $(A_2)_\partial$-condition is properly contained in the (A_2)-condition. The essential part of the following theorem is proved in proposition 5 and lemma 2.

Theorem 6. Let $\{b_j\}$ be a finite sequence of complex numbers such that $b_i \neq b_j (i \neq j)$ and $\{a(j)\}$ be a finite sequence of real numbers. Put $p_j(z) = |z - b_j|$ and $u = \prod p_j^{a(j)}$, then the following are true.

(1) If $a(j) \geqq 0$ for all j or $a(j) \leqq 0$ for all j, then u satisfies the $(A_2)_\partial$-condition if and only if $a(j) < 2$ or $a(j) > -2$ when b_j is in $D \cup \partial D$, respectively.

(2) u satisfies the (A_2)-condition if and only if $-2 < a(j) < 2$ when b_j is in D.

Proof. (1) By (2) of proposition 5 and the remark above this theorem, it is enough to prove that u^{-1} is not integrable on D when $a(j) \geqq 2$ for some b_j in $D \cup \partial D$. Suppose that there is a j such that b_j in $D \cup \partial D$ and $a(j) \geqq 2$, then there exists a L^∞-function h such that $u(z) = |z - b_j|^2 \cdot h(z)$. It is easy to see that u^{-1} is not integrable on $U = \{z \in D; |z - b_j| < \mathrm{dist}(b_j, \partial D)\}$ when b_j is in D, therefore we consider the case when $b_j = 1$. Put $M_2 = \| h \|_\infty$, then

$$\int u^{-1} dm \geq M_2^{-1} \int_0^1 2r \int_0^{2\pi} |1 - r e^{i\theta}|^{-2} d\theta / 2\pi \ dr$$

$$= M_2^{-1} \int_0^1 2r(1-r^2)^{-1}\, dr$$

$$= M_2^{-1} \int_0^1 t^{-1} dt.$$

Hence we obtain that u^{-1} is not integrable.

(2) Suppose that $-2< a(j)<2$ when b_j is in D, then apparently lemma 2 implies that u satisfies the (A_2)-condition. Conversely, suppose that there exist $r>0$ and $A_r>0$ such that

$$\hat{u}_r(a)\times(u^{-1})\hat{}_r(a) \leqq A_r$$

for all a in D. Since $\hat{u}_r$ is non-zero on D, therefore $(u^{-1})\hat{}_r(a)<\infty$ for all a in D. By the same argument in (1), we have that $a(j)$ must be less than 2 when b_j is in D. In fact, if $a(j)\geqq 2$ for some b_j in D, then there exists a function h such that $u(z)=|z-b_j|^2\cdot h(z)$. Put

$$\varepsilon = \min\{ \mathrm{dist}(b_i, b_j)/2\ ; i \neq j\}$$

and

$$U(j)=\{z\in D;\ |z-b_j|<\varepsilon\},$$

then obviously h is bounded on $U(j)$. Since there exists a_j such that a center of the Bergman disk $D_r(a_j)$ is just equal to b_j, therefore we have that u^{-1} is not integrable on $D_r(a_j)\cap U(j)$, and thus, it follows that the average of u^{-1} on $D_r(a_j)$ is infinite. This contradicts the above fact. Consequently, we obtain that $a(j)$ must lie in $(-\infty,2)$ when b_j is in D. Applying the same argument to u^{-1}, we have that $a(j)$ must lie in $(-2,\infty)$ when b_j is in D. Therefore, we conclude that $-2< a(j)<2$ when b_j is in D. ∎

§ 4. Uniformly absolutely continuous

Recall that

$$\varepsilon_r(\mu) = \sup_{a\in D} \left(\int_{D\setminus D_r(a)} |k_a|^2\, d\mu\right)\times\left(\int_D |k_a|^2\, d\mu\right)^{-1},$$

where μ is a finite positive measure on D (see lemma 1 and proposition 2). Using the quantity ε_r, we give a necessary condition on ν and μ which satisfy the (ν, μ)-Carleson inequality.

Theorem 7. Suppose that $d\nu = v\, dm$, $\varepsilon_t(\nu) \to 0 (t \to \infty)$, and that v satisfies the (A_2)-condition, furthermore μ and ν satisfy the (μ, ν)-Carleson inequality. If there is a constant $C > 0$ such that

$$\int_D |f|^2 \, d\nu \leqq C \int_D |f|^2 \, d\mu$$

for all f in P, then there exist $r > 0$ and $\gamma > 0$ such that

$$\hat{\nu}_r(a) \leqq \gamma \ \hat{\mu}_r(a)$$

for all a in D.

