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and Uniform Algebras**

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Invertible Toeplitz Operators

and

Uniform Algebras

by

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Abstract. Toeplitz operators T_ϕ^M are defined on invariant subspaces M of an arbitrary uniform algebra A . We give a necessary and sufficient condition of uniformly invertible T_ϕ^M with respect to some family \mathcal{F} of invariant subspaces M . This condition is the same to a classical one in case A is the disc algebra. In special uniform algebras, we can choose a small family, in fact, if A is a disc algebra then \mathcal{F} can be a single set. Then this generalizes the Widom-Devinatz theorem. As an application, we study a \mathcal{F} -union of spectrums of $\{T_\phi^M ; M \in \mathcal{F}\}$.

§1. Introduction

Let X be a compact Hausdorff space, let $C(X)$ be the algebra of complex-valued continuous functions on X , and let A be a uniform algebra on X . Let τ be a nonzero complex homomorphism of A and let N_τ be the set of representing measures for τ whose support is contained in X and $m \in N_\tau$. The abstract Hardy space $H^p = H^p(m)$, $1 \leq p \leq \infty$, determined by A is defined to be the closure of A in $L^p = L^p(m)$ when p is finite and to be the weak* - closure of A in $L^\infty = L^\infty(m)$ when p is infinite. Put $H_0^p = \{f \in H^p ; \int f dm = 0\}$, $K_0^p = \{f \in L^p ; \int f g dm = 0 \text{ for all } g \in A\}$ and $K^p = K_0^p + C$. Then $H_0^p \subset K_0^p$ and $H_0^p \subset K^p$. The abstract Hardy spaces sometimes coincide with the concrete Hardy spaces or the concrete Bergman spaces.

A closed subspace M for A of $L^2 = L^2(m)$ is said to be invariant if fM is contained in M for all f in A . $\text{lat } A$ denotes the set of all invariant subspaces of A in L^2 . For ϕ in L^∞ , the Toeplitz operator T_ϕ^M is the operator on M defined by

$$T_\phi^M(f) = P_M(\phi f)$$

where $M \in \text{lat } A$ and P_M is the orthogonal projection onto M . If $M = H^2$ then we will write $T_\phi = T_\phi^M$. In this paper we are interested in the equivalence of the following four statements.

- (1) T_ϕ is invertible.
- (2) For each $M \in \text{lat } A$, T_ϕ^M is invertible.
- (3) For each $M \in \text{lat } A$, T_ϕ^M is invertible and $\sup \{\|(T_\phi^M)^{-1}\| ; M \in \text{lat } A\}$ is finite.
- (4) $\text{ess. inf } |\phi| > 0$ and there exists a function g in $(H^\infty)^{-1}$ such that $\text{Re}(\phi g) \geq \delta$ a.e. for some constant $\delta > 0$.

In the four statements, (3) \Rightarrow (2) and (2) \Rightarrow (1) are clear. If (4) is valid, then there exist positive constants ε and ε_0 such that $\|\varepsilon \phi g - 1\|_\infty \leq 1 - \varepsilon_0$. Hence $T_{\varepsilon \phi g}^M$ is invertible for any $M \in \text{lat } A$ and

$$\|(T_{\varepsilon \phi g}^M)^{-1}\| \leq \frac{1}{1 - \|T_{\varepsilon \phi g}^M - 1\|} \leq \frac{1}{\varepsilon_0}.$$

Thus $\sup \{\|(T_\phi^M)^{-1}\| ; M \in \text{lat } A\} < \infty$. This implies (3). In this paper we will show (3) \Rightarrow (4) under a condition: $H^\infty = H^1 \cap L^\infty$. This condition is satisfied by five natural examples in §2.

When A is the disc algebra of Example 1 in §2, Devinatz and Widom (see [4]) showed (1) \Leftrightarrow (4). Hence the four statements (1) \sim (4) are equivalent. When A is the rational function algebra of Example 3 in §2, Abrahamse [1] showed (2) \Leftrightarrow (4) in case the symbols of Toeplitz operators are unimodular. In §4 of this paper, we will show (2) \Leftrightarrow (4) without the condition on the symbols. Hence our result solves the problem which was proposed by Abrahamse [1, p294]. When A is the subalgebra of the disc algebra of Example 4 in §2, Anderson and Rochberg [2] gave a generalization of the theorem of

Devinatz and Widom. Their definition of Toeplitz operators is different from ours but their results are closed to (2) \Leftrightarrow (4). In §4, we will study their result in a more general setting. By Corollary 1 in §3, (3) \Leftrightarrow (4) is still true for Toeplitz operators in the Bergman space (see Example 2 in §2) and in the Hardy space of the polydisc (see Example 5 in §2). In Examples 2, 3 and 4, it is known that (1) does not imply (4).

