Mathematics for Nonlinear Phenomena: Analysis and Computation

- International Conference in honor of Professor Yoshikazu Giga on his 60th birthday -

Organizers:
S. Jimbo, S. Goto, Y. Kohsaka,
H. Kubo, Y. Maekawa, M. Ohnuma

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Mathematics for Nonlinear Phenomena: Analysis and Computation

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Professor Yoshikazu Giga on his 60th birthday -

Organizers:
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Sapporo Convention Center

August 16 – 18, 2015

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Preface

We welcome all the participants to the conference: Mathematics for Nonlinear Phenomena: Analysis and Computation – International Conference in honor of Professor Yoshikazu Giga's 60th birthday. This volume is intended as the proceeding of this conference, held for the period of August 16 -18, 2015 in Sapporo.

Importance of mathematics is significantly increasing in various areas of sciences. Particularly, a lot of interesting nonlinear phenomena take place in many research fields and mathematics are expected to be applicable to these subjects. Accordingly, the nonlinear analysis and nonlinear PDE theories are now more and more active in the stream of such movement of sciences. For this reason we are motivated to organize this conference, to deepen the discussions and communications among active participants based on the lectures by strong leaders in various different subjects of nonlinear mathematical phenomena. Professor Yoshikazu Giga has been making a huge contribution to these research fields for several decades. We would like to take this occasion to recognize his feat in mathematical researches.

We hope you enjoy the conference Mathematics for Nonlinear Phenomena and your stay in the nice weather of summer in Sapporo.

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Institute for Mathematics in Advanced Interdisciplinary Study

Hokkaido Kaihatsu Kokusai Kouryu Kikin

Program

<u>Aug. 16 (Sun.)</u>

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13:20 - 14:00 **Robert V. Kohn (Courant Institute)**
Prediction without probability: a PDE approach to a model problem from machine learning

14:10 - 14:50 **Piotr Rybka (University of Warsaw)**
The method of viscosity solutions for analysis of singular diffusion problems appearing in crystal growth problems

15:20 - 15:40 **Takeshi Ohtsuka (Gunma University)**
Evolution of spirals by crystalline curvature and eikonal equation

15:40 - 16:00 **Yasunori Maekawa (Tohoku University)**
Stability of scale-critical circular flows in a two-dimensional exterior domain

16:15 - 17:30 Poster Session

19:00 - Banquet

<u>Aug. 17 (Mon.)</u>

10:30 - 11:10 **Charles M. Elliott (University of Warwick)**
Partial differential equations on evolving domains

11:30 - 12:10 **Zensho Yoshida (The University of Tokyo)**
Topological constraints and structures in macro (fluid and plasma) systems

14:00 - 14:40 **Matthias Hieber (TU Darmstadt)**
A journey through the world of incompressible viscous fluid flows: an evolution equation perspective

14:50 - 15:30 **Alex Mahalov (Arizona State University)**
Stochastic three-dimensional rotating Navier-Stokes equations: averaging, convergence, regularity and 3D nonlinear dynamics

16:00 - 16:40 **Y.-H.Richard Tsai (University of Texas)**
Boundary integral methods for implicitly defined interfaces

16:50 - 17:30 **Ryo Kobayashi (Hiroshima University)**
Locomotion of animals, design of robots and mathematics

<u>Aug. 18 (Tue.)</u>

10:00 - 10:40 **Chun Liu (Penn State University)**
Transport of charged particles in biological environments

11:00 - 11:40 **Jürgen Saal (Düsseldorf University)**
Fluid flow and rotation: a fascinating interplay

11:40 Closing

Abstracts

Prediction without probability: a PDE approach to a model problem from machine learning

Robert V Kohn, Courant Institute, NYU

Abstract

In the machine learning literature, one approach to "prediction" assumes that there are two or more "experts"; the best prediction in this setting is the one that "minimizes regret", i.e. minimizes the shortfall relative to the best-performing expert. My talk focuses on a model problem involving the prediction of a binary sequence (loosely speaking: a stock whose price is restricted to a binomial tree) when there are just two experts. I'll discuss a continuum limit in which the optimal prediction is determined by solving a 2nd order parabolic PDE. This is joint work with Kangping Zhu (for two very simple experts) and Nadejda Drenska (for more realistic experts).

The method of viscosity solutions for analysis of singular diffusion problems appearing in crystal growth problems

Piotr Rybka, Warszawa, Poland

An important ingredient of the modified Stefan problem is Gibbs-Thomson law on a moving interface,

$$\beta V = \sigma + \kappa_\gamma \qquad \text{on } \Gamma(t). \tag{1}$$

Since κ_γ is the weighted mean curvature, then (1) is a driven weighted mean curvature (wmc) flow, where σ is temperature (supersaturation or pressure depending upon the model). Here, we assume that σ is given. We are interested in γ which is convex but not C^1. An important point is how to interpret κ_γ, which is formally defined by

$$\kappa_\gamma = -\mathrm{div}_S\,(\nabla_X \gamma)(X)|_{X=\mathbf{n}(x)},$$

where $\mathbf{n}(x)$ is the normal to $\Gamma(t)$ at x. The issue is that γ may not be differentiable at the normals to Γ, hence κ_γ may not be defined on a large subset of Γ.

An important step of analysis is writing (1) for graphs. If Γ is the graph of u, then we can show,

Proposition 1. *Let us suppose that $u : (0, L) \times \mathbb{R}_+ \to \mathbb{R}$ and $\Gamma(t)$ is the graph of $u(\cdot, t)$. In this case, $\mathbf{n}(x) = (-1, u_x)/\sqrt{1 + u_x^2}$. Then, there is $W : \mathbb{R}^2 \to \mathbb{R}$ such that for all $x \in (0, L)$ function $p \mapsto W(p, x)$ is convex such that the operator $V - \kappa_\gamma/\beta$ on $\Gamma(t)$ takes the following form,*

$$\frac{v_t(1+\kappa v)}{\sqrt{v_s^2 + (1+\kappa v)^2}} - \frac{a(v_s, v, s)}{\sqrt{(1+\kappa v)^2 + v_s^2}}\frac{\partial}{\partial x}(W_p(v_x, x)), \tag{2}$$

where κ is the curvature of $\Gamma(t)$.

First, we would like to study simpler problems, where $W = W(p)$, $a = a(p)$, i.e. these two functions depend just on the derivative of the unknown function. After such simplifications (1) takes the form,

$$\begin{array}{ll} u_t = a(u_x)(W_p(u_x)_x + \tilde{\sigma}), & (x,t) \in (0, L) \times \mathbb{R}_+, \\ u(x, 0) = u_0(x), & x \in (0, L), \end{array} \tag{3}$$

augmented with boundary conditions.

We will recall the definition of viscosity solutions for (3) after [1]. We will also state a Comparison Principle for viscosity solutions, [1]. We will make comments on solvability of (3), different notions of solutions and their relationship.

We studied (1) for closed curves called bent-rectangles, [2], when the anisotropy function γ given by the following formula

$$\gamma(p_1, p_2) = |p_1|\gamma_\Lambda + |p_2|\gamma_R. \tag{4}$$

By definition, a *bent rectangle* is a Lipschitz curve, which is a small perturbation of a rectangle with sides parallel to the axes. For the sake of simplicity, we assume that bent rectangles have the symmetry center at the origin. The deformed sides are graphs of Lipschitz functions which are constant near the origin and at a distance from the origin. These flat parts, parallel to the axes are called *facets*. On facets the derivative of γ given by (4) is not defined, what makes problem (1) interesting.

In a series of papers, including [2], we constructed so-called *variational* solutions to (1), when the initial datum Γ_0 is a bent rectangle. This construction has drawbacks: a) the verteces moved as intersections of facets, not by (1), b) a uniqueness result was missing. The idea is to use the tools of viscosity theory to resolve these issues. In order to do this we show that bent-rectangles are graphs over a reference manifold. We can show, see [3]:

Theorem 1. *If a bent rectangle $\Gamma(0)$ satisfies an additional geometric condition, then we can construct a reference manifold $\mathcal{M}$ such that $\Gamma(0)$ and its 'small perturbations' are graphs. That is, there is function $u : [0, 2\pi L) \times [0, T) \to \mathbb{R}$, called a profile function, such that*

$$\Gamma(t) = \{\psi(s) + \nu(s)u(s,t), \quad s \in [0, 2\pi L)\},$$

where $\psi : [0, 2\pi L) \to \mathbb{R}^2$ is a parametrization of $\mathcal{M}$ and $\nu(s)$ is the outer normal to $\mathcal{M}$ at $\psi(s)$.

Theorem 1 applies to a class of bent rectangles, so that we can subsequently use Proposition 1. The resulting equation is,

$$u_t = a(u_s, u, s)(\frac{\partial}{\partial x}(W_p(u_x), x) + \tilde{\sigma}), \quad (s,t) \in [0, 2\pi L) \times \mathbb{R}_+ \tag{5}$$

with the initial condition $u(x, 0) = u_0(x)$ and periodic boundary conditions. The main difference, in comparison with (3), is that now the coefficients a and W depend on s and the unknown u. In particular the singular slopes change from point to point. This makes us adapt the definition of the viscosity solution and we have to prove a new version of the comparison principle. Having these tools at hand we are able to show a result, which may be expressed roughly as follows, (see [3] for more details):

Theorem 2. *Let us suppose that $\Gamma(\cdot)$, a family of bent rectangles, which is a variation solution to (1) and $\Gamma(0)$ is a bent rectangle satisfying assumptions of Theorem 1. Then,*
(a) the corresponding profile function u is a viscosity solution of (5) iff an additional condition holds;
(b) u is a unique solution to (5), hence $\Gamma(\cdot)$ is a unique variational solution to (1).

