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GINZBURG LANDAU EQUATION AND STABLE SOLUTIONS IN A ROTATIONAL DOMAIN

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GINZBURG LANDAU EQUATION AND STABLE SOLUTIONS IN A ROTATIONAL DOMAIN

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ABSTRACT. The Ginzburg Landau equations in a rotational domain in $\mathbb{R}^3$ are studied. Rotational solutions are constructed and proved to be stable by the spectral analysis on the linearized equation.

§1 Introduction.

We deal with the Ginzburg Landau (GL) equations and existence of stable solutions in a rotational domain. GL equation is well-known as a model of the super-conductivity phenomena in a low temperature (cf. [11]) and has been playing an important role in the mathematical physics. Several important features arising in the super-conductivity phenomena have been understood through the GL equations. The experiment in physics shows that in a ring shaped material (superconductor) a permanent circular current of electrons exists with no energy dissipation under no external magnetic field. Our purpose in this paper is to construct an adequate stable solution describing this phenomenon.

There have been many mathematical works from the PDE point of view on the GL equations in several situations. The GL equations have an important parameter (the Landau parameter) which gives a crucial influence on the situations. Odeh [20] dealt with a bifurcation from zero solution with respect to this parameter. Caroll and Glick [6] proved a unique existence of solution in a certain range of the parameter by a fixed point theorem. Jaffe and Taubes [12] gave a method to prove the magnetic screening effect and gave the quantization of the total vorticies in $\mathbb{R}^2$. Moreover for the critical value of the Landau parameter, they constructed a solution with the arbitrarily prescribed vortices in $\mathbb{R}^2$. Klimov [15], Bobylev [5] obtained multiple solutions by a topological method in a large range of the parameter. Berger and Chen [3] constructed a radially symmetric solution (vortex solution) in $\mathbb{R}^2$ and obtained an elaborate

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asymptotic behavior when the parameter goes to ∞. Chen [7] constructed a nonsymmetric solution in a 2-dim bounded domain with the boundary condition $|\Phi| = 1$. Yang [25], [26] constructed smooth solutions in $\mathbb{R}^3$ and in a bounded domain with boundary conditions of several kinds under an outer magnetic force. Monvel-Berthier, Georgescu and Pruce [19] gave a detailed characterization of the configuration space with prescribed total vorticity under the boundary condition $|\Phi| = 1$ in 2 dimension. There are also studies on the zero set of the solutions to the model without magnetic effect (cf. (1.6)) for boundary condition of several kinds (cf. Baumann, Carlson and Phillips [2], Elliot, Matano and Tang [9], Bethuel, Brezis and Helein [4]). However, it seems to be still obscure about the stability of solutions for a general domain. In our previous paper [13], we dealt with the GL equation without magnetic effect (cf. (1.6)) and consider the existence of stable non-constant solutions and proved that there exist stable non-constant solutions in a thin annulus, while no stable one in any convex domain. In this paper we will construct a rotational solution (non-constant) to the full GL equation in a ring shaped domain (cf. Fig. 1) and prove their stability, which means that the solutions correspond to realizable phenomena.

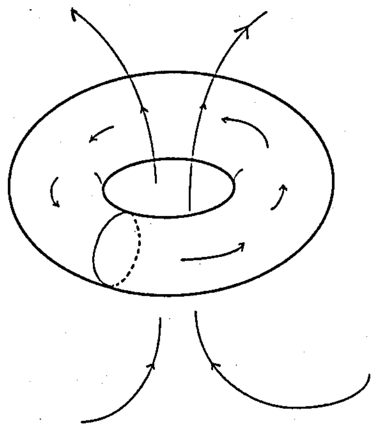

Figure 1 : Ring Shaped Domain Ω

Now we formulate the problem. Let Ω be a bounded domain in $\mathbb{R}^3$ with C^2 boundary and consider the following functional:

$$\mathcal{H}(\Phi, A) = \int_\Omega \left\{ \frac{1}{2}|(\nabla - iA)\Phi|^2 + \frac{\alpha}{4}(1 - |\Phi|^2)^2 \right\} dx + \int_{\mathbb{R}^3} \frac{1}{2}|\mathrm{rot} A|^2 dx. \tag{1.1}$$

$\alpha > 0$ is a positive parameter and Φ is a $\mathbb{C}$ valued function in Ω (which is the wave function of the electrons) and A is a real vector valued function in $\mathbb{R}^3$ (which is a vector potential of the magnetic field). The first term corresponds to the energy of the electrons confined in Ω and the second term corresponds to that of the magnetic field which exist over $\mathbb{R}^3$. We suppose $A \in L^2_{loc}(\mathbb{R}^3; \mathbb{R}^3)$, $\nabla A \in L^2(\mathbb{R}^3; \mathbb{R}^{3\times 3})$ and $\Phi \in H^1(\Omega; \mathbb{C})$. GL equation is the variational equation of the above functional, that is,

$$\begin{cases} (\nabla - iA)^2\Phi + \alpha(1 - |\Phi|^2)\Phi = 0 \quad \text{in} \quad \Omega, \\ \dfrac{\partial \Phi}{\partial \nu} - i\langle A \cdot \nu \rangle \Phi = 0 \quad \text{on} \quad \partial\Omega, \\ \mathrm{rot}\,\mathrm{rot}\, A + \left(i(\overline{\Phi}\nabla\Phi - \Phi\nabla\overline{\Phi})/2 + |\Phi|^2 A \right) \Lambda_\Omega = 0 \quad \text{in} \quad \mathbb{R}^3, \end{cases} \tag{1.2}$$

where $\Lambda_\Omega(x) = 1$ for $x \in \Omega$ and $\Lambda_\Omega(x) = 0$ for $x \in \mathbb{R}^3 \setminus \Omega$. $\langle \cdot, \cdot \rangle$ is the standard inner product in $\mathbb{R}^3$. A realizable physical state corresponds to a certain local minimizer of the above energy functional. To see the stability of a critical point (Φ, A), we need to consider the second variation of the functional,

$$\mathcal{L}(\Phi, A, \Psi, B) = \frac{d^2}{d\epsilon^2}\mathcal{H}(\Phi + \epsilon\Psi, A + \epsilon B)_{|\epsilon=0}. \tag{1.3}$$

However it is well known that $\mathcal{H}$ is invariant under the following (gauge) transformation: for a real valued function ρ in $\mathbb{R}^3$,

$$\Phi' = e^{i\rho}\Phi, \qquad A' = A + \nabla\rho \qquad \text{(Gauge transformation)}. \tag{1.4}$$

In view of this any critical point (Φ, A) of $\mathcal{H}$ can not be a "strict" local minimizer because the above transformation makes a continuum containing (Φ, A) (∞ dimensional set) whose elements give the same value as (Φ, A). This is a natural because all (Φ', A')'s in this continuum corresponds to one physical state and hence we need to compare the energy among only gauge-distinct-states around (Φ, A). For this purpose we consider the tangent space to the continuum at (Φ, A), which can be expressed as follows:

$$T(\Phi, A) = \{(i\xi\Phi, \nabla\xi) \mid \xi : \mathbb{R} \text{ valued function on } \mathbb{R}^3\}.$$

and choose a space $N(\Phi, A)$ which is transversal to $T(\Phi, A)$ such that $T(\Phi, A) \cap N(\Phi, A) = \{0\}$. We will consider strong coercivity of the second variation (1.3) in $N(\Phi, A)$ to investigate the stability of the solution.

In some situation of the superconductivity, the magnetic field is very small in the interior of Ω and the model without A is also used. We also deal with the following functional:

$$\mathcal{H}_0(\Phi) = \int_\Omega \left\{ \frac{1}{2}|\nabla\Phi|^2 + \frac{\alpha}{4}(1 - |\Phi|^2)^2 \right\} dx, \tag{1.5}$$

whose variational equation is also called the GL equation. It can be written as follows,

$$\begin{cases} \Delta\Phi + \alpha(1 - |\Phi|^2)\Phi = 0 & \text{in} \quad \Omega, \\ \dfrac{\partial\Phi}{\partial\nu} = 0 & \text{on} \quad \partial\Omega. \end{cases} \tag{1.6}$$

As in the case of (1.1)-(1.2), (1.5)-(1.6) has a similar invariance, that is, if $c \in \mathbb{R}$, the transformation

$$\Phi' = e^{ic}\Phi \tag{1.7}$$

leaves (1.5) and (1.6) invariant. We also consider the existence of stable solutions to (1.6) as well as (1.2). The second variation of $\mathcal{H}_0$ is defined by

$$\mathcal{L}_0(\Phi,\Psi) = \frac{d^2}{d\epsilon^2}\mathcal{H}_0(\Phi+\epsilon\Psi)_{|\epsilon=0}. \tag{1.8}$$

Similary to the case of (1.1), we prove the stability of solutions by taking account of the invariance (1.7).

In §2 we give the definitions of several spaces for the formulation for the stability. In §3 we present our main results. In §4 $\sim$ §6 we prove the theorems.

§2 Formulation.

In this section we will formulate $T(\Phi, A)$ and $N(\Phi, A)$, precisely. Let (Φ, A) be a solution of (1.2) such that $\Phi \in C^1(\overline{\Omega};\mathbb{C})$ and $A \in C^1(\mathbb{R}^3;\mathbb{R}^3)$ and $\nabla A \in L^2(\mathbb{R}^3;\mathbb{R}^{3\times 3})$. Acutually the solution we will construct later satisfies this condition. For convenience, we also deal with Φ-component in terms of real functions by taking the real and imaginary parts. We put $\Phi = u+vi$ and $\Psi = \phi+\psi i$. Hereafter we identify $\mathcal{H}(\Phi, A)$ and $\mathcal{L}(\Phi, A, \Psi, B)$ with $\mathcal{H}(u,v,A)$ and $\mathcal{L}(u,v,A,\phi,\psi,B)$, respectively. The tangent space $T(\Phi, A)$ can be expressed as follows,

$$T(u,v,A) = \{(-v\xi, u\xi, \nabla\xi) \mid \xi \in L^2_{loc}(\mathbb{R}^3),\ \nabla\xi \in H^1(\mathbb{R}^3;\mathbb{R}^3)\} \tag{2.1}$$

To define a subspace $N(u,v,A)$ in an adequate form, we prepare the operators P and $\widetilde{P}$. It is known that exists an orthogonal decomposition $L^2(\Omega;\mathbb{R}^3) = X_1 \oplus X_2$, $L^2(\mathbb{R}^3;\mathbb{R}^3) = Y_1 \oplus Y_2$ where

$$\begin{aligned}
X_1 &= \{\nabla\xi \mid \xi \in L^2(\Omega),\ \nabla\xi \in L^2(\Omega;\mathbb{R}^3)\},\\
X_2 &= \{B \in L^2(\Omega;\mathbb{R}^3) \mid \operatorname{div} B = 0 \text{ in } H^{-1}(\Omega),\ \langle B\cdot\nu\rangle = 0 \text{ in } H^{-1/2}(\partial\Omega)\},\\
Y_1 &= \{\nabla\xi \mid \xi \in L^2_{loc}(\mathbb{R}^3),\ \nabla\xi \in L^2(\mathbb{R}^3;\mathbb{R}^3)\},\\
Y_2 &= \{B \in L^2(\mathbb{R}^3;\mathbb{R}^3) \mid \operatorname{div} B = 0 \text{ in } H^{-1}(\mathbb{R}^3)\}.
\end{aligned}$$

See [24]. Let P and $\widetilde{P}$ be the orthogonal projections of $L^2(\Omega;\mathbb{R}^3)$ and $L^2(\mathbb{R}^3;\mathbb{R}^3)$ onto X_2 and Y_2, respectively. We define

$$\begin{aligned}
\overline{N}(u,v,A) &= \left\{(\phi,\psi,B) \in H^1(\Omega)\times H^1(\Omega)\times H^1(\mathbb{R}^3;\mathbb{R}^3) \mid \int_\Omega (v\phi - u\psi)dx = 0,\ B_{|\Omega} \in X_2\right\},\\
N(u,v,A) &= \left\{(\phi,\psi,B) \in H^1(\Omega)\times H^1(\Omega)\times H^1(\mathbb{R}^3;\mathbb{R}^3) \mid \int_\Omega (v\phi - u\psi)dx = 0,\ B \in Y_2\right\}.
\end{aligned}$$

Using the above, we have the following decomposition.

Proposition 1.

$$H^1(\Omega)\times H^1(\Omega)\times H^1(\mathbb{R}^3;\mathbb{R}^3) = T(u,v,A) + \overline{N}(u,v,A).$$

(Proof of Proposition 1) Assume $(u,v) \not\equiv (0,0)$. Otherwise the proof is much easier. Let (ϕ,ψ,B) be any element in the left hand side. By $B_{|\Omega} = P(B_{|\Omega}) + (I-P)(B_{|\Omega})$, there exists $\xi \in H^2(\Omega)$ such that $\nabla\xi = (I-\widetilde{P})(B_{|\Omega})$. Let $\overline{\xi}$ be such that $\overline{\xi} \in H^2(\mathbb{R}^3)$ and $\overline{\xi}_{|\Omega} = \xi$. Put $B_1 = B - \nabla\overline{\xi}$ and

$$c = \int_\Omega (\phi v - u\psi + \overline{\xi}(u^2+v^2))dx / \int_\Omega (u^2+v^2)dx.$$

Hence if we define $\widehat{\xi} = \overline{\xi} - c$, then we have a decomposition,

$$(\phi,\psi,B) = (-\widehat{\xi}v, \widehat{\xi}u, \nabla\widehat{\xi}) + (\phi+\widehat{\xi}v, \psi-\widehat{\xi}u, B_1) \in T(u,v,A) + N(u,v,A). \quad \square$$

Proposition 2.

(2.2)
$$H^1(\Omega)\times H^1(\Omega)\times H^1(\mathbb{R}^3;\mathbb{R}^3) = T(u,v,A)\oplus N(u,v,A).$$

(Proof of Proposition 2) Assume $(u,v) \not\equiv (0,0)$. Otherwise the conclusion is known. Let (ϕ,ψ,B) be any element in the left hand side. By the aid of the projection $\widetilde{P}$ we have the decomposition $B = \nabla\xi + B_1 \in Y_1 + Y_2$, hence

$$(\phi,\psi,B) = (-\xi v, \xi u, \nabla\xi) + (\phi+\xi v, \psi-\xi u, B_1)$$

We can adjust ξ by adding an constant to ξ as in the proof of Propostion 1 so that the second term of the right hand side belongs to $N(u,v,A)$. If $(-\xi v, \xi u, \nabla\xi)$ belongs to $N(u,v,B)$, ξ is a harmonic function in $\mathbb{R}^3$ and so is $\partial\xi/\partial x_i$ $(1 \leqq i \leqq 3)$. Since if it is assumed to be in $L^2(\mathbb{R}^3)$, and hence it is identical to 0 from the Liouville type theorem. This implies that ξ is a constant function in $\mathbb{R}^3$. On the other hand we have $\int_\Omega (u^2+v^2)\xi dx = 0$, and so $\xi \equiv 0$. This deduces that $T(u,v,A)\cap N(u,v,A) = \{0\}$. $\square$

By a direct calculation we can derive a concrete expression of the second variation (1.3).