Proof. By hypotheses on ν and lemma 1, there exist $t > 0$, $\rho > 0$ and $A > 0$ such that

$$\tilde{\nu} \leqq \rho \cdot \hat{\nu}_t \leqq A \rho \cdot (\widehat{v^{-1}})_t^{-1}.$$

Moreover, lemma 4.3.3 in [9;p60] and the (μ, ν)-Carleson inequality imply that there exist $L > 0$ and $C' > 0$ such that

$$L \cdot \hat{\mu}_t \leqq \tilde{\mu} \leqq C' \cdot \tilde{\nu}.$$

Thus, a desired result follows from (2) of proposition 2. ▮

Luecking [5] shows the above theorem when ν is the Lebesgue area measure m. It is clear that $\varepsilon_r(m) \to 0 \ (r \to \infty)$ and m satisfies the (A_2)-condition. Now, we are interested in measures μ such that $\varepsilon_r(\mu) < 1$ or $\varepsilon_r(\mu) \to 0 (r \to \infty)$.

Proposition 8. Suppose that $d\mu = u\, dm$, and u is a non-negative function in L^1. If u is the function such that (1) or (2), then there exists $0 < r < \infty$ such that $\varepsilon_r(\mu) < 1$.

(1) u satisfies the $(A_2)_\partial$-condition.

(2) $u(z)=(1-|z|^2)^\alpha$ for some $1\leqq \alpha<2$.

Proof. If u has the property in (1), then by the remark above theorem 3, for any $r>0$ there is a positive constant $\rho=\rho_r$ such that $\tilde{\mu}(a)\leqq \rho\,\hat{\mu}_r(a)$ for all a in D and hence $\varepsilon_r(\mu)<1$ by lemma 1. Suppose that u has the form of (2). For any fixed $1\leqq \alpha<2$, put $u(z)=(1-|z|^2)^\alpha$. Then, Rudin's lemma (cf.[9;p53]) shows that

$$\tilde{u}(a) = (1-|a|^2)^\alpha \int_D (1-|z|^2)^\alpha |1-\bar{a}z|^{-2\alpha}\, dm(a)$$

$$\leqq \gamma(1-|a|^2)^\alpha,$$

where $\gamma>0$ is finite. On the other hand, lemma 4.3.3 in [9;p60] implies that

$$\hat{u}_r(a) \geqq M^{-1}\times(1-|a|^2)^\alpha \int_{D_r(0)} (1-|z|^2)^\alpha|1-\bar{a}z|^{-2\alpha}\, dm(z)$$

$$\geqq M^{-1}\times(1-|a|^2)^\alpha(1-\tanh^2 r)^\alpha\times 2^{-2\alpha},$$

therefore, by (3) of lemma 1, we obtain that $\varepsilon_r(\mu)<1$. ∎

Proposition 9. Suppose that $d\mu=u\,dm$, and u is a non-negative function in L^1. If u is one of the following functions (1)~(7), then $\varepsilon_r(\mu)\to 0(r\to\infty)$.

(1) There exists $\varepsilon_0>0$ such that $\tilde{u}\geqq\varepsilon_0$ on D,and $\{u\circ\phi_a\,dm; a\in D\}$ is uniformly absolutely continuous with respect to the Lebesgue area measure m.

(2) There exists $\varepsilon_0>0$ such that $\tilde{u}\geqq\varepsilon_0$ on D, and there is a constant $C>0$ such that $(u^{1+\beta})^{\sim}\leqq C$ on D for some $\beta>0$.

(3) u is in L^∞, and there exist $r>0$ and $\delta>0$ such that $u\geqq\delta$ on $D\setminus D_r(0)$.

(4) $u=|p|$, where p is an analytic polynomial which has no zeros on ∂D.

(5) $u(z)=(1-|z|^2)^\alpha$ for some $-1<\alpha\leqq 1$.

(6) $u = \prod p_j^{\alpha(j)}$, where $p_j(z) = |z - b_j|$, $b_i \neq b_j (i \neq j)$, and $0 < \alpha(j) < 2$ for b_j in $D \cup \partial D$, or $-2 < \alpha(j) < 0$ for b_j in $D \cup \partial D$.

(7) $u = \prod p_j^{\alpha(j)}$, where $p_j(z) = |z - b_j|$, $b_i \neq b_j (i \neq j)$, and $-1 < \alpha(j) < 1$ for b_j in $D \cup \partial D$.

Proof. Firstly, we show that the assertion is true when u has the property of (1). Since $\{u \circ \phi_a dm; a \in D\}$ is uniformly absolutely continuous, for any $\varepsilon > 0$ there exists $r > 0$ such that $\int_{D_r(0)^c} u \circ \phi_a dm < \varepsilon_0 \cdot \varepsilon$ for all a in D. Therefore, making a change of variable, let r be sufficiently large,then $\varepsilon_r(\mu) < \varepsilon_0^{-1} \cdot \varepsilon_0 \cdot \varepsilon = \varepsilon$. Hence, we obtain that $\varepsilon_r(\mu) \to 0 (r \to 0)$.