We stress that the set of singular slopes in (5) changes from point to point. On facets intersecting the axis (1) takes the form,

$$u_t = (\operatorname{sgn} u_x)_x + f,$$

while near the verteces it looks like

$$u_t = (\operatorname{sgn}(u_x + 1) + \operatorname{sgn}(u_x - 1))_x + f. \tag{6}$$

We may study the above equation for its own sake, especially that we may expect competition of facets with different slopes. We showed, (see [4]):

Theorem 3. *Let us consider (6) with $f \equiv 0$ for $(x,t) \in (0, L) \times \mathbb{R}_+$ and initial condition $u(\cdot, 0) = u_0 \in BV$. We assume either periodic or homogeneous Neumann or Dirichlet boundary conditions. Then, there exists a unique viscosity solution to (6).*

We also study of solutions to (6) with the help of the Comparison Principle.

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EVOLUTION OF SPIRALS BY CRYSTALLINE CURVATURE AND EIKONAL EQUATION

TAKESHI OHTSUKA

Dedicated to Professor Yoshikazu Giga on the occasion of his 60th birthday

Burton et al. [1] proposed a theory of crystal growth with aid of screw dislocations in 1951. Screw dislocations provide spiral steps (discontinuity) in the crystal height. Atoms on the surface bond with the crystal structure at the step, and thus results in an evolution of the steps. The dynamics of the step in this setting is well studied in [1]. The normal velocity V of the step is given as the curvature equation

$$V = C - \kappa,$$

where C is a constant denoting a driving force of the evolution.

One often see the spiral-shaped polygonal pattern, which is drawn by steps, on growing crystal surface. Such an anisotropic pattern should be caused by the anisotropic surface energy density whose equilibrium shape $\mathcal{W}_\gamma$ is a N_γ sided convex polygon. We call $\mathcal{W}_\gamma$ Wulff shape. In such an evolution, the normal velocity V_j of j-th facet of the spiral step is given as

$$\beta_j V_j = U - H_j, \tag{1}$$

where β_j is a constant denoting the mobility, H_j is the crystalline curvature defined by the length of j-th facet of $\mathcal{W}_\gamma$ for $j \in \mathbb{Z}/(N_\gamma\mathbb{Z})$. Note that the facet number j is considered as the generalized number; we regard $j + nN_\gamma$ is equivalent for $n \in \mathbb{Z}$. We call the evolution of spiral-shaped polygonal curve by (1) crystalline motion.

When $\mathcal{W}_\gamma$ is a convex polygon, the surface energy density is possibly not convex, thus partial differential equation approach for tracking the evolution does not work well. Taylor [7] introduced an ordinary differential equation(ODE) approach to the crystalline motion of interface. Ishiwata [4] proposed an ODE approach to the crystalline motion of a spiral step with a pre-determined trajectory of the center (which is called tip trajectory). However, one often find evolution of spiral steps which seems to be associated with a fixed center in the in situ observation of crystal surface, or theory of crystal growth.

Then, in this talk we shall give a scheme for crystalline motion of polygonal spiral step with a fixed center. We also compare the evolution of the polygonal spiral step by ODE approach and a formal level set approach by [5] numerically.

The crucial difference between earlier work by [4] and ours are the scheme of the generation of new facet around the center. In [4] the new facet generates when the center turns the vertex of the tip trajectory. On the other hand, the new facet will be generated by our scheme when the facet associated with the fixed center has suitable length for the evolution. Therefore, we prove not only the existence and uniqueness of the solution, but also there exists an countably infinite sequence of

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times when new facets are generated. We also prove that the solution of polygonal spiral step evolving by our scheme has no self-intersections for whole time.

This is a joint work with T. Ishiwata.

1. Main results

We prepare some notations for the Wulff shape $\mathcal{W}_\gamma$. Let $\mathcal{W}_\gamma$ be a N_γ sided convex polygon. We now give a number of facet of $\mathcal{W}_\gamma$ as a generalized number $j \in \mathbb{Z}/(N_\gamma\mathbb{Z})$ with conter-clockwise rotating orientation. Then, let $\mathbf{N}_j$ be the outer unit normal vector of j-th facet of $\mathcal{W}_\gamma$ with the angle $\varphi_j \in [0, 2\pi)$, i.e., $\mathbf{N}_j = (\cos\varphi_j, \sin\varphi_j)$. Since $\mathcal{W}_\gamma$ is a convex polygon we have

(W1) $0 = \varphi_0 < \varphi_1 < \varphi_2 < \cdots < \varphi_{N_\gamma-1} < 2\pi$,

(W2) $\varphi_j < \varphi_{j+1} < \varphi_j + \pi$ for $j \in \mathbb{Z}/(N_\gamma\mathbb{Z})$.

Let $\mathbf{T}_j$ be a unit tangential vector of j-th facet of $\mathcal{W}_\gamma$ such that $\mathbf{T}_j = (\sin\varphi_j, -\cos\varphi_j)$ for $j \in \mathbb{Z}/(N_\gamma\mathbb{Z})$. In other words, $\mathbf{T}_j$ is the rotation of $\mathbf{N}_j$ with the rotation angle $-\pi/2$. Let $\ell_j > 0$ be a length of j-th facet of $\mathcal{W}_\gamma$.

We introduce a new scheme for the evolution of a polygonal spiral by (1). The scheme is composed by the evolution of a polygonal spiral curve, and generation of a new facet. We first deduce an evolution equation for a polygonal spiral curve $\Gamma(t)$ by (1). Let $\Gamma(t)$ be given by $\Gamma(t) = \bigcup_{j=0}^{k} L_j(t)$ with

$$L_j(t) = \{\lambda y_j(t) + (1-\lambda)y_{j-1}(t) |\ \lambda \in [0,1]\}$$

for $j = 1, 2, \ldots, k$. We call $L_j(t)$ j-th facet of $\Gamma(t)$. We may assume that $L_0(t)$ is parallel to 0-th facet of $\mathcal{W}_\gamma$, i.e., $L_0(t)$ is given as

$$L_0(t) := \{y_0(t) + r\mathbf{T}_0 |\ r > 0\}.$$

In this talk we only consider the case that $\Gamma(t)$ is convex in the following sense; we say $\Gamma(t)$ is convex if

- $(y_{j-1}(t) - y_j(t))/|y_{j-1}(t) - y_j(t)| = \mathbf{T}_j$ for $j = 1, 2, \ldots, k$,
- the direction of evolution of $L_j(t)$ is same as $\mathbf{N}_j$, i.e., $V_j(t) = \dot{s}_j(t)$ with $s_j(t) = y_j(t) \cdot \mathbf{N}_j$.

If $\Gamma(t)$ is convex, then the crystalline curvature H_j of $L_j(t)$ is given as

$$H_j = \frac{\ell_j}{d_j},$$

where $d_j = d_j(t) = |y_j(t) - y_{j-1}(t)|$ denotes the length of $L_j(t)$. See [7] or [3] for details.

Under the above hypothesis, when the facet $L_j(t)$ and $L_{j\pm1}(t)$ evolve with the normal velocity $V_j(t)$ and $V_{j\pm1}(t)$, respectively, then the length $d_j(t)$ satisfies

$$\dot{d}_j = -\left(\frac{1}{\tan(\varphi_{j+1} - \varphi_j)} + \frac{1}{\tan(\varphi_j - \varphi_{j-1})}\right) V_j + \frac{1}{\sin(\varphi_{j+1} - \varphi_j)} V_{j+1} + \frac{1}{\sin(\varphi_j - \varphi_{j-1})} V_{j-1}.$$

Thus, if $\Gamma(t)$ evolves by (1), then d_j should be imposed

$$\dot{d}_j = -b_j\left(U - \frac{\ell_j}{d_j}\right) + c_j^+\left(U - \frac{\ell_{j+1}}{d_{j+1}}\right) + c_j^-\left(U - \frac{\ell_{j-1}}{d_{j-1}}\right) \quad \text{for } t > T_{k-1},\ j = 2, 3, \ldots, k-1, \tag{2}$$

where T_{k-1} is the time when $L_k(t)$ is generated and added to $\Gamma(t)$, and b_j, $c_j^{\pm}$ are the constants given by

$$b_j = \frac{1}{\beta_j}\left(\frac{1}{\tan(\varphi_{j+1}-\varphi_j)} + \frac{1}{\tan(\varphi_j - \varphi_{j-1})}\right), \quad c_j^{\pm} = \pm\frac{1}{\beta_{j\pm1}\sin(\varphi_{j\pm1}-\varphi_j)}$$

for $j \in \mathbb{Z}/(N_\gamma\mathbb{Z})$. Moreover, we impose that the center of spiral, which is denoted by $y_k(t)$ provided that $\Gamma(t)$ is given by $\Gamma(t) = \bigcup_{j=0}^{k} L_j(t)$, stays at the origin. It means $V_k = 0$, so that we now impose

$$\left.\begin{aligned} \dot{d}_k &= c_k^-\left(U - \frac{\ell_{k-1}}{d_{k-1}}\right), \\ \dot{d}_{k-1} &= -b_{k-1}\left(U - \frac{\ell_{k-1}}{d_{k-1}}\right) + c_{k-1}^-\left(U - \frac{\ell_{k-2}}{d_{k-2}}\right) \end{aligned}\right\} \text{ for } t > T_{k-1}. \tag{3}$$

Note that d_k has no influence to determine d_j for $j = 1, 2, \ldots, k-1$. On the other hand, we have $d_0(t) = \infty$ for every $t \in \mathbb{R}$ by the definition of $L_0(t)$, so that $V_0(t) = U/\beta_0$. Then we now impose

$$\dot{d}_1 = -b_1\left(U - \frac{\ell_1}{d_1}\right) + c_1^+\left(U - \frac{\ell_2}{d_2}\right) + c_1^- U \quad \text{for } t > T_{k-1}. \tag{4}$$

Hence, we obtain the system (2)–(4) of length d_j for $\Gamma(t)$ evolving by (1). When we obtain the solution $d_k, \ldots, d_1$ of the above, then draw $\Gamma(t)$ by setting

$$y_{j-1}(t) = y_j(t) + d_j(t)\mathbf{T}_j \quad \text{for } j = k, \ldots, 1 \tag{5}$$

with $y_k(t) = O$.