Formula of second variation of $\mathcal{H}$.

(2.3)
$$\mathcal{L}(u,v,A,\phi,\psi,B) = \frac{1}{2}\frac{d^2}{d\epsilon^2}\mathcal{H}(u+\epsilon\phi, v+\epsilon\psi, A+\epsilon B)_{|\epsilon=0} =$$

$$\int_\Omega \left\{|\nabla\phi|^2 + |\nabla\psi|^2 - \alpha(1-u^2-v^2)(\phi^2+\psi^2) + 2\alpha(u\phi+v\psi)^2\right\} dx$$

$$+\int_\Omega \left\{A^2(\phi^2+\psi^2) - 2(\phi\langle\nabla\psi\cdot A\rangle - \psi\langle\nabla\phi\cdot A\rangle)\right\} dx$$

$$+\int_{\mathbb{R}^3} |\mathrm{rot}B|^2 dx + \int_\Omega (u^2+v^2)B^2 dx$$

$$+4\int_\Omega \langle A\cdot B\rangle(u\phi+v\psi)dx - 2\int_\Omega \left\{\phi\langle\nabla v\cdot B\rangle - \psi\langle\nabla u\cdot B\rangle + u\langle\nabla\psi\cdot B\rangle - v\langle\nabla\phi\cdot B\rangle\right\} dx$$

Remark. If $\operatorname{div} B = 0$ in Ω and $\langle B \cdot \nu \rangle = 0$ on $\partial\Omega$, then

$$\int_\Omega (u\langle \nabla\psi \cdot B\rangle - v\langle \nabla\phi \cdot B\rangle)\, dx = \int_\Omega (\phi\langle \nabla v \cdot B\rangle - \psi\langle \nabla u \cdot B\rangle)\, dx.$$

This equality will be used in §6. In the next proposition, we see that the second variation of $\mathcal{H}$ does not depend on the tangential component $T(u, v, A)$.

Proposition 3. If $(\phi, \psi, B), (\phi', \psi', B') \in H^1(\Omega) \times H^1(\Omega) \times H^1(\mathbb{R}^3; \mathbb{R}^3)$, $(\phi - \phi', \psi - \psi', B - B') \in T(u, v, A)$, then

$$\mathcal{L}(u, v, A, \phi, \psi, B) = \mathcal{L}(u, v, A, \phi', \psi', B'). \tag{2.4}$$

Now let Φ be a C^1 solution of (1.6). Similarly as above we put $\Phi = u + vi$ and $\Psi = \phi + \psi i$ again and denote $\mathcal{H}_0(\Phi)$ and $\mathcal{L}_0(\Phi, \Psi)$ by $\mathcal{H}_0(u, v)$ and $\mathcal{L}_0(u, v, \phi, \psi)$, respectively. Let us define

$$T_0(u, v) = \{(-tv, tu) \in H^1(\Omega) \times H^1(\Omega) \mid t \in \mathbb{R}\},$$

$$N_0(u, v) = \{(\phi, \psi) \in H^1(\Omega) \times H^1(\Omega) \mid \int_\Omega (v\phi - u\psi) dx = 0\}.$$

Accordingly the next property holds.

Proposition 4.

$$H^1(\Omega) \times H^1(\Omega) = T_0(u, v) \oplus N_0(u, v).$$

Formula of the second variation of $\mathcal{H}_0$

$$\mathcal{L}_0(u, v, \phi, \psi) = \frac{1}{2}\frac{d^2}{d\epsilon^2}\mathcal{H}_0(u + \epsilon\phi, v + \epsilon\psi)_{|\epsilon=0} = \tag{2.5}$$

$$\int_\Omega \left\{(|\nabla\phi|^2 + |\nabla\psi|^2) - \alpha(1 - u^2 - v^2)(\phi^2 + \psi^2) + 2\alpha(u\phi + v\psi)^2\right\} dx.$$

Proposition 5. If $(\phi, \psi), (\phi', \psi') \in H^1(\Omega) \times H^1(\Omega)$ and $(\phi - \phi', \psi - \psi') \in T_0(u, v)$, then

$$\mathcal{L}_0(u, v, \phi, \psi) = \mathcal{L}_0(u, v, \phi', \psi').$$

§3 Main Results.

In this section we will present the main results. Let D be a domain defined by

$$D \equiv \{(r,z) \in \mathbb{R}^2 \mid r > 0\}$$

and let Σ be a bounded domain in D with C^3 boundary (cf. Fig.2). Throughout this paper we assume the next assumption on Σ.

Assumption. Σ is convex and $\overline{\Sigma} \subset D$.

Now we give a domain $\Omega \subset \mathbb{R}^3$ where we consider the equations (1.2) and (1.6) as follows,

$$\Omega = \left\{(r\cos\theta, r\sin\theta, z) \in \mathbb{R}^3 \mid (r,z) \in \Sigma,\ 0 \leqq \theta < 2\pi\right\} \simeq \Sigma \times S^1 \qquad \text{(cf. Fig.1)}. \tag{3.1}$$

We will construct a rotational (stable) solutions to (1.2) and (1.6) in this domain Ω defined above. In the present section and the later sections, we sometimes use the cylindrical coordinate system (r,z,θ) in $\mathbb{R}^3$. Considering the symmetry of the domain, we find a solution (Φ, A) to (1.2) in the particular form:

$$A(r,z,\theta) = Y(r,z)\left(\frac{-\sin\theta}{r}, \frac{\cos\theta}{r}, 0\right), \quad \Phi(r,z,\theta) = W(r,z)e^{im\theta} \tag{3.2}$$

which corresponds to a phenomena of rotational eternal current of electrons. Here $W = W(r,z) > 0$ and $Y = Y(r,z)$ are real valued functions in Σ and D, respectively.

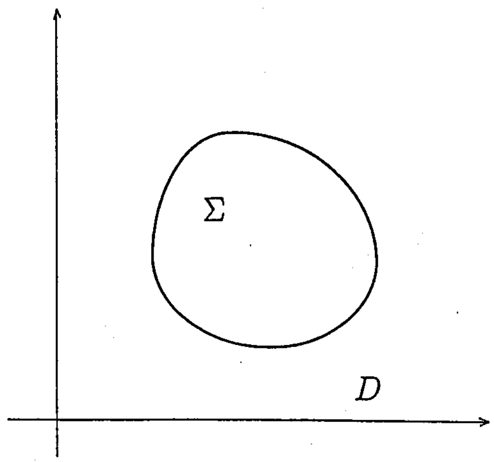

Figure 2: $\overline{\Sigma} \subset D$

We present the main theorems of this paper.

Theorem 6. Let m be an integer. There exists a $\alpha_0 > 0$ such that for any $\alpha \geqq \alpha_0$, there exists a solution (Φ_α, A_α) to (1.2) with

$$\Phi_\alpha(x) = W_\alpha(r,z)e^{im\theta}, \quad A_\alpha(x) = Y_\alpha(r,z)\left(\frac{-\sin\theta}{r}, \frac{\cos\theta}{r}, 0\right)$$

where $W_\alpha \in C^2(\overline{\Sigma})$ and $Y_\alpha \in C^1(D)$ and

$$\lim_{\alpha\to\infty} \sup_{(r,z)\in\Sigma} |W_\alpha(r,z) - 1| = 0. \tag{3.3}$$

Moreover it is stable in the sense that there exists a constant $\delta > 0$ such that

$$\mathcal{L}(\Phi_\alpha, A_\alpha, \Psi, B) \geqq \delta\left(\|\Psi\|^2_{L^2(\Omega;\mathbb{C})} + \|B\|^2_{L^2(\Omega)} + \|\nabla B\|^2_{L^2(\mathbb{R}^3;\mathbb{R}^{3\times3})}\right) \tag{3.4}$$

for $(\Psi, B) \in N(\Phi_\alpha, A_\alpha)$ and $\alpha \geqq \alpha_0$.

We can similarly find a stable solution to (1.6) in the form:

$$\Phi(x) = Z(r,z)e^{im\theta}, \quad (Z(r,z) > 0). \tag{3.5}$$

Theorem 7. Let m be an integer. Then there exists a constant $\alpha_1 > 0$ such that for any $\alpha \geqq \alpha_1$, there exists a solution $\Phi_\alpha(x) = Z_\alpha(r,z)e^{im\theta}$ to (1.6) with

$$\lim_{\alpha\to\infty} \sup_{(r,z)\in\Sigma} |Z_\alpha(r,z) - 1| = 0. \tag{3.6}$$

Moreover Φ_α is stable in the sense there exists a constant $\delta_0 > 0$ such that

$$\mathcal{L}_0(\Phi_\alpha, \Psi) \geqq \delta_0\|\Psi\|^2_{L^2(\Omega;\mathbb{C})} \tag{3.7}$$

for any $\Psi \in N_0(\Phi_\alpha)$ and $\alpha \geqq \alpha_1$.

We will prove these results in the following sections.

Remark. We can obtain similar theorems for the 2 dimensional case, namely for an annulus. The proof will be done in the same manner.

Remark. In our previous paper [13], we constructed a stable solution to (1.6) for a "thin" annulus. Hence, Theorem 7 is in an extended line of the study in [13].

§4. Construction of solutions

In this section we will prove the existence of solutions to the GL equations (1.2) and (1.6) in the forms of (3.2) and (3.5), respectively. We prove the asymptotic behaviors of solutions as $\alpha \to \infty$ (cf. (4.2) and (4.4)), which play essential roles to investigate the linearized eigenvalue problems of the solutions in §5 and §6. We deal with (1.6) first and (1.2) next in this section and §4. By a direct calculation with (3.5), we get the equation for Z.

$$(4.1) \qquad \begin{cases} L_1 Z - \dfrac{m^2}{r^2} Z + \alpha Z(1 - Z^2) = 0 \quad \text{in} \quad \Sigma, \\ \dfrac{\partial Z}{\partial \mathbf{n}} = 0 \quad \text{on} \quad \partial\Sigma, \end{cases}$$

where

$$L_1 = \frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial}{\partial r}) + \frac{\partial^2}{\partial z^2}$$

and $\mathbf{n}$ is the outward unit normal vector on $\partial\Sigma$ in D.

The next proposition shows the existence of a positive solution and the asymptotic behavior of it as $\alpha \to \infty$.

Proposition 8. There exists $\alpha_0 > 0$ such that there exists a unique solution $Z = Z_\alpha(r, z) > 0$ to (4.1) for any $\alpha \geqq \alpha_0$ with the following asymptotic properties:

$$(4.2) \qquad \begin{cases} \limsup\limits_{\alpha \to \infty} \sup\limits_{(r,z)\in\Sigma} \alpha |Z_\alpha(r, z) - 1| < \infty, \qquad \limsup\limits_{\alpha \to \infty} \sup\limits_{(r,z)\in\Sigma} \alpha |\nabla Z_\alpha(r, z)| < \infty, \\ \lim\limits_{\alpha \to \infty} \sup\limits_{(r,z)\in\Sigma} |\alpha(1 - Z_\alpha(r, z)^2) - \dfrac{m^2}{r^2}| = 0. \end{cases}$$

Similarly we have the following equation (4.3) for (W, Y) from (3.2):

$$(4.3) \qquad \begin{cases} L_1 W - \dfrac{1}{r^2}(m - Y)^2 W + \alpha W(1 - W^2) = 0 \quad \text{in} \quad \Sigma, \\ L_2 Y + (m - Y) W^2 \Lambda_\Sigma = 0 \quad \text{in} \quad D, \\ \dfrac{\partial W}{\partial \mathbf{n}} = 0 \quad \text{on} \quad \partial\Sigma, \quad Y = 0 \quad \text{on} \quad \partial D, \end{cases}$$

where

$$L_2 = \frac{\partial^2}{\partial r^2} - \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}$$

and $\Lambda_\Sigma(r, z) = 1$ for $(r, z) \in \Sigma$ and $\Lambda_\Sigma(r, z) = 0$ for $(r, z) \in D \setminus \Sigma$.

We remark that in the case $m = 0$, (4.3) reduces to (4.1) if one puts $Y = 0$. If (W, Y) is a solution to (4.3), then $(W, -Y)$ with $-m$ is a one to (4.3). Hence for a negative integer we also have the same assumption provided that (W,Y) and m are replaced by $(W,-Y)$ and $-m$. Hence in the proof of Proposition 9 (below), it is enough to to deal with the case $m \in \mathbb{N}$. For $m \in \mathbb{N}$, (4.3) becomes a so-called cooperation system for functions as long as $0 \leqq W \leqq 1$, $0 \leqq Y \leqq m$. Indeed because the nonlinear term in the former equation is in Y while so is the one in the latter equation in W if W and Y is in the above region. We will make use of the technique of the comparison method (upper-lower solution method). We have the next result.

Proposition 9. There exists a solution $(W,Y) = (W_\alpha(r,z), Y_\alpha(r,z))$ to (4.3) with the following asymptotic properties:

$$\text{(4.4)} \qquad \begin{cases} \limsup\limits_{\alpha\to\infty} \sup\limits_{(r,z)\in\Sigma} \alpha|W_\alpha(r,z)-1| < \infty, \quad \limsup\limits_{\alpha\to\infty} \sup\limits_{(r,z)\in\Sigma} \alpha|\nabla W_\alpha(r,z)| < \infty, \\ \lim\limits_{\alpha\to\infty} \sup\limits_{(r,z)\in\Sigma} |\alpha\,(1-W_\alpha(r,z)^2) - \dfrac{1}{r^2}(m - Y_\alpha(r,z))^2| = 0. \end{cases}$$

In the rest of this section we prove these propositions after presenting several auxiliary lemmas.

Lemma 10. Let ρ be a real valued function which is C^3 in a open set $E \subset \mathbb{R}^n$. Then we have

$$\text{(4.5)} \qquad |\nabla\,\rho|\,\Delta\,|\nabla\rho| \geqq \operatorname{grad}\rho\,(\Delta\rho) \quad \text{in} \quad \{x \in E \mid |\nabla\rho(x)| \neq 0\}$$

where $\operatorname{grad}\rho$ is the differential operator which is expressed as

$$\operatorname{grad}\rho = \sum_{k=1}^{n} \frac{\partial\rho}{\partial x_k}\frac{\partial}{\partial x_k}$$

in terms of the orthogonal coordinate $(x_1, x_2, ..., x_n)$.

