第 21 回　関数空間セミナー報告集

2012 年 12 月 24 日㈰ ～ 12 月 26 日㈫

（会場：東京理科大学森戸記念館）

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Seminar on Function Spaces, 2012

CONTENTS

Fractional calculi on parabolic Hardy spaces

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Let $n \geq 1$ and H the upper half-space of the $(n+1)$-dimensional Euclidean space, that is, $H = \{X = (x,t) \in \mathbb{R}^{n+1} : x = (x_1, \dots, x_n) \in \mathbb{R}^n,\ t > 0\}$. For $0 < \alpha \leq 1$, the parabolic operator $L^{(\alpha)}$ is defined by

$$L^{(\alpha)} := \partial_t + (-\Delta_x)^\alpha,$$

where $\partial_t = \partial/\partial t$, $\partial_j = \partial/\partial x_j$, and $\Delta_x = \partial_1^2 + \cdots + \partial_n^2$. Let $C(H)$ be the set of all real-valued continuous functions on H. A function $u \in C(H)$ is said to be $L^{(\alpha)}$-harmonic if $L^{(\alpha)}u = 0$ in the sense of distributions. For $1 \leq p < \infty$, the Lebesgue space $L^p = L^p(\mathbb{R}^n)$ is defined to be the Banach space of Lebesgue measurable (real-valued) functions f on $\mathbb{R}^n$ with

$$\|f\|_{L^p} := \left(\int_{\mathbb{R}^n} |f(x)|^p dV_n(x) \right)^{\frac{1}{p}} < \infty,$$

where dV_n is the Lebesgue volume measure on $\mathbb{R}^n$. The parabolic Hardy space $\boldsymbol{h}^p_\alpha$ is the set of all $L^{(\alpha)}$-harmonic functions u on H with

$$\|u\|_{\boldsymbol{h}^p_\alpha} := \sup_{t>0} \|u(\cdot,t)\|_{L^p} < \infty.$$

We remark that $\boldsymbol{h}^p_{1/2}$ coincide with the harmonic Hardy spaces of [1, Chapter 7].

Our aim is the study of fractional calculi on parabolic Hardy spaces. In [2], we study fractional calculi on parabolic Bergman spaces, which are the Banach spaces consisting of all $L^p(H)$-solutions of the parabolic operator $L^{(\alpha)}$. Parabolic Bergman spaces are often studied by using fractional calculi (see [2], [3], and [4]). In this talk, we present properties of fractional calculi on parabolic Hardy spaces. As an application, we present $L^{(\alpha)}$-conjugates of parabolic Hardy functions, which is the extension of the harmonic conjugates.

To state our results of this paper, we give some notations. For a real number κ, let $\mathcal{D}_t^\kappa = (-\partial_t)^\kappa$ be the fractional differential operator with respect to t, and $\mathcal{FC}^\kappa$ the class of functions φ on $\mathbb{R}_+ = (0,\infty)$ such that $\mathcal{D}_t^\kappa \varphi$ is well defined. For a multi-index $\gamma = (\gamma_1, \cdots, \gamma_n) \in \mathbb{N}_0^n$, let $\partial_x^\gamma := \partial_1^{\gamma_1} \cdots \partial_n^{\gamma_n}$. Theorem 1 shows fundamental properties of fractional calculi on parabolic Hardy spaces.

THEOREM 1. *Let* $0 < \alpha \leq 1$, $1 \leq p < \infty$, $\gamma \in \mathbb{N}_0^n$, *and* $\nu > -(n/2\alpha)(1/p) - |\gamma|/2\alpha$. *If* $u \in \boldsymbol{h}^p_\alpha$, *then the following statements hold*:

(1) *The derivative* $\mathcal{D}_t^\nu \partial_x^\gamma u(x,t)$ *is well defined, and there exists a constant* $C = C(n,\alpha,p,\gamma,\nu) > 0$ *such that*

$$|\mathcal{D}_t^\nu \partial_x^\gamma u(x,t)| \leq C t^{-(n/2\alpha)(1/p) - |\gamma|/2\alpha - \nu} \|u\|_{\boldsymbol{h}^p_\alpha}$$

for all $(x,t) \in H$. *Moreover, if* $\nu > -(n/2\alpha)(1/p)$, *then the derivative* $\partial_x^\gamma \mathcal{D}_t^\nu u(x,t)$ *is well defined, and the equation* $\partial_x^\gamma \mathcal{D}_t^\nu u(x,t) = \mathcal{D}_t^\nu \partial_x^\gamma u(x,t)$ *holds.*

(2) *If* $\beta \in \mathbb{N}_0^n$, *then the derivative* $\partial_x^\beta \mathcal{D}_t^\nu \partial_x^\gamma u(x,t)$ *is well defined, and*

$$\partial_x^\beta \mathcal{D}_t^\nu \partial_x^\gamma u(x,t) = \mathcal{D}_t^\nu \partial_x^{\beta+\gamma} u(x,t).$$

(3) *If a real number* κ *satisfies* $\kappa + \nu > -(n/2\alpha)(1/p) - |\gamma|/2\alpha$, *then the derivative* $\mathcal{D}_t^\kappa \mathcal{D}_t^\nu \partial_x^\gamma u(x,t)$ *is well defined, and*

$$\mathcal{D}_t^\kappa \mathcal{D}_t^\nu \partial_x^\gamma u(x,t) = \mathcal{D}_t^{\kappa+\nu} \partial_x^\gamma u(x,t).$$

(4) *The derivative* $\mathcal{D}_t^\nu \partial_x^\gamma u(x,t)$ *is* $L^{(\alpha)}$*-harmonic on H.*

We present the definition of an $L^{(\alpha)}$-conjugate of functions on H, which is introduced in [4].

DEFINITION 1 ([4, Definition 1]). Let $0 < \alpha \leq 1$ and u a function on H. We shall say that an n-tuple of functions $(v_1, \ldots, v_n)$ on H is an $L^{(\alpha)}$-conjugate of u if $v_j(x,\,\cdot\,), u(x,\,\cdot\,) \in \mathcal{FC}^{1/2\alpha}$ and $(n+1)$-tuple $(v_1, \ldots, v_n, u)$ satisfies the following equations:

$$\partial_j v_k = \partial_k v_j, \qquad 1 \leq j, k \leq n, \tag{N.1}$$

$$\partial_j u = -\mathcal{D}_t^{1/2\alpha} v_j, \qquad 1 \leq j \leq n, \tag{N.2}$$

and

$$\mathcal{D}_t^{1/2\alpha} u = \sum_{j=1}^n \partial_j v_j. \tag{N.3}$$

We note that when $\alpha = 1/2$, the equations of Definition 1 coincide with the generalized Cauchy-Riemann equations for harmonic functions in [7]. As we see below, $u(x,\,\cdot\,) \in \mathcal{FC}^{1/2\alpha}$ for all $u \in \boldsymbol{h}_\alpha^p$. Theorem 2 shows the existence and the norm estimates of $L^{(\alpha)}$-conjugates of $\boldsymbol{h}_\alpha^p$-functions.

THEOREM 2. *Let* $0 < \alpha \leq 1$ *and* $1 < p < \infty$, *then the following statements hold*:

(1) *If* $u \in \boldsymbol{h}_\alpha^p$, *then there exists a unique* $L^{(\alpha)}$*-conjugate* $(v_1, \ldots, v_n)$ *of* u *such that* $v_j \in \boldsymbol{h}_\alpha^p$.

(2) *If an* n*-tuple of functions* $(v_1, \ldots, v_n)$ *with* $v_j \in \boldsymbol{h}_\alpha^p$ *satisfies Equation (N.1), then there exists a unique function* $u \in \boldsymbol{h}_\alpha^p$ *such that* $(v_1, \ldots, v_n)$ *is the* $L^{(\alpha)}$*-conjugate of* u.

(3) *There exists a constant* $C > 0$ *independent of* $u \in \boldsymbol{h}_\alpha^p$ *such that*

$$C^{-1} \|u\|_{\boldsymbol{h}_\alpha^p} \leq \sum_{j=1}^n \|v_j\|_{\boldsymbol{h}_\alpha^p} \leq C \|u\|_{\boldsymbol{h}_\alpha^p},$$

where $(v_1, \ldots, v_n)$ *is the* $L^{(\alpha)}$*-conjugate of* u *with* $v_j \in \boldsymbol{h}_\alpha^p$.

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NORMAL SINGULAR INTEGRAL OPERATORS WITH CAUCHY KERNEL ON L^2 (THIS IS THE ABSTRACT OF OUR ORIGINAL PAPER)

TAKAHIKO NAKAZI AND TAKANORI YAMAMOTO

1. Introduction

We will publish the paper which contains the proofs of new results of this abstract. Let $L^p = L^p(\mathbb{T})$ be the Lebesgue space on the unit circle $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$, and let H^p be the corresponding Hardy space for $1 \le p \le \infty$. Then $\|f\|_p = \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(e^{ix})|^p dx\right)^{1/p}$. If $p = 2$, then $\langle f, g\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(e^{ix})\bar{g}(e^{ix})dx$ and $\|f\| = \|f\|_2$. Let $H^{2\perp} = L^2 \ominus H^2 = \overline{zH^2}$. Let P denote the orthogonal projection of L^2 onto H^2. Let Q denote the orthogonal projection of L^2 onto H^2 (resp. $H^{2\perp}$). Let I denote the identity operator on L^2. Then $P + Q = I$.