We next introduce a rule of generation of a new facet. Define

$$T_k = \sup\{T > T_{k-1};\ d_k(t) \le \ell_k/U \text{ for } t \in [T_{k-1}, T)\},$$

i.e., T_k is the first time when $d_k = \ell_k/U$. We call T_k the generation time of L_{k+1} (or $(k+1)$-th facet). At $t = T_k$ we add a new vertex $y_{k+1}(t) = O$ and a facet $L_{k+1}(t)$ with the following rule.

(O_k^+) If $s_k(t) \ge 0$, then the direction of evolution of $L_{k+1}(t)$ is $\mathbf{N}_{k+1}$, so that $y_k(t) = y_{k+1}(t) + d_k(t)\mathbf{T}_{k+1}$ for $t \ge T_k$.

(O_k^-) If $s_k(t) < 0$, then the direction of evolution of $L_{k+1}(t)$ is $\mathbf{N}_{k-1}$, so that $y_k(t) = y_{k+1}(t) + d_k(t)\mathbf{T}_{k-1}$ for $t \ge T_k$.

The crucial difference on the scheme of generation of new facet between [4] and ours is that the generation of new facet is resultant of the solution d_k of the system to (2)–(4). for $\Gamma(t)$ evolving by (1). In fact, the scheme of the generation in [4] is built-in to the "tip trajectory" which is a convex polygonal curve related to $\mathcal{W}_\gamma$ where the center moves on. On the other hand, we have to prove the existence of a sequence $\{T_k\}$ satisfying $\lim_{k\to\infty} T_k = \infty$ in our scheme. If the sequence $\{T_k\}$ is finite then the spiral step does not pile up, or if $\lim_{k\to\infty} T_k < \infty$ then the height of the growing crystal blows up at the center of the spiral step at $t = \lim_{k\to\infty} T_k$.

Finally, we introduce a class of the initial curve. Assume that the initial curve $\Gamma(T_{k_0-1}) = \bigcup_{j=0}^{k_0}(T_{k_0})$ is a convex polygonal curve satisfying either the following (I1) or (I2) holds.

(I1) For $k_0 = 1$; $\Gamma(T_0) = \bigcup_{j=0}^{1} L_j(T_0)$, $y_1(T_0) = y_0(T_0) = O$.

(I2) For $k_0 \geq 2$; $\Gamma(T_{k_0-1}) = \bigcup_{j=0}^{k_0} L_j(T_{k_0-1})$ is a convex spiral satisfying

$$d_{k_0}(T_{k_0-1}) = 0, \quad d_j(T_{k_0-1}) = \delta_j \geq \ell_j/U \quad \text{for } j = 1, 2, \ldots, k_0 - 1, \tag{6}$$

$$-b_{k_0-1}\left(U - \frac{\ell_{k_0-1}}{\delta_{k_0-1}}\right) + c_{k_0-1}^{-}\left(U - \frac{\ell_{k_0-2}}{\delta_{k_0-2}}\right) > 0, \tag{7}$$

$$-b_j\left(U - \frac{\ell_j}{\delta_j}\right) + c_j^{+}\left(U - \frac{\ell_j}{\delta_j}\right) + c_j^{-}\left(U - \frac{\ell_j}{\delta_j}\right) > 0 \quad \text{for } j = 2, \ldots, k_0 - 2, \tag{8}$$

$$-b_1\left(U - \frac{\ell_1}{\delta_1}\right) + c_1^{+}\left(U - \frac{\ell_2}{\delta_2}\right) + c_1^{-}U > 0. \tag{9}$$

Then, we are now in the position to propose a new scheme to the evolution of polygonal spiral curve by a crystalline curvature equation (1).

Summary of the scheme (SP).

Step 1. Give an imitial spiral curve $\Gamma(T_{k_0-1})$ satisfying either (I1) or (I2), and an initial time $T_{k_0-1} \in \mathbb{R}$.

Step 2. Solve the system (2)–(4) for given $\Gamma(T_{k-1})$, and draw $\Gamma(t)$ for $t \geq T_{k-1}$ with (5), where $k \geq k_0$.

Step 3. If $T_k < \infty$, then generate $L_{k+1}(T_k)$ and $y_{k+1}(T_k)$ with the rule (O_k^+) or (O_k^-). Return to Step 2 with updating the initial data by $\Gamma(T_k) \cup L_{k+1}(T_k)$ and the initial time T_k.

Definition 1. We say $\Gamma(t)$ is a semi-solution to (1) with the scheme (SP) if there exists a convex polygonal spiral curve $\Gamma(t)$ for $t \geq T_{k_0-1}$ and an increasing sequence T_k for $k \geq k_0 - 1$ (which is possibly infinite) such that

(i) $d_j(t) = |y_j(t) - y_{j-1}(t)| > 0$ and is continuous provided that $t > T_j$ for $j = 1, 2, \ldots$,

(ii) $(d_1, \ldots, d_k) \in C^1(T_{k-1}, T_k)^k \cap C^0[T_{k-1}, T_k]^k$ is a solution to (2)–(4) in $(T_{k-1}, T_k]$,

(iii) the generation rule either (O_k^+) or (O_k^-) holds at $t = T_k$ for every $k \geq k_0$.

We say $\Gamma(t)$ is a solution to (1) with the scheme (SP) if $\Gamma(t)$ is a semi-solution to (1) with the scheme (SP) and has no self-intersections for $t \geq T_{k_0-1}$.

Then, we obtain the following results.

Theorem 2. *Let $\Gamma(T_{k_0-1}) = \bigcup_{j=0}^{k_0} L_j(T_{k_0-1})$ be a convex polygonal spiral curve satisfying either (I1) or (I2). Then, there exists a solution $\Gamma(t)$ to (1) with the scheme (SP), and the infinite sequence of generation time T_k for $k \geq k_0 - 1$ satisfying $\lim_{k\to\infty} T_k = \infty$.*

The strategy of the proof of Theorem 2 is dividing the proof into the two steps; existence of semi-solution, and intersection-free result on the semi-solution. By the theory of ordinary differential equations, one can find the existence of local solution to the system (2)–(4) in a neighborhood of $t = T_{k-1}$ for the initial data satisfying (I2). Then, we prove the following a priori estimates to the solution of (2)–(4) on $[T_{k-1}, \infty)$;

(i) $d_j > \ell_j/U$, $\dot{d}_j > 0$ in (T_{k-1}, ∞) for $j = 1, 2, \ldots, k - 1$,

(ii) $\sup_{[T_{k-1},\infty)} \dot{d}_j < \infty$ for $j = 1, 2, \dots, k$.

Then, we obtain the global solution to (2)–(4) on $[T_{k-1}, \infty)$ and the generation time $T_k < \infty$, and there exists R_k for $k \in \mathbb{Z}/(N_\gamma\mathbb{Z})$ satisfying $T_k - T_{k-1} \geq R_k$. Moreover, we find $s_j \geq 0$ for $j \in \mathbb{Z}/(N_\gamma\mathbb{Z})$ as long as $L_j(t)$ exists. This monotonicity result and the properties of the crystalline curvature equation yield the intersection-free result.

A level set approach for spirals with a single auxiliary function is proposed by [5]. Let $\Omega \subset \mathbb{R}^2$ be a bounded domain satisfying $O \in \Omega$. According to their method, a continuous spiral curve with counter-clockwise rotational orientation whose center is the origin is described by

$$\Gamma(t) := \{x \in \overline{W};\ u(t,x) - \theta(x) \equiv 0 \mod 2\pi\mathbb{Z}\}, \quad \mathbf{n} = -\frac{\nabla(u-\theta)}{|\nabla(u-\theta)|},$$

where $\mathbf{n} \in S^1$ is the unit normal vector field of $\Gamma(t)$ denoting a direction of the evolution, $W = \Omega \setminus B_\rho(0)$ with a small constant $\rho > 0$, and $\theta(x) = \arg(x)$ is a multiple valued function getting the value of the argument of the vector $x \in \mathbb{R}^2$.

According to [2], an anisotropic curvature equation for a curve Γ surrounding D is described as an L^2-gradient flow of the surface energy

$$\Gamma \mapsto \int_\Gamma \gamma_0(\mathbf{n})\mathrm{d}S + \int_D U\mathrm{d}x$$

with a function $\gamma_0 \colon S^1 \to (0,\infty)$, where $\mathrm{d}S$ denotes a surface element. Then, the anisotropic curvature equation with a constant driving force is represented as

$$\beta(\nabla(u-\theta))u_t - \gamma(\nabla(u-\theta))\{\mathrm{div} D\gamma(\nabla(u-\theta)) + U\} = 0 \quad \text{on } (0,T) \times W$$

with $\gamma(p) = |p|\gamma_0(p/|p|)$ for $p \in \mathbb{R}^2 \setminus \{0\}$ and a positive function β on $\mathbb{R}^2 \setminus \{0\}$. Its mathematical analysis with an isotropic Neumann boundary condition was done in [6] when γ and β are at least smooth. Then, we shall present some numerical results comparing between (SP) and the level set approach approximating the situation such that its surface energy density gives an Wulff shape approximating a convex polygonal $\mathcal{W}_\gamma$.