(Proof of Lemma 10) From a direct calculation

$$\frac{\partial}{\partial x_k}\left(\sum_{j=1}^{n}(\frac{\partial\rho}{\partial x_j})^2\right)^{1/2} = |\nabla\rho|^{-1}\sum_{j=1}^{n}\frac{\partial\rho}{\partial x_j}\frac{\partial^2\rho}{\partial x_j\partial x_k}$$

$$\frac{\partial^2}{\partial x_k^2}|\nabla\rho| = |\nabla\rho|^{-1}\left(\sum_{j=1}^{n}\frac{\partial\rho}{\partial x_j}\frac{\partial^3\rho}{\partial x_j\partial x_k^2} + \sum_{j=1}^{n}(\frac{\partial^2\rho}{\partial x_j\partial x_k})^2\right)$$

$$- |\nabla\rho|^{-3}\left(\sum_{j=1}^{n}\frac{\partial\rho}{\partial x_j}\frac{\partial^2\rho}{\partial x_j\partial x_k}\right)^2 \geqq |\nabla\rho|^{-1}\sum_{j=1}^{n}\frac{\partial\rho}{\partial x_j}\frac{\partial^3\rho}{\partial x_j\partial x_k^2}.$$

Here we used the Schwarz's inequality. Summing up the above in $k = 1, 2, ..., n$, we have the desired inequality. □

Lemma 11. Let $E \subset \mathbb{R}^n$ be a domain with C^2 boundary and Γ is a relatively open subset of ∂E. Let $\rho \in C^2(\overline{E})$ be a real valued function with $\partial\rho/\partial\nu = 0$ on Γ where ν is the unit outward normal vector on ∂E. Then

$$\frac{1}{2}\frac{\partial}{\partial\nu}|\nabla\rho|^2 = -h(\operatorname{grad}\rho, \operatorname{grad}\rho) \quad \text{on} \quad \Gamma \tag{4.6}$$

where $h(\cdot,\cdot)$ is the second fundamental form of the inclusion $\partial E \subset \mathbb{R}^n$ with respect to $-\nu$. (cf. [15 ; Chap. 7]).

By the Neumann boundary condition of ρ, $\operatorname{grad}\rho$ can be identified with a first order differential operator on Γ and also a vector field on Γ.

(Proof of Lemma 11) First extend $\nu(x) = (\nu_1(x), ..., \nu_n(x))$ as a C^2 vector field on a neighborhood of Γ.

$$\frac{1}{2}\frac{\partial}{\partial\nu}|\nabla\rho|^2 = \frac{1}{2}\sum_{k=1}^{n}\nu_k\frac{\partial}{\partial x_k}\sum_{j=1}^{n}(\frac{\partial\rho}{\partial x_j})^2 = \sum_{k=1}^{n}\sum_{j=1}^{n}\nu_k\frac{\partial\rho}{\partial x_j}\frac{\partial^2\rho}{\partial x_j\partial x_k}$$

On the other hand we have $\sum_{k=1}^{n}\nu_k(x)\partial\rho/\partial x_k = 0$ on Γ. Since $\operatorname{grad}\rho$ is a differential along Γ, we can operate $\operatorname{grad}\rho$ to the above equation (recall Neumann B.C.):

$$0 = \operatorname{grad}\rho\,(\sum_{k=1}^{n}\nu_k(x)\partial\rho/\partial x_k) = \sum_{k,j}\left(\nu_k\frac{\partial\rho}{\partial x_j}\frac{\partial^2\rho}{\partial x_j\partial x_k} + \frac{\partial\nu_k}{\partial x_j}\frac{\partial\rho}{\partial x_j}\frac{\partial\rho}{\partial x_k}\right)$$

Using this, we have

$$\frac{1}{2}\frac{\partial}{\partial\nu}|\nabla\rho|^2 = -\sum_{j,k}\frac{\partial\nu_k}{\partial x_j}\frac{\partial\rho}{\partial x_j}\frac{\partial\rho}{\partial x_k}$$

This completes the proof of Lemma 11. □

Proof of Proposition 8

The equation (4.1) is a specific case dealt with in the classical work [1] and [23], and so we briefly discuss the existence of the solution. Define two constant functions as follows:

$$Z_{+,\alpha}(r,z) = 1, \quad Z_{-,\alpha}(r,z) = 1 - \frac{d}{\alpha} \quad \text{in} \quad \Sigma.$$

It is easy to check that if $d > 0$ is large enough, $Z_{-,\alpha} \leqq Z_{+,\alpha}$ are lower-upper solutions pair of (4.1) for a $\alpha > d$. Consequently there exists at least one solution Z_α such that $Z_{-,\alpha} \leqq Z_\alpha \leqq Z_{+,\alpha}$ in Σ. The uniqueness can be proved in the same manner as in that of Lemma 3.1 in [13], so we omit it. The first estimate in (4.2) directly follows from this inequality:

$$1 - \frac{d}{\alpha} \leqq Z_\alpha(r,z) \leqq 1 \quad \text{in} \quad \Sigma \tag{4.7}$$

for any $\alpha > d$. We will prove the remaning estimates of (4.2). Regarding Z_α as a function defined in Ω by $Z(x) = Z(\sqrt{x_1^2 + x_2^2}, x_3)$, we have

$$\Delta_x Z_\alpha - \frac{m^2 Z_\alpha}{x_1^2 + x_2^2} + \alpha Z_\alpha(1 - Z_\alpha^2) = 0 \quad \text{in} \quad \Omega, \tag{4.8}$$

with the Neumann B.C. on $\partial\Omega$. The nonlinear term in (4.8) is bounded in $C^0(\overline{\Omega})$ for $\alpha > d$ by (4.7) and the Schauder estimate for the elliptic boundary value problem yields that $\{Z_\alpha\}_{\alpha > d}$ is bounded in $C^{1+\gamma}(\overline{\Omega})$ where $\gamma \in [0, 1)$ is an arbitrary constant. This means $|\nabla Z_\alpha(x)|$ is relatively compact in $C^0(\overline{\Omega})$ and hence it converges to 0 uniformly in $\overline{\Omega}$ as $\alpha \to \infty$. Operating grad Z_α on the equation (4.8) and applying Lemma 10, we obtain the following differential inequality:

$$\begin{cases} |\nabla Z_\alpha| \Delta |\nabla Z_\alpha| + g_\alpha(x) \geqq 0 & \text{in} \quad \Omega \cap G_\alpha, \\ |\nabla Z_\alpha| \dfrac{\partial}{\partial \nu} |\nabla Z_\alpha| = -h(\operatorname{grad} Z_\alpha, \operatorname{grad} Z_\alpha) \leqq 0 & \text{on} \quad \partial\Omega \cap G_\alpha, \end{cases} \tag{4.9}$$

where

$$G_\alpha = \{x \in \overline{\Omega} \mid |\nabla Z_\alpha(x)| > 0\}$$

$$g_\alpha(x) = -\frac{m^2 |\nabla Z_\alpha|^2}{x_1^2 + x_2^2} - \sum_{j=1}^{2} \frac{\partial}{\partial x_j}\left(\frac{m^2}{x_1^2 + x_2^2}\right) \frac{\partial Z_\alpha}{\partial x_j} Z_\alpha + \alpha(1 - 3Z_\alpha^2)|\nabla Z_\alpha|^2.$$

The first inequality in (4.9) is deduced by a direct calculation. The inequality of second one follows from Lemma 11. Remember that grad Z_α is normal to the longitudinal direction of $\partial\Omega$ (i.e. $Z_\alpha = Z_\alpha(\sqrt{x_1^2 + x_2^2}, x_3)$ is constant in the longitudinal direction). Considering the sign of the second fundamental form and that the cross section Σ of Ω is convex, we obtain $h(\operatorname{grad} Z_\alpha, \operatorname{grad} Z_\alpha) \geqq 0$ on $\partial\Omega$.
Z_α is not a constant function and the set F_α defined by

$$F_\alpha = \{x \in \overline{\Omega} \mid 0 < |\nabla Z_\alpha(x)| = \max_{\overline{\Omega}} |\nabla Z_\alpha|\},$$

is not empty for $\alpha > d$. By virtue of (4.7) there exists $c > 0$ such that $0 \leqq \alpha(1 - Z_\alpha^2) \leqq c$ in Σ for $\alpha > d$. We will estimate $|\nabla Z_\alpha|$ in F_α from the upper. We divide the argument into the following two cases of α : (I) and (II).

(Case I) $F_\alpha \not\subset \partial\Omega$. Take any point $x_0 \in F_\alpha \setminus \partial\Omega$. We have $\Delta|\nabla Z_\alpha| \leqq 0$ at $x = x_0$ and consequently we have $g_\alpha(x_0) \geqq 0$. By a simple calculation, we get

$$-\frac{m^2 |\nabla Z_\alpha|^2}{x_1^2 + x_2^2} - \sum_{j=1}^{2} \frac{\partial}{\partial x_j}\left(\frac{m^2}{x_1^2 + x_2^2}\right) \frac{\partial Z_\alpha}{\partial x_j} Z_\alpha + \alpha(1 - 3Z_\alpha^2)|\nabla Z_\alpha|^2 \geqq 0 \quad \text{in} \quad F_\alpha$$

$$-\frac{1}{|\nabla Z_\alpha|} \sum_{j=1}^{2} \frac{\partial}{\partial x_j}\left(\frac{m^2}{x_1^2 + x_2^2}\right) \frac{\partial Z_\alpha}{\partial x_j} Z_\alpha + \alpha(1 - Z_\alpha^2)|\nabla Z_\alpha| \geqq 2\alpha Z_\alpha^2 |\nabla Z_\alpha| \quad \text{in} \quad F_\alpha,$$

$$2\alpha Z_\alpha^2|\nabla Z_\alpha| \leqq \left(\sum_{j=1}^{2}(\frac{\partial}{\partial x_j}(\frac{m^2}{x_1^2+x_2^2}))^2\right)^{1/2} + 2c|\nabla Z_\alpha|$$

For $\alpha \geqq 2d + 8c$,

$$\alpha|\nabla Z_\alpha| \leqq 4\left(\sum_{j=1}^{2}(\frac{\partial}{\partial x_j}(\frac{m^2}{x_1^2+x_2^2}))^2\right)^{1/2} + 2c|\nabla Z_\alpha| \quad \text{in} \quad F_\alpha. \tag{4.10}$$

(Case II) $F_\alpha \subset \partial\Omega$. From Lemma 11,

$$\partial|\nabla Z_\alpha|/\partial\nu \leqq 0 \quad \text{on} \quad \partial\Omega \cap G_\alpha \tag{4.11}$$

First we show that $g_\alpha(x) \geqq 0$ in F_α. If $g_\alpha(x_0) < 0$ for some $x_0 \in F_\alpha$, we see that $\Delta|\nabla Z_\alpha| > 0$ in some neighbood of x_0 and we have $\partial|\nabla Z_\alpha|/\partial\nu > 0$ at $x = x_0$ from the Hopf's boundary point lemma (cf. [21]). This is contrary to (4.11). We conclude that g_α is non-negative in F_α. By this fact we can show (4.10) as in the case (I).

This completes the proof of the first two asymptotic properties of (4.2). From these estimates, the $C^1(\overline{\Sigma})$ norm of the nonlinear term of (4.1) is bounded when $\alpha \to \infty$. Applying again the Schader estimate to (4.1), we get that $\|Z_\alpha\|_{C^{2+\gamma}(\overline{\Sigma})}$ is bounded when $\alpha \to \infty$ for any fixed $0 \leqq \gamma < 1$. This implies that ΔZ_α uniformly converges to 0 in Σ as $\alpha \to \infty$ and we obtain the last one of (4.2). This completes the proof of Proposition 8. □

Proof of Proposition 9

As is mentioned before, we can assume without loss of generality that m is a positive integer. Since (4.2) is a cooperation system, we can again use a comparison argument to construct a solution. However, in this turn, we have a difficulty that D is a unbounded domain and the coefficients of the equation of Y is singular on ∂D. We deal with such difficulty by considering an approximation problem by taking a bounded subdomain D_p where the coefficients are bounded and we can thereby get the desired solution in D by taking the limit $p \to \infty$. First we introduce some auxiliary comparison functions.

$$\begin{cases} W_1(r,z) = 1 - \dfrac{d_1}{\alpha}, \quad W_2(r,z) = 1, \\ Y_1(r,z) = d_2 r^2 e^{-\eta((r-a)^2+(z-b)^2)}, \quad Y_2'(r,z) = \dfrac{r^2}{1+r^2+z^2}, \\ Y_3'(r,z) = \dfrac{r}{(r^2+z^2)^s}, \quad Y_2(r,z) = \min(d_3Y_2'(r,z), d_3Y_3'(r,z), m) \end{cases} \tag{4.12}$$

where $d_1 > 0$, $d_2 > 0$, $d_3 > 0$ and $\eta > 0$ are positive constants and s is a constant such that $4s^2 - 2s - 1 < 0$, $1/2 < s$ (for instance $s = 3/4$) and (a, b) is an arbitrarily fixed point in Σ. It is easy to calculate and get

$$(4.13)\qquad \begin{cases} L_2Y_1(r,z) = d_2\eta r e^{-\eta((r-a)^2+(z-b)^2)}\left(4\eta r(r-a)^2 + 4\eta r(z-b)^2 - 10r + 6a\right) \\ L_2Y_2'(r,z) = -\dfrac{10r^2+2r^2z^2+2r^4}{(1+r^2+z^2)^3} \\ L_2Y_3'(r,z) = \dfrac{(4s^2-2s-1)r^2 - z^2}{r(r^2+z^2)^{s+1}} \end{cases}$$

Lemma 12. There are numbers $d_1 > 0$, $d_2 > 0$, $d_3 > 0$, $\eta > 0$, for which the following inequalities are valid if $\alpha > 0$ is large.

$$(4.14)\qquad 0 < W_1 \leqq W_2 \leqq 1 \quad \text{in} \quad \Sigma, \quad 0 < Y_1 \leqq Y_2 \leqq m \quad \text{in} \quad D,$$

$$(4.15)\qquad \begin{cases} L_1W_1 - \dfrac{1}{r^2}(m-Y_1)^2W_1 + \alpha W_1(1-W_1^2) \geqq 0 \quad \text{in} \quad \Sigma, \\ L_2Y_1 + (m-Y_1)W_1^2\Lambda_\Sigma \geqq 0 \quad \text{in} \quad D, \\ \dfrac{\partial W_1}{\partial \mathbf{n}} = 0 \quad \text{on} \quad \partial\Sigma, \quad Y_1 = 0 \quad \text{on} \quad \partial D, \end{cases}$$

$$(4.16)\qquad \begin{cases} L_1W_2 - \dfrac{1}{r^2}(m-Y_2)^2W_2 + \alpha W_2(1-W_2^2) \leqq 0 \quad \text{in} \quad \Sigma, \\ L_2Y_2 + (m-Y_2)W_2^2\Lambda_\Sigma \leqq 0 \quad \text{in} \quad D, \\ \dfrac{\partial W_2}{\partial \mathbf{n}} = 0 \quad \text{on} \quad \partial\Sigma, \quad Y_2 = 0 \quad \text{on} \quad \partial D, \end{cases}$$

We remark that Y_2 is not $C^2(D)$ and we take the differential inequality of Y_2 in (3.16) in the sense of distribution, that is,

$$(4.17)\qquad \int_D (Y_2L_2S + (m-Y_2)W_2^2\Lambda_\Sigma S)r\,drdz \leqq 0$$

for any $S = S(r,z) \in C^\infty(D)$ satisfying $S \geqq 0$ in D and $\operatorname{supp} S \subset D$.