In §5, as an application of the equivalence of (2) \sim (4) we will study $\cup \{\sigma(T_\phi^M); M \in \text{lat } A\}$ where $\sigma(T_\phi^M)$ denotes the spectrum of T_ϕ^M . In this paper, we use the theory of abstract Hardy spaces (see [3] and [6]). It is powerful to study the problem above.

§2. Concrete examples

(1) Let Δ be the unit disc of \mathbb{C} and let A' be an algebra consisting of all functions with continuous extensions to the closure $\bar{\Delta}$ of Δ which are analytic in Δ . Put $A = A'|X$ and $X = \partial\Delta$. A is called the disc algebra which is a uniform algebra on X . Suppose $\tau(f) = \tilde{f}(0)$, where \tilde{f} denotes the holomorphic extension of f in A , then τ is a nonzero complex homomorphism. The normalized Lebesgue measure m on the unit circle is a representing measure for τ and $N_\tau = \{m\}$. H^2 is the classical Hardy space and $H_0^2 = K_0^2$, sometimes we will write $H^p = H^p(\Delta)$.

(2) In (1), put $A = A'$ and $X = \bar{\Delta}$. Suppose $\tau(f) = f(0)$ and m is the normalized area measure on Δ , then $m \in N_\tau$ and $\dim N_\tau = \infty$. H^2 is the Bergman space.

(3) Let D be a bounded connected open subset of \mathbb{C} whose boundary consists of $n + 1$ non-intersecting, analytic Jordan curves and let A' be an algebra consisting of functions with continuous extensions to the closure \bar{D} of D which are analytic in D . Put $A = A'|X$ and $X = \partial D$. A is uniform algebra on X and it is called an annulus algebra when D is an annulus. Suppose $\tau(f) = \tilde{f}(t)$, where \tilde{f} denotes the holomorphic extension of f in A and $t \in D$, then τ is a nonzero complex homomorphism of A . If m is a harmonic measure of t then m is the unique logmodular measure of N_τ and $\dim N_\tau = n < \infty$ [6, p116]. Sometimes we will write $H^p = H^p(D)$.

(4) Let \mathcal{A} be the disc algebra and A be a subalgebra of \mathcal{A} which contains the constants and which has finite codimension in \mathcal{A} . If $\tau(f) = \tilde{f}(0)$ for $f \in A$ and m is the normalized Lebesgue measure on the unit circle $\partial\Delta$, then it is easy to check that m is a core point of N_τ , $\dim N_\tau < \infty$ and N_τ has a lot of logmodular measures (see [7, p154]).

(5) The unit polydisc Δ^n and the torus $(\partial\Delta)^n$ are cartesian products of n copies of Δ and of $\partial\Delta$, respectively. A' denotes the class of all continuous functions on the closure $\bar{\Delta}^n$ of Δ^n with holomorphic restrictions to Δ^n . Let $A = A'|X$ and $X = (\partial\Delta)^n$. This is the so-called polydisc algebra. Let m be the normalized Lebesgue measure, then m is a representing measure for τ on X where $\tau(f) = \tilde{f}(0)$ and $0 \in \Delta^n$. Then $\dim N_\tau = \infty$.

§3. The Inversion Theorem

Put $\mathcal{L} = \{v \in L^\infty; v^{-1} \in L^\infty \text{ and } v \geq 0\}$ and $\mathcal{F} = \{vH^2; v \in \mathcal{L}\}$. Then \mathcal{F} is a subfamily of lat A . In [9, Proposition 7], the following theorem was proved when the symbols are unimodular. In this section, we show it for arbitrary symbols using a lifting theorem in a uniform algebra due to the first author and Yamamoto [11, Theorem 2']. In fact we use Theorem 2' in case $\mu = (\mu_{ij})$ is absolutely continuous with respect to m .

Theorem 1. Let ϕ be a nonzero function in L^∞ . For each M in \mathcal{F} there exists a nonzero positive constant $\varepsilon(M)$ such that

$$\|T_\phi^M f\|_2 \geq \varepsilon(M)\|f\|_2, f \in M$$

and $\inf \{\varepsilon(M); M \in \mathcal{F}\} = \varepsilon > 0$ if and only if there exists a function g in $H^1 \cap L^\infty$ such that

$$|\phi|^2 \geq \varepsilon^2 + |\phi + g|^2 \quad \text{a.e.}$$

Proof. If $\|T_\phi^M f\|_2 \geq \varepsilon\|f\|_2$ for all $M \in \mathcal{F}$ then it is easy to see that $|\phi| \geq \varepsilon$ a.e. and for all $M \in \mathcal{F}$