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T. OHTSUKA

Division of Pure and Applied Science, Faculty of Science and Technology, Gunma University, Aramaki-machi 4-2, Maebashi, 371-8510 Gunma, Japan
E-mail address: tohtsuka@gunma-u.ac.jp

Stability of scale-critical circular flows in a two-dimensional exterior domain

Yasunori Maekawa (Tohoku University)

The circular flows are typical flows we often observe in our lives; typhoon exhibits a circular flow pattern of clouds, tornado is an updraft with a strong swirling flow, a rotating disk leads to a circular flow around it. In this talk we discuss a special class of two-dimensional circular flows for viscous incompressible fluids, having a critical decay in space or time in view of scaling. The first one we consider is the Lamb-Oseen vortex, denoted by $\alpha U^G(t,x)$, where α is a given real number which represents a circulation at spatial infinity, while U^G is the velocity field defined as

$$U^G(t,x) = \frac{x^\perp}{2\pi|x|^2}\big(1 - e^{-\frac{|x|^2}{4t}}\big)\,, \qquad x^\perp = (-x_2, x_1)\,. \tag{1}$$

For each α the Lamb-Oseen vortex αU^G is an exact forward self-similar solution to the Navier-Stokes equations in $\mathbb{R}^2$:

$$\partial_t u - \Delta u + u\cdot\nabla u + \nabla p = 0\,, \quad \operatorname{div} u = 0\,, \qquad t>0\,, \quad x\in\mathbb{R}^2\,. \tag{NS}$$

Here $u = u(t,x) = (u_1(t,x), u_2(t,x))$ and $p = p(t,x)$ are the velocity field and the pressure field, respectively. We have used the standard notation for derivatives: $\partial_t = \frac{\partial}{\partial t}$, $\partial_j = \frac{\partial}{\partial x_j}$, $\Delta = \sum_{j=1}^2 \partial_j^2$, $\operatorname{div} u = \sum_{j=1}^2 \partial_j u_j$, $u\cdot\nabla u = \sum_{j=1}^2 u_j \partial_j u$. It is well known that (NS) is invariant under the scaling:

$$u_\lambda(t,x) = \lambda u(\lambda^2 t, \lambda x)\,, \quad p_\lambda(t,x) = \lambda^2 p(\lambda^2 t, \lambda x)\,, \qquad \lambda>0\,. \tag{2}$$

One can easily check that $U^G(t,x)$ satisfies the invariant property with respect to the scaling (2), $U^G_\lambda(t,x) = U^G(t,x)$ for any $\lambda>0$, and the norm

$$\sup_{t>0}\|u(t)\|_{L^{n,\infty}(\Omega)} + \sup_{t>0} t^{\frac14}\|u(t)\|_{L^{2n}(\Omega)} \tag{3}$$

with $n=2$ and $\Omega=\mathbb{R}^2$ of U^G is finite. Here $L^{n,\infty}(\Omega)$ is the weak-L^n space. Note that (3) with $\Omega = \mathbb{R}^n$ is an invariant norm under the scaling (2). The unique existence of solutions to the Navier-Stokes equations possessing the small invariant norm (3) was proved by Kozono-Yamazaki [14] when Ω is an exterior domain in $\mathbb{R}^n$ with $n \geq 2$ under the no-slip boundary condition on u.

The asymptotic stability of the Lamb-Oseen vortex was firstly studied by Giga-Kambe [7] in 1988, and their result was extended by Carpio [3] and Gallay-Wayne [5]. In particular, it is shown in [5] that for any α the Lamb-Oseen vortex αU^G is asymptotically stable with respect to arbitrary initial perturbation u_0 such that its vorticity $\omega_0 = \partial_1 u_{0,2} - \partial_2 u_{0,1}$ is integrable and satisfies the zero mass condition $\int_{\mathbb{R}^2} \omega_0 \, \mathrm{d}x = 0$; see Giga-Giga-Saal [6] for details on this problem and related topics.

Although the velocity in (1) does not satisfy the prescribed boundary condition in general, it is possible to formulate the stability problem of the Lamb-Oseen vortex even in exterior domains. However, the approach used in [7, 3, 5], which is for the case $\Omega = \mathbb{R}^2$, is not applied to the case of exterior domains. The reason is that the vorticity formulation is essentially used there, while in the presence of nontrivial boundary the no-slip boundary condition on u leads to a production of vorticity near the boundary and it is hard in general to obtain useful information on this vorticity production. Recently, based on a new energy estimate for the perturbed velocity the global stability of αU^G is proved by Gallay-M. [4] for sufficiently small $|\alpha|$ also in the exterior problem, and this result is further extended by the author [15] to the small scale-critical flow satisfying (3) (with $n = 2$); see also Iftimie-Karch-Lacave [11] for asymptotic behaviors of exterior flows. In particular, we have the following result for two-dimensional exterior flows.

Theorem 1 ([4, 15]) *If $|\alpha|$ is sufficiently small then the Lamb-Oseen vortex αU^G is asymptotically stable with respect to arbitrary L^2 initial perturbations for the Navier-Stokes equations in two-dimensional exterior domains (under the no-slip boundary condition on the velocity fields).*

The L^2 stability of the two-dimensional scale-critical flow as in Theorem 1 is nontrivial even if the norm (3) (with $n = 2$) is assumed to be sufficiently small. This is because the Hardy-type inequality

$$|\langle u \cdot \nabla v, v \rangle_{L^2(\Omega)}| \leq C \|u\|_{L^{n,\infty}(\Omega)} \|\nabla v\|^2_{L^2(\Omega)}, \qquad v \in \dot{H}^1_0(\Omega) \tag{4}$$

is not available in general when Ω is a domain in $\mathbb{R}^n$ with $n = 2$, which leads to a serious difficulty in obtaining the uniform bound for the kinetic energy of the perturbation such as $\sup_{t>0} \|v(t)\|_{L^2(\Omega)} < \infty$. This is contrastive to the higher dimensional case, where a unified approach has been established for the global L^2 stability of small scale-critical flows by using the Hardy-type inequality (4);

see Karch-Pilarczyk-Schonbeck [12] and Hishida-Schonbeck [10]. In [4, 15] the difficulty in the two-dimensional case has been overcome by establishing the logarithmic growth energy estimate for the perturbation flow, which is then combined with the low frequency analysis used in Borcher-Miyakawa [1] and Kozono-Ogawa [13]. The stability of the Lamb-Oseen vortex for not small $|\alpha|$ is still open in the case of exterior domains.

Next let us take $t \to 0$ in (1), which yields a steady circular flow

$$U(x) = \frac{x^{\perp}}{2\pi|x|^2}\,, \qquad x^{\perp} = (-x_2, x_1)\,, \quad x \neq 0\,. \tag{5}$$

For each α the flow αU is a steady self-similar solution to the two-dimensional Navier-Stokes equations in $\mathbb{R}^2 \setminus \{0\}$. Another important aspect of αU is that it defines a stationary flow around a rotating disk. Indeed, αU is a stationary solution to the following Navier-Stokes equations in the exterior disk $\Omega = \{x \in \mathbb{R}^2 \mid |x| > 1\}$, which is regarded as a simplest model of the flow around a rotating obstacle in two dimensions:

$$\begin{cases} \partial_t u + u\cdot\nabla u \;=\; \Delta u - \nabla p\,, & \text{for } x\in\Omega\,, \quad t>0\,,\\ \ \operatorname{div} u \;=\; 0\,, & \text{for } x\in\Omega\,, \quad t\geq 0\,,\\ u(x,t) \;=\; \frac{\alpha}{2\pi}x^{\perp}\,, & \text{for } x\in\partial\Omega\,, \quad t>0\,,\\ u(x,0) \;=\; u_0(x)\,, & \text{for } x\in\Omega\,. \end{cases} \tag{NS$_\alpha$}$$

Note that U has a scale-critical decay $O(|x|^{-1})$ as $|x| \to \infty$. For the three-dimensional exterior problem Borchers-Miyakawa [2] established the existence and the stability of small stationary solutions decaying in the scale-critical order $O(|x|^{-1})$; see also [12, 10] for recent stability results in the three-dimensional case. As for the two-dimensional exterior problem, Yamazaki [17] proved the existence of small stationary solutions having $O(|x|^{-1})$ decay under some symmetry conditions on both domains and given data, and Hillairet-Wittwer [8] constructed stationary solutions near the circular flow αU when $|\alpha|$ is large. Recently, the asymptotic behavior of the two-dimensional steady Stokes flow around a rotating obstacle is investigated in details by Hishida [9], where it is shown that the leading profile is given by a constant multiple of the circular flow U. On the other hand, in the two-dimensional case, little seems to be known about the stability of stationary flows decaying in the critical order $O(|x|^{-1})$, again due to the absence of the Hardy-type inequality (4) for $\Omega \subset \mathbb{R}^n$ with $n = 2$. In fact, the argument in [4, 15] essentially uses the bound $\sup_{t>0} t^{1/4}\|u(t)\|_{L^4(\Omega)} \ll 1$ in showing the L^2 stability of u, and therefore, it does not work for the stability problem of stationary flows. As far as the author knows, so far there is no general stability result for two-dimensional stationary flows when they decay like $O(|x|^{-1})$ as

$|x| \to \infty$, even under the smallness condition on both stationary flows and initial perturbations.

In this talk we will discuss the local L^2 stability of αU in the exterior disk. Although the result is limited to such a specific flow and a domain, the next theorem seems to be the first contribution to the stability problem of the two-dimensional exterior flow in the situation such that the Hardy-type inequality is not available.