(Proof of Lemma 12) By taking $\eta > 0$ large, we see
$\{(r,z) \in D \mid 4\eta r(r-a)^2 + 4\eta r(z-b)^2 - 10r + 6a < 0\} \subset \Sigma$.
We take $d_2 > 0$ so small such that $0 < Y_1 \leqq m$ in D. Then

$$L_1W_1 - \frac{1}{r^2}(m-Y_1)^2W_1 + \alpha W_1(1-W_1^2) \geqq W_1(d_1 - \frac{m^2}{r^2}) \geqq 0$$

for large fixed $d_1 > 0$. We can retake $d_2 > 0$ smaller such that the second inequality of (4.15) is valid. Next we prove (4.16). The first inequality is trivial. We see that $L_2 Y_2'(r,z)$ and $L_2 Y_3'(r,z)$ are negative in D. Taking $d_3 > 0$ large so that $Y_2 = m$ in Σ and $Y_1 \leqq Y_2$ in D. The first inequality in the sense of distribution is can be checked from the definition of Y_2. □

We approximate the domain D. Let $D_p = \{(r,z) \in D \mid 1/p < r,\ r^2 + z^2 < p^2\}$ where $p \in \mathbb{N}$ is a parameter. We consider the following boundary value problem of a elliptic system,

$$(4.18)\qquad \begin{cases} L_1 W - \dfrac{1}{r^2}(m-Y)^2 W + \alpha W(1-W^2) = 0 & \text{in} \quad \Sigma, \\ L_2 Y + (m-Y)W^2 \Lambda_\Sigma = 0 & \text{in} \quad D_p, \\ \dfrac{\partial W}{\partial \mathbf{n}} = 0 \quad \text{on} \quad \partial\Sigma, \quad Y = Y_1 & \text{on} \quad \partial D_p. \end{cases}$$

This is a cooporation system in the region $0 < W < 1$, $0 < Y < m$. If we have a lower solution and an upper solution, we can conclude that there exists a solution between them by using a standard theory (cf. [17], [18], [23]). In our case the situation is a little different from those dealt with in the above literature because the domains of definition of W and Y are defferent. However we can carry out a completely similar argument and get a solution (W,Y) such that $W_1 \leqq W \leqq W_2$ in Σ and $Y_1 \leqq Y \leqq Y_2$ in D_p. Thus we have the following approximation sequence of solution to (4.2).

Lemma 13. For large $p \in \mathbb{N}$, there exists a solution $(W^{(p)}, Y^{(p)}) \in C^{2+\gamma}(\overline{\Sigma}) \times C^{1+\gamma}(\overline{D}_p)$ to (4.18) such that

$$(4.19)\qquad \begin{cases} W_1(r,z) \leqq W^{(p)}(r,z) \leqq W_2(r,z) & \text{in} \quad \Sigma, \\ Y_1(r,z) \leqq Y^{(p)}(r,z) \leqq Y_2(r,z) & \text{in} \quad D_p, \end{cases}$$

where $0 \leqq \gamma < 1$ is an arbitrarily fixed constant.

The Schauder estimates of the elliptic equations together with (4.19) yields that for any large k the set of approximate solutions $\{(W^{(p)}, Y^{(p)})\}_{p \geqq k+1}$ are relatively compact in $C^2(\overline{\Sigma}) \times C^1(\overline{D}_k)$. Applying the diagonal argument, we get a convergent subsequence and consequently a solution $(W_\alpha, Y_\alpha) \in C^2(\overline{\Sigma}) \times C^1(D)$ to the equation (4.2) with the same estimate as (4.19), that is,

$$(4.20)\qquad \begin{cases} W_1(r,z) \leqq W_\alpha(r,z) \leqq W_2(r,z) & \text{in} \quad \Sigma, \\ Y_1(r,z) \leqq Y_\alpha(r,z) \leqq Y_2(r,z) & \text{in} \quad D, \end{cases}$$

for large $\alpha > 0$. The former estimate in (4.20) implies the first properties of (4.4). We will prove the last one of (4.4). We remark that the nonlinear term of (4.2) is bounded uniformly in large α from the estimate proved just now. Using the Schauder estimate of the elliptic equations, we have a constant $c > 0$ such that

$$(4.21)\qquad |\nabla W_\alpha| + |\nabla Y_\alpha| \leqq c \quad \text{in} \quad \overline{\Omega} \quad (\text{large } \alpha).$$

We can carry out quite a similar argument as in the proof of Proposition 8. Return to the original orthonormal coordinate (x_1, x_2, x_3) and denote $W_\alpha(x) = W_\alpha((x_1^2 + x_2^2)^{1/2}, x_3)$. The former equation of (4.2) becomes

$$\text{(4.22)} \qquad \begin{cases} \Delta W_\alpha - \dfrac{1}{x_1^2 + x_2^2}(m - Y_\alpha)^2 W_\alpha + \alpha W_\alpha(1 - W_\alpha^2) = 0 \quad \text{in} \quad \Omega, \\ \dfrac{\partial W_\alpha}{\partial \nu} = 0 \quad \text{on} \quad \partial\Omega. \end{cases}$$

Applying grad W_α to the above equation and using Lemma 10 and Lemma 11, we get

$$\text{(4.23)} \qquad \begin{cases} |\nabla W_\alpha| \Delta |\nabla W_\alpha| + g_\alpha(x) \geqq 0 \quad \text{in} \quad \Omega \cap G_\alpha, \\ |\nabla W_\alpha| \dfrac{\partial}{\partial \nu} |\nabla W_\alpha| \leqq 0 \quad \text{on} \quad \partial\Omega \cap G_\alpha. \end{cases}$$

where $G_\alpha = \{x \in \overline{\Omega} \mid |\nabla W_\alpha| > 0\}$ and

$$g_\alpha(x) = -\frac{1}{x_1^2 + x_2^2}(m - Y_\alpha)^2 |\nabla W_\alpha|^2 - \langle \nabla W_\alpha \nabla \left(\frac{(m - Y_\alpha)^2}{x_1^2 + x_2^2} \right) \rangle W_\alpha$$
$$+ \alpha |\nabla W_\alpha|^2 (1 - 3W_\alpha^2).$$

Consider the set

$$F_\alpha = \{x \in \Omega \mid 0 < |\nabla W_\alpha(x)| = \max_{\overline{\Omega}} |\nabla W_\alpha|\}$$

and apply the same argument used to get the differential inequalities (4.23), we have $g_\alpha(x) \geqq 0$ on F_α for large $\alpha > 0$. Thus we obtain a uniform bound for $\alpha |\nabla W_\alpha|$ in Ω. From this estimate, $C^1(\overline{\Omega})$ norm of the nonlinear term of (4.22) when $\alpha \to \infty$ and again from the Schauder estimate, $\{W_\alpha\}_\alpha$ is bounded in $C^{2+\gamma}$ for $0 \leqq \gamma < 1$. Therefore ΔW_α uniformly converges to 0 in Ω as $\alpha \to \infty$ and we obtain the desired convergence in the last property of (4.4). □

§5 Stability of Φ_α in Theorem 7

In this section, we will complete the proof of Theorem 7. We will prove the stability of Φ_α which was constructed through (4.1) by Proposition 8,

$$\text{(5.1)} \qquad \Phi_\alpha(x) = Z_\alpha(r, z) e^{im\theta}.$$

To prove the stability of Φ_α in (1.6) for large $\alpha > 0$, we will show the positivity of $\mathcal{L}_0$ in $N_0(\Phi_\alpha)$. For this purpose we consider the linearized eigenvalue problem of (1.6). Hereafter we will argue in terms of real functions. Let u_α and v_α be the the real and the imaginary part of Φ_α, i.e. $u_\alpha(x) = Z_\alpha(r, z) \cos m\theta$, $v_\alpha(x) = Z_\alpha(r, z) \sin m\theta$. Thus we consider the following eigenvalue problem,

$$\text{(5.2)} \qquad \begin{cases} \Delta \begin{pmatrix} \phi \\ \psi \end{pmatrix} + \alpha \begin{pmatrix} 1 - 3u_\alpha^2 - v_\alpha^2 & -2u_\alpha v_\alpha \\ -2u_\alpha v_\alpha & 1 - u_\alpha^2 - 3v_\alpha^2 \end{pmatrix} \begin{pmatrix} \phi \\ \psi \end{pmatrix} \\ \qquad + \mu \begin{pmatrix} \phi \\ \psi \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \text{in } \Omega, \\ \dfrac{\partial \phi}{\partial \nu} = \dfrac{\partial \psi}{\partial \nu} = 0 \quad \text{on} \quad \partial\Omega, \end{cases}$$

where

$$\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2} = \frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial}{\partial r}) + \frac{\partial^2}{\partial z^2} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}$$

and ϕ, ψ are real valued functions in Ω. One can show that (5.2) is a self-adjoint eigenvalue problem with the real (countable) eigenvalues $\{\mu_\ell(\alpha)\}_{\ell=1}^{\infty}$ which are arranged in increasing order (counting multiplicity). It is easy to see that the set of eigenvalues contain 0 because $(\phi,\psi) = (-v_\alpha, u_\alpha)$ satisfies (5.11) with $\mu = 0$. This is due to the invariance (1.7). We will prove that $\mu_1(\alpha) = 0$ and $\mu_2(\alpha) > 0$ is bounded away from 0 when $\alpha > 0$ is large. We change the variable as follows

$$\begin{pmatrix} \widehat{\phi}(r,\theta,z) \\ \widehat{\psi}(r,\theta,z) \end{pmatrix} = R(-m\theta)\begin{pmatrix} \phi(r,\theta,z) \\ \psi(r,\theta,z) \end{pmatrix} \text{ where } R(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}. \tag{5.3}$$

The eigenvalue problem is written in tems of $\widehat{\phi}$, $\widehat{\psi}$ as follows.

$$\begin{cases} \Delta\begin{pmatrix} \widehat{\phi} \\ \widehat{\psi} \end{pmatrix} - \frac{2m}{r^2}\begin{pmatrix} \partial\widehat{\psi}/\partial\theta \\ -\partial\widehat{\phi}/\partial\theta \end{pmatrix} + (\alpha(1-Z_\alpha^2) - \frac{m^2}{r^2})\begin{pmatrix} \widehat{\phi} \\ \widehat{\psi} \end{pmatrix} \\ \qquad\qquad -2\alpha Z_\alpha^2\begin{pmatrix} \widehat{\phi} \\ 0 \end{pmatrix} + \mu\begin{pmatrix} \widehat{\phi} \\ \widehat{\psi} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{ in } \Omega, \\ \frac{\partial\widehat{\phi}}{\partial\nu} = \frac{\partial\widehat{\psi}}{\partial\nu} = 0 \quad \text{on} \quad \partial\Omega, \end{cases} \tag{5.4}$$

We express $\begin{pmatrix} \widehat{\phi} \\ \widehat{\psi} \end{pmatrix}$ in the form of the Fourier expansion as follows,

$$\begin{pmatrix} \widehat{\phi}(r,\theta,z) \\ \widehat{\psi}(r,\theta,z) \end{pmatrix} = \frac{1}{\sqrt{2}}\xi_0(r,z) + \sum_{k=1}^{\infty}(\xi_k(r,z)\cos k\theta + \zeta_k(r,z)\sin k\theta) \tag{5.5}$$

where the vector valued functions

$$\xi_k(r,z) = \begin{pmatrix} \xi_{k,1}(r,z) \\ \xi_{k,2}(r,z) \end{pmatrix}, \quad \zeta_k(r,z) = \begin{pmatrix} \zeta_{k,1}(r,z) \\ \zeta_{k,2}(r,z) \end{pmatrix} \tag{5.6}$$

are defined in Σ. Substitute (5.5) into (5.4), we can decompose the eigenvalue problem into a series of infinitely many elliptic eigenvalue problems.

$$\begin{cases} L_1\xi_0 + (\alpha(1-Z_\alpha^2) - \frac{m^2}{r^2})\xi_0 - 2\alpha Z_\alpha^2\begin{pmatrix} \xi_{0,1} \\ 0 \end{pmatrix} + \mu\xi_0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{ in } \Sigma, \\ \frac{\partial}{\partial n}\xi_0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \text{on} \quad \partial\Sigma. \end{cases} \tag{5.7}$$

$$
(5.8)\quad \begin{cases}
L_1\xi_k - \dfrac{k^2}{r^2}\xi_k - \dfrac{2mk}{r^2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\zeta_k + (\alpha(1-Z_\alpha^2) - \dfrac{m^2}{r^2})\xi_k \\
\qquad\qquad - 2\alpha Z_\alpha^2 \begin{pmatrix} \xi_{k,1} \\ 0 \end{pmatrix} + \mu\xi_k = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{ in } \Sigma, \\
L_1\zeta_k - \dfrac{k^2}{r^2}\zeta_k + \dfrac{2mk}{r^2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\xi_k + (\alpha(1-Z_\alpha^2) - \dfrac{m^2}{r^2})\zeta_k \\
\qquad\qquad - 2\alpha Z_\alpha^2 \begin{pmatrix} \zeta_{k,1} \\ 0 \end{pmatrix} + \mu\zeta_k = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \text{in } \Sigma, \\
\dfrac{\partial}{\partial \mathbf{n}}\xi_k = \dfrac{\partial}{\partial \mathbf{n}}\zeta_k = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \text{on} \quad \partial\Sigma \quad (k \geqq 1).
\end{cases}
$$

(5.8) are rewritten as the sequences of the following 2×2 systems of elliptic eigenvalue problems.