$$(T_\phi^M)^*(T_\phi^M) \geq \varepsilon^2.$$

Put $H_\phi^M f = (I - P_M)(\phi f)$ for $f \in M$. Since $(T_\phi^M)^* T_\phi^M + (H_\phi^M)^* H_\phi^M = T_{|\phi|^2}^M$

$$\|H_\phi^M f\|_2 \leq \|(|\phi|^2 - \varepsilon^2)^{1/2} f\|_2, f \in M$$

and hence

$$\sup \{|(H_\phi^M f, g)|^2; g \in M^\perp \text{ and } \int |g|^2 dm \leq 1\} \leq \int (|\phi|^2 - \varepsilon^2) |f|^2 dm$$

where (\cdot, \cdot) is an inner product with respect to m . Thus for $f \in M$ and $g \in M^\perp$

$$|\int \phi f \bar{g} dm|^2 \leq \int (|\phi|^2 - \varepsilon^2) |f|^2 dm \int |g|^2 dm. \quad (\text{a})$$

Since $M = vH^2$ and $M^\perp = v^{-1}\bar{K}_0^2$ for some $v \in \mathcal{L}$, for $F \in H^2$ and $G \in \bar{K}_0^2$

$$|\int \phi F \bar{G} dm|^2 \leq \int (|\phi|^2 - \varepsilon^2) |F|^2 v^2 dm \int |G|^2 v^{-2} dm.$$

Hence for $F \in H^\infty$ and $G \in \bar{K}_0^\infty$

$$\begin{aligned} & -2\text{Re} \int \phi F \bar{G} dm \leq 2|\int \phi F \bar{G} dm| \\ & \leq 2\{\int (|\phi|^2 - \varepsilon^2) |F|^2 v^2 dm\}^{1/2} \{\int |G|^2 v^{-2} dm\}^{1/2} \\ & \leq \int (|\phi|^2 - \varepsilon^2) |F|^2 v^2 dm + \int |G|^2 v^{-2} dm. \end{aligned} \quad (\text{b})$$

Let \tilde{X} be the maximal ideal space of L^∞ , then \tilde{X} is a compact Hausdorff space and L^∞ is isometrically isomorphic to $C(\tilde{X})$ by the Gelfand transform. Put B be the image of H^∞ by the transform and let \tilde{m} be the Radonization of m . Then the measure \tilde{m} on \tilde{X} is multiplicative on B and $H^p(\tilde{m})$ or $L^p(\tilde{m})$ is isometrically isomorphic to $H^p(m)$ or $L^p(m)$, respectively, where $H^p(\tilde{m})$ is the abstract Hardy space determined by B . The inequality (b) implies that the measure matrices for all $v \in \mathcal{L}$

$$\begin{pmatrix} v^2(|\phi|^2 - \varepsilon^2) & \phi \\ \bar{\phi} & v^{-2} \end{pmatrix}$$

are positive on $H^\infty \times \bar{K}_0^\infty$ as $A = H^\infty$ and $K_0 = K_0^\infty$ in [11, p93]. By the absolutely continuous case of the lifting theorem [11, Theorem 2'] and the isomorphism above, there exists a nonzero function g in $(K_0^\infty)^\perp \cap L^1 = H^1$ such that the measure matrix

$$\begin{pmatrix} |\phi|^2 - \varepsilon^2 & \phi + g \\ \bar{\phi} + \bar{g} & 1 \end{pmatrix}$$

is positive on $L^\infty \times L^\infty$. Therefore

$$|\phi + g|^2 + \varepsilon^2 \leq |\phi|^2 \quad \text{a.e..}$$

Conversely suppose that for some $\varepsilon > 0$ there exists a function g in $H^1 \cap L^\infty$ such that $|\phi|^2 \geq \varepsilon^2 + |\phi + g|^2$ a.e.. Then for arbitrary $v \in \mathcal{L}$

$$\int \phi F \bar{G} dm = \int (\phi + g) F v \cdot \bar{G} v^{-1} dm$$

where $F \in H^\infty$ and $G \in \bar{K}_0^\infty$. Hence from the Schwarz's inequality, (b) and hence (a) follow. This implies $(T_\phi^M)^*(T_\phi^M) \geq \varepsilon^2 > 0$ for all $M \in \mathcal{F}$.

Corollary 1. Suppose $H^\infty = H^1 \cap L^\infty$. Let ϕ be a nonzero function in L^∞ . T_ϕ^M is invertible for every M in \mathcal{F} and $\sup \{ \|(T_\phi^M)^{-1}\| ; M \in \mathcal{F} \} < \infty$ if and only if both

- (1) $\text{ess.inf } |\phi| > 0$ and
- (2) there exists a function g in $(H^\infty)^{-1}$ such that $\text{Re}(\phi g) \geq \delta$ a.e. for some constant $\delta > 0$.