Theorem 2 ([16]) *If $|\alpha|$ is sufficiently small then the stationary flow αU in* (5) *to the Navier-Stokes equations* (NS_α) *in the exterior disk $\Omega = \{x \in \mathbb{R}^2 \mid |x| > 1\}$ is asymptotically stable with respect to small L^2 initial perturbations.*

It should be emphasized here that no symmetry condition is imposed on the perturbations in Theorem 2. The key step of the proof is the spectral analysis for the linearized operator

$$\begin{aligned} D_{L^2_\sigma}(A_\alpha) &= W^{2,2}(\Omega)^2 \cap W^{1,2}_0(\Omega)^2 \cap L^2_\sigma(\Omega)\,, \\ A_\alpha v &= -\mathbb{P}\Delta v + \alpha \mathbb{P}\big(U \cdot \nabla v + v \cdot \nabla U\big)\,, \qquad v \in D_{L^2_\sigma}(A_\alpha)\,. \end{aligned} \tag{6}$$

Here $L^2_\sigma(\Omega) = \overline{\{f \in C_0^\infty(\Omega)^2 \mid \mathrm{div}\, f = 0 \ \text{ in } \Omega\}}^{\|\cdot\|_{L^2(\Omega)}}$ is the space of solenoidal vector fields in $L^2(\Omega)^2$, and $\mathbb{P} : L^2(\Omega)^2 \to L^2_\sigma(\Omega)$ is the Helmholtz projection. Some details on the spectrum of A_α will be presented in the talk.

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Partial Differential Equations on Evolving Domains

Charles M. Elliott, University of Warwick

Abstract

We present an abstract framework for treating the theory of well-posedness of solutions to abstract parabolic partial differential equations on evolving Hilbert spaces. This theory is applicable to variational formulations of PDEs on evolving spatial domains including moving hyper-surfaces. We formulate an appropriate time derivative on evolving spaces called the material derivative and define a weak material derivative in analogy with the usual time derivative in fixed domain problems; our setting is abstract and not restricted to evolving domains or surfaces. Then we show well-posedness to a certain class of parabolic PDEs under some assumptions on the parabolic operator and the data. Specifically, we study in turn a surface heat equation, an equation posed on a bulk domain, a novel coupled bulk-surface system and an equation with a dynamic boundary condition. We give some background to applications, primarily in cell biology. We describe how the theory may be used in the numerical analysis of evolving surface finite element methods and give some computational examples involving the coupling of surface evolution to processes on the surface.

Topological constraints and structures in macro (fluid and plasma) systems

Z. Yoshida *
Graduate School of Frontier Sciences, University of Tokyo

1 Casimir invariants

The theory of mechanics is built from two elements: matter and space; the former is formulated by an energy = Hamiltonian, while the latter is mathematically a geometry. A Hamiltonian is a function on a phase space X, and the geometry of X is dictated by a Poisson bracket $\{F, G\}$ (F and G are functions on X), and is called a Poisson manifold.

A complex form of the Hamiltonian (for example, the Ginzburg-Landau free energy that has multiple equilibrium points) is often the root cause of nontrivial structures or dynamics. But this is not the case for a *weakly coupled system* like a usual fluid or a plasma, in which the Hamiltonian is equivalent to the norm of the phase space. Then, the equilibrium point is just the "vacuum" that bears no structure. However, we do observe diverse structures created in a fluid or a plasma (which are typically "vortical" like a tyhoon or a galaxy). It must be, then, some structure of the space that imparts interesting structures to such a system. When the Poisson manifold is foliated by *topological constraints* so that the state vector can move only on a *leaf* embedded in X, the effective energy is the restriction of the Hamiltonian on the leaf, which may be appreciably distorted by the curvature of the leaf.

Topological constraints are caused by the degeneracy of the Poisson bracket (mathematically the *center* of the Poisson algebra $C^\infty_{\{\ ,\ \}}(X)$). We call a function (observable) C a *Casimir invariant*, if $\{C, F\} = 0$ for all F. In fact, such C is invariant: Given a Hamiltonian H, $\mathrm{d}C/\mathrm{d}t = \{C, H\} \equiv 0$ (notice that the constancy of C is independent of the choice of H, which is

*This work was done in collaboration with P J Morrison of Department of Physics, University of Texas at Austin.

in marked contrast to usual invariants that pertain to some symmetries of a specific Hamiltonian, i.e., Noether charges).

Here we proffer the following visions:

1. A Casimir invariant may be viewed as an *adiabatic invariant*, which is an action variable separated by coarse-graining a “microscopic” angle variable.

2. By extending the phase space, a Casimir invariant can be converted to a Noether charge corresponding to a gauge symmetry of “macroscopic” variables.

Connecting 1 and 2, the coarse-grained microscopic variable is the gauge freedom of the macroscopic variables.

The point is the incorporation of the notion of “scale” —the merit of doing so is not only in providing Casimir invariants (topological constraints) with physical interpretations, but also in formulating a systematic and physically meaningful method of singular perturbations to “unfreeze” the topological constraints.

In this talk, we put a simple example of fluid equations into the perspective. We draw heavily on the previous works [1, 2, 3, 4].

2 Ideal vortex dynamics

We consider an incompressible ideal fluid on $\Omega = T2$, which obeys Euler's equation of motion:

$$\partial_t \boldsymbol{v} + (\boldsymbol{v} \cdot \nabla)\boldsymbol{v} = -\nabla p, \tag{1}$$

$$\nabla \cdot \boldsymbol{v} = 0, \tag{2}$$

where $\boldsymbol{v}$ is a 2-dimensional vector field (1-form) representing the velocity field, and p is a scalar field representing the fluid pressure. On $T2$, we may represent $\boldsymbol{v} = \mathrm{d}\varphi = {}^t(\partial_y\varphi, -\partial_x\varphi)$ with a scalar field φ. The vorticity ω is the exterior derivative of $\boldsymbol{v}$, which reads $\omega = -\Delta\varphi$. Inverting the Laplacian, we will write $\varphi = \mathcal{K}\omega$. The exterior derivative of (1) gives the *vorticity equation*

$$\partial_t \omega = [\omega, \mathcal{K}\omega], \tag{3}$$

where $[a, b] = \partial_y a \cdot \partial_x b - \partial_x a \cdot \partial_y b$.

We may cast (3) into a Hamiltonian form (see [1] for a mathematical justification). We define a Poisson bracket

$$\{F, G\}_\omega = \langle \partial_\omega F, [\omega, \partial_\omega G] \rangle, \tag{4}$$

where $\langle \ , \ \rangle$ is the inner product of the phase space $X_\omega = \{\omega \in C(\Omega)\}$. With a Hamiltonian

$$H(\omega) = \frac{1}{2}\int_\Omega (\mathcal{K}\omega)\cdot\omega \, \mathrm{d}^2 x, \tag{5}$$

the adjoint equation $\mathrm{d}F/\mathrm{d}t = \{H, F\}_\omega$ is equivalent to (3). Evidently, $C_h = \int h(\omega)\,\mathrm{d}^2 x$ (h is an arbitrary smooth function) is a Casimir invariant (especially, $\int \omega^2\,\mathrm{d}^2 x$ is the appreciated *enstrophy*).

We extend the phase space by including a *phantom field* ψ, and define an extended Poisson algebra $C^\infty_{\{\ ,\ \}_\zeta}(X_\zeta)$ by

$$\begin{cases} \zeta = \begin{pmatrix} \omega \\ \psi \end{pmatrix} \in X_\zeta, \\ \\ \{F, G\}_\zeta = \langle \partial_\zeta F, \mathcal{J}_\zeta \partial_\zeta G\rangle, \quad J_\zeta = \begin{pmatrix} [\omega, \circ] & [\psi, \circ] \\ [\psi, \circ] & 0 \end{pmatrix}. \end{cases} \tag{6}$$

The extended system (6) has two different types of Casimir invariants:

$$C_f = \int \omega f(\psi)\,\mathrm{d}^2 x, \tag{7}$$

$$C_g = \int g(\psi)\,\mathrm{d}^2 x, \tag{8}$$

where f and g are arbitrary smooth functions.

Remark 1 (phantom field) *As far as the Hamiltonian H is independent of ψ, the phantom ψ co-moves with ω without causing any change in the evolution of ω. If we include ψ into H, however, it influences the dynamics (then, we say that ψ is actualized). For example, when we consider*

$$H(\omega, \psi) = \frac{1}{2}\int_\Omega [(\mathcal{K}\omega)\cdot\omega + (-\Delta\psi)\cdot\psi]\,\mathrm{d}^2 x, \tag{9}$$

the corresponding Hamilton's equation represents the ideal magnetohydrodynamics (ψ is the magnetic flux) [5, 6].

Let us consider a canonical Poisson algebra $C^\infty_{\{\ ,\ \}_z}(X_z)$ with

$$\begin{cases} z = \begin{pmatrix} q \\ p \end{pmatrix} \in X_z, \\ \\ \{F, G\}_z = \langle \partial_z F, \mathcal{J}_z \partial_z G\rangle, \quad \mathcal{J}_z = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}, \end{cases} \tag{10}$$

We relate the phase spaces X_ζ and X_z by

$$\omega = [p,q], \quad \psi = p. \tag{11}$$

One may regard that writing $\omega = [p,q]$ is a kind of parameterization of ω by two fields q and p. By "chain rule", we obtain

Lemma 1 *For all $F(\omega,\psi) = F([p,q],p)$ and $G(\omega,\psi) = G([p,q],p)$,*

$$\{F,G\}_\zeta = \{F,G\}_z. \tag{12}$$

Under the parameterization (11), $C_f = \int [p,q] f(p)\,\mathrm{d}^2x = \int [\Phi(p),q]\,\mathrm{d}^2x = 0$ (Φ is the primitive function of f), implying that this parameterization restricts the Poisson manifold X_z to the leaves of $C_f = 0$. However, the other set of invariants C_g is not trivial. The invariance of C_g in the canonical system $C^\infty_{\{\ ,\ \}_z}(X_z)$ is due to the symmetry of a Hamiltonian (and all other observables) forced by (11). In fact, $C_g = \int g(p)\,\mathrm{d}^2x$ is a Noether charge corresponding to the gauge symmetry of the parameterization (11):

$$\mathrm{ad}^*_{C_g} = \mathcal{J}_z \partial_z C_g = \begin{pmatrix} g'(p) \\ 0 \end{pmatrix}$$

generates the infinitesimal gauge transformation $q \mapsto q + \epsilon g'(p)$, where g' is the derivative of g. The co-adjoint orbit $\mathrm{Ad}^*_{C_g}(\tau)$ ($\tau \in \mathbb{R}$) defines an *angle variable* Θ_g conjugate to the *action variable* C_g; solving $\{\Theta_g, C_g\}_z = 1$, we obtain

$$\Theta_g = \frac{\langle q, g'(p)\rangle}{\|g'(p)\|^2}.$$

Notice that the set of canonical action-angle pairs C_g and Θ_g span an infinite dimension.