$$
(5.9)\quad \begin{cases}
L_1\begin{pmatrix} \xi_{k,1} \\ \zeta_{k,2} \end{pmatrix} - \dfrac{k^2}{r^2}\begin{pmatrix} \xi_{k,1} \\ \zeta_{k,2} \end{pmatrix} - \dfrac{2mk}{r^2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \xi_{k,1} \\ \zeta_{k,2} \end{pmatrix} \\
+(\alpha(1-Z_\alpha^2) - \dfrac{m^2}{r^2})\begin{pmatrix} \xi_{k,1} \\ \zeta_{k,2} \end{pmatrix} - 2\alpha Z_\alpha^2\begin{pmatrix} \xi_{k,1} \\ 0 \end{pmatrix} + \mu\begin{pmatrix} \xi_{k,1} \\ \zeta_{k,2} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{ in } \Sigma, \\
\dfrac{\partial}{\partial \mathbf{n}}\begin{pmatrix} \xi_{k,1} \\ \zeta_{k,2} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \text{on} \quad \partial\Sigma, \qquad (k \geqq 1).
\end{cases}
$$

$$
(5.10)\quad \begin{cases}
L_1\begin{pmatrix} \zeta_{k,1} \\ -\xi_{k,2} \end{pmatrix} - \dfrac{k^2}{r^2}\begin{pmatrix} \zeta_{k,1} \\ -\xi_{k,2} \end{pmatrix} - \dfrac{2mk}{r^2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \zeta_{k,1} \\ -\xi_{k,2} \end{pmatrix} \\
+(\alpha(1-Z_\alpha^2) - \dfrac{m^2}{r^2})\begin{pmatrix} \zeta_{k,1} \\ -\xi_{k,2} \end{pmatrix} - 2\alpha Z_\alpha^2\begin{pmatrix} \zeta_{k,1} \\ 0 \end{pmatrix} + \mu\begin{pmatrix} \zeta_{k,1} \\ -\xi_{k,2} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{ in } \Sigma, \\
\dfrac{\partial}{\partial \mathbf{n}}\begin{pmatrix} \zeta_{k,1} \\ -\xi_{k,2} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \text{on} \quad \partial\Sigma, \qquad (k \geqq 1).
\end{cases}
$$

We see that (5.9) and (5.10) are identical as an eigenvalue problem. We deal with at once in the following form.

$$
(5.11)\quad \begin{cases}
L_1\begin{pmatrix} \tau \\ \sigma \end{pmatrix} - \dfrac{k^2}{r^2}\begin{pmatrix} \tau \\ \sigma \end{pmatrix} - \dfrac{2mk}{r^2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \tau \\ \sigma \end{pmatrix} \\
+(\alpha(1-Z_\alpha^2) - \dfrac{m^2}{r^2})\begin{pmatrix} \tau \\ \sigma \end{pmatrix} - 2\alpha Z_\alpha^2\begin{pmatrix} \tau \\ 0 \end{pmatrix} + \mu\begin{pmatrix} \tau \\ \sigma \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \text{in } \Sigma, \\
\dfrac{\partial}{\partial \mathbf{n}}\begin{pmatrix} \tau \\ \sigma \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \text{on} \quad \partial\Sigma.
\end{cases}
$$

where σ and τ are real valued functions in Σ. We can regard (5.7) as the case $k = 0$ in (5.11). We consider (5.11) for $k \geqq 0$. Let

$$
(5.12)\qquad \{\mu_\ell^{(k)}(\alpha)\}_{\ell=1}^\infty \quad \text{and} \quad \left\{\begin{pmatrix} \tau_{\ell,\alpha}^{(k)} \\ \sigma_{\ell,\alpha}^{(k)} \end{pmatrix}\right\}_{\ell=1}^\infty \subset L^2(\Omega)\times L^2(\Omega)
$$

be the set of the eigenvalues arranged in increasing order (counting multiplicity) and the complete system of the corresponding eigenfunctions orthonormalized in $L^2(\Sigma) = L^2(\Sigma; rdrdz)$ to (5.11). Thus we obtain

$$\frac{1}{\sqrt{2\pi}}\begin{pmatrix}\tau_{\ell,\alpha}^{(0)}(r,z)\\ \sigma_{\ell,\alpha}^{(0)}(r,z)\end{pmatrix}, \quad \frac{1}{\sqrt{2\pi}}\begin{pmatrix}\tau_{\ell,\alpha}^{(k)}(r,z)\cos k\theta\\ \sigma_{\ell,\alpha}^{(k)}(r,z)\sin k\theta\end{pmatrix}, \quad \frac{1}{\sqrt{2\pi}}\begin{pmatrix}\tau_{\ell,\alpha}^{(k)}(r,z)\sin k\theta\\ -\sigma_{\ell,\alpha}^{(k)}(r,z)\cos k\theta\end{pmatrix} \quad (k \geqq 1,\, \ell \geqq 1)$$

which form a complete orthonormal system of eigenfunctions of (5.4) in $L^2(\Omega) \times L^2(\Omega)$.

We need to study the asymptotic property of these eigenvalues and eigenfunctions when $\alpha \to \infty$.

Lemma 14. For each non-negative integer k, we have

$$\lim_{\alpha\to\infty} \mu_\ell^{(k)}(\alpha) = \mu_\ell^{(k)} \quad (k \geqq 0,\, \ell \geqq 1). \tag{5.13}$$

$$\lim_{\alpha\to\infty} \left(\|\nabla\tau_{\ell,\alpha}^{(k)}(\alpha)\|^2_{L^2(\Sigma)} + \alpha\|\tau_{\ell,\alpha}^{(k)}(\alpha)\|^2_{L^2(\Sigma)}\right) = 0. \tag{5.14}$$

where $\{\mu_\ell^{(k)}\}_{\ell=1}^\infty$ is the set of the eigenvalues arranged in increasing order (counting multiplicity) of the following eigenvalue problem

$$\begin{cases} L_1\sigma - \dfrac{k^2}{r^2}\sigma + \mu\sigma = 0 & \text{in} \quad \Sigma, \\ \dfrac{\partial\sigma}{\partial \mathbf{n}} = 0 & \text{on} \quad \partial\Sigma. \end{cases} \tag{5.15}$$

(Proof of Lemma 14.) We prove this lemma by using the variational characterization of the eigenvalues. Let $\{\sigma_\ell^{(k)}\}_{\ell=1}^\infty$ be the orthonormal system of eigenfunctions of (5.15) corresponding to $\{\mu_\ell^{(k)}\}_{\ell=1}^\infty$, i.e., $\int_\Sigma \sigma_i^{(k)}\sigma_j^{(k)} rdrdz = \delta_{i,j}$ for $i,j \geqq 1$. For given k, we prove (5.13) and (5.14). For simplicity of notation we drop the sufix k and denote $\mu_\ell^{(k)}$, $\mu_\ell^{(k)}(\alpha)$, $\tau_{\ell,\alpha}^{(k)}$, $\sigma_{\ell,\alpha}^{(k)}$, $\sigma_\ell^{(k)}$ by μ_ℓ, $\mu_\ell(\alpha)$, $\tau_{\ell,\alpha}$, $\sigma_{\ell,\alpha}$, σ_ℓ, respectively. We prove that for any sequence $\{\alpha_j\}_{j=1}^\infty$ which goes to ∞ as $j \to \infty$, there exists a subsequence on which the limits in (5.13) and (5.14) hold for any ℓ. From (5.11) and the variational characterization of the eigenvalue of the selfadjoint operator (cf. [22]), we have

$$\mu_1(\alpha) = \inf\{J_\alpha(\tau,\sigma) \mid \sigma,\tau \in H^1(\Sigma), \int_\Sigma (\sigma^2+\tau^2) rdrdz = 1\} \tag{5.16}$$

where

$$J_\alpha(\tau,\sigma) = \int_\Sigma \Bigg(\left(\frac{\partial \sigma}{\partial r}\right)^2 + \left(\frac{\partial \sigma}{\partial z}\right)^2 + \left(\frac{\partial \tau}{\partial r}\right)^2 + \left(\frac{\partial \tau}{\partial z}\right)^2 + \frac{k^2}{r^2}(\sigma^2+\tau^2) + \frac{4mk}{r^2}\sigma\tau$$
$$+(\alpha(1-Z_\alpha^2) - \frac{m^2}{r^2})(\sigma^2+\tau^2) + 2\alpha Z_\alpha^2 \tau^2 \Bigg) r dr dz$$

Using test function $(\tau,\sigma) = (0,\sigma_1)$, we have $\mu_1(\alpha) \leqq J(0,\sigma_1)$,

$$\limsup_{\alpha\to\infty} \mu_1(\alpha) \leqq \mu_1. \tag{5.17}$$

On the other hand the eigenfunction $(\tau_{1,\alpha}, \sigma_{1,\alpha})$ satisfies

$$\begin{aligned} &\int_\Sigma \Bigg(\left(\frac{\partial \tau_{1,\alpha}}{\partial r}\right)^2 + \left(\frac{\partial \tau_{1,\alpha}}{\partial z}\right)^2 + \frac{k^2}{r^2}\tau_{1,\alpha}^2 + \frac{2mk}{r^2}\sigma_{1,\alpha}\tau_{1,\alpha} \\ &+ (\alpha(1-Z_\alpha^2) - \frac{m^2}{r^2})\tau_{1,\alpha}^2 + 2\alpha Z_\alpha^2 \tau_{1,\alpha}^2 \Bigg) r dr dz = \mu_1(\alpha) \|\tau_{1,\alpha}\|^2_{L^2(\Sigma)}, \end{aligned} \tag{5.18}$$

$$\begin{aligned} &\int_\Sigma \Bigg(\left(\frac{\partial \sigma_{1,\alpha}}{\partial r}\right)^2 + \left(\frac{\partial \sigma_{1,\alpha}}{\partial z}\right)^2 + \frac{k^2}{r^2}\sigma_{1,\alpha}^2 + \frac{2mk}{r^2}\sigma_{1,\alpha}\tau_{1,\alpha} \\ &+ (\alpha(1-Z_\alpha^2) - \frac{m^2}{r^2})\sigma_{1,\alpha}^2 \Bigg) r dr dz = \mu_1(\alpha) \|\sigma_{1,\alpha}\|^2_{L^2(\Sigma)}. \end{aligned} \tag{5.19}$$

From Proposition 8 and the boundedness of $\mu_1(\alpha)$ (when $\alpha \to \infty$), it follows that

$$\|\tau_{1,\alpha}\|^2_{L^2(\Sigma)} \sim O(1/\alpha) \qquad (\alpha \to \infty).$$

Consdiering this fact in (5.18), we obtain

$$\alpha \|\tau_{1,\alpha}\|^2_{L^2(\Sigma)} \to 0, \quad \|\nabla \tau_{1,\alpha}\|^2_{L^2(\Sigma)} \to 0 \quad \text{as} \quad \alpha \to \infty,$$

and

$$\lim_{\alpha\to\infty} \|\sigma_{1,\alpha}\|^2_{L^2(\Sigma)} = 1.$$

In view of (5.19), $\sigma_{1,\alpha}$ is bounded in $H^1(\Sigma)$ and also relatively compact in the weak topology. Thus there exists a subsequence $\{\eta_j\}_j$ of $\{\alpha_j\}_j$ such that σ_{1,η_j} weakly converges to a certain $\widehat{\sigma}_1 \in H^1(\Sigma)$ and $\mu_1(\eta_j)$ converges to a μ' ($\leqq \mu_1$). From the upper semicontinuity of the weak convergence in $H^1(\Sigma)$, we see

$$\|\widehat{\sigma}_1\|_{L^2(\Sigma)} = 1, \quad \liminf_{j\to\infty} \|\nabla \sigma_{1,\eta_j}\|^2_{L^2(\Sigma)} \geqq \|\nabla \widehat{\sigma}_1\|^2_{L^2(\Sigma)}. \tag{5.20}$$

Taking the limit $j \to \infty$ in the second component of (5.11) for $\alpha = \eta_j$ and the regularity argument of the solution in the weak formulation yield that $\widehat{\sigma}_1$ belongs to $C^2(\overline{\Sigma})$ and satisfies the Nemann boundary condition on $\partial\Sigma$.

$$
(5.21) \qquad \begin{cases} L_1\widehat{\sigma}_1 - \dfrac{k^2}{r^2}\widehat{\sigma}_1 + \mu'\widehat{\sigma}_1 = 0 \quad \text{in} \quad \Sigma, \\ \dfrac{\partial\widehat{\sigma}_1}{\partial \mathrm{n}} = 0 \quad \text{on} \quad \partial\Sigma. \end{cases}
$$

This implies that μ' is an eigenvalue of (5.15) and $\mu' = \mu_1$ by (5.17). $\lim_{j\to\infty} \mu(\eta_j) = \mu_1$ is also true. Moreover we see from (5.19) that

$$
\lim_{j\to\infty} \|\nabla\sigma_{1,\eta_j}\|^2_{L^2(\Sigma)} \geqq \|\nabla\widehat{\sigma}_1\|^2_{L^2(\Sigma)}.
$$

This concludes the first step of the induction. Next take an element $\widetilde{\sigma} \in$ L.h.$[\sigma_1, \sigma_2]$ such that $(\widetilde{\sigma}\cdot\widehat{\sigma}_1)_{L^2(\Sigma)} = 0$ and $\|\widetilde{\sigma}\|_{L^2(\Sigma)} = 1$, where L.h.$[G]$ is the subspace generated by the set G. Recall

$$
\mu_2(\alpha) = \inf\{J_\alpha(\tau,\sigma) \mid \|\sigma\|^2_{L^2(\Sigma)} + \|\tau\|^2_{L^2(\Sigma)} = 1,\ (\sigma\sigma_{1,\alpha})_{L^2(\Sigma)} + (\tau\tau_{1,\alpha})_{L^2(\Sigma)} = 0\}.
$$

Taking the test function $(\tau,\sigma) = (0,\widetilde{\sigma})$, we have

$$
(5.22) \qquad \mu_2(\alpha) \leqq \frac{J_\alpha(-(\widetilde{\sigma}\cdot\sigma_{1,\alpha})_{L^2(\Sigma)}\tau_{1,\alpha}, \widetilde{\sigma} - (\widetilde{\sigma}\cdot\sigma_{1,\alpha})_{L^2(\Sigma)}\sigma_{1,\alpha})}{\|-(\widetilde{\sigma}\cdot\sigma_{1,\alpha})_{L^2(\Sigma)}\tau_{1,\alpha}\|^2_{L^2(\Sigma)} + \|\widetilde{\sigma} - (\widetilde{\sigma}\cdot\sigma_{1,\alpha})_{L^2(\Sigma)}\sigma_{1,\alpha}\|^2_{L^2(\Sigma)}}
$$