For $\alpha \in L^\infty$, let M_α denote the multiplication operator of L^2 to L^2 such that $M_\alpha f = \alpha f$, $(f \in L^2)$, let T_α denote the Toeplitz operator of H^2 to H^2 such that $T_\alpha f = P(\alpha f)$, $(f \in H^2)$, let $\tilde{T}_\alpha$ denote the operator of $H^{2\perp}$ to $H^{2\perp}$ such that $\tilde{T}_\alpha f = Q(\alpha f)$, $(f \in H^{2\perp})$, let H_α denote the Hankel operator of H^2 to $H^{2\perp}$ such that $H_\alpha f = Q(\alpha f)$, $(f \in H^2)$ and let $\tilde{H}_\alpha$ denote the operator of $H^{2\perp}$ to H^2 such that $\tilde{H}_\alpha f = P\alpha f$, $(f \in H^{2\perp})$. Then $T_z^* T_\alpha T_z = T_\alpha$, $\tilde{T}_z \tilde{T}_\alpha \tilde{T}_z^* = \tilde{T}_\alpha$ and $\tilde{H}_\phi = H_{\bar{\phi}}^*$.

For $\alpha, \beta \in L^\infty$, let $S_{\alpha,\beta}$ denote the singular integral operator of L^2 to L^2 such that $S_{\alpha,\beta} f = \alpha P f + \beta Q f$, $(f \in L^2)$. Then $M_\alpha = S_{\alpha,\alpha}$, and

$$(S_{\alpha,\beta} f)(z) = \frac{\alpha(z) + \beta(z)}{2} f(z) + \frac{\alpha(z) - \beta(z)}{2} \frac{1}{\pi i} \int_{\mathbb{T}} \frac{f(\tau)}{\tau - z} d\tau,$$

where the integral is understood in the sense of Cauchy's principal value (cf. [4], Vol. I, p.12). If $f \in L^1$, then $(S_{\alpha,\beta} f)(z)$ exists for almost all $z \in \mathbb{T}$.

It is well known that T_ϕ is self-adjoint if and only if ϕ is real-valued. Brown and Halmos [1] established that T_ϕ is normal if and only if $\phi = au + b$ for some real-valued function $u \in L^\infty$ and $a, b \in \mathbb{C}$ (cf. [5], p.107). It is well known that M_ϕ is self-adjoint if and only if ϕ is a real-valued function, and that M_ϕ is always normal. Martínez-Avendaño and Rosenthal [5] established the following. Then $PKM_\alpha|_{H^2}$ is a Hankel operator which maps H^2 to H^2. They proved that $PKM_\alpha|_{H^2}$ is self-adjoint if and only if $\alpha(z) = \overline{\alpha(\bar{z})}$, and that $PKM_\alpha|_{H^2}$ is normal if and only if $PKM_\alpha|_{H^2}$ is a constant multiple of a self-adjoint Hankel operator $PKM_\beta|_{H^2}$ for some $\beta \in L^\infty$. In this paper, we study the normal operator $S_{\alpha,\beta}$. Since $S_{\alpha,\alpha} = M_\alpha$, $S_{\alpha,\alpha}$ is always normal.

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Lemma 1.1. *Let α and β be functions in L^∞. Then $L^2 = H^2 \oplus H^{2\perp}$ is a decomposition such that, as matrices relative to this decomposition,*

$$S_{\alpha,\beta} = \begin{pmatrix} T_\alpha & \tilde{H}_\beta \\ H_\alpha & \tilde{T}_\beta \end{pmatrix}, \qquad S^*_{\alpha,\beta} = \begin{pmatrix} T_{\bar\alpha} & \tilde{H}_{\bar\alpha} \\ H_{\bar\beta} & \tilde{T}_{\bar\beta} \end{pmatrix}.$$

Definition 1.1. *If $\alpha = au + b$ for some real-valued function $u \in L^\infty$ and some $a, b \in \mathbb{C}$, then α is said to be a Brown-Halmos function. Let (BH) denote the set of all Brown-Halmos functions.*

By the Brown and Halmos theorem [1] (cf. [5], p.107), T_α is normal if and only if α is a Brown-Halmos function.

2. Normal Operator $S_{\alpha,\beta}$

Let α and β be functions in L^∞. In this section, we give the general descriptions of symbols of normal operators $S_{\alpha,\beta}$.

Theorems 2.1, 2.2 and 2.3 are the main theorems.

Lemmas 2.1 - 2.5 are used to prove Lemma 2.6.

Lemmas 2.1 and 2.6 are used to prove Theorem 2.1.

Theorem 2.1 is used to prove Theorems 2.2 and 2.3.

Lemma 2.1. *Let α and β be functions in L^∞, and let $\phi = \alpha - \beta$. Then the following hold.*

(1) $S^*_{\alpha,\beta}S_{\alpha,\beta} - S_{\alpha,\beta}S^*_{\alpha,\beta}$

$$= \begin{pmatrix} \tilde{H}_\alpha H_{\bar\alpha} - \tilde{H}_\beta H_{\bar\beta} & \tilde{H}_\beta \tilde{T}_{\bar\phi} - T_\phi \tilde{H}_{\bar\alpha} \\ \tilde{T}_\phi H_{\bar\beta} - H_\alpha T_{\bar\phi} & H_\beta \tilde{H}_{\bar\beta} - H_\alpha \tilde{H}_{\bar\alpha} \end{pmatrix}.$$

(2) *$S_{\alpha,\beta}$ is normal if and only if*

$$\tilde{H}_\alpha H_{\bar\alpha} - \tilde{H}_\beta H_{\bar\beta} = H_\beta \tilde{H}_{\bar\beta} - H_\alpha \tilde{H}_{\bar\alpha} = \tilde{T}_\phi H_{\bar\beta} - H_\alpha T_{\bar\phi} = 0.$$

By the proof of the Cowen theorem ([2]), the first author and Takahashi [6] proved the following lemma by the Sarason's lifting theorem ([8] and p.230 of [7]). Although the equivalence of (1) and (2) is easy, but the equivalence of (1), (2) and (3) is not so easy.

Lemma 2.2. *Let α and β be functions in L^∞. Then the following are equivalent.*

(1) $H^*_\alpha H_\alpha = H^*_\beta H_\beta$ *(That is $\tilde{H}_{\bar\alpha} H_\alpha = \tilde{H}_{\bar\beta} H_\beta$).*
(2) $H_\alpha H^*_\alpha = H_\beta H^*_\beta$ *(That is $H_\alpha \tilde{H}_{\bar\alpha} = H_\beta \tilde{H}_{\bar\beta}$).*
(3) *$\alpha - c\beta \in H^\infty$ for some constant c with $|c| = 1$.*

Lemma 2.3. *Let α be functions in L^∞. Suppose $F, f \in zH^2$ satisfy $\alpha - \hat\alpha(0) = F + \bar f$. Then $F, f \in \cap_{1<p<\infty} L^p$.*

Proof. Well known. □

Lemma 2.4. *If $g \in L^2$, then $\overline{Q\bar g} = zP(\bar z g) = Pg - \hat g(0)$.*

Lemma 2.5. *Let α and β be functions in L^∞ and let $\phi = \alpha - \beta$. Suppose $F, f, G, g \in zH^2$ satisfy $\alpha - \hat\alpha(0) = F + \bar f$ and $\beta - \hat\beta(0) = G + \bar g$. If $H_\alpha T_{\bar\phi} = \tilde{T}_\phi H_{\bar\beta}$, then $Gg = Ff$.*

Lemma 2.6. *Let α and β be functions in L^∞ and let $\phi = \alpha - \beta$. Suppose $F, f, G, g \in zH^2$ satisfy $\alpha - \hat{\alpha}(0) = F + \bar{f}$ and $\beta - \hat{\beta}(0) = G + \bar{g}$. If $S_{\alpha,\beta}$ is normal, then there exists $c, d \in \mathbb{T}$ such that $g = cf$, $G = dF$, $(1 - cd)Ff = 0$ and either $\alpha - c\beta \in \mathbb{C}$ or $\alpha - \bar{d}\beta \in \mathbb{C}$ holds.*

The following theorem is used to prove Theorems 3.2 and 3.3.

Theorem 2.1. *Let α and β be functions in L^∞. Then the following are equivalent.*

1. *$S_{\alpha,\beta}$ is normal.*
2. *There exist constants c, d such that $|c| = 1$, $\alpha = c\beta + d$ and $\tilde{T}_\phi H_{\bar{\beta}} - H_\alpha T_{\bar{\phi}} = 0$, where $\phi = \alpha - \beta$.*
3. *There exist constants c, d such that $|c| = 1$, $\alpha = c\beta + d$ and $(c - 1)|\beta|^2 + d\bar{\beta} - c\bar{d}\beta \in H^\infty$.*
4. *There exist constants c, d such that $|c| = 1$, $\alpha = c\beta + d$ and $(c - 1)|\alpha|^2 + d\bar{\alpha} - c\bar{d}\alpha \in H^\infty$.*

If $\alpha - \beta$ is a non-constant function, then the following theorem gives the descriptions of symbols of normal operators $S_{\alpha,\beta}$.