We call X_ζ a "macroscopic" phase space, and X_z a "microscopic" phase space. The quotient of X_z by $\mathrm{Ad}^*_{C_g}(\tau)$ ($\tau \in \mathbb{R}$) mod-outs (or, coarse-grains) the microscopic angle variable Θ_g, by which the macroscopic quantity C_g is frozen.

We may refine Lemma 1 as

Theorem 1 *The reduction of $C^\infty_{\{\ ,\ \}_z}(X_z)$ by the gauge symmetry group $G = \{\mathrm{Ad}^*_{C_g}(\tau);\ \tau \in \mathbb{R}, \forall g\}$ yields $C^\infty_{\{\ ,\ \}_\zeta}(X_\zeta)$, i.e.,*

$$C^\infty_{\{\ ,\ \}_z}(X_z/G) = C^\infty_{\{\ ,\ \}_\zeta}(X_\zeta).$$

One may compare the present argument with the well-known story of "reduction for canonization" [7] —given a leaf as a co-adjoint orbit of some Casimir invariants, one may produce a symplectic leaf by mod-outing the conjugate angle variables. The canonical system is, then, of a smaller phase space. Here, we are exploring the opposite direction, i.e., "extension for canonization". First, we delineate how a reduction of "microscopic variable" yields a noncanonical (degenerate) Poisson bracket, and then, changing the perspective, we recover the "microscopic variables" to canonize the system. We can unfreeze each C_g by including the corresponding Θ_g into the Hamiltonian: $\mathrm{d}C_g/\mathrm{d}t = \{C_g, H([q,p],p,\Theta_g)\}_z$.

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A Journey through the World of Incompressible Viscous Fluid Flows: an Evolution Equation Perspective

Matthias Hieber

TU Darmstadt, Germany

Incompressible fluids are subject to the following system of balance laws

$$\begin{aligned} \varrho(\partial_t + u\cdot\nabla)u + \nabla\pi &= \text{div } S && \text{in } \Omega,\\ \text{div } u &= 0 && \text{in } \Omega,\\ \varrho(\partial_t + u\cdot\nabla)\epsilon + \text{ div } q &= S:\nabla u && \text{in } \Omega, \end{aligned}$$

where u, ϱ denote the velocity and density of the fluid, π its pressure, S the stress tensor, ϵ the internal energy, q the heat flux and $\Omega \subset \mathbb{R}^n$ a bounded domain with smooth boundary. The above equations represent the balance laws for the momentum, mass and energy of the fluid, respectively.

Neglecting the balance law for the energy and choosing $S = 0$, one obtains *Euler's equations*, whereas choosing in this case $S = S_{Newton} = 2\mu D(u)$, we obtain the equations of *Navier-Stokes*. Here μ denotes the viscosity of the fluid and $D(u)$ its deformation tensor given by $D(u) = 1/2[\nabla u + (\nabla u)^T]$.

In this talk we discuss various models for *incompressible viscous flows* including the equations of Navier-Stokes, the primitive equations of ocean dynamics, viscoelastic fluids of Oldroyd-B type as well as the Ericksen-Leslie model for the flow of nematic liquid crystals.

Starting with the equations of *Navier-Stokes*, we mainly concentrate on strong solutions within the L^p-setting. Our strategy for obtaining strong solutions is to rewrite the Navier-Stokes as an evolution equation of the form

$$u'(t) - Au(t) = -P[u(t)\cdot\nabla)u(t)], \quad u(0) = u_0,$$

where A denotes the Stokes operator and P the Helmholtz projection. We then convert this equation into an integral equation of the form

$$u(t) = e^{tA}u_0 - \int_0^t e^{(t-s)A}P[(u(s)\cdot\nabla)u(s)]ds,$$

and aim to solve this integral equation via fixed point methods. Of central importance in this context are properties of *Stokes operator* A and the *Stokes semigroup* e^{tA}. Pioneering key results in this direction are due to Y. Giga, see [Gig81] and [Gig85]. We survey several results on strong solvability of the Navier-Stokes equations in scaling invariant spaces, in particular in

$$\dot{H}^{\frac{1}{2}}(\mathbb{R}^3) \hookrightarrow L^3(\mathbb{R}^3) \hookrightarrow B_{p,\infty}^{-1+\frac{3}{p}}(\mathbb{R}^3) \hookrightarrow BMO^{-1}(\mathbb{R}^3) \hookrightarrow B_{\infty,\infty}^{-1}(\mathbb{R}^3),$$

and observe that $L^p - L^q$-smoothing properties of the Stokes semigroup T and gradient estimates of the form $\|\nabla e^{tA} f\|_p \leq Ct^{-1/2}\|f\|_p$ for $t > 0$ as described by Y. Giga in [Gig86] are of crucial importance.

Abe and Giga introduced in [AG13] a blow up argument which implies that the Stokes operator generates an *analytic semigroup also on* $L^\infty_\sigma(\Omega)$ for a large class of domains $\Omega \subset \mathbb{R}^n$. We present here a direct approach to L^∞-a priori estimates for the Stokes equation, which is described in detail in [AGH15]. Our approach to L^∞ -estimates for the solution of the Stokes equation is inspired by the Masuda-Stewart technique for elliptic operators and allows to obtain a rather general picture of the Stokes semigroup acting on spaces of bounded functions; see also [HM14], [BH15].

The *primitive equations of ocean dynamics* are a fundamental model for many geophysical flows. They are described by a system of equations which are derived from the equations of viscous incompressible flows by assuming that the vertical motion is modeled by the hydrostatic balance. Starting from a fundamental well-posedness result due to Cao and Titi [CT07], we describe a new strategy for obtaining global strong well-posedness of the three dimensional primtive equations in L^p-spaces for a rather general class of initial data, see [HK14]. Our approach is based on the fact that the the newly defined *hydrostatic Stokes operator* generates an analytic semigroup on a certain subspace of L^p associated with the newly defined hydrostatic Helmholtz projection as well as on H^2 a priori bounds.

We then turn our attention to *viscoelastic fluids of Oldroyd-B type* and their stability properties. In this case, the stress tensor S is determined by $S = S_N + S_e$, where $S_N = 2\mu\frac{\lambda_2}{\lambda_1}D(u)$ corresponds to the Newtonian part and S_e to the purely elastic part, which is described by a differential equation. Here λ_1 and λ_2 denote the relaxation and retardation time of the fluid and $\alpha = 1 - \lambda_2/\lambda_1 \in (0, 1)$ the retardation parameter of the system. We are interested in stability questions for this type of fluids in exterior domains $\Omega \subset \mathbb{R}^3$, see [GHN14]. Since 0 lies in the the spectrum of the linearized problem, questions of this type are delicate. We show first that the solution of linearized equation is governed by a bounded analytic semigroup T on $L^p(\Omega) \times W^{1,p}(\Omega)$. If α is close to 0, then the angle φ of analyticity of T is close to $\pi/2$ representing the parabolic character of the fluid equation. On the other hand, if α is close to 1, then φ is close to 0, representing the hyperbolic character of the transport equation. Showing that T is strongly stable, we see moreover that the trivial solution of this system is asymptotically stable in the sense that any solution starting in a small ball around the origin converges towards 0 as $t \to \infty$.

Finally, we discuss the *general Ericksen-Leslie model describing the flow of nematic liquid crystals in a thermodynamically consistent way*. The model reads as

$$\begin{aligned}
\rho\mathcal{D}_t u + \nabla\pi &= \operatorname{div} S && \text{in } \Omega,\\
\operatorname{div} u &= 0 && \text{in } \Omega,\\
\rho\kappa\mathcal{D}_t\theta + \operatorname{div} q &= S : \nabla u + \operatorname{div}(\lambda\nabla)d \cdot \mathcal{D}_t d + (\theta\partial_\theta\lambda)\nabla d\nabla\mathcal{D}_t d && \text{in } \Omega,\\
\gamma\mathcal{D}_t d - \mu_V V d - \operatorname{div}[\lambda\nabla]d &= \lambda|\nabla d|^2 d + \mu_D P_d D d && \text{in } \Omega,
\end{aligned}$$

The variables θ, d denote the temperature and the so called *director*, $\mathcal{D}_t = \partial_t + u \cdot \nabla$ the Lagrangian derivative, $P_d = I - d \otimes d$ and V the vorticity tensor. These equations are supplemented by the thermodynamical laws for the internal energy ϵ, the entropy η, the heat capacity κ and the Ericksen tension λ and by the *constitutive laws* for

$$S = S_N + S_E + S_L,$$

where S_E and S_L denotes the Ericksen and Leslie stress, respectively, see [Eri62].

Our strategy for obtaining strong global well-posedness of the above system for data close to equilibria points is to consider the system as a quasilinear parabolic evolution equation within the L^p-setting and to apply methods from maximal L^p-regularity; see [HP15]. We explain key points of our analysis at various simplifications of the above model (see e.g., [HNPS13]) and develop a rather complete understanding of the underlying dynamics of the full model.