Next, we prove the implications (2)⇨(1), (3)⇨(2), and (4)⇨(3). Then $\varepsilon_r(\mu) \to 0$ when u is a function such that (2), (3) or (4). In fact, suppose that there exists $\beta > 0$ such that the Berezin transform of the function $u^{1+\beta}$ is bounded, then a set of functions $\{u \circ \phi_a; a \in D\}$ is uniformly integrable (cf.[1;p120]), therefore it follows that $\{u \circ \phi_a dm; a \in D\}$ is uniformly absolutely continuous with respect to m. Hence, (2) implies (1). If there exist $r > 0$ and $\delta > 0$ such that $u \geqq \delta$ on $D \backslash D_r(0)$, then

$$\begin{aligned} \tilde{u}(a) &\geqq \delta - \delta \int_{D_r(0)} |k_a|^2 \, dm \\ &= \delta[1 - m(D_r(a))] \\ &\geqq \delta(1 - \tanh^2 r) \\ &> 0. \end{aligned}$$

Hence (3) implies (2) because $(u^{1+\beta})^{\sim}(a) \leqq \|u\|_\infty^{1+\beta}$ for all a in D and any $\beta > 0$. Next, let p be an analytic polynomial which has no zeros on ∂D, then there are $r > 0$ and $\delta > 0$ such that $u = |p| \geqq \delta$ on $D \backslash D_r(0)$, therefore (4)⇨(3).

We prove that the assertion is true when u has the form of (5). For any fixed $-1 < \alpha \leqq 1$, put $u(z) = (1 - |z|^2)^\alpha$ and making a change of variable, then

$$\varepsilon_r(\mu) = \sup \left(\int_D (1 - |z|^2)^\alpha |1 - \bar{a} z|^{-2\alpha} \, dm(z) \right)^{-1}$$

$$\times\left(\int_{D\setminus D_r(0)}(1-|z|^2)^{\alpha}|1-\bar{a}z|^{-2\alpha}\,dm(z)\right).$$

When $0\leqq\alpha\leqq1$, since $0<1-|z|^2\leqq1$, we have that

$$\int_D(1-|z|^2)^{\alpha}|1-\bar{a}z|^{-2\alpha}\,dm$$

$$\geqq 2^{-2\alpha}\int_D(1-|z|^2)\,dm$$

$$=\text{constant}.$$

If $-1<\alpha<0$, then the familiar inequality between the harmonic and arithmetic means shows that

$$\int_D(1-|z|^2)^{\alpha}|1-\bar{a}z|^{-2\alpha}\,dm$$

$$\geqq\left(\int_D(1-|z|^2)^{-\alpha}|1-\bar{a}z|^{2\alpha}\,dm\right)^{-1}$$

$$\geqq\text{constant}.$$

Here, the last inequality follows from Rudin's lemma (cf.[9;p53]). Again using Rudin's lemma, since $-1<\alpha\leqq1$, there exists $\beta>0$ such that a set of functions $\{\ [(1-|z|^2)^{\alpha}|1-az|^{-2\alpha}]^{1+\beta};a\in D\ \}$ is bounded in L^1. This implies that the set of these functions are uniformly integrable (cf.[1;p120]), therefore it follows that $\varepsilon_r(\mu)\to0(r\to\infty)$.

We show that $\varepsilon_r(\mu)\to0$ when u has the form of (6). As in the proof of (2) of proposition 5, we only prove that $\varepsilon_r(\mu)\to0(r\to\infty)$ when $u=p_1^{\alpha(1)}\cdot p_2^{\alpha(2)}$, where $p_1(z)=|z-b_1|$, $p_2(z)=|z-b_2|$, $0<\alpha(1)$, $\alpha(2)<2$, and b_1 is in D, b_2 is in ∂D. We suppose that B_j, M_1, and ε are as in the proof of (2) of proposition 5. By the definition of $\varepsilon_r(\mu)$, we have that

$$\varepsilon_r(\mu)=\sup\,(u\chi_{D_r(a)^c})^{\sim}(a)\times\tilde{u}(a)^{-1}.$$

Moreover,

$$(u\chi_{D_r(a)^c})^{\sim}(a)\times\tilde{u}(a)^{-1}$$

$$\leqq (u\chi_{D_r(a)^c})^{\sim}(a)\times(u^{-1})^{\sim}(a)$$

$$\leqq (u\chi_{D_r(a)^c})^{\sim}(a)\ \times\ \varepsilon^{-a(1)-a(2)}\int_{B_0}|k_a|^2\ dm$$

$$+(u\chi_{D_r(a)^c})^{\sim}(a)\ \times\ \varepsilon^{-a(2)}\cdot(p_1^{-a(1)})^{\sim}(a)$$

$$+M_1\times\varepsilon^{-a(1)}\ \times C\int_{D\setminus D_r(0)}|1-\bar{a}z|^{-a(2)}\ dm,$$

where

$$C\ =\ \|\phi_a(b_2)-z\|_\infty^{a(2)}\times\ \|1-\bar{a}z\|_\infty^{a(2)}\times\int_{2D}|w|^{-a(2)}\ dm.$$