Theorem 2.2. *Let α and β be functions in L^∞. Suppose $\alpha - \beta$ is a non-constant function. Then the following are equivalent.*

1. *$S_{\alpha,\beta}$ is normal.*
2. *There exist a non-constant unimodular function ψ in L^∞ and constants a, b, c such that $a \neq 0$, $|c| = 1$, $c \neq 1$, $\alpha = ca\psi + b$ and $\beta = a\psi + b$.*

Lemma 2.7. *Let α be a function in L^∞ and let $a \in \mathbb{T}$. Then the following are equivalent.*

1. *$\alpha = a\bar{\alpha} + b$ where and $b \in \mathbb{C}$.*
2. *$\alpha = af + \bar{f} + b$ where $f \in zH^2$ and $b \in \mathbb{C}$.*
3. *$\alpha = a^{1/2}u + b$ where u is a real-valued function in L^∞, and $b \in \mathbb{C}$.*

Lemma 2.8. *Let α be a non-constant function in L^∞. If $\alpha = af + \bar{f} + b = a_1 g + \bar{g} + b_1$ for some $a, a_1 \in \mathbb{T}$, $b, b_1 \in \mathbb{C}$ and some $f, g \in zH^2$, then $a = a_1$ and $f = g$.*

Definition 2.1. *For $a \in \mathbb{T}$, the subset $(BH)(a)$ is defined by*

$$(BH)(a) = \{\alpha : \alpha = af + \bar{f} + b,\ f \in zH^2,\ b \in \mathbb{C}\}.$$

By Lemma 3.7, $\mathbb{C} \subset (BH)(a) \subset (BH)$. By Lemma 3.8, if $a, b \in \mathbb{T}$ and $a \neq b$, then $(BH)(a) \cap (BH)(b) = \mathbb{C}$. If $\alpha - \beta$ is a constant, then the following theorem gives the descriptions of symbols of normal operators $S_{\alpha,\beta}$.

Theorem 2.3. *Let α and β be non-constant functions in L^∞. Suppose $\alpha - \beta$ is a non-zero constant. Then the following are equivalent.*

1. *$S_{\alpha,\beta}$ is normal.*
2. *$\alpha \in (BH)((\alpha - \beta)/(\bar{\alpha} - \bar{\beta}))$.*
3. *There exist $f \in zH^2$ and $p, q \in \mathbb{C}$ with $p \neq q$ such that*

$$\alpha = \frac{p - q}{\bar{p} - \bar{q}} f + \bar{f} + p, \quad \beta = \frac{p - q}{\bar{p} - \bar{q}} f + \bar{f} + q.$$

Lemma 2.9. *Let α and β be functions in L^∞. Suppose there exists $c \in \mathbb{T}$ such that $c \neq 1$ and $\alpha - c\beta = 0$. Then the following are equivalent.*

(1) $S_{\alpha,\beta}$ *is normal.*
(2) *Both* $|\alpha|$ *and* $|\beta|$ *are constant.*

Lemma 2.10. *Let* α *and* β *be functions in* L^∞*. Then the following are equivalent.*

(1) $S_{\alpha,\beta}$ *is an isometry.*
(2) $|\alpha| = |\beta| = 1$ *and* $\alpha\bar{\beta} \in H^\infty$*.*

Theorem 2.4. *Let* α *and* β *be functions in* L^∞*. Then the following are equivalent.*

(1) $S_{\alpha,\beta}$ *is unitary.*
(2) $|\alpha| = |\beta| = 1$ *and there exists* $d \in \mathbb{T}$ *such that* $\alpha - d\beta = 0$*.*

Lemma 2.11. *Let* α *and* β *be functions in* L^∞*. Suppose there exists* $c \in \mathbb{T}$ *such that* $c \neq 1$ *and* $\alpha - c\beta = 0$*. Then the following are equivalent.*

(1) $S_{\alpha,\beta}$ *is normal.*
(2) *Both* $|\alpha|$ *and* $|\beta|$ *are constant.*

Lemma 2.12. *Let* α *and* β *be functions in* L^∞*. Then the following are equivalent.*

(1) $S_{\alpha,\beta}$ *is an isometry.*
(2) $|\alpha| = |\beta| = 1$ *and* $\alpha\bar{\beta} \in H^\infty$*.*

Theorem 2.5. *Let* α *and* β *be functions in* L^∞*. Then the following are equivalent.*

(1) $S_{\alpha,\beta}$ *is unitary.*
(2) $|\alpha| = |\beta| = 1$ *and there exists* $d \in \mathbb{T}$ *such that* $\alpha - d\beta = 0$*.*

Theorem 2.6. *Let* α *and* β *be functions in* L^∞*. Then the following are equivalent.*

(1) $S_{\alpha,\beta}$ *is self-adjoint and unitary.*
(2) *Either* $(\alpha, \beta) = (1, 1), (1, -1), (-1, 1), (-1, -1)$*, or there exists a measurable subset* $E \subset \mathbb{T}$ *such that* $0 < m(E) < 1$ *and* $\alpha = \beta = 2\chi_E - 1$*.*

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Extremal Problems In H^1 For Continuous Kernels On A Bidisc

Takahiko Nakazi (Hokusei Gakuen University)

Let D be the open unit disc in the complex plane $\mathbb{C}$ and $T = \partial D$. D^n and T^n denote the Cartesian products of n copies of D and T, respectively. Let Z be the set of all integers and Z_+ the set of all nonnegative integers. Z^n and Z_+^n denote the set of all $\alpha = (\alpha_1, \cdots, \alpha_n)$ with $\alpha_i \in Z$ and Z_+^n the set of all $\alpha \in Z^n$ with $\alpha_i \in Z_+$ for $1 \leq i \leq n$, respectively.

Let m_n be the normalized Lebesgue measure on T^n. For $1 \leq p \leq \infty$, let $L^p(T^n) = L^p(T^n, m_n)$ denote the Lebesgue space, $H^p(T^n)$ the usual Hardy space and $K^p(T^n) = \{f \in L^p(T^n) : \hat{f}(\alpha) = 0 \text{ if } -\alpha \notin Z_+^n\}$. Then for $1 < p < \infty$ $L^p(T^n) = H^p(T^n) \oplus \overline{K^p}(T^n)$. For ϕ in $L^\infty(T^n)$, a linear functional B_ϕ on $H^1(T^n)$ is defined by the following :

$$B_\phi(f) = \int_{T^n} f(\zeta)\phi(\zeta)dm_n(\zeta) \quad (f \in H^1(T^n)).$$

Then $\|B_\phi\| = \|\phi + K^\infty(T^n)\|$.

Let $S = \{f \in H^1(T^n) : \|f\|_1 = 1\}$. We are interested in $S_\phi = \{f \in S : B_\phi(f) = \|B_\phi\|\}$ when ϕ is continuous. Such a extremal problem was solved by the present author [5] for $n = 1$. In this lecture we describe S_ϕ for $n = 2$ when S_ϕ contains an outher function (see [8]).

I. Definition

$\alpha, \beta \in Z_+^n$ のとき $\alpha \leq \beta$ は $\alpha_j \leq \beta_j$ $(1 \leq j \leq n)$ を意味し、$|\alpha| = \alpha_1 + \cdots + \alpha_n$ とする。$N_+(T^n)$ は Smirnov class を示し、$H^p(T^n)_\ell = \{f \in L^p(T^n) : \hat{f}(\alpha_1, \cdots, \alpha_\ell, \cdots, \alpha_n) = 0$ if $\alpha_\ell < 0\}$ $(1 \leq \ell \leq n)$ とする。このとき $\bigcap_{\ell=1}^n H^p(T^n)_\ell = H^p(T^n)$ となる。

$\psi \in L^\infty(T^n)$ が extremal kernel とは $\psi = \phi + k, k \in K^\infty(T^n)$ かつ $\|B_\phi\| = \|\psi\|_\infty$ となるときである。$f \in H^1(T^n)$ が rigid (strongly outer) とは $S_\phi = \{f/\|f\|_1\}$ となることをいう。$f \in N_+(T^n)$ が outer とは

$$\int_{T^n} \log|f| dm_n = \log\left|\int_{T^n} f dm_n\right| > -\infty$$

のときのことである。

II. One dimension

$S_{\bar{z}^n} = \{\gamma \prod_{j=1}^n (z - a_j)(1 - \bar{a}_j z) \in S : \gamma > 0, a_j \in \bar{D}\}$ は良く知られていたと思われるが、ϕ が rational のときは 1950 年に Macintyre-Rogosinski により、ϕ が T で analytic

のときは 1958 年に deLeeuw-Rudin により、ϕ が T 上で continuous のときは 1983 年に Nakazi により S_ϕ は $S_{\bar{z}^n}$ を用いて描かれた。次の Theorem 1 と Corollary 1 は Nakazi [5] による。

Theorem 1

$\phi \in L^\infty(T), \phi \notin H^\infty(T)$ のとき次の (1)~(5) は同値である。

(1) S_ϕ は weak*compact で empty ではない。

(2) $f \in S_\phi, f = qh$ が inner-outer 分解であるとき、q は degree n の finite Blaschke product である。