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Stochastic Three-Dimensional Rotating Navier-Stokes Equations: Averaging, Convergence, Regularity and 3D Nonlinear Dynamics

Alex Mahalov[1,*]
The Wilhoit Foundation Dean's Distinguished Professor
[1] Arizona State University, USA, *mahalov@asu.edu

Dedicated to Professor Yoshikazu Giga on his 60th birthday

Abstract

We consider stochastic three-dimensional rotating Navier-Stokes equations and prove averaging theorems for stochastic problems in the case of strong rotation. Regularity results are established by bootstrapping from global regularity of the limit stochastic equations and convergence theorems. The effective covariance operator is computed using Ito's stochastic calculus and averaging theorems for operator valued processes. The energy injected in the system by the noise is large, the initial conditions have large energy, and the regularization time horizon is long for the 3D stochastic dynamics (infinite time regularity is proven in the deterministic case). Regularization is the consequence of precise mechanisms of relevant three-dimensional nonlinear interactions. We establish multiscale averaging and convergence theorems for the stochastic dynamics. These stochastic averaging, convergence and regularity results hold for many important physical systems described by three-dimensional Navier-Stokes and Maxwell PDEs coupled with fast wave dynamics.

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Boundary integral methods for implicitly defined interfaces

Y.-H. Richard Tsai, University of Texas

Abstract

I will present a new approach for computing boundary integrals that are defined on implicit interfaces, without the need of explicit parameterization. A key component of this approach is a volume integral which is identical to the integral over the interface. I will show results applying this approach to simulate interfaces that evolve according to Mullins-Sekerka dynamics used in certain phase transition problems. I will also discuss our latest results in generalization of this approach to summation of unstructured point clouds.

Locomotion of animals, design of robots and mathematics

Ryo Kobayashi

JST, CREST

Department of Mathematical Life Sciences, Hiroshima University

Why can animals show amazingly sinuous and robust motion under unpredictable complex environments ? It is because animals have a large number of degrees of freedom in their bodies and can orchestrate them very well. Even for the most advanced robots today, such abilities are difficult to attain. In order to create animal-like robots, *autonomous decentralized control* (ADC) is the key concept that facilitates real-time control of a large number of degrees of freedom corresponding to the changing surroundings. We propose a simple design principle of ADC, which is termed as *discrepancy control*; then, we test it by implementing it in various types of robots.

It is known that animals control their large number of degrees of freedom in a well-coordinated manner by means of distributed neural networks called central pattern generators (CPGs)[1] which generates rhythmic signals. Adopting CPG as a controller seems to be an advantageous approach to realizing ADC in robots; however, its design principle has not been established thus far. Our strategy for obtaining the design principle of ADC is to learn from the most primitive living system, *true slime mold*. Every part of this creature exhibits a contraction oscillation with the period about 2 minutes, and the protoplasmic flow is caused by the pressure difference. Distributed oscillators are considered to be *mechanically* coupled through the protoplasm.

We proposed a mathematical model of the oscillatory motion of true slime mold (Fig.1)[2]. In this model, the oscillators receives the feedback signals to make the phase shift so as to decrease $p^2/2$ where p is a pressure. The term $p^2/2$ was named *discrepancy function* because it is an indicator of discrepancy between the real state and the state specified by the controller. We applied this idea to design the feedback signal in the ADC of the robot as indicated in Fig.2. The robot has actuators S_i controlled by the phase oscillator ϕ_i, and the controller ϕ_i forms a coupled oscillator system through the direct communication $g_{ij}(\phi_i, \phi_j)$. Each controller ϕ_i also receives a feedback signal obtained from the discrepancy function. The equation of the controller is given as

$$\partial_t \phi_i = \omega_i + \sum_j g_{ij}(\phi_i, \phi_j) - \partial_{\phi_i} I_i, \tag{1}$$

where I_i is a discrepancy function defined by the locally accessible variables. Though the discrepancy function is designed for each individual case, it expresses the locally accumulated stress in general. It is essential that the controllers interacts not only through

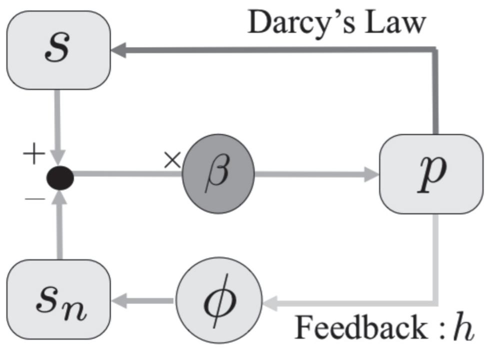

Figure 1: Diagram of the model of the motion of true slime mold [2]. ϕ : distributed phase oscillator, s : thickness of plasmodium, s_n : target thickness of plasmodium controlled by the phase oscillator, p : pressure given by $p = \beta(s - s_n)$, h : feedback signal given by $h = -\sigma\partial_\phi(p^2/2)$.

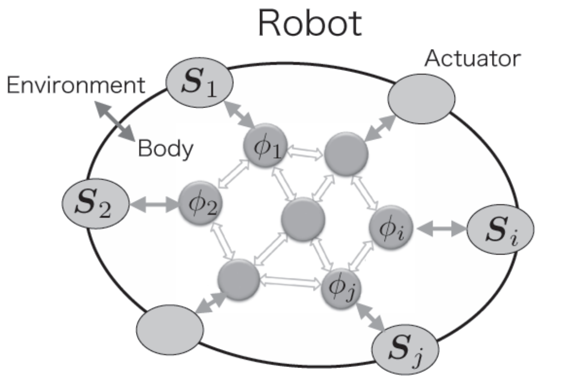

Figure 2: Schematic description of the robot whose actuators S_i are controlled by the local oscillators ϕ_i. Each oscillators receives the feedback signal which makes the discrepancy (locally defined quantity) decrease. Good design of the discrepancy function makes well coordinated motion.

the direct communication, but also through the mechanical coupling between actuators. They can also get the information of environments through the feedback signal. We implemented this ADC scheme to several type of robots, *e.g.* amoeboid robot Slimy[3], snake robot HAUBOT[4] and quadruped robot OSCILLEX[5]. In this presentation, we will demonstrate HAUBOT and OSCILLEX.

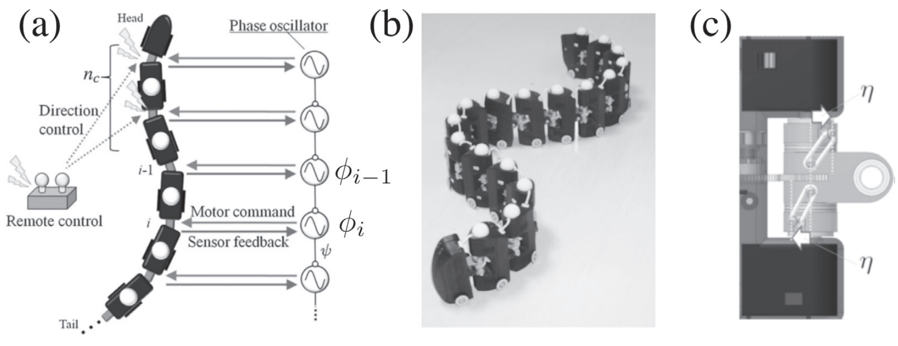

Figure 3: (a) Schematic of HAUBOT (b) HAUBOT (c) Distortion caused by opposite rotations of the upper and the lower motor can adjust the degree of muscle tonus.

HAUBOT has a one-dimensional link mechanism as a backbone, and it generates a motion similar to snakes' lateral undulation by giving torque to each joint. Let us set the variables as follows, θ_i: angle of the ith joint, $\bar{\theta}_i$: target angle of the ith joint, ϕ_i : oscillator which controls the target angle. Target angle is given by $\bar{\theta}_i = \theta_0 \sin\phi_i$ for $i > n_c$ and $\bar{\theta}_i = \theta_0 \sin\phi_i + \theta_d$ for $1 \leq i \leq n_c$, where θ_d is a direction control signal which is given remotely. Phase oscillators is driven by the formula

$$\partial_t\phi_i = \omega + \epsilon \sin(\phi_{i-1} - \phi_i - \psi) - \sigma\partial_{\phi_i} I_i \tag{2}$$

where the discrepancy function is given by $I_i = |\theta_i - \bar{\theta}_i|$. In the design of HAUBOT, not only a phasic feedback but also a tonic feedback is taken into account. Tonic feedback is given by the policy "strengthen the stiffness in more stressed actuators". The variable η_i characterizes the muscle tonus by setting the upper and lower target angles by $\bar{\theta}_i^u = \bar{\theta}_i + \eta_i$ and $\bar{\theta}_i^l = \bar{\theta}_i - \eta_i$. The variable itself follows the equation

$$\partial_t \eta_i = \alpha(\beta I_i - \eta_i). \tag{3}$$

The experiments shows that the phasic feedback enhances the energy efficiency and the tonic feedback achieves powerful motion, and both of them can collaborate to make a good performance.

OSCILLEX is a quadruped robot. The most characteristic feature is that its four legs are controlled not by central controller but by the local phase oscillators ϕ_i $(i = 1, 2, 3, 4)$ independently. In addition, the controllers has no direct communication ($g_{ij} = 0$), thus they interact only through mechanical coupling through their body. By defining the discrepancy function as $I_i = N_i \sin \phi_i$, the equation of each oscillator is given by the simple equation

$$\partial_t \phi_i = \omega - \sigma N_i \cos \phi_i, \tag{4}$$

where N_i is a load to the toe of the ith leg. The biggest advantage of this robot is that no time is needed to get into the stationary walking state, while it usually takes some time for the initial transition if the oscillators are used for the controller. This good property is achieved by the fact that the controllers goes into the excitable state from the oscillatory state when the load surpass the critical value ω/σ. Also the gait pattern is automatically generated corresponding to the weight balance of the body automatically.