Since u is bounded, therefore $\{u\circ\phi_a; a\in D\}$ is uniformly integrable (cf.[1;p120]), moreover applying the same argument in the proof of this proposition when u has the form of (5), Rudin's lemma implies that a set of functions $\{|1-\bar{a}z|^{-a(2)}; a\in D\}$ is also uniformly integrable, hence we conclude that $\varepsilon_r(\mu)\to 0(r\to\infty)$. The proof of the latter half of (6) of this proposition is similar that in the above.

If u has the form of (7), then by the similar arguments in the proof of (3) of proposition 5, set $j(+)=\{j; a(j)\geqq 0\}$, $j(-)=\{j; a(j)<0\}$. And put $u_1=\prod_{j(+)}p_j^{a(j)}$, $u_2=\prod_{j(-)}p_j^{a(j)}$, then

$$(u\chi_{D_r(a)^c})^{\sim}(a)\times\tilde{u}(a)^{-1}$$

$$\leqq (u\chi_{D_r(a)^c})^{\sim}(a)\times(u^{-1})^{\sim}(a)$$

$$=\ (u_1u_2\chi_{D_r(a)^c})^{\sim}(a)\times(u_1^{-1}u_2^{-1})^{\sim}(a).$$

Therefore, the desired result follows from the Cauchy-Schwarz's inequality and (6) of this proposition. ∎

Corollary 3. Suppose that $d\nu = v\ dm$ and there is a constant $C>0$ such that

$$\int_D |f|^2\, d\nu \leqq C \int_D |f|^2\, d\mu$$

for all a in D, then the following are true.

(1) If $v(z)=(1-|z|^2)^\alpha$ for some $-1<\alpha\leqq 1$, and there exist $l>0$ and $\gamma'=\gamma_l'>0$ such that

$$\hat{\mu}_l(a) \leqq \gamma'(1-|a|^2)^\alpha$$

for all a in D, then there exist $r>0$ and $\gamma=\gamma_r>0$ such that

$$(1-|a|^2)^\alpha \leqq \gamma\, \hat{\mu}_r(a)$$

for all a in D.

(2) If $v=\prod p_j{}^{\alpha(j)}$, where $p_j(z)=|z-b_j|$, $b_i\neq b_j (i\neq j)$, and $0<\alpha(j)<2$ for b_j in $D\cup\partial D$ or $-2<\alpha(j)<0$ for b_j in $D\cup\partial D$, and if there exist $l>0$ and $\gamma'=\gamma_l'>0$ such that

$$\hat{\mu}_l(a) \leqq \gamma' \prod_{j\in\Lambda} |a-b_j|^{\alpha(j)}$$

for all a in D, then there exist $r>0$ and $\gamma=\gamma_r>0$ such that

$$\prod_{j\in\Lambda} |a-b_j|^{\alpha(j)} \leqq \gamma\, \hat{\mu}_r(a)$$

for all a in D, where $\Lambda=\{j\,;b_j \text{ is in } \partial D\}$.

(3) If $v=\prod p_j{}^{\alpha(j)}$, where $p_j(z)=|z-b_j|$, $b_i\neq b_j (i\neq j)$, and $-1<\alpha(j)<1$ for b_j in $D\cup\partial D$, and if there exist $l>0$ and $\gamma'=\gamma_l'>0$ such that

$$\hat{\mu}_l(a) \leqq \gamma' \prod_{j\in\Lambda} |a-b_j|^{\alpha(j)}$$

for all a in D, then there exist $r>0$ and $\gamma=\gamma_r>0$ such that

$$\prod_{j\in\Lambda} |a-b_j|^{\alpha(j)} \leqq \gamma\, \hat{\mu}_r(a)$$

for all a in D, where $\Lambda=\{j\,;b_j \text{ is in } \partial D\}$.

Proof. We show that (1) is true. By the fact in the proof of corollary 1, and the fact that $u(a)=(1-|z|^2)^\alpha$ satisfies the (A_2)-condition for all $\alpha>-1$ (see[6]), the hypothesis in (1) of the corollary and proposition 1 imply the (μ, ν)-Carleson inequality. Hence, theorem 7 and proposition 9 show that the assertion is true.

Similarly, (2) and (3) follow from proposition 1, lemma 2, (5) of proposition 4, theorem 6, theorem 7, and proposition 9. ∎

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