By the result obtained in the first step and taking the sequence $\alpha = \eta_j$ $(j = 1,2,3,...)$, we get by a direct calculation

$$
\limsup_{j\to\infty} \mu_2(\eta_j) \leqq \mu_2.
$$

The same arugument works in the equalities like (5.18) and (5.19) for $(\tau_{2,\eta_j}, \sigma_{2,\eta_j})$ to show that

$$
\eta_j\|\tau_{2,\eta_j}\|^2_{L^2(\Sigma)} \to 0, \quad \|\nabla\tau_{2,\eta_j}\|^2_{L^2(\Sigma)} \to 0 \quad \|\sigma_{2,\eta_j}\|^2_{L^2(\Sigma)} \to 1 \quad \text{as} \quad j\to\infty,
$$

and that $\{\sigma_{2,\eta_j}\}_j$ is bounded in $H^1(\Sigma)$. There exist a subsequence $\{\kappa_j\}_{j=1}^\infty \subset \{\eta_j\}_{j=1}^\infty$, $\mu'' \leqq \mu_2$ and $\widehat{\sigma}_2 \in H^1(\Sigma)$ such that

$$
(5.23) \qquad \|\widehat{\sigma}_2\|_{L^2(\Sigma)} = 1, \quad (\widehat{\sigma}_2, \widehat{\sigma}_1)_{L^2(\Sigma)} = 0.
$$

$$
(5.24) \qquad \begin{cases} L_1\widehat{\sigma}_2 - \dfrac{k^2}{r^2}\widehat{\sigma}_2 + \mu''\widehat{\sigma}_2 = 0 \quad \text{in} \quad \Sigma, \\ \dfrac{\partial\widehat{\sigma}_2}{\partial \mathrm{n}} = 0 \quad \text{on} \quad \partial\Sigma. \end{cases}
$$

By this we conclude that $\mu'' = \mu_2^{(2)}$ and

$$\lim_{j\to\infty} \|\nabla \sigma_{1,\kappa_j}\|^2_{L^2(\Sigma)} = \|\nabla \widehat{\sigma}_2\|^2_{L^2(\Sigma)}.$$

We can apply the similar argument inductively and as a consequense, for any ℓ, there exists a subsequence $\{\alpha_{\ell,j}\}_{j=1}^\infty \subset \{\alpha_j\}_{j=1}^\infty$ such that

$$\lim_{j\to\infty} \mu_\ell(\alpha_{\ell,j}) = \mu_\ell. \tag{5.25}$$

$$\lim_{j\to\infty} \left(\|\nabla \tau_{\ell,\alpha_{\ell,j}}\|^2_{L^2(\Sigma)} + \alpha_{\ell,j}\|\tau_{\ell,\alpha_{\ell,j}}\|^2_{L^2(\Sigma)} \right) = 0. \tag{5.26}$$

From the arbitrariness of the sequence $\{\alpha_j\}_{j=1}^\infty$ (which goes to ∞) it follows that (5.13) and (5.14) hold. □

Now we in a position to prove the stability of Φ_α.

Proof of Theorem 7

From (5.15) it is easy to see that

$$\mu_1^{(k)} \geqq \frac{k^2}{r_0^2} \quad (k \geqq 0),$$

where $r_0 = \inf\{r \mid (r,z) \in \Sigma\} > 0$, and $\mu_2^{(0)} > 0$. Hence, by Lemma 14, all $\mu_{\ell,\alpha}^{(k)}$ except for $\mu_{1,\alpha}^{(0)}$ are positive and bounded away from 0 when $\alpha \to \infty$. We see that $\mu_{1,\alpha}^{(0)} = 0$ because we can take $(\tau, \sigma) = (0, Z_\alpha)$ in (5.11) with $\mu = 0$ and $k = 0$. Then there exists constants $\delta_0 > 0$ and $\alpha_* > 0$ such that

$$\mathcal{L}_0(\phi, \psi) \geqq \delta_0 \left(\|\phi\|^2_{L^2(\Omega)} + \|\psi\|^2_{L^2(\Omega)} \right)$$

for $(\phi, \psi) \in H^1(\Omega)$ such that $\int_\Omega (\phi v_\alpha - \psi u_\alpha)dx = 0$ and $\alpha > \alpha_*$. This completes the proof of Theorem 7. □

§6. Stability of (Φ_α, A_α) in Theorem 6

In this section we will prove the stability of (Φ_α, A_α), which we constructed in §4. Recall

$$\begin{cases} A_\alpha(x) = Y_\alpha(r,z)\left(\dfrac{-\sin\theta}{r}, \dfrac{\cos\theta}{r}, 0 \right), \\ \Phi_\alpha(x) = W_\alpha(r,z)e^{im\theta}. \end{cases} \tag{6.1}$$

As is the case in §5, we express Φ_α in terms of real valued functions, i.e., we put $u_\alpha(x) = W_\alpha(r,z)\cos m\theta$, $v_\alpha(x) = W_\alpha(r,z)\sin m\theta$. First we estimate the second variation $\mathcal{L}(\phi, \psi, B)$ on $\overline{N}(\Phi_\alpha, A_\alpha) = \overline{N}(u_\alpha, v_\alpha, A_\alpha)$ from below. We change the variables ϕ, ψ into $\widehat{\phi}, \widehat{\psi}$ by

$$\begin{pmatrix} \widehat{\phi} \\ \widehat{\psi} \end{pmatrix} = R(-m\theta) \begin{pmatrix} \phi \\ \psi \end{pmatrix} \quad (cf.\ (5.3)).$$

Using the formula of the second variation of $\mathcal{H}$ (cf. (2.8)), we can express $\mathcal{L}(u_\alpha, v_\alpha, A_\alpha, \phi, \psi, B)$ in terms of $\widehat{\phi}$, $\widehat{\psi}$ for the solution $(\Phi_\alpha, A_\alpha) = (u_\alpha, v_\alpha, A_\alpha)$ constructed in §4 (cf. Proposition 9). If $\operatorname{div} B = 0$ and $\langle B \cdot \nu \rangle = 0$ on $\partial\Omega$ (this is valid for $(\phi, \psi, B) \in \overline{N}(u_\alpha, v_\alpha, A_\alpha)$), we have

$$\mathcal{L}(\phi, \psi, B) = I_1(\widehat{\phi}, \widehat{\psi}) + I_2(B) + I_3(\widehat{\phi}, \widehat{\psi}, B)$$

$$I_1(\widehat{\phi}, \widehat{\psi}) = \int_\Omega \left((\frac{\partial \widehat{\phi}}{\partial r})^2 + (\frac{\partial \widehat{\phi}}{\partial z})^2 + (\frac{\partial \widehat{\psi}}{\partial r})^2 + (\frac{\partial \widehat{\psi}}{\partial z})^2 - \alpha(1 - W_\alpha^2)(\widehat{\phi}^2 + \widehat{\psi}^2) + 2\alpha W_\alpha^2 \widehat{\phi}^2 \right) dx$$

$$+ \int_\Omega \left(\frac{(Y_\alpha - m)^2}{r^2}(\widehat{\phi}^2 + \widehat{\psi}^2) + \frac{1}{r^2}((\frac{\partial \widehat{\phi}}{\partial \theta})^2 + (\frac{\partial \widehat{\psi}}{\partial \theta})^2) + \frac{2(Y_\alpha - m)}{r^2}(\widehat{\psi}\frac{\partial \widehat{\phi}}{\partial \theta} - \widehat{\phi}\frac{\partial \widehat{\psi}}{\partial \theta}) \right) dx$$

$$I_2(B) = \int_{\mathbb{R}^3} |\operatorname{rot} B|^2 dx + \int_\Omega W_\alpha^2 B^2 dx$$

$$I_3(\widehat{\phi}, \widehat{\psi}, B) = 4 \int_\Omega \left(\frac{(Y_\alpha - m) S_1 W_\alpha \widehat{\phi}}{r^2} + S_2 \frac{\partial W}{\partial r} \widehat{\psi} + S_3 \frac{\partial W}{\partial z} \widehat{\psi} \right) dx$$

where we put

$$B = (-S_1 \frac{\sin\theta}{r} + S_2 \cos\theta, S_1 \frac{\cos\theta}{r} + S_2 \sin\theta, S_3).$$

To investigate the coerciveness of I_1, we consider the eigenvalue problem.

$$\text{(6.2)} \qquad \begin{cases} \Delta \begin{pmatrix} \widehat{\phi} \\ \widehat{\psi} \end{pmatrix} + \frac{2h}{r^2}(Y_\alpha - m) \begin{pmatrix} \partial\widehat{\psi}/\partial\theta \\ -\partial\widehat{\phi}/\partial\theta \end{pmatrix} - \frac{1}{r^2}(Y_\alpha - m)^2 \begin{pmatrix} \widehat{\phi} \\ \widehat{\psi} \end{pmatrix} \\ + \alpha(1 - W_\alpha^2) \begin{pmatrix} \widehat{\phi} \\ \widehat{\psi} \end{pmatrix} - 2\alpha W_\alpha^2 \begin{pmatrix} \widehat{\phi} \\ 0 \end{pmatrix} + \mu \begin{pmatrix} \widehat{\phi} \\ \widehat{\psi} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \text{in } \Omega, \\ \frac{\partial \widehat{\phi}}{\partial \nu} = \frac{\partial \widehat{\psi}}{\partial \nu} = 0 \quad \text{on} \quad \partial\Omega. \end{cases}$$

As in the proof of Theorem 7, we express $\begin{pmatrix} \widehat{\phi} \\ \widehat{\psi} \end{pmatrix}$ in the Fourier expansion as follows,

$$\text{(6.3)} \qquad \begin{pmatrix} \widehat{\phi}(r,\theta,z) \\ \widehat{\psi}(r,\theta,z) \end{pmatrix} = \frac{1}{\sqrt{2}} \xi_0(r,z) + \sum_{k=1}^\infty (\xi_k(r,z)\cos k\theta + \zeta_k(r,z)\sin k\theta)$$

where the real vector functions:

$$\text{(6.4)} \qquad \xi_k(r,z) = \begin{pmatrix} \xi_{k,1}(r,z) \\ \xi_{k,2}(r,z) \end{pmatrix} \quad (k \geqq 0), \quad \zeta_k(r,z) = \begin{pmatrix} \zeta_{k,1}(r,z) \\ \zeta_{k,2}(r,z) \end{pmatrix} \quad (k \geqq 1)$$

satisfy

$$
(6.5)\qquad \begin{cases} L_1\begin{pmatrix}\xi_{k,1}\\ \zeta_{k,2}\end{pmatrix}-\dfrac{k^2}{r^2}\begin{pmatrix}\xi_{k,1}\\ \zeta_{k,2}\end{pmatrix}+(\alpha(1-W_\alpha^2)-\dfrac{1}{r^2}(Y_\alpha-m)^2)\begin{pmatrix}\xi_{k,1}\\ \zeta_{k,2}\end{pmatrix} \\ -\dfrac{2k}{r^2}(Y_\alpha-m)F\begin{pmatrix}\xi_{k,1}\\ \zeta_{k,2}\end{pmatrix}-2\alpha W_\alpha^2\begin{pmatrix}\xi_{k,1}\\ 0\end{pmatrix}+\mu\begin{pmatrix}\xi_{k,1}\\ \zeta_{k,2}\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix} \text{ in } \Sigma \\ \dfrac{\partial}{\partial n}\begin{pmatrix}\xi_{k,1}\\ \zeta_{k,2}\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix} \quad \text{on} \quad \partial\Sigma. \end{cases}
$$

and

$$
(6.6)\qquad \begin{cases} L_1\begin{pmatrix}\zeta_{k,1}\\ -\xi_{k,2}\end{pmatrix}-\dfrac{k^2}{r^2}\begin{pmatrix}\zeta_{k,1}\\ -\xi_{k,2}\end{pmatrix}+(\alpha(1-W_\alpha^2)-\dfrac{1}{r^2}(Y_\alpha-m)^2)\begin{pmatrix}\zeta_{k,1}\\ -\xi_{k,2}\end{pmatrix} \\ -\dfrac{2k}{r^2}(Y_\alpha-m)F\begin{pmatrix}\zeta_{k,1}\\ -\xi_{k,2}\end{pmatrix}-2\alpha W_\alpha^2\begin{pmatrix}\zeta_{k,1}\\ 0\end{pmatrix}+\mu\begin{pmatrix}\zeta_{k,1}\\ -\xi_{k,2}\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix} \text{ in } \Sigma \\ \dfrac{\partial}{\partial n}\begin{pmatrix}\zeta_{k,1}\\ -\xi_{k,2}\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix} \quad \text{on} \quad \partial\Sigma. \end{cases}
$$

respectively, where $F=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$. Both of (6.5) and (6.6) are the same eigenvalue problem:

$$
(6.7)\qquad \begin{cases} L_1\begin{pmatrix}\tau\\ \sigma\end{pmatrix}-\dfrac{k^2}{r^2}\begin{pmatrix}\tau\\ \sigma\end{pmatrix}+(\alpha(1-W_\alpha^2)-\dfrac{1}{r^2}(Y_\alpha-m)^2)\begin{pmatrix}\tau\\ \sigma\end{pmatrix} \\ -\dfrac{2k}{r^2}(Y_\alpha-m)F\begin{pmatrix}\tau\\ \sigma\end{pmatrix}-2\alpha W_\alpha^2\begin{pmatrix}\tau\\ 0\end{pmatrix}+\mu\begin{pmatrix}\tau\\ \sigma\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix} \text{ in } \Sigma, \\ \dfrac{\partial}{\partial n}\begin{pmatrix}\tau\\ \sigma\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix} \quad \text{on} \quad \partial\Sigma. \end{cases}
$$

Let

$$
\{\mu_\ell^{(k)}(\alpha)\}_{\ell=1}^\infty \quad \text{and} \quad \left\{\begin{pmatrix}\tau_{\ell,\alpha}^{(k)}(r,z)\\ \sigma_{\ell,\alpha}^{(k)}(r,z)\end{pmatrix}\right\}_{\ell=1}^\infty \subset L^2(\Omega)\times L^2(\Omega)
$$

be the eigenvalues arranged in increasing order (with counting multiplicity) and the complete system of the corresponding orthonormal eigenfunctions of (6.7). We can apply the completely similar argument as in Lemma 14 and obtain the following asymptotic behaviors of the eigenvalues and eigenfunctions.