Proof. We may assume $\|\phi\|_\infty = 1$. Suppose $\sup \{ \|(T_\phi^M)^{-1}\| ; M \in \mathcal{F} \} < \infty$, then

$$\|T_\phi^M(f)\|_2 \geq \varepsilon \|f\|_2, \quad f \in M$$

where $\varepsilon = \{ \sup \|(T_\phi^M)^{-1}\| \}^{-1}$. By Theorem 1, there exists a function g in $H^1 \cap L^\infty = H^\infty$ such that $|\bar{\phi}|^2 \geq \varepsilon^2 + |\bar{\phi} - g|^2$ a.e.. Hence $\text{ess.inf } |\phi| > 0$, and

$$\operatorname{Re}\phi g \geq \delta > 0 \text{ a.e. and } \delta = \varepsilon^2/2.$$

Therefore there exist positive constants α and ε_0 such that

$$\|a\phi g - 1\|_\infty \leq 1 - \varepsilon_0.$$

This implies $T_{\phi g}$ is invertible. By hypothesis on T_ϕ, T_g is invertible. Hence $gH^2 = H^2$ and $g^{-1} \in L^\infty$ because $\operatorname{Re}\phi g \geq \delta > 0$ a.e.. Thus $g^{-1} \in H^2 \cap L^\infty = H^\infty$.

In Corollary 1, if ϕ is unimodular then (1) and (2) hold if and only if to the distance from ϕ to the invertible elements in H^∞ is less than one. Results in this section apply to Toeplitz operators in Examples 1, 2 and 5.

§4. Some special cases

In this section, assuming that the set of representing measures for τ is finite dimensional, we give the inversion theorems. These appear to be more useful than Theorem 1 and Corollary 1. They apply to Toeplitz operators in Examples 3 and 4.

Suppose $n = \dim N_\tau < \infty$. Let m be a core point of N_τ and let N^∞ be the real annihilator of A in $L^\infty_{\mathbb{R}}$. Then $\dim N^\infty = n$ (cf. [6, p109]). Set $\mathcal{E} = \exp N^\infty$, then \mathcal{E} is a subgroup of \mathcal{L} . Put $\mathcal{F}_1 = \{vH^2 : v \in \mathcal{E}\}$ then \mathcal{F}_1 is a subfamily of \mathcal{F} . Then it can be shown that Theorem 1 and Corollary 1 are true for \mathcal{F}_1 instead of \mathcal{F} . These give inversion theorems in Example 4 which are related to [2, Theorems 2 and 3]. If $n = 0$ then $\mathcal{E} = \{1\}$ and hence $\mathcal{F}_1 = \{H^2\}$. Therefore if $n = 0$ then Corollary 1 shows the equivalence of (1) \sim (4) in Introduction and gives the theorem of Widom and Devinatz (see [4]).

For any v in \mathcal{L} . let P_v be the projection operator which takes L^2 onto H^2 and is self-adjoint as an operator on the weighted Lebesgue space $L^2(v^2 dm)$. Define the associated Toeplitz operator R_ϕ^v mapping H^2 to itself by $R_\phi^v(f) = P_v(\phi f)$. This definition is due to Anderson and Rochberg [2]. $(\cdot, \cdot)_v$ and (\cdot, \cdot) denote the inner products in $L^2(v^2 dm)$ and L^2 , respectively. Put $M = vH^2$. For any f and g in H^2 ,

$$(R_\phi^v(f), g)_v = (P^v(\phi f), g)_v = (\phi f, g)_v = (\phi v f, v g) = (P^M(\phi v f), v g) = (T_\phi^M(v f), v g)$$

It is clear that R_ϕ^v is (left) invertible if and only if T_ϕ^M is (left) invertible. Hence Theorem 2 and the remark above Theorem 3 in [2] show the equivalence of (2) \sim (4) in Introduction when A is a subalgebra of the disc algebra in Example 4.

If we assume that m is the unique logmodular measure for τ then the linear span of $N^\infty \cap \log|(H^\infty)^{-1}|$ is N^∞ (cf. [6, p114]). Choose $h_1, \dots, h_n \in (H^\infty)^{-1}$ so that $\{\log|h_j|\}_{j=1}^n$ is a basis in N^∞ . Put $u_j = \log|h_j|$ ($1 \leq j \leq n$) and $\mathcal{E}_0 = \{\exp(\sum_{j=1}^n s_j u_j) : 0 \leq s_j \leq 1\}$. Then $\mathcal{E}_0 \subset \mathcal{E}$. Put $\mathcal{F}_0 = \{vH^2 : v \in \mathcal{E}_0\}$ then $\mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \mathcal{F} \subseteq \operatorname{lat} A$. Note that the unique logmodular measure is a core point of N_τ when N_τ is finite dimensional.