(3) ψ が ϕ の extremal kernel のとき、$\psi = \bar{z}^n |f_0|/f_0$ と書ける。ここで f_0 は rigid であり、$n \in Z_+$ である。

(4) $S_\phi = (\{\gamma : \gamma > 0\} \times S_{\bar{z}^n} \times f_0) \cap S$ と書ける。ここで f_0 は rigid かつ $n \in Z_+$ である。

(5) $\dim\langle S_\phi \rangle = 2n+1$ である。

Corollary 1

ϕ が T 上で continuous ならば、$S_\phi = (\{\gamma : \gamma > 0\} \times S_{\bar{z}^n} \times f_0) \cap S$ と書ける。ここで f_0 は rigid かつ $n \in Z_+$ である。

Ⅲ. Two dimension

$f \in H^1(T^2)$ のとき、z を fix して $f_z(w) = f(z, w)$ かつ w を fix して $f_w(z) = f(z, w)$ と書くと、$f_z \in H^1(T_w)$ かつ $f_w \in H^1(T_z)$ となる。

f が w-rigid、z-rigid とは、それぞれ f_z は rigid a.e.z、f_w は rigid a.e.w となることである。f が separately rigid とは f が w-rigid かつ z -rigid となることである。f が separately rigid ならば f は rigid であるが逆は成立しない。outer function についても同様な定義ができるが、f が separately outer でも f は outer とならないときがあるが、逆は成立する。

§1. Separately finite dimensional solution set

Hasumi により、$\dim\langle S_{\phi_z} \rangle = 1$ a.e.z が $\dim\langle S_{\phi_w} \rangle = 1$ a.e.w となる必要十分条件は $\dim\langle S_\phi \rangle = 1$ が示された。我々は $\dim\langle S_{\phi_z} \rangle = 1$ a.e.z かつ $\dim\langle S_{\phi_w} \rangle < \infty$ a.e.w ならば $\dim\langle S_\phi \rangle < \infty$ を知っている (see [7])。よって次の問題は自然である。

Problem

$\dim\langle S_{\phi_z} \rangle \leq 2n+1$ a.e.z かつ $\dim\langle S_{\phi_w} \rangle \leq 2m+1$ a.e.w ならば $\dim\langle S_\phi \rangle \leq (2n+1)(2m+1)$ が成立するか。

Lemma 1

$f \in N_+(T^2)$ かつ $E_j \subset T$ を $m(E_j) > 0$ $(j = 1, 2)$ とする。f_z は w-rational function、degree $\leq r_1$ $(z \in E_1)$ かつ f_w は z-rational function、degree $\leq r_2$ ならば f は rational function でありかつ degree $\leq (r_1, r_2)$ となる。

Proof. Quo-Wang [2] は lemma を $f \in H^\infty(T^2)$ のとき示したが、$f \in N_+(T^2)$ の場合へのその証明は通じる。

次の Theorem 1 は Problem の特別な場合に肯定的に解いている。

Theorem 1

S_ϕ が少なくとも一つの outer function を含むとする。もし $\dim\langle S_{\phi_z}\rangle < \infty$ a.e.z かつ $\dim\langle S_{\phi_w}\rangle < \infty$ a.e.w ならば、次の様な非負な整数 n, m が存在する。即ち、$\dim\langle S_{\phi_z}\rangle \leq 2n+1$ a.e.z、 $\dim\langle S_{\phi_w}\rangle \leq 2m+1$ a.e.w かつ $\dim\langle S_\phi\rangle \leq (2n+1)(2m+1)$.

§2. Continuous kernel

この § はこの講演の主要な結果についてである。それらは Ⅱ の Theorem 1 と Ⅱ の Lemma 1 を用いて証明される。

Theorem 2

ϕ が T^2 上で continuous function かつ S_ϕ が少なくとも一つの outer function を含むならば $\dim\langle S_\phi\rangle < \infty$ である。

Theorem 3

S_ϕ が少なくとも一つの rational function を含むならば $\dim\langle S_\phi\rangle < \infty$ である。

$\phi = z - w$ は outer function ではないが、$S_\phi = \{(z-w)/\|z-w\|_1\}$ である。

§3. Some extremal kernel

この § では ϕ がある特別な extremal kernel の場合に、S_ϕ の表現についての one dimension の結果（Ⅱ の Theorem 1）を two dimension へ拡張することを考える。それで次の Problem は自然である。

Problem

$\phi = \bar{\zeta}^\alpha |f_0|/f_0$、$\alpha \in Z_+^2$ かつ f_0 は rigid ならば $S_\phi = (\{\gamma : \gamma > 0\} \times S_{\bar{\zeta}^\alpha} \times f_0) \cap S$ が成立するか。

$\alpha = (\alpha_1, \alpha_2) \in Z_+^2$ ならば $S_{\bar{\zeta}^\alpha} = \{f \in S : f = \zeta^\alpha F$ かつ $F = \sum_{-\alpha \leq \beta \leq \alpha} a(\beta)\zeta^\beta \geq 0\}$ となり、$\dim\langle S_{\bar{\zeta}^\alpha}\rangle = (2\alpha_1+1)(2\alpha_2+1)$ である。

Theorem 4

$\phi = \bar{\zeta}^{\alpha}|f_0|/f_0$、$\alpha \in Z_+^2$ かつ f_0 は separately rigid ならば $S_\phi = (\{\gamma : \gamma > 0\} \times S_{\bar{\zeta}^{\alpha}} \times f_0) \cap S$ となる。

上の Theorem 4 で separately rigid を単に rigid とはできなかった。

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Compound matrix and its applications to the numerical ranges

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Abstract

For a triple of $n \times n$ Hermitian matrices (H_1, H_2, H_3) and a real sequence $c \in \mathbf{R}^n$, the c-joint c-numerical range $W_c(H_1, H_2, H_3)$ is realized as the joint numerical range $W(K_1, K_2, K_3)$ of some triple of $N \times N$ Hermitian matrices (K_1, K_2, K_2) for $N = \prod_{k=1}^{n} {}_nC_k$.

1. Compound matrix

Let A be an $n \times n$ complex matrix and k a positive integer $\leq n$.

The kth *compound matrix* $C_k(A)$ of A is an N-by-N matrix whose α, β entry is $\det A[\alpha|\beta]$, where $N = {}_nC_k = n!/(k!(n-k)!)$, $\alpha, \beta \in Q_{k,n}$ and

$$Q_{k,n} = \{\alpha = (\alpha_1, \ldots, \alpha_k) : 1 \leq \alpha_1 < \cdots < \alpha_k \leq n\}$$

with usual lexicographical order. The formula $C_k(AB) = C_k(A)C_k(B)$ was proved in [4]. We shall define $C_k(A)$ in another way. Denote by e_i the i-th column of the identity matrix I_n. Then the vectors $\{e_1, e_2, \ldots, e_n\}$ is an orthonormal basis of $\mathbf{C}^n$. If the integers $(1 \leq)i_1, i_2, \ldots, i_k(\leq n)$ are k-distinct numbers, we define

$$e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_k} = \frac{1}{\sqrt{k!}} \sum_{\sigma \in S_k} \mathrm{Sgn}(\sigma) e_{i_{\sigma(1)}} \otimes e_{i_{\sigma(2)}} \otimes \cdots \otimes e_{i_{\sigma(k)}}.$$

For k vectors

$$\xi_1 = \sum_{j_1=1}^{n} \xi_{j_1,1} e_{j_1}, \xi_2 = \sum_{j_2=1}^{n} \xi_{j_2,2} e_{j_2}, \ldots, \xi_k = \sum_{j_k=1}^{n} \xi_{j_k,k} e_{j_k}$$

the vector $\xi_1 \wedge \xi_2 \wedge \cdots \wedge \xi_k$ is defined as

$$\sum_{1 \leq j_1 < j_2 < \cdots < j_k \leq n} \mathrm{Det}(\begin{pmatrix} \xi_{j_1,1} & \ldots & \xi_{j_k,k} \\ \ldots & \ldots & \ldots \\ \xi_{j_k,1} & \ldots & \xi_{j_k,k} \end{pmatrix})\, e_{j_1} \wedge \cdots \wedge e_{j_k}.$$

We denote by $V_k(\mathbf{C}^n)$ the ${}_nC_k$-dimenentional vector space spanned by

$$\{e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_k} : 1 \leq i_1 < i_2 < \ldots < i_k \leq n\}.$$

An element of $V_k(\mathbf{C}^n)$ is called a *decomposable* k -vector if it is expressed as $\xi_1 \wedge \xi_2 \wedge \cdots \xi_k$ by some vectors $\xi_1, \xi_2, \ldots, \xi_k$ in $\mathbf{C}^n$. For instance $e_1 \wedge e_2 + e_1 \wedge e_3 + e_1 \wedge e_4 + e_2 \wedge e_3 + e_2 \wedge e_4 + e_3 \wedge e_4$ is not a decomposable 2-vector. This fact follows is verified by the following proposition.