Lemma 15. For each non-negative integer k,

$$
(6.8)\qquad \lim_{\alpha\to\infty}\mu_\ell^{(k)}(\alpha)=\mu_\ell^{(k)} \quad (k\geqq 0,\ \ell\geqq 1).
$$

$$(6.9) \qquad \lim_{\alpha\to\infty}\left(\|\nabla\tau_{\ell,\alpha}^{(k)}(\alpha)\|_{L^2(\Sigma)}^2+\alpha\|\tau_{\ell,\alpha}^{(k)}(\alpha)\|_{L^2(\Sigma)}^2\right)=0.$$

where $\{\mu_\ell^{(k)}\}_{\ell=1}^{\infty}$ is the set of the eigenvalues arranged in increasing order (with counting multiplicity) of the following eigenvalue problem

$$(6.10) \qquad \begin{cases} L_1\sigma-\dfrac{k^2}{r^2}\sigma+\mu\sigma=0 & \text{in} \quad \Sigma,\\ \dfrac{\partial\sigma}{\partial n}=0 & \text{on} \quad \partial\Sigma. \end{cases}$$

Lemma 16. The family of functions

$$(6.11) \qquad \frac{1}{\sqrt{2\pi}}\begin{pmatrix}\tau_{\ell,\alpha}^{(0)}(r,z)\\ \sigma_{\ell,\alpha}^{(0)}(r,z)\end{pmatrix},\quad \frac{1}{\sqrt{2\pi}}\begin{pmatrix}\tau_{\ell,\alpha}^{(k)}(r,z)\cos k\theta\\ \sigma_{\ell,\alpha}^{(k)}(r,z)\sin k\theta\end{pmatrix},\quad \frac{1}{\sqrt{2\pi}}\begin{pmatrix}\tau_{\ell,\alpha}^{(k)}(r,z)\sin k\theta\\ -\sigma_{\ell,\alpha}^{(k)}(r,z)\cos k\theta\end{pmatrix}$$

$(k\geqq 1,\ell\geqq 1)$ form a complete orthonormal basis in $L^2(\Omega)\times L^2(\Omega)$.

Now we can expand any $\begin{pmatrix}\widehat{\phi}\\ \widehat{\psi}\end{pmatrix}\in L^2(\Omega)\times L^2(\Omega)$ in terms of the above basis:

$$(6.12) \qquad \begin{pmatrix}\widehat{\phi}\\ \widehat{\psi}\end{pmatrix}=\frac{1}{\sqrt{2\pi}}\sum_{k\geqq1,\ell\geqq1}\left(c_{k,\ell}\begin{pmatrix}\tau_{\ell,\alpha}^{(k)}\cos k\theta\\ \sigma_{\ell,\alpha}^{(k)}\sin k\theta\end{pmatrix}+d_{k,\ell}\begin{pmatrix}\tau_{\ell,\alpha}^{(k)}\sin k\theta\\ -\sigma_{\ell,\alpha}^{(k)}\cos k\theta\end{pmatrix}\right)$$

$$+\frac{1}{\sqrt{2\pi}}\sum_{\ell\geqq1}g_\ell\begin{pmatrix}\tau_{\ell,\alpha}^{(0)}\\ \sigma_{\ell,\alpha}^{(0)}\end{pmatrix},\quad g_\ell,\ c_{k,\ell},\ d_{k,\ell},\in\mathbb{R}.$$

Here $g_\ell, c_{k,\ell}, d_{k,\ell}$ are related with $\widehat{\phi}$, $\widehat{\psi}$ through

$$(6.13) \qquad \begin{cases} g_\ell=\dfrac{1}{\sqrt{2\pi}}\left((\widehat{\phi}\cdot\tau_{\ell,\alpha}^{(0)})_{L^2(\Omega)}+(\widehat{\psi}\cdot\sigma_{\ell,\alpha}^{(0)})_{L^2(\Omega)}\right),\\ c_{k,\ell}=\dfrac{1}{\sqrt{2\pi}}\left((\widehat{\phi}\cdot\tau_{\ell,\alpha}^{(k)}\cos k\theta)_{L^2(\Omega)}+(\widehat{\psi}\cdot\sigma_{\ell,\alpha}^{(k)}\sin k\theta)_{L^2(\Omega)}\right),\\ d_{k,\ell}=\dfrac{1}{\sqrt{2\pi}}\left((\widehat{\phi}\cdot\tau_{\ell,\alpha}^{(k)}\sin k\theta)_{L^2(\Omega)}-(\widehat{\psi}\cdot\sigma_{\ell,\alpha}^{(k)}\cos k\theta)_{L^2(\Omega)}\right),\\ \|\widehat{\phi}\|_{L^2(\Omega)}^2+\|\widehat{\psi}\|_{L^2(\Omega)}^2=\displaystyle\sum_{\ell=1}^{\infty}g_\ell^2+\sum_{k=1}^{\infty}\sum_{\ell=1}^{\infty}(c_{k,\ell}^2+d_{k,\ell}^2). \end{cases}$$

We prepare an auxiliary property concerning a complete orthonormal basis of a product of Hilbert spaces, which we will use in the proof of Lemma 19.

Lemma 17. Let H_1 and H_2 be two real Hilbert spaces with inner products $(\cdot,\cdot)_{H_1}$ and $(\cdot,\cdot)_{H_2}$, respectively and Let H be the product Hilbert space $H_1 \times H_2$ with inner product:

$$\left(\begin{pmatrix}\phi\\ \psi\end{pmatrix},\begin{pmatrix}\phi'\\ \psi'\end{pmatrix}\right)_H \equiv (\phi,\phi')_{H_1} + (\psi,\psi')_{H_2} \quad \text{for} \quad \begin{pmatrix}\phi\\ \psi\end{pmatrix},\begin{pmatrix}\phi'\\ \psi'\end{pmatrix} \in H.$$

If there exists an orthonormal basis $\left\{\begin{pmatrix}\phi_n\\ \psi_n\end{pmatrix}\right\}_{n=1}^{\infty} \subset H$, then

$$\text{(6.14)} \qquad \begin{cases} \displaystyle\sum_{n=1}^{\infty}(\phi,\phi_n)_{H_1}(\psi,\psi_n)_{H_2} = 0 \quad \text{for any} \quad \begin{pmatrix}\phi\\ \psi\end{pmatrix} \in H, \\ \displaystyle\|\phi\|_{H_1}^2 = \sum_{n=1}^{\infty}(\phi,\phi_n)_{H_1}^2, \quad \|\psi\|_{H_2}^2 = \sum_{n=1}^{\infty}(\psi,\psi_n)_{H_2}^2. \end{cases}$$

(Proof of Lemma 17) Take any $\begin{pmatrix}\phi\\ \psi\end{pmatrix} \in H$ and expand with respect to the given orthonormal basis. Then

$$\text{(6.15)} \qquad \begin{pmatrix}\phi\\ \psi\end{pmatrix} = \sum_{n=1}^{\infty}((\phi,\phi_n)_{H_1} + (\psi,\psi_n)_{H_2})\begin{pmatrix}\phi_n\\ \psi_n\end{pmatrix}$$

$$\text{(6.16)} \qquad \left(\begin{pmatrix}\phi\\ \psi\end{pmatrix},\begin{pmatrix}\phi\\ \psi\end{pmatrix}\right)_H = \|\phi\|_{H_1}^2 + \|\psi\|_{H_2}^2 = \sum_{n=1}^{\infty}((\phi,\phi_n)_{H_1} + (\psi,\psi_n)_{H_2})^2$$

Taking $\phi = 0 \in H_1$ or $\psi = 0 \in H_2$, we get the second and third equalities of (6.14), with which (6.16) yields the first equality of (6.14). □

The following lemma directly follows from the above lemmas.

Lemma 18. For any $(\widehat{\phi},\widehat{\psi}) \in L^2(\Omega) \times L^2(\Omega)$,

$$\sum_{\ell=1}^{\infty}(\widehat{\phi}\cdot\tau_{\ell,\alpha}^{(0)})_{L^2(\Omega)}(\widehat{\psi}\cdot\sigma_{\ell,\alpha}^{(0)})_{L^2(\Omega)} + \sum_{k,\ell\geqq 1}(\widehat{\phi}\cdot\tau_{\ell,\alpha}^{(k)}\cos k\theta)_{L^2(\Omega)}(\widehat{\psi}\cdot\sigma_{\ell,\alpha}^{(k)}\sin k\theta)_{L^2(\Omega)}$$

$$- \sum_{k,\ell\geqq 1}(\widehat{\phi}\cdot\tau_{\ell,\alpha}^{(k)}\sin k\theta)_{L^2(\Omega)}(\widehat{\psi}\cdot\sigma_{\ell,\alpha}^{(k)}\cos k\theta)_{L^2(\Omega)} = 0 \qquad (\alpha \geqq \alpha_0).$$

$$\|\widehat{\phi}\|_{L^2(\Omega)}^2 = \frac{1}{2\pi}\left(\sum_{\ell=1}^{\infty}(\widehat{\phi}\cdot\tau_{\ell,\alpha}^{(0)})_{L^2(\Omega)}^2 + \sum_{\ell,k\geqq 1}(\widehat{\phi}\cdot\tau_{\ell,\alpha}^{(k)}\cos k\theta)_{L^2(\Omega)}^2 + \sum_{\ell,k\geqq 1}(\widehat{\phi}\cdot\tau_{\ell,\alpha}^{(k)}\sin k\theta)_{L^2(\Omega)}^2\right)$$

$$\|\widehat{\psi}\|^2_{L^2(\Omega)} = \frac{1}{2\pi}\left(\sum_{\ell=1}^{\infty}(\widehat{\psi}\cdot\sigma^{(0)}_{\ell,\alpha})^2_{L^2(\Omega)} + \sum_{\ell,k\geqq 1}(\widehat{\psi}\cdot\sigma^{(k)}_{\ell,\alpha}\sin k\theta)^2_{L^2(\Omega)} + \sum_{\ell,k\geqq 1}(\widehat{\psi}\cdot\sigma^{(k)}_{\ell,\alpha}\cos k\theta)^2_{L^2(\Omega)}\right)$$

(Proof of Lemma 18) Putt $H_1 = H_2 = L^2(\Omega)$, $(\cdot,\cdot)_{H_1} = (\cdot,\cdot)_{H_2} = (\cdot,\cdot)_{L^2(\Omega)}$ and $(\widehat{\phi}\cdot\widehat{\psi})_{L^2(\Omega)} = \int_\Omega \widehat{\phi}(r,z)\widehat{\psi}(r,z)rdrdzd\theta$ for $\widehat{\phi},\widehat{\psi}\in L^2(\Omega)$. Combining Lemma 16 and Lemma 17 yields the conclusion. □

$I_1(\widehat{\phi},\widehat{\psi})$ is expressed in terms the Fourier coefficients of $\widehat{\phi},\widehat{\psi}$:

$$I_1(\widehat{\phi},\widehat{\psi}) = \sum_{\ell=1}^{\infty}\mu^{(0)}_\ell(\alpha)g_\ell^2 + \sum_{k=1}^{\infty}\sum_{\ell=1}^{\infty}\mu^{(k)}_\ell(\alpha)(c^2_{k,\ell}+d^2_{k,\ell}). \tag{6.17}$$

We remark that $\tau^{(0)}_{1,\alpha}(r,z) = 0$, $\sigma^{(0)}_{1,\alpha}(r,z) = e_\alpha W_\alpha(r,z)$, $\mu^{(0)}_1(\alpha) = \mu^{(0)}_1 = 0$, $\mu^{(0)}_2 > 0$, $e_\alpha \neq 0$ is a certain real number, which satisfies $\lim_{\alpha\to\infty} e^2_\alpha = 1/|\Omega|$.
We have the following coercive inequality.

Lemma 19. For any $c>0$ and $\eta>0$, there exists a constant $\alpha_1>0$ and $c'>0$ such that

$$I_1(\widehat{\phi},\widehat{\psi}) \geqq c\|\widehat{\phi}\|^2_{L^2(\Omega)} + \left(\min(\mu^{(0)}_2(\alpha),\mu^{(1)}_1(\alpha)) - \eta\right)\|\widehat{\psi}\|^2_{L^2(\Omega)} - c'(\widehat{\psi}\cdot W_\alpha)^2_{L^2(\Omega)} \tag{6.18}$$

for any $\widehat{\phi},\widehat{\psi}\in H^1(\Omega)$ and $\alpha \geqq \alpha_1$.

(Proof of Lemma 19) In view of the eigenvalues of (6.10) and Lemma 15, given $c>0$, we can take a natural number N so that $\mu^{(k)}_\ell(\alpha)\geqq c+1$ for $k+\ell>N$, $k\geqq 0$, $\ell\geqq 1$ and for any large $\alpha>0$,

$$I_1(\widehat{\phi},\widehat{\psi}) \geqq \sum_{\ell=1}^{N}\mu^{(0)}_\ell(\alpha)g^2_\ell + \sum_{k+\ell\leqq N}\mu^{(k)}_\ell(\alpha)(c^2_{k,\ell}+d^2_{k,\ell})$$

$$+(c+1)\left(\sum_{\ell>N}g^2_\ell + \sum_{k\geqq 1,\ell\geqq 1,k+\ell>N}(c^2_{k,\ell}+d^2_{k,\ell})\right)$$