Abrahamse [1, Part 4] studied T_ϕ^M for simply invariant subspaces M . \mathcal{F} is a proper subset of the set of simply invariant subspaces.

Theorem 2. Suppose N_τ is finite dimensional and m is a core point of N_τ . Let ϕ be a nonzero function in L^∞ .

(1) For each M in \mathcal{F}_1 there exists a nonzero positive constant $\varepsilon(M)$ such that

$$\|T_\phi^M f\|_2 \geq \varepsilon(M)\|f\|_2, \quad f \in M$$

and $\inf\{\varepsilon(M); M \in \mathcal{F}_1\} = \varepsilon > 0$ if and only if there exists a function g in H^∞ such that

$$|\phi|^2 \geq \varepsilon^2 + |\phi + g|^2 \quad \text{a.e.}$$

(2) When m is the unique logmodular measure for τ , T_ϕ^M is left invertible for any M in \mathcal{F}_0 if and only if there exists a positive constant ε and a function g in H^∞ such that

$$|\phi|^2 \geq \varepsilon^2 + |\phi + g|^2 \quad \text{a.e.}$$

Proof. It is known that $H^\infty = H^1 \cap L^\infty$ (cf. [6, p109]). (1) By Theorem 1 it is sufficient to show that if $\inf\{\varepsilon(M); M \in \mathcal{F}_1\} > 0$ then $\inf\{\varepsilon(M); M \in \mathcal{F}\} > 0$. In order to prove it we will show that

$$\inf\{\varepsilon(M); M \in \mathcal{F}\} = \inf\{\varepsilon(M); M \in \mathcal{F}_1\}. \quad (c)$$

If $v \in \mathcal{L}$ then $\log v = u + u_1$ where $u_1 \in N^\infty$ and u is in the weak*-closure of $\text{Re}A$ (cf. [6, p109]). Then $u = \log |h|$ for h in $(H^\infty)^{-1}$ and so $v = |h|v_1$ where $v_1 = e^{u_1}$ and $v_1 \in \mathcal{E}$. Hence $M = vH^2 = b(v_1H^2) = bN$, $M \in \mathcal{F}$ and $N \in \mathcal{F}_1$, where $b = |h|/h$. Then $T_\phi^M = M_b T_\phi^N M_b^*$ where M_b is a multiplication operator on L^2 . This implies (c) because M_b is unitary.

(2) By (1) it is sufficient to show that if T_ϕ^M is left invertible for every M in \mathcal{F}_0 , then for any M in \mathcal{F}_1

$$\|T_\phi^M f\|_2 \geq \varepsilon(M)\|f\|_2 \quad f \in M$$

and $\inf\{\varepsilon(M); M \in \mathcal{F}_1\} > 0$. We will show that

$$\inf\{\varepsilon(M); M \in \mathcal{F}_1\} = \inf\{\varepsilon(M); M \in \mathcal{F}_0\} \quad (d)$$

and

$$\inf\{\varepsilon(M); M \in \mathcal{F}_0\} > 0 \quad (e)$$

If $v \in \mathcal{E}$ then $v = |h|v_0$ for some h in $(H^\infty)^{-1}$ and some $v_0 \in \mathcal{E}_0$ by the remark above Theorem 2. By the same argument as in the proof of (1), we can show (d). Suppose $v_\ell \in \mathcal{E}_0$ and $M_\ell = v_\ell H^2 \in \mathcal{F}_0$ with $\varepsilon(M_\ell) \rightarrow 0$. Since $v_\ell \in \mathcal{E}_0$, $v_\ell = \exp(\sum_{j=1}^n s_{j\ell} u_j)$ and $0 \leq s_{j\ell} \leq 1$ ($1 \leq j \leq n$). By passing to a subsequence, if necessary, we can assume that $s_{j\ell}$ converges to a constant s_j for each j , and $|s_j| \leq 1$ ($1 \leq j \leq n$). Put $t = \exp(\sum_{j=1}^n s_j u_j)$ then $t \in \mathcal{E}_0$ and $\varepsilon(t) = 0$. This contradicts that $T_\phi^{M_t}$ is left invertible. Thus (e) follows.

Corollary 2. Suppose N_τ is finite dimensional and m is a core point of N_τ . Let ϕ be a nonzero function in L^∞ .