Proposition 1.1 An element

$$X_1 e_1 \wedge e_2 + X_2 e_1 \wedge e_3 + X_3 e_1 \wedge e_4 + X_4 e_2 \wedge e_3 + X_5 e_2 \wedge e_4 + X_6 e_3 \wedge e_4$$

of $V_2(\mathbf{C}^4)$ in the case $V_1(\mathbf{C}^n) = \mathbf{C}^4$ is a decomposable 2-vector if and only if

$$X_1 X_6 - X_2 X_5 + X_3 X_4 = 0.$$

For $1 \leq i_1 < i_2 < \ldots < i_k \leq n$, $1 \leq j_1 < j_2 < \ldots < j_k \leq n$, the entry $C_k(A)(i_1, i_2, \ldots, i_k : j_1, j_2, \ldots, j_k)$ is chacterrized as

$$C_k(A)(i_1, i_2, \ldots, i_k : j_1, j_2, \ldots, j_k) = \langle (T \otimes T \otimes \cdots \otimes T)(e_{j_1} \wedge e_{j_2} \wedge \cdots \wedge e_{j_k}), e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_k}$$

For instance, $C_1(A) = A$, $C_n(A) = \det(A)$. Let $K = \mathrm{diag}((-1)^j : j = 1, 2, 3. \ldots) = \mathrm{diag}(-1, 1, -1, 1, \ldots)$. Then the matrix $KC_{n-1}K$ is called the cofactor matrix of A. If A is invertible, then

$$A^{-1} = \frac{1}{\det(A)} [KC_{n-1}K]^T.$$

The kth additive compound matrix $D_k(A)$ of A is defined as

$$D_k(A) = \frac{d}{dt}\, C_k(I + tA)|_{t=0},$$

equivalently, $D_k(A)$ is the linear term in

$$C_k(I + tA) = I + tD_k(A) + t^2R + \cdots .$$

For $1 \leq i_1 < i_2 < \ldots < i_k \leq n$, $1 \leq j_1 < j_2 < \ldots < j_k \leq n$, the entry $D_k(A)(i_1, i_2, \ldots, i_k : j_1, j_2, \ldots, j_k)$ is chacterrized as

$$D_k(A)(i_1, i_2, \ldots, i_k : j_1, j_2, \ldots, j_k)$$

$$= \langle([T\otimes I\otimes\cdots\otimes I + I\otimes T\otimes I\otimes\cdots\otimes I + \cdots I\otimes\cdots\otimes I\otimes T)(e_{j_1}\wedge e_{j_2}\wedge\cdots\wedge e_{j_k}), e_{i_1}\wedge e_{i_2}\wedge\cdots\wedge e_{i_k}$$

Proposition 1.2 Suppose that A is an $n\times n$ complex matrix with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$. Then the matrices $C_k(A)$, $D_k(A)$ ($1 \leq k \leq n$) satisfy

$$det(tI_{{}_nC_k} - C_k(A)) = \prod_{1\leq i_1<i_2<\cdots<c_k\leq n} (t - \lambda_{i_1}\lambda_{i_2}\cdots\lambda_{i_k}),$$

$$det(tI_{{}_nC_k} - D_k(A)) = \prod_{1\leq i_1<i_2<\cdots<c_k\leq n} (t - [\lambda_{i_1} + \lambda_{i_2} + \cdots + \lambda_{i_k}]).$$

2. Joint c-numerical range of an ordered m-tule of Hermitian matrices

For an ordered m-tuple of Hermitian $n\times n$ matrices $(H_1, H_2, \ldots, H_m)$ and a real sequence $c = (c_1, c_2, \ldots, c_n)$, the joint c-numerical range $W_c(H_1, \ldots, H_m)$ is defined by

$$W_c(H_1, H_2, \ldots, H_m) = \{(\sum_{j=1}^{n} c_j\langle H_1\xi_j, \xi_j\rangle, \sum_{j=1}^{n} c_j\langle H_2\xi_j, \xi_j\rangle, \ldots, \sum_{j=1}^{n} c_j\langle H_m\xi_j, \xi_j\rangle) :$$

$$\{\xi_1, \xi_2, \ldots, \xi_n\} \text{ is an orthonormal basis of } \mathbb{C}^n\}.$$

If $(c_1, c_2, \ldots, c_n) = (1, 0, \ldots, 0)$, then the range $W_c(H_1, H_2, , H_m)$ is reduced to the clasical joint numerical range $W(H_1, H_2, \ldots, H_m)$.

In 1919, Hausdorff [8] proved the convexity of $W(H_1, H_2)$ which is identified with $W(A)$ for $A = H_1 + iH_2$. In 1918, Toeplitz [10] introduced $W(A)$ by

$$W(A) = \{\langle A\xi, \xi\rangle : \xi \in \mathbb{C}^n, ||\xi|| = 1\}.$$

In 1975, Westwick [11] proved the convexity of $W_c(H_1, H_2)$. An indefinite analogue of this result and and its appllications are given in [3, 2].

In 2011, Chien and N [5] proved the following reduction theorem.

Theorem 2.1 Let (H_1, H_2) be an arbitrary pair of $n \times n$ Hermitian matrices and $c = (c_1, c_2, \ldots, c_n)$ a real sequence in $\mathbb{R}^n$. Then there exists a pair of $n! \times n!$ Hermitian matrices (K_1, K_2) satisfying

$$W_c(H_1, H_2) = W(K_1, K_2).$$

The proof of the above theorem depends on a deep result of W. Helton and V. Vinnikov [9] on hyperbolic forms using Riemann theta functions. In [6] their result is used.

In 1983, Y. H. Au-Yeung and N. K. Tsing [1] proved the convexity of $W_c(H_1, H_2, H_3)$ under the condition $n \geq 3$. Some convexity theorem of $W_c(H_1, \ldots, H_m)$ is obtained for $m \geq 4$ under some restricted condition in [7].

Theorem 2.2[Chien-N] Let (H_1, H_2, H_3) be a triple of $n \times n$ Hermitan matrices with $n \geq 3$ and $c = (c_1, c_2, \ldots, c_n)$ a real sequence in $\mathbb{R}^n$. Then there exists a triple of $N \times N$ Hermitian matrices (K_1, K_2, K_3) satisfying

$$W_c(H_1, H_2, H_3) = W(K_1, K_2, K_3)$$

for

$$N = \frac{(n!)^{n-1}}{\{(n-1)!}^2 = \prod_{k=1}^{n-1} {}_nC_k.$$

We shall give a sketch of the proof of Theorem 2.2. By remarking the equation

$$W_{(c_1+c_0, c_2+c_0, \ldots, c_n+c_0)}(H_1, H_2, H_3) = W_{(c_1, c_2, \ldots, c_n)}(H_1, H_2, H_3) + c_0(\mathrm{tr}(H_1), \mathrm{tr}(H_2), \mathrm{tr}(H_3))$$

we assume that $c_n = 0$. Under this assumption we set

$$b_{n-1} = c_{n-1}, b_{n-2} = c_{n-2} - c_{n-1}, b_{n-3} = c_{n-3} - c_{n-2}, \ldots, b_1 = c_1 - c_2.$$

We also set

$$K_j = b_1 H_j \otimes I_{m_2} \otimes I_{m_3} \otimes \cdots \otimes I_{m_{n-1}} + b_2 I_{m_1} \otimes D_2(H_j) \otimes I_{n_3} \otimes \cdots \otimes I_{m_{n-1}}$$

$$+ b_3 I_{m_1} \otimes I_{m_2} \otimes D_3(H_j) \otimes \cdots \otimes I_{m_{n-1}}$$

$$+ \ldots + b_{n-1} I_{m_1} \otimes I_{m_2} \otimes \cdots \otimes I_{m_{n-2}} \otimes D_{n-1}(H_j),$$

where $m_k = {}_nC_k = n!/(k!(n-k)!)$, $j = 1, 2, 3$, then the equation

$$\sum_{j=1}^{n} c_j \lambda_j (x_1 H_1 + x_2 H_2 + x_3 H_3) = \lambda_1 (x_1 K_1 + x_2 K_2 + x_3 K_3),$$

holds for every $(x_1, x_2, x_3) \in \mathbf{R}^3$. By the convexity of $W_c(A_1, A_2, A_3)$, this equation implies that $W_c(H_1, H_2, H_3) = W(K_1, K_2, K_3)$.

In the case $m \geq 4$, we have the following.

Proposition 2.3 Let $(H_1, H_2, \ldots, H_m)$ be an arbityrary m-tuple of $n \times n$ Hermitian matrices. Then for every $1 \leq k \leq n$, the following equation holds:

$$W_k(H_1, H_2, \ldots, H_m) = \{(\langle D_k(H_1)\xi, \xi\rangle, \langle D_k(H_2)\xi, \xi\rangle, \ldots, \langle D_k(H_m)\xi, \xi\rangle) :$$

$$\xi \in V_k(\mathbf{C}), ||\xi|| = 1, \xi \,\text{is decomposable}\, k - \text{vector}\}.$$

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Nonlinear Analytic Methods for Linear Operators in Banach Spaces

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1 Introduction

Let E be a real Banach space and let C be a nonempty closed convex subset of E. For a mapping $T : C \to C$, we denoted by $F(T)$ the set of fixed points of T. A mapping $T : C \to C$ is called *nonexpansive* if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. In particular, a nonexpansive mapping $T : E \to E$ is called *contractive* if it is linear. From [19] we know a weak convergence theorem by Mann's iteration for nonexpansive mappings in a Hilbert space.