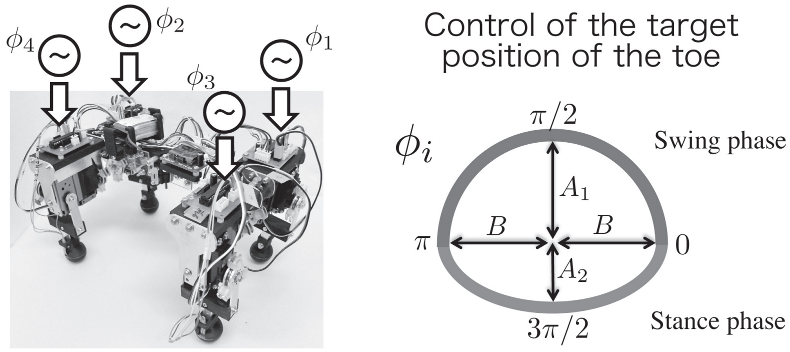

Figure 4: Quadruped robot OSCILLEX [5]

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Transport of Charged Particles in Biological Environments

Chun Liu, Penn State University

Abstract

Almost all biological activities involve transport of special particles or molecules in complicated environments. In this talk, I will discuss the diffusion of those particles with electric interaction, and those in undiluted solutions, where size effects and relative drags become important. In particular, I will make connections between these generalized diffusion and the classic systems such as porous media equations and cross diffusion systems.

Fluid flow and rotation: a fascinating interplay

Jürgen Saal

Rotating fluid phenomena appear numerously in applications. Corresponding models have undergone a substantial mathematical development in the recent two decades. Starting from groundbreaking works of Babin, Mahalov, and Nicolaenko around 2000 on the rotating Navier-Stokes equations

$$\begin{cases} \partial_t v - \mu\Delta v + \omega \mathbf{e}_3 \times v + (v\cdot\nabla)v &= -\nabla p \quad \text{in } (0,T)\times G, \\ \text{div } v &= 0 \quad \text{in } (0,T)\times G, \\ v &= 0 \quad \text{on } (0,T)\times \partial G, \\ v|_{t=0} &= v_0 \quad \text{in } G, \end{cases} \tag{0.1}$$

as the basic model for a rotating fluid, since then many related models have beed considered. For instance, for fluid flow past a rotating obstacle the linearly growing drift term $(\omega \times x)\cdot\nabla u$ has to be added, which can change regularity and stability behavior completely [7, 8]. In technological applications such as the spin-coating process even a free boundary part enters in the model, i.e., then we have $G = G(t)$ [4]. Another wide branch of rotating fluids is given by the field of geostrophic boundary layers. One of the most basic examples here is represented by the Ekman boundary layer. Setting $G = \mathbb{R}^3_+$ (half-space) and $v_0 = \mathbf{U}^E + \tilde{v}_0$ by, (0.1) turns into a commonly accepted model for the Ekman boundary layer [10]. Here $\mathbf{U}^E$ is the famous Ekman spiral given by

$$\mathbf{U}^E(x_3) = U_\infty(1 - e^{-x_3/\delta}\cos(x_3/\delta),\ e^{-x_3/\delta}\sin(x_3/\delta),\ 0)^T, \quad x_3 \geq 0. \tag{0.2}$$

where U_∞ denotes the total velocity of the flow. The parameter δ denotes the layer thickness given by $\delta = \sqrt{2\mu/|\omega|}$. The couple $(\mathbf{U}^E, p^E)$ with pressure

$$p^E(x_2) = -\omega U_\infty x_2$$

represents an exact steady state solution of the Ekman boundary layer problem.

Whereas in generator systems rotation is produced through fluid flow, in other technological processes such as spin-coating or in geostrophic layers rotation is the driving force to influence fluid properties. For instance, it is known that rapid oscillation can regularize fluid flow. Considering e.g. a rotating cylinder filled with water, at high angular velocity ω there is no variation of the fluid velocity parallel to the axis of rotation. Thus the fluid flow becomes two-dimensional and hence regular. The physical principle behind that phenomenon is called Taylor-Proudman-theorem and has been known since roughly a century. It took more than 80 years until a first rigorous analytical proof of the Taylor-Proudman theorem has been derived in the celebrated papers of Babin, Mahalov, and Nicolaenko [1, 2, 9]. In other words, Babin, Mahalov, and Nicolaenko proved the striking result of global-in-time regularization of a flow in periodic domains, if the rotation is sufficiently fast. Based on their works, subsequently many authors generalized their results in various directions. For an alternative proof in $\mathbb{R}^n$ based on dispersive effects, see e.g. [3]. The results obtained by Babin, Mahalov, and Nicolaenko are not only mathematically of great interest. They could also play a significant role in applied situations. This is justified by the fact that in applications the angular velocity of rotation is often much higher than other appearing parameters. This is true in geophysical situations, e.g. for the rotating earth, but also in technological applications such as the spin-coating-process. However, there is no rigorous proof of the Taylor-Proudman theorem on domains with boundary so far.

The key ingredient in the approach of Babin, Mahalov, and Nicoleanko is uniformness in ω of appearing quantities such as an existence interval or a bound for solutions. On the other hand, rotating boundary layer flows usually display an oscillating behavior, i.e., they are nondecaying at space infinity. These two requirements, that is

(i) uniformness in ω,
(ii) nondecaying flows,

i.g. cannot be satisfied simultaneously by a treatment in standard function spaces. Therefore, in [5, 6] an approach in spaces of Fourier transformed vector Radon measures is developed. Besides giving account to the nature of boundary layer problems, this approach offers a couple of further features:

(1) the computations are rather elementary and as a consequence we can find explicit dependence of the solution on related parameters;
(2) the eigenvalues producing unstable eigenmodes belong to the point spectrum of the linearized operator;

As a consequence results on linear and nonlinear stability/instability of the Ekman spiral can be derived. The results are indeed uniformly in ω which, as mentioned before, is the essential pre-condition for regularization induced by rapid oscillation.

It is the purpose of my talk to give an outline of fundamental results obtained during the last decade, but also to present recent developments on the topic of rotating fluids, in particular concerning the Ekman boundary layer problem.

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Poster Session Listing

Koichi Anada (Waseda University)
Behavior of type II blow-up solutions to a quasi-linear parabolic partial differential equation

Tomoro Asai (Hiroshima City University)
The self-similar solution for fourth order curvature flow equation

I-Kun Chen (Kyoto University)
Singularity of macroscopic variables near boundary for gases with cutoff hard potential

Kiyoko Furuya (Ochanomizu University)
On formally self-adjoint Schrödinger operators with measurable potential

Mitsuo Higaki (Tohoku University)
Navier wall law for nonstationary viscous incompressible flows

Takefumi Igarashi (Nihon University)
Blow-up and critical exponents in a degenerate parabolic equation

Tetsuya Ishiwata (Shibaura Institute of Technology)
Behavior of polygons by area-preserving crystalline curvature flow

Takashi Kagaya (Tokyo Institute of Technology)
Exponential stability of a traveling wave for an area preserving curvature motion

Yuki Kaneko (Waseda University)
Spreading and vanishing phenomena for a free boundary problem of reaction-diffusion equations

Kota Kumazaki (Tomakomai National College of Technology)

A mathematical model for concrete carbonation process

Michal Lasica (University of Warsaw)

On the anisotropic curvature flow of planar curves in the uniformly convex case

Tatsuhiko Miura (The University of Tokyo)

Zero width limit of the heat equation on moving thin domains

Tatsuya Miura (The University of Tokyo)

Singular perturbation by bending for an adhesive obstacle problem

Masashi Mizuno (Nihon University)

Convergence of the Allen-Cahn equation with Neumann boundary conditions

Atsushi Nakayasu (The University of Tokyo)

On one-dimensional singular diffusion equations with spatially inhomogeneous driving force

Tokinaga Namba (The University of Tokyo)

On cell problems for Hamilton-Jacobi equations with non-coercive Hamiltonians and its application to homogenization problems

Masaki Ohnuma (Tokushima University)

TBA

Michiaki Onodera (Kyushu University)

Dynamical approach to an elliptic overdetermined problem

Eugene B. Postnikov (Kursk State University),
Anastasia Lavrova (Immanuel Kant Baltic Federal University)

The continuous wavelet transform as an analysis tool for non-linear oscillations [co-authored by E.B. Postnikov, A.I. Lavrova and E.A. Lebedeva]

Motohiko Sato (Wakkanai Hokusei Gakuen University)
Dynamic boundary conditions for singular degenerate parabolic equations

Yukihiro Seki (Kyushu University)
Recent results on blow-up for nonlinear heat equations

Masahiko Shimojo (Okayama University of Science)
On a free boundary problem of a curvature flow with a driving force

Ken Shirakawa (Chiba University),
Hiroshi Watanabe (Salesian Polytechnic)
Energy-dissipations in non-isothermal phase-field systems associated with grain boundary motions

Takuya Suzuki (The University of Tokyo)
On the Stokes resolvent estimates for cylindrical domains

Kohtaro Tadaki (Chubu University)
An operational characterization of the notion of probability by algorithmic randomness

Kazutoshi Taguchi (The University of Tokyo)
On the discrete 1-harmonic flows

Go Takahashi (Waseda University)
Extension criterion via Morrey type functional on solutions to the Navier-Stokes equations

Shuji Takahashi (Tokyo Denki University)
On decay rate estimates in subspaces for the Navier-Stokes equations

Keisuke Takasao (The University of Tokyo)
Existence of weak solution for volume preserving mean curvature flow via phase field method

Kota Uriya (Tohoku University)

Final state problem for a system of nonlinear Schrödinger equations with mass resonance

Hiroshi Watanabe (Salesian Polytechnic)

Well-posedness for strongly degenerate parabolic equations