(1) T_ϕ^M is invertible for every M in \mathcal{F}_1 and $\sup\{\|(T_\phi^M)^{-1}\|; M \in \mathcal{F}_1\} < \infty$ if and only if $\text{ess.inf}|\phi| > 0$ and there exists a function g in $(H^\infty)^{-1}$ such that $\text{Re}(\phi g) \geq \delta$ a.e. for some constant $\delta > 0$.

(2) When m is the unique logmodular measure for τ , T_ϕ^M is invertible for every M in \mathcal{F}_0 if and only if both $\text{ess.inf}|\phi| > 0$ and there exists a function g in $(H^\infty)^{-1}$ such that $\text{Re}(\phi g) \geq \delta$ a.e. for some constant $\delta > 0$.

Proof. Since for any subset S of lat A

$$(\inf\{\varepsilon(M); M \in S\})^{-1} = \sup\{\|(T_\phi^M)^{-1}\|; M \in S\},$$

both (1) and (2) are clear by Theorem 2.

If ϕ is unimodular, then (2) of Theorem 2 shows Theorem 4.1 in [1] and (2) of Corollary 2 shows Theorem 4.6. in [1]. Our results apply to more general uniform algebras than that of [1] by [7, p157]. (1) of Corollary 2 does not show Theorem 2 in [2] for general functions ϕ in L^∞ . However by the remark above Theorem 2 in this paper we can get the invertibility theorems about a family $\{R_\phi^v, v \in \mathcal{E}_1\}$.

§5. Spectrums of selfadjoint Toeplitz operators

$\sigma(T_\phi^M)$ denotes the spectrum of T_ϕ^M for each M in lat A . Hartman and Wintner [8] showed $\sigma(T_\phi) = [\text{ess.inf } \phi, \text{ess. sup } \phi]$ for a real valued function ϕ in L^∞ where A is the disc algebra. This theorem is not valid in general. For example, it is not true in Examples 2 and 3. In the case of Example 2, if ϕ is a real valued continuous function on $\bar{\Delta}$ and $\phi = 0$ on $\partial\Delta$ then T_ϕ is compact (cf. [14, p107]). Hence $\sigma(T_\phi) \neq [\text{ess.inf } \phi, \text{ess. sup } \phi]$. In the case of Example 3, see [1, p295].

For each M in \mathcal{F} , let $\varepsilon(M) = \varepsilon(M, \phi)$ be the maximum of non-negative constants δ such that

$$\|T_\phi^M f\|_2 \geq \delta \|f\|_2 \text{ for all } f \in M.$$

Put $\Sigma_\phi = \{s \in C : \inf[\varepsilon(M, \phi - s), M \in \mathcal{F}] = 0\}$, then

$$\Sigma_\phi \cup \bar{\Sigma}_\phi \supseteq \bigcup_{M \in \mathcal{F}} \sigma(T_\phi^M).$$

The above two sets sometimes coincide.

Proposition 3. Let ϕ be a function in L^∞ .

(1) If A is a uniform algebra in Example 4, then

$$\bigcup_{M \in \mathcal{F}_1} \sigma(T_\phi^M) = \Sigma_\phi \cup \bar{\Sigma}_\phi.$$

(2) If N_τ is finite dimensional and m is the unique logmodular measure, then

$$\bigcup_{M \in \mathcal{F}_0} \sigma(T_\phi^M) = \Sigma_\phi \cup \bar{\Sigma}_\phi.$$

Proof. We have to prove that $\Sigma_\phi \cup \bar{\Sigma}_\phi \subset \bigcup \{\sigma(T_\phi^M) ; M \in S\}$ for $S = \mathcal{F}_1$ in (1) and $S = \mathcal{F}_0$ in (2).

(1) If $s \notin \bigcup \{\sigma(T_\phi^M) ; M \in \mathcal{F}_1\}$, then for any $M \in \mathcal{F}_1$ $T_{\phi-s}^M$ is invertible. By the proof of (1) of Theorem 2, for any $M \in \mathcal{F}$ there exists $N \in \mathcal{F}_1$ such that $T_{\phi-s}^M$ is unitarily equivalent to $T_{\phi-s}^N$. Hence for any $M \in \mathcal{F}$ $T_{\phi-s}^M$ is invertible. By the remark above Theorem 2, for any $v \in \mathcal{L}$ $R_{\phi-s}^v$ is invertible. By Theorem 2 and the remark above Theorem 3 in [2], $\inf |\phi - s| > 0$ a.e., and there exists a function g in $(H^\infty)^{-1}$ such that $\operatorname{Re}(\phi - s)g \geq \delta$ a.e. for some constant $\delta > 0$. By Corollary 1, $\sup \{\|(T_{\phi-s}^M)^{-1}\| ; M \in \mathcal{F}\} < \infty$ and hence $s \notin \Sigma_\phi \cup \bar{\Sigma}_\phi$.