Theorem 1.1. *Let H be a real Hilbert space, let C be a nonempty closed convex subset of H and let $T : C \to C$ be a nonexpansive mapping with $F(T) \neq \emptyset$. Define a sequence $\{x_n\}$ in C by $x_1 = x \in C$ and*

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)Tx_n, \quad \forall n \in \mathbb{N},$$

where $\{\alpha_n\}$ is a real sequence in $[0, 1]$ such that $\sum_{n=1}^{\infty} \alpha_n(1 - \alpha_n) = \infty$. Then, $\{x_n\}$ converges weakly to an element z of $F(T)$, where $z = \lim_{n\to\infty} Px_n$ and P is the metric projection of H onto $F(T)$.

By Reich [14], such a theorem was extended to a uniformly convex Banach space with a Fréchet differentiable norm. However, we have not known whether the limit point z is characterized under any projections in a Banach space. Recently, using nonlinear analytic methods obtained by [8], [9] and [4], Takahashi and Yao [22] solved such a problem for positively homogeneous nonexpansive mappings in a Banach space. In 1938, Yosida [25] also proved the following mean ergodic theorem for linear operators; see also Kido and Takahashi [11].

Theorem 1.2. *Let E be a real Banach space and let T be a linear operator of E into itself such that there exists a constant C with $\|T^n\| \leq C$ for $n \in \mathbb{N}$, and T is weakly completely continuous, i.e., T maps the closed unit ball of E into a weakly compact subset of E. Then, for each $x \in E$, the Cesàro means $S_n x$ converge strongly as $n \to \infty$ to a fixed point of T.*

In this article, motivated by these theorems, we study nonlinear analytic methods for linear contractive operators in Banach spaces and obtain some new strong convergence theorems for commutative families of linear contractive operators in Banach spaces. We also extend Bauschk, Deutsch, Hundal and Park's theorem for linear contractive operators in Hilbert spaces to commutative families of linear contractive operators in Banach spaces. In our results, the limit points are characterized by sunny generalized nonexpansive retractions.

2 Preliminaries

Throughout this talk, we assume that a Banach space E with the dual space E^* is real. We also denote by $\langle x, x^* \rangle$ the dual pair of $x \in E$ and $x^* \in E^*$. A Banach space E is said to be *strictly convex* if $\|x + y\| < 2$ for $x, y \in E$ with $\|x\| \leq 1$, $\|y\| \leq 1$ and $x \neq y$. A Banach space E is said to be *smooth* provided

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t}$$

exists for each $x, y \in E$ with $\|x\| = \|y\| = 1$. Let E be a Banach space. With each $x \in E$, we associate the set $J(x) = \{x^* \in E^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\}$. The multi-valued operator $J : E \to E^*$ is called the *normalized duality* mapping of E. From the Hahn-Banach theorem, $Jx \neq \emptyset$ for each $x \in E$. We know that E is smooth if and only if J is single-valued; see [17]. Let E be a smooth Banach space and let J be the normalized duality mapping of E. We define the function $\phi : E \times E \to \mathbb{R}$ by

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy \rangle + \|y\|^2$$

for all $x, y \in E$. It is easy to see that $(\|x\| - \|y\|)^2 \leq \phi(x, y) \leq (\|x\| + \|y\|)^2$ for all $x, y \in E$. Let C be a nonempty closed convex subset of a smooth, strictly convex and reflexive Banach space E. For an arbitrary point x of E, the set $\{z \in C : \phi(z, x) = \min_{y \in C} \phi(y, x)\}$ is always a singleton. Let us define the mapping Π_C of E onto C by $z = \Pi_C x$ for every $x \in E$, i.e., $\phi(\Pi_C x, x) = \min_{y \in C} \phi(y, x)$ for every $x \in E$. Such Π_C is called the *generalized projection* of E onto C; see Alber [1]. Let D be a nonempty closed subset of a smooth Banach space E, let T be a mapping from D into itself and let $F(T)$ be the set of fixed points of T. Then, T is said to be *generalized nonexpansive* if $F(T)$ is nonempty and $\phi(Tx, u) \leq \phi(x, u)$ for all $x \in D$ and $u \in F(T)$. Let C be a nonempty subset of E and let R be a mapping from E onto C. Then R is said to be a *retraction*, or a *projection* if $Rx = x$ for all $x \in C$. It is known that if a mapping P of E into E satisfies $P^2 = P$, then P is a projection of E onto $\{Px : x \in E\}$. A mapping $T : E \to E$ with $F(T) \neq \emptyset$ is a retraction if and only if $F(T) = R(T)$, where $R(T)$ is the range of T. The mapping R is also said to be *sunny* if $R(Rx + t(x - Rx)) = Rx$ whenever $x \in E$ and $t \geq 0$. A nonempty subset C of a smooth Banach space E is said to be a *generalized nonexpansive retract* (resp. *sunny generalized nonexpansive retract*) of E if there exists a generalized nonexpansive retraction (resp. sunny generalized nonexpansive retraction) R from E onto C. The following lemmas are in [12].

Lemma 2.1 (Kohsaka and Takahashi [12]). *Let E be a smooth, strictly convex and reflexive Banach space, let C^* be a nonempty closed convex subset of E^* and let Π_{C^*} be the generalized projection of E^* onto C^*. Then the mapping R defined by $R = J^{-1}\Pi_{C^*}J$ is a sunny generalized nonexpansive retraction of E onto $J^{-1}C^*$.*

Lemma 2.2 (Kohsaka and Takahashi [12]). *Let E be a smooth, strictly convex and reflexive Banach space and let D be a nonempty closed subset of E. Then, the following are equivalent.*

(1) D is a sunny generalized nonexpansive retract of E;
(2) D is a generalized nonexpansive retract of E;
(3) JD is closed and convex.

Let C be a closed convex subset of a strictly convex and reflexive Banach space E. For an arbitrary point x of E, the set $\{z \in C : \|z - x\| = \min_{y \in C} \|y - x\|\}$ is always a singleton.

Let us define the mapping P_C of E onto C by $z = P_Cx$ for every $x \in E$, i.e., $\|P_Cx - x\| = \min_{y\in C} \|y - x\|$ for every $x \in E$. Such P_C is called the *metric projection* of E onto C. Let E be a Banach space and let K be a closed convex cone of E. Then, $T : K \to K$ is called a *positively homogeneous* mapping if $T(\alpha x) = \alpha Tx$ for all $\alpha \geq 0$ and $x \in K$. Let M be a closed linear subspace of E. Then, $T : M \to M$ is called a *homogeneous* mapping if $T(\beta x) = \beta Tx$ for all $\beta \in \mathbb{R}$ and $x \in M$. In L^p spaces, $1 < p < \infty$, we know examples of nonexpansive and positively homogeneous mappings; see, for instance, Wittmann [24]. From Takahashi and Yao [22] we have the follwing result; see also Honda, Takahashi and Yao [4].

Lemma 2.3. *Let E be a smooth Banach space and let K be a closed convex cone in E. If $T : K \to K$ is a positively homogeneous nonexpansive mapping, then T is generalized nonexpansive. In particular, if $T : E \to E$ is a linear contractive mapping, then T is generalized nonexpansive.*

Let S be a semitopological semigroup, i.e., S is a semigroup with a Hausdorff topology such that for each $a \in S$ the mappings $s \mapsto a \cdot s$ and $s \mapsto s \cdot a$ from S to S are continuous. In the case when S is commutative, we denote st by $s + t$. A commutative semigroup S with identity is a directed system when the binary relation is defined by $s \preceq t$ if and only if $\{t\} \cup (S + t) \subset \{s\} \cup (S + s)$. Let $B(S)$ be the Banach space of all bounded real valued functions on S with supremum norm and let $C(S)$ be the subspace of $B(S)$ of all bounded real valued continuous functions on S. Let μ be an element of $C(S)^*$ (the dual space of $C(S)$). We denote by $\mu(f)$ the value of μ at $f \in C(S)$. Sometimes, we denote by $\mu_t(f(t))$ or $\mu_t f(t)$ the value $\mu(f)$. For each $s \in S$ and $f \in C(S)$, we define two functions $l_s f$ and $r_s f$ as follows: $(l_s f)(t) = f(st)$ and $(r_s f)(t) = f(ts)$ for all $t \in S$. An element μ of $C(S)^*$ is called a *mean* on $C(S)$ if $\mu(e) = \|\mu\| = 1$, where $e(s) = 1$ for all $s \in S$. We know that $\mu \in C(S)^*$ is a mean on $C(S)$ if and only if

$$\inf_{s\in S} f(s) \leq \mu(f) \leq \sup_{s\in S} f(s), \quad \forall f \in C(S).$$

A mean μ on $C(S)$ is called *left invariant* if $\mu(l_s f) = \mu(f)$ for all $f \in C(S)$ and $s \in S$. Similarly, a mean μ on $C(S)$ is called *right invariant* if $\mu(r_s f) = \mu(f)$ for all $f \in C(S)$ and $s \in S$. A left and right invariant invariant mean on $C(S)$ is called an *invariant* mean on $C(S)$.