Substituting (6.13), we have

$$2\pi I_1 \geqq \sum_{\ell=1}^{N}\mu^{(0)}_\ell(\alpha)(\widehat{\phi}\tau^{(0)}_{\ell,\alpha})^2_{L^2(\Omega)} + \sum_{k+\ell\leqq N}\mu^{(k)}_\ell(\alpha)\left((\widehat{\phi}\tau^{(k)}_{\ell,\alpha}\cos k\theta)^2_{L^2(\Omega)} + (\widehat{\phi}\tau^{(k)}_{\ell,\alpha}\sin k\theta)^2_{L^2(\Omega)}\right)$$

$$+(c+1)\left\{\sum_{\ell>N}(\widehat{\phi}\tau^{(0)}_{\ell,\alpha})^2_{L^2(\Omega)} + \sum_{k+\ell>N}\left((\widehat{\phi}\tau^{(k)}_{\ell,\alpha}\cos k\theta)^2_{L^2(\Omega)} + (\widehat{\phi}\tau^{(k)}_{\ell,\alpha}\sin k\theta)^2_{L^2(\Omega)}\right)\right\}$$

$$+2\sum_{\ell=1}^{N}\mu_\ell^{(0)}(\alpha)(\widehat{\phi}\tau_{\ell,\alpha}^{(0)})_{L^2(\Omega)}(\widehat{\psi}\sigma_{\ell,\alpha}^{(0)})_{L^2(\Omega)}+2\sum_{k+\ell\leqq N}\mu_\ell^{(k)}(\alpha)(\widehat{\phi}\tau_{\ell,\alpha}^{(k)}\cos k\theta)_{L^2(\Omega)}(\widehat{\psi}\sigma_{\ell,\alpha}^{(k)}\sin k\theta)_{L^2(\Omega)}$$

$$-2\sum_{k+\ell\leqq N}\mu_\ell^{(k)}(\alpha)(\widehat{\phi}\tau_{\ell,\alpha}^{(k)}\sin k\theta)_{L^2(\Omega)}(\widehat{\psi}\sigma_{\ell,\alpha}^{(k)}\cos k\theta)_{L^2(\Omega)}+2(c+1)\sum_{\ell>N}(\widehat{\phi}\tau_{\ell,\alpha}^{(0)})_{L^2(\Omega)}(\widehat{\psi}\sigma_{\ell,\alpha}^{(0)})_{L^2(\Omega)}+$$

$$2(c+1)\sum_{k+\ell>N}\Big((\widehat{\phi}\tau_{\ell,\alpha}^{(k)}\cos k\theta)_{L^2(\Omega)}(\widehat{\psi}\sigma_{\ell,\alpha}^{(k)}\sin k\theta)_{L^2(\Omega)}-(\widehat{\phi}\tau_{\ell,\alpha}^{(k)}\sin k\theta)_{L^2(\Omega)}(\widehat{\psi}\sigma_{\ell,\alpha}^{(k)}\cos k\theta)_{L^2(\Omega)}\Big)$$

$$+\sum_{\ell=1}^{N}\mu_\ell^{(0)}(\alpha)(\widehat{\psi}\sigma_{\ell,\alpha}^{(0)})^2_{L^2(\Omega)}+\sum_{k+\ell\leqq N}\mu_\ell^{(k)}(\alpha)\Big((\widehat{\psi}\sigma_{\ell,\alpha}^{(k)}\sin k\theta)^2_{L^2(\Omega)}+(\widehat{\psi}\sigma_{\ell,\alpha}^{(k)}\cos k\theta)^2_{L^2(\Omega)}\Big)$$

$$+(c+1)\Big\{\sum_{\ell>N}(\widehat{\psi}\sigma_{\ell,\alpha}^{(0)})^2_{L^2(\Omega)}+\sum_{k+\ell>N}\Big((\widehat{\psi}\sigma_{\ell,\alpha}^{(k)}\sin k\theta)^2_{L^2(\Omega)}+(\widehat{\psi}\sigma_{\ell,\alpha}^{(k)}\cos k\theta)^2_{L^2(\Omega)}\Big)\Big\}$$

From Lemma 18,

$$I_1(\widehat{\phi},\widehat{\psi})\geqq(c+1)\|\widehat{\phi}\|^2_{L^2(\Omega)}+\frac{1}{2\pi}\sum_{\ell=1}^{N}(\mu_\ell^{(0)}(\alpha)-c-1)(\widehat{\phi}\tau_{\ell,\alpha}^{(0)})^2_{L^2(\Omega)}$$
$$+\sum_{k+\ell\leqq N}(\mu_\ell^{(k)}(\alpha)-c-1)\Big((\widehat{\phi}\tau_{\ell,\alpha}^{(k)}\cos k\theta)^2_{L^2(\Omega)}+(\widehat{\phi}\tau_{\ell,\alpha}^{(k)}\sin k\theta)^2_{L^2(\Omega)}\Big)$$
$$+\frac{1}{2\pi}\sum_{\ell=1}^{N}2(\mu_\ell^{(0)}(\alpha)-c-1)(\widehat{\phi}\tau_{\ell,\alpha}^{(0)})_{L^2(\Omega)}(\widehat{\psi}\sigma_{\ell,\alpha}^{(0)})_{L^2(\Omega)}$$
$$+\frac{1}{2\pi}\sum_{k+\ell\leqq N}2(\mu_\ell^{(k)}(\alpha)-c-1)(\widehat{\phi}\tau_{\ell,\alpha}^{(k)}\cos k\theta)_{L^2(\Omega)}(\widehat{\psi}\sigma_{\ell,\alpha}^{(k)}\sin k\theta)_{L^2(\Omega)}$$
$$+\frac{1}{2\pi}\sum_{k+\ell\leqq N}(-2)(\mu_\ell^{(k)}(\alpha)-c-1)(\widehat{\phi}\tau_{\ell,\alpha}^{(k)}\sin k\theta)_{L^2(\Omega)}(\widehat{\psi}\sigma_{\ell,\alpha}^{(k)}\cos k\theta)_{L^2(\Omega)}$$
$$+\min(\mu_2^{(0)}(\alpha),\mu_1^{(1)}(\alpha))\left(\|\widehat{\psi}\|^2_{L^2(\Omega)}-\frac{1}{2\pi}(\widehat{\psi}\sigma_{1,\alpha}^{(0)})^2_{L^2(\Omega)}\right).\tag{6.19}$$

We used $\mu_1^{(0)}(\alpha)=0$, $\tau_1^{(0)}(\alpha)=0$, $\sigma_1^{(0)}(\alpha)=e_\alpha W_\alpha$. Apply Lemma 15 to the right hand side of (6.19). Then the terms including $\tau_{\ell,\alpha}^{(k)}$ can be abosrbed in the ones including $\|\widehat{\phi}\|^2_{L^2(\Omega)}$ and $\|\widehat{\psi}\|^2_{L^2(\Omega)}$ by taking large α, so

$$I_1\geqq c\|\widehat{\phi}\|^2_{L^2(\Omega)}+(\min(\mu_2^{(0)}(\alpha),\mu_1^{(1)}(\alpha))-\eta)\|\widehat{\psi}\|^2_{L^2(\Omega)}-c'(\widehat{\psi}\cdot W_\alpha)^2_{L^2(\Omega)}\quad\text{for large}\quad\alpha.$$

This gives (6.18). $\square$

Now we estimate $\mathcal{L}(\widehat{\phi}, \widehat{\psi}, B)$ from the lower.

Proof of Theorem 6
Assume that $(\phi, \psi, B) \in N(u_\alpha, v_\alpha, A_\alpha)$.

$$\mathcal{L}(\phi, \psi, B) = I_1(\widehat{\phi}, \widehat{\psi}) + I_2(B) + I_3(\widehat{\phi}, \widehat{\psi}, B)$$

We prove that $|I_3(\widehat{\phi}, \widehat{\psi}, B)|$ is dominated by I_1 and I_2.

$$|I_3(\widehat{\phi}, \widehat{\psi}, B)| \leqq |\int_\Omega \frac{(Y_\alpha - m)S_1 W_\alpha \widehat{\phi}}{r^2} dx| + |\int_\Omega (S_2 \frac{\partial W}{\partial r} \widehat{\psi} + S_3 \frac{\partial W}{\partial z} \widehat{\psi}) dx|$$

$$\leqq \int_\Omega \frac{|mS_1 \widehat{\phi}|}{r^2} dx + \sup_\Omega |\nabla W_\alpha| \int_\Omega |\widehat{\psi}|(|S_2| + |S_3|) dx$$

$$\leqq \frac{|m|}{r_0}(\epsilon \int_\Omega \frac{S_1^2}{r^2} dx + \frac{1}{4\epsilon} \int_\Omega \widehat{\phi}^2 dx) + \sup_\Omega |\nabla W_\alpha| (\int_\Omega \widehat{\psi}^2 + \frac{S_2^2 + S_3^2}{2}) dx$$

$$= \sup_\Omega |\nabla W_\alpha| \cdot \|\widehat{\psi}\|^2_{L^2(\Omega)} + \frac{|m|}{4\epsilon r_0} \|\widehat{\phi}\|^2_{L^2(\Omega)} + \frac{|m|\epsilon}{r_0} \int_\Omega \frac{S_1^2}{r^2} dx + \sup_\Omega |\nabla W_\alpha| \int_\Omega \frac{S_2^2 + S_3^2}{2} dx.$$

From $|B|^2 = S_1^2/r^2 + S_2^2 + S_3^2$ and Proposition 9, take $\epsilon > 0$ so that $\epsilon h|m|/r_0 = 1/2$. Next take c in Lemma 19 such that $c = |m|/(4\epsilon r_0) + 1$. From Lemma 19 and Proposition 9, we can take α_1 large so that the following inequality is true for $\delta = \min(\mu_2^{(0)}/2, \mu_1^{(1)}/2, 1) > 0$:

$$\text{(6.20)} \qquad \mathcal{L}(\phi, \psi, B) \geqq \delta \left(\|\widehat{\phi}\|^2_{L^2(\Omega)} + \|\widehat{\psi}\|^2_{L^2(\Omega)} + \|B\|^2_{L^2(\Omega)} + \|\mathrm{rot}\, B\|^2_{L^2(\mathbb{R}^3)} \right)$$

$$= \delta \left(\|\phi\|^2_{L^2(\Omega)} + \|\psi\|^2_{L^2(\Omega)} + \|B\|^2_{L^2(\Omega;\mathbb{R}^3)} + \|\mathrm{rot}\, B\|^2_{L^2(\mathbb{R}^3;\mathbb{R}^{3\times 3})} \right) \quad \text{for} \quad \alpha \geqq \alpha_1.$$

We used $(\widehat{\psi} \cdot W_\alpha)_{L^2(\Omega)} = 0$ for $(\phi, \psi, B) \in \overline{N}(u_\alpha, v_\alpha, A_\alpha)$. We will get a similar inequality on $N(u_\alpha, v_\alpha, A_\alpha)$. First we recall the following inequality:

$$\int_{\mathbb{R}^3} \frac{\varphi(x)^2}{|x-y|^2} dx \leqq 4 \int_{\mathbb{R}^3} |\nabla \varphi(x)|^2 dx \quad (\forall y \in \mathbb{R}^3, \forall \varphi \in H^1(\mathbb{R}^3)) \quad (cf. [16]).$$

Since $\Omega \subset \mathbb{R}^3$ is a bounded domain, by fixing y outside of Ω, we see that there exists a constant $R_1 > 0$ such that

$$\text{(6.21)} \qquad R_1 \int_\Omega \varphi^2 dx \leqq \int_{\mathbb{R}^3} |\nabla \varphi|^2 dx \quad (\forall \varphi \in H^1(\mathbb{R}^3)).$$

From Propostition 9-(4.4), there exist a constant $R_2 > 0$ and $\alpha_2 > 0$ such that for any $\alpha > \alpha_2$,

$$R_2 \int_\Omega \varphi^2 dx \leqq \int_\Omega |\nabla\varphi|^2 dx \quad \left(\forall\varphi \in H^1(\Omega);\ \int_\Omega (u_\alpha^2 + v_\alpha^2)\varphi dx = 0\right). \tag{6.22}$$

It is also true that there exists a $R_3 > 0$ such that

$$\int_{\partial\Omega} \varphi^2 dS \leqq R_3 \int_\Omega (|\varphi|^2 + |\nabla\varphi|^2) dx \quad (\forall\varphi \in H^1(\Omega)). \tag{6.23}$$

Take any $(\phi, \psi, B) \in N(u_\alpha, v_\alpha, A_\alpha)$, and we have

$$(\phi, \psi, B) = (-v\xi, u\xi, \nabla\xi) + (\overline{\phi}, \overline{\psi}, \overline{B}) \in T(u, v, A) + \overline{N}(u, v, A)$$

and which is equivalently, $\phi = -v\xi + \overline{\phi}$, $\psi = u\xi + \overline{\psi}$, $B = \nabla\xi + \overline{B}$ and

$$\int_\Omega (u_\alpha^2 + v_\alpha^2)\xi(x) dx = 0, \quad \Delta\xi = 0 \text{ in } \Omega, \quad \frac{\partial\xi}{\partial\nu} = \langle B \cdot \nu\rangle \text{ on } \partial\Omega. \tag{6.24}$$

From these equations we see

$$\operatorname{rot} B = \operatorname{rot} \overline{B} \quad \text{in} \quad \mathbb{R}^3, \tag{6.25}$$

$$\phi^2 + \psi^2 = (-v\xi + \overline{\phi})^2 + (u\xi + \overline{\psi})^2 \leqq 2(\overline{\phi}^2 + \overline{\psi}^2) + 2\xi^2 \quad \text{in} \quad \Omega. \tag{6.26}$$

By (6.22) and (6.24) we obtain

$$R_2 \int_\Omega \xi^2 dx \leqq \int_\Omega |\nabla\xi|^2 dx. \tag{6.27}$$

On the other hand (6.24) yields

$$0 = \int_\Omega \xi\Delta\xi dx = \int_{\partial\Omega} \xi \frac{\partial\xi}{\partial\nu} dS - \int_\Omega |\nabla\xi|^2 dx$$

subsequently,

$$\begin{aligned} \int_\Omega |\nabla\xi|^2 dx &= \int_{\partial\Omega} \xi\langle B \cdot \nu\rangle dS \leqq \frac{\epsilon}{2} \int_{\partial\Omega} \xi^2 dS + \frac{1}{2\epsilon} \int_{\partial\Omega} |B|^2 dS \\ &\leqq \frac{\epsilon R_3}{2} \int_\Omega (\xi^2 + |\nabla\xi|^2) dx + \frac{R_3}{2\epsilon} \int_\Omega (|B|^2 + |\nabla B|^2) dx. \end{aligned} \tag{6.28}$$

Combining (6.27) and (6.28) and taking $\epsilon = \frac{R_2}{R_3(R_2+1)}$, we have

$$\int_\Omega |\nabla \xi|^2 dx \leqq \frac{R_3^2(1+1/R_2)}{2h^2} \int_\Omega \left(|B|^2 + |\nabla B|^2\right) dx.$$

Using (6.26), (6.27), (6.28), (6.29), we conclude that there exists a constant $c > 0$, (which is independent of $(\phi, \psi, B) \in N(u_\alpha, v_\alpha, A_\alpha)$) such that

$$\int_\Omega (\phi^2 + \psi^2) dx + \int_\Omega |B|^2 dx + \int_{\mathbb{R}^3} |\nabla B|^2 dx \leqq c \int_\Omega (\overline{\phi}^2 + \overline{\psi}^2) dx + \int_\Omega |\overline{B}|^2 dx + \int_{\mathbb{R}^3} |\nabla \overline{B}|^2 dx$$

On the other hand from $\operatorname{div} B = 0$ in $\mathbb{R}^3$, it is true that

$$\|\nabla B\|_{L^2(\mathbb{R}^3;\mathbb{R}^{3\times 3})} = \|\operatorname{rot} B\|_{L^2(\mathbb{R}^3;\mathbb{R}^3)}$$

and $\mathcal{L}(\phi, \psi, B) = \mathcal{L}(\overline{\phi}, \overline{\psi}, \overline{B})$ (cf. Proposition 3). Hence we obtain the desired inequality (3.4) from (6.20), which completes the proof of Theorem 6. □

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