(2) If $s \notin \bigcup \{\sigma(T_\phi^M) ; M \in \mathcal{F}_0\}$ then Corollary 1 and (2) of Corollary 2 $s \notin \Sigma_\phi \cup \bar{\Sigma}_\phi$.

If ϕ is a real valued function in L^∞ then by Theorem 1

$$[\operatorname{ess. inf} \phi, \operatorname{ess. sup} \phi] \supseteq \Sigma_\phi \supseteq \bigcup_{M \in \mathcal{F}} \sigma(T_\phi^M).$$

The following theorem shows the relations of the three sets above.

Theorem 4. Let ϕ be a real valued function in L^∞ . Then the following are valid.

(1) $\Sigma_\phi = [\operatorname{ess. inf} \phi, \operatorname{ess. sup} \phi]$.

(2) If A is a uniform algebrs in example (4) then

$$\bigcup_{M \in \mathcal{F}_1} \sigma(T_\phi^M) = [\operatorname{ess. inf} \phi, \operatorname{ess. sup} \phi].$$

(3) If N_τ is finite dimensional and m is the unique logmodular measure for τ , then

$$\bigcup_{M \in \mathcal{F}_0} \sigma(T_\phi^M) = [\text{ess.inf } \phi, \text{ess.sup } \phi].$$

Proof. (1) Put $a = \text{ess.inf } \phi$ and $b = \text{ess.sup } \phi$. It is easy to see that $\Sigma_\phi = \Sigma_\phi \cup \bar{\Sigma}_\phi \subset [a, b]$. In fact, if s is a non-real complex number, that is, $s = x + iy$ and $y \neq 0$, then

$$|\phi - s| \geq \varepsilon + |\phi - s + (s - x)| \quad \text{a.e.}$$

where ε is a positive number with $\varepsilon^2 + 2\varepsilon\|\phi - x\| \leq y^2$. Hence $s \notin \Sigma_\phi$ by Theorem 1. If s is a real number with $s < a$ then $|\phi - s| = |a - s| + |\phi - s + (s - a)|$ a.e. and if $s > b$ then $|\phi - s| = |s - b| + |\phi - s + (s - b)|$ a.e.. Hence $s \notin \Sigma_\phi$ by Theorem 1.

$R(\phi) \subset \Sigma_\phi$ by Theorem 1 where $R(\phi)$ is the essential range of ϕ . We will prove that $[a, b] \subset \Sigma_\phi$. Suppose $a < s < b$ and $s \notin \Sigma$. Then we will get one contradiction. Put $s_a = \text{ess.sup min } (\phi, s)$ and $s_b = \text{ess.inf max } (\phi, s)$, then $s_a < s < s_b$. Put

$$E_a = \{x \in X : a \leq \phi(x) \leq s_a\} \text{ and } E_b = \{x \in X : s_b \leq \phi(x) \leq b\}.$$

then $m(E_a \cup E_b) = 1$ and $m(E_a \cap E_b) = 0$. Since $s \in \Sigma_\phi$, by Theorem 1 there exist a nonzero function g in H^∞ and $\varepsilon > 0$ such that

$$|\phi - s| \geq \varepsilon + |\phi + g| \quad \text{a.e..}$$

Put $S_a = \{t \in R : \text{dist}(g(E_a), t) \leq \varepsilon/2\}$ and $S_b = \{t \in R : \text{dist}(g(E_b), t) \leq \varepsilon/2\}$ where R is a real line and $g(E_\ell)$ is an essential range of g in E_ℓ with $\ell = a, b$. Then S_a and S_b are compact subsets in R . If $S_a \cap S_b$ is essentially nonempty then there exist $x \in E_a$ and $x' \in E_b$ such that $|g(x) - g(x')| \leq \varepsilon$. Hence

$$\begin{aligned} \phi(x') - \phi(x) &= |\phi(x) - \phi(x')| \\ &\leq |\phi(x) + g(x)| + |\phi(x') + g(x')| + |g(x) - g(x')| \\ &\leq |\phi(x) - s| - \varepsilon + |\phi(x') - s| - \varepsilon + \varepsilon \\ &= \phi(x') - \phi(x) - \varepsilon. \end{aligned}$$

because $\phi(x) \leq s_a < s < s_b \leq \phi(x')$. This contradiction shows that $S_a \cap S_b$ is essentially empty. By a theorem of Runge we can show that $\chi_E \in H^\infty$. Thus $m(E_a) = 0$ or 1. This contradiction shows that $[a, b] \subset \Sigma_\phi$. (2) and (3) are results of (1) and (2) in Proposition 3 and (1) in this theorem.