Let E be a Banach space and let C be a nonempty, closed and convex subset of E. Let S be a semitopological semigroup and let $\mathcal{S} = \{T_s : s \in S\}$ be a family of nonexpansive mappings of C into itself. Then $\mathcal{S} = \{T_s : s \in S\}$ is called a *continuous representation* of S as nonexpansive mappings on C if $T_{st} = T_sT_t$ for all $s, t \in S$ and $s \mapsto T_sx$ is continuous for each $x \in C$. The following definition [15] is crucial in the nonlinear ergodic theory of abstract semigroups. Let S be a topological space and Let $C(S)$ be the Banach space of all bounded real valued continuous functions on S with supremum norm. Let E be a reflexive Banach space. Let $u : S \to E$ be a continuous function such that $\{u(s) : s \in S\}$ is bounded and let μ be a mean on $C(S)$. Then there exists a unique element z_0 of E such that

$$\mu_s\langle u(s), x^*\rangle = \langle z_0, x^*\rangle, \quad \forall x^* \in E^*.$$

We call such z_0 the *mean vector* of u for μ. In particular, if $\mathcal{S} = \{T_s : s \in S\}$ is a continuous representation of S as nonexpansive mappings on C and $u(s) = T_sx$ for all $s \in S$, then there exists $z_0 \in C$ such tat

$$\mu_s\langle T_sx, x^*\rangle = \langle z_0, x^*\rangle, \quad \forall x^* \in E^*.$$

We denote such z_0 by $T_\mu x$.

3 Strong Convergence Theorems and Applications

In this section, we state two strong convergence theorems for commutative families of linear contractive operators in Banach spaces. Let Y be a nonempty subset of a Banach space E and let Y^* be a nonempty subset of the dual space E^*. Then, we can define the annihilator $Y^*_\perp$ of Y^* and the annihilator $Y^\perp$ of Y as follows:

$$Y^*_\perp = \{x \in E : f(x) = 0,\ \forall f \in Y^*\} \ \text{ and } \ Y^\perp = \{f \in E^* : f(x) = 0,\ \forall x \in Y\}.$$

Theorem 3.1 (Takahashi, Wong and Yao [21])**.** *Let S be a commutative semitopological semigroup with identity. Let E be a strictly convex, smooth and reflexive Banach space, let $\mathcal{S} = \{T_s : s \in S\}$ be a continuous representation of S as linear contractive operators of E into itself and let $\{S_\alpha : \alpha \in I\}$ be a net of contractive linear operators of E into itself such that $F(\mathcal{S}) \subset F(S_\alpha)$ for all $\alpha \in I$. Suppose $T_s \circ S_\alpha = S_\alpha \circ T_s$ for all $\alpha \in I$ and $s \in S$. Then, the following are equivalent:*

(1) *$\{S_\alpha x\}$ converges to an element of $F(\mathcal{S})$ for all $x \in E$;*
(2) *$\{S_\alpha x\}$ converges to 0 for all $x \in (JF(\mathcal{S}))_\perp$;*
(3) *$\{S_\alpha x \quad T_s \circ S_\alpha x\}$ converges to 0 for all $x \in E$ and $s \in S$.*

Furthermore, if (1) holds, then $\{S_\alpha x\}$ converges to $R_{F(\mathcal{S})}x \in F(\mathcal{S})$, where $R_{F(\mathcal{S})} = J^{-1}\Pi_{JF(\mathcal{S})}J$ and $\Pi_{JF(\mathcal{S})}$ is the generalized projection of E^* onto $JF(\mathcal{S})$.

Theorem 3.2 (Takahashi, Wong and Yao [21])**.** *Let S be a commutative semitopological semigroup with identity. Let E be a strictly convex, smooth and reflexive Banach space, let $\mathcal{S} = \{T_s : s \in S\}$ be a continuous representation of S as linear contractive operators of E into itself and let $\{T_n : n \in \mathbb{N}\}$ be a sequence of linear contractive operators of E into itself such that $F(\mathcal{S}) \subset F(T_n)$ for all $n \in \mathbb{N}$. Let $S_n = T_n \circ T_{n-1} \circ \quad \circ T_1$ for all $n \in \mathbb{N}$ and suppose that $T_s \circ S_n = S_n \circ T_s$ for all $n \in \mathbb{N}$ and $s \in S$. Then, the following are equivalent:*

(1) *$\{S_n x\}$ converges to an element of $F(\mathcal{S})$ for all $x \in E$;*
(2) *$\{S_n x\}$ converges to 0 for all $x \in (JF(\mathcal{S}))_\perp$;*
(3) *$S_n x \quad T_s \circ S_n x \to 0$ for all $x \in E$ and $s \in S$.*

Furthermore, if (1) holds, then $\{S_n x\}$ converges to $R_{F(\mathcal{S})}x \in F(\mathcal{S})$, where $R_{F(\mathcal{S})} = J^{-1}\Pi_{JF(\mathcal{S})}J$ and $\Pi_{JF(\mathcal{S})}$ is the generalized projection of E^* onto $JF(\mathcal{S})$.

Using Theorem 3.1, we obtain the following theorem to commutative families of linear contractive mappings in Banach spaces.

Theorem 3.3 (Takahashi, Wong and Yao [21])**.** *Let E be a strictly convex, smooth and reflexive Banach space and let M be a closed linear subspace of E such that there exists a sunny generalized nonexpansive retraction R of E onto M. Let S be a commutative semitopological semigroup with identity and let $\mathcal{S} = \{T_s : s \in S\}$ be a continuous representation of S as linear contractive operators of E into itself. Then the following are equivalent:*

(1) *$\{T_s x\}$ converges to the element Rx of M for all $x \in E$;*
(2) *$M = F(\mathcal{S})$ and $\{T_s x\}$ converges to 0 for all $x \in (JM)_\perp$;*
(3) *$M = F(\mathcal{S})$ and $T_s x \quad T_{s+t}x \to 0$ for all $x \in E$ and $t \in S$.*

Furthermore, if (1) holds, then $R = R_{F(\mathcal{S})} = J^{-1}\Pi_{JF(\mathcal{S})}J$, where $\Pi_{JF(\mathcal{S})}$ is the generalized projection of E^* onto $JF(\mathcal{S})$.

From Theorem 3.3, we get Bauschk, Deutsch, Hundal and Park's theorem [2] in 2003.

Theorem 3.4 ([2]). *Let T be a contractive linear operator on a Hilbert space H; i.e. $\|T\| \leq 1$, and let M be a closed linear subspace of H. Consider the following statements;*

(1) $\lim_{n\to\infty} \|T^n x - P_M x\| = 0$ *for all* $x \in H$;
(2) $M = F(T)$ *and* $T^n x$ *converges to* 0 *for all* $x \in M^{\perp}$;
(3) $M = F(T)$ *and* $T^n x - T^{n+1}x \to 0$ *for all* $x \in E$.

Then, all statements are equivalent.

If M is a closed linear subspace of a Hilbert space H, then there exists the metric projection P of H onto M. In a Hilbert space, the metric projection P of H onto M is coincident with the sunny generalized nonexpansive retraction R_M of H onto M. From Theorem 3.1, we can also show the following mean strong convergence theorem for commutative semigroups of contractive linear operators in a Banach space; see Yosida [25].

Theorem 3.5. *Let S be a commutative semitopological semigroup with identity. Let E be a strictly convex, smooth and reflexive Banach space, let $\mathcal{S} = \{T_s : s \in S\}$ be a continuous representation of S as linear contractive operators of E into itself. Let $\{\mu_\alpha\}$ be a net of strongly asymptotically invariant means on $C(S)$, i.e., for each $s \in S$, $\|l_s^*\mu_\alpha - \mu_\alpha\| \to 0$, where l_s^* is the adjoint operator of l_s. Then, for each $x \in E$, $\{T_{\mu_\alpha}x\}$ converges strongly to the element Rx of $F(\mathcal{S})$, where $R = R_{F(\mathcal{S})} = J^{-1}\Pi_{JF(\mathcal{S})}J$ and $\Pi_{JF(\mathcal{S})}$ is the generalized projection of E^* onto $JF(\mathcal{S})$.*

In Theorem 3.5, note that the point $z = \lim_\alpha T_{\mu_\alpha}x$ is characterlized by the sunny generalized nonexpansive retraction $R = R_{F(\mathcal{S})} = J^{-1}\Pi_{JF(\mathcal{S})}J$ of E onto $F(\mathcal{S})$. Such a result is still new even if the operators T_s are linear. Applying Theorem 3.2, we obtain the following strong convergence theorem of Mann's type for commutative semigroups of linear contractive operators in a Banach space.