Widom [13] proved that $\sigma(T_\phi)$ is connected for arbitrary symbol ϕ when A is the disc algebra. This theorem is not valid in Examples 2, 3 and 4. Here we assume that A is a uniform algebra in Example 3. Abrahamse [1, p295] announced without proof that if ϕ is unimodular then $\cup\{\sigma(T_\phi^M) ; M \in S\}$ is connected when A is an annulus algebra. S denotes the set of simply invariant subspaces.

The universal covering surface of D is conformally equivalent to the open unit disc Δ and an analytic projection map π from Δ onto D can be chosen so that $\pi(0) = t$. Abrahamse [1, p294], using his inversion theorem, when ϕ is unimodular and $\tilde{\phi} = \phi \circ \pi$, if T_ϕ^M is invertible for each $M \in S$ then $T_{\tilde{\phi}}$ is invertible and hence $\sigma(T_{\tilde{\phi}}) \subset \cup\{\sigma(T_\phi^M) ; M \in S\}$. The following proposition generalizes the Abrahamse's result to arbitrary symbols. Recall that \mathcal{F} is a proper subset of S .

Proposition 5. Suppose A is a uniform algebra in Example 3. Let ϕ be a nonzero function in L^∞ . If T_ϕ^M is invertible for each M in \mathcal{F}_0 . then $T_{\tilde{\phi}}$ is invertible. Hence

$$\sigma(T_{\tilde{\phi}}) \subset \bigcup_{M \in \mathcal{F}_0} \sigma(T_\phi^M).$$

If $n = 1$ then $\sigma(T_{\tilde{\phi}}) = \cup\{\sigma(T_\phi^M) ; M \in \mathcal{F}_0\}$ and hence $\cup\sigma(T_\phi^M)$ is connected.

Proof. If T_ϕ^M is invertible for each $M \in \mathcal{F}_0$, then by Corollary 2 there is $g \in (H^\infty)^{-1}$ such that $Re(\phi g) \geq \delta$ a.e. for some constant $\delta > 0$. This implies that $Re(\tilde{\phi} \tilde{g}) \geq \delta$ a.e. and $\tilde{g} \in H^\infty(\Delta)^{-1}$. By Corollary 2 $T_{\tilde{\phi}}$ is invertible. If $n = 1$, we will show that the converse is valid. Suppose $T_{\tilde{\phi}}$ is invertible. By Theorem 2, there exists a positive constant ε and a function g in $H^\infty(\Delta)$ such that

$$|\tilde{\phi}|^2 \geq \varepsilon^2 + |\tilde{\phi} + g|^2 \quad \text{a.e.}$$

If we can find h in H^∞ such that

$$|\tilde{\phi}|^2 \geq \varepsilon^2 + |\tilde{\phi} + h|^2 \quad \text{a.e.}$$

then by Theorem 2 T_ϕ^M is left invertible for every $M \in \mathcal{F}_0$. The same proof of the complex conjugate $\bar{\phi}$ shows that T_ϕ^M is invertible for every $M \in \mathcal{F}_0$.

We will show the existence of such an h in H^∞ . We can regard \tilde{H}^∞ as the closed subalgebra of $H^\infty(\Delta)$ consisting of those $f \in H^\infty(\Delta)$ which are invariant under a certain group of conformal maps of Δ onto itself. This group, which we shall denote by G , is in an infinite cyclic group. Put $S = \{g \in H^\infty(\Delta) : |\tilde{\phi}|^2 \geq \varepsilon^2 + |\tilde{\phi} + g|^2 \quad \text{a.e.}\}$ then S is a convex subset of $H^\infty(\Delta)$. We define a group operators $\{\Phi_\gamma\}_{\gamma \in G}$ on $H^\infty(\Delta)$ by means of $\Phi_\gamma h = h \circ \gamma$ for all $\gamma \in G$. Let σ be the topology of almost uniform convergence in $H^\infty(\Delta)$. Then σ is metrizable, and a normal families argument shows that S is σ -compact. Since they commute, the fixed point theorem of Markov and Kakutani ([5, p456]), [12, Theorem 2.1]) affirms the existence of k in S which is invariant under the group $\{\Phi_\gamma\}_{\gamma \in G}$. Hence $k = \tilde{h}$ for some $h \in H^\infty$ and

$$|\tilde{\phi}|^2 \geq \varepsilon^2 + |\tilde{\phi} + \tilde{h}|^2 \quad a.e..$$

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