Theorem 3.6. *Let S be a commutative semitopological semigroup with identity. Let E be a strictly convex, smooth and reflexive Banach space and let $\mathcal{S} = \{T_s : s \in S\}$ be a continuous representation of S as linear contractive operators of E into itself. Let $\{\mu_n\}$ be a sequence of means on $C(S)$ which is strongly asymptotically invariant, i.e., for each $s \in S$, $\|l_s^*\mu_n - \mu_n\| \to 0$, where l_s^* is the adjoint operator of l_s. Let $\{\alpha_n\}$ be a sequence of real numbers such that $0 \leq \alpha_n \leq 1$ and $\sum_{n=1}^{\infty}(1 - \alpha_n) = \infty$. Then, a sequence $\{x_n\}$ generated by $x_1 = x \in E$ and*

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)T_{\mu_n}x_n, \quad n \in \mathbb{N}$$

converges strongly to the element Rx of $F(\mathcal{S})$, where $R = R_{F(\mathcal{S})} = J^{-1}\Pi_{JF(\mathcal{S})}J$ and $\Pi_{JF(\mathcal{S})}$ is the generalized projection of E^* onto $JF(\mathcal{S})$.

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REPRESENTATION OF SCHRÖDINGER OPERATOR VIA SHORT-TIME FOURIER TRANSFORM

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Abstract. 短時間フーリエ変換を用いて自由粒子のシュレディンガー作用素 $e^{\frac{1}{2}it\Delta}$ の新たな表現を与える. その応用としてモジュレーション空間 $M^{p,q}$ におけるシュレディンガー方程式の解の評価式を与える (より詳しい証明は Kato-Kobayashi-Ito [3] 参照).

1. Introduction

まず始めに, フーリエ変換を用いた自由粒子のシュレディンガー方程式の初期値問題の解き方について簡単に復習しよう. 次の方程式を考える:

$$
\begin{cases} i\partial_t u(t,x) + \frac{1}{2}\Delta u(t,x) = 0, & (t,x) \in \mathbb{R}\times\mathbb{R}^n, \\ u(0,x) = u_0(x), & x \in \mathbb{R}^n. \end{cases} \tag{1.1}
$$

ただし, $i = \sqrt{-1}$, $u(t,x)$ は $(t,x) \in \mathbb{R}\times\mathbb{R}^n$ の複素数値関数, $u_0(x)$ は $x \in \mathbb{R}^n$ の複素数値関数, $\partial_t u = \partial u/\partial t$, $\Delta u = \sum_{i=1}^n \partial^2 u/\partial x_i^2$ とする.

フーリエ変換を用いて方程式 (1.1) を解く場合, まず x に関するフーリエ変換を行う. すると (1.1) は

$$
\begin{cases} i\partial_t \widehat{u}(t,\xi) - \frac{1}{2}|\xi|^2 \widehat{u}(t,\xi) = 0, \\ \widehat{u}(0,\xi) = \widehat{u_0}(\xi) \end{cases} \tag{1.2}
$$

に変換される. ただし, $\widehat{f}(\xi) = \int_{\mathbb{R}^n} f(x)e^{-ix\cdot\xi}dx$ でもって $f(x)$ のフーリエ変換を表す. (1.2) は ξ をパラメータにもつ t に関する 1 階の常微分方程式なので, これを解くと

$$
\widehat{u}(t,\xi) = e^{-\frac{1}{2}it|\xi|^2}\widehat{u_0}(\xi)
$$

を得る. ここで, フーリエ変換は可逆な変換なので ξ に関する逆フーリエ変換を行うと (1.1) の解

$$
u(t,x) = (e^{\frac{1}{2}it\Delta}u_0)(x) = [e^{-\frac{1}{2}it|\cdot|^2}\widehat{u_0}(\cdot)]^\vee(x)
$$

を得る. 以上のことからフーリエ変換を用いた自由粒子のシュレディンガー方程式の解の表示 (またはフーリエ変換を用いたシュレディンガー作用素の表現) を得たことになる. 本稿では, 他の変換を用いて得られる (1.1) の解の新たな表示を与えるのが目標である.

2. Preliminaries

以下で必要となるいくつかの記号を準備しよう. $\mathcal{S}(\mathbb{R}^n)$ でもって $\mathbb{R}^n$ 上の急減少関数全体のなす集合, $\mathcal{S}'(\mathbb{R}^n)$ でもって $\mathbb{R}^n$ 上の緩増加超関数全体のなす集合を表す.

定義 1 (短時間フーリエ変換 V_ϕ)**.** $\phi \in \mathcal{S}(\mathbb{R}^n)$ とする. このとき ϕ を窓とする $f \in \mathcal{S}'(\mathbb{R}^n)$ の短時間フーリエ変換 $V_\phi f$ を

$$V_\phi f(x,\xi) := \langle f(y), \phi(y-x)e^{iy\cdot\xi}\rangle = \int_{\mathbb{R}^n} f(y)\overline{\phi(y-x)}e^{-iy\cdot\xi}dy$$

で定める. ここで, ϕ は窓関数とよばれる.

注意 2. $V_\phi f(x,\xi)$ は $\mathbb{R}^n \times \mathbb{R}^n$ 上の関数である. また, 形式的に $\phi \equiv 1$ とすると $V_\phi f(x,\xi) = \widehat{f}(\xi)$, すなわち, 短時間フーリエ変換とフーリエ変換は一致する.

当然「フーリエ変換と同じように短時間フーリエ変換が可逆な変換なのか?」という疑問が生じるが, 次の補題から分かるように短時間フーリエ変換は可逆の変換である.

定義 3 (V_ϕ の (形式的な) 共役作用素)**.** $\phi \in \mathcal{S}(\mathbb{R}^n)$ とする. このとき, $\mathbb{R}^n \times \mathbb{R}^n$ 上の関数 $F(y,\xi)$ に対して

$$V_\phi^* F(x) := \iint_{\mathbb{R}^{2n}} F(y,\xi)\phi(x-y)e^{ix\cdot\xi}dy\,đ\xi, \quad đ\xi = (2\pi)^{-n}d\xi.$$

と定める.

この共役作用素を用いて次の反転公式を得る.

補題 4 ([2, Corollary 11.2.7])**.** $\langle\psi,\phi\rangle = \int_{\mathbb{R}^n}\psi(x)\overline{\phi(x)}dx \neq 0$ をみたす $\phi, \psi \in \mathcal{S}(\mathbb{R}^n)$ に対して

$$\frac{1}{\langle\psi,\phi\rangle}V_\psi^* V_\phi f = f, \quad f \in \mathcal{S}'(\mathbb{R}^n)$$

がなりたつ.

3. Main Topic

本題に戻ろう. 短時間フーリエ変換を用いてシュレディンガー方程式の初期値問題

$$\begin{cases} i\partial_t u(t,x) + \frac{1}{2}\Delta u(t,x) = 0, & (t,x) \in \mathbb{R}\times\mathbb{R}^n, \\ u(0,x) = u_0(x), & x \in \mathbb{R}^n. \end{cases} \tag{3.1}$$

を解く. まず, $\varphi_0 \in \mathcal{S}(\mathbb{R}^n)\setminus\{0\}$ を任意に取る. (注意：φ_0 を窓とする短時間フーリエ変換は考えない!! 実際, そのようにすると上手くいかない. その代わりに)

$$\begin{cases} i\partial_t\varphi + \frac{1}{2}\Delta\varphi = 0, \\ \varphi(0,x) = \varphi_0(x). \end{cases}$$

の解である

$$\varphi(t,\cdot)=(e^{\frac{1}{2}it\Delta}\varphi_0)(\cdot)$$

を窓とする短時間フーリエ変換 $V_{\varphi(t,\cdot)}$ により方程式 (3.1) がどのように変換されるか考えよう．短時間フーリエ変換の定義と部分積分の公式より

$$\begin{aligned}
&V_{\varphi(t,\cdot)}\left(\frac{1}{2}\Delta u(t,\cdot)\right)(x,\xi)\\
&=\frac{1}{2}\int_{\mathbb{R}^n}\overline{\varphi(t,y-x)}\Delta_y u(t,y)e^{-iy\cdot\xi}dy\\
&=\int_{\mathbb{R}^n}\overline{\frac{1}{2}\Delta_y\varphi(t,y-x)}u(t,y)e^{-iy\cdot\xi}dy+\int_{\mathbb{R}^n}(-i\xi\cdot\nabla_y)\overline{\varphi(t,y-x)}u(t,y)e^{-iy\cdot\xi}dy\\
&\quad-\frac{1}{2}|\xi|^2\int\overline{\varphi(t,y-x)}u(t,y)e^{-iy\cdot\xi}dy\\
&=V_{\frac{1}{2}\Delta\varphi(t,\cdot)}(u(t,\cdot))(x,\xi)+\left(i\xi\cdot\nabla_x-\frac{1}{2}|\xi|^2\right)V_{\varphi(t,\cdot)}(u(t,\cdot))(x,\xi)
\end{aligned}$$

かつ

$$\begin{aligned}
i\partial_t V_{\varphi(t,\cdot)}(u(t,\cdot))(x,\xi)&=i\partial_t\left(\int_{\mathbb{R}^n}\overline{\varphi(t,y-x)}u(t,y)e^{-iy\cdot\xi}dy\right)\\
&=V_{-i\partial_t\varphi(t,\cdot)}(u(t,\cdot))(x,\xi)+V_{\varphi(t,\cdot)}(i\partial_t u(t,\cdot))(x,\xi)
\end{aligned}$$

となるので, これらを合わせれば方程式 (3.1) は

$$\begin{aligned}
&i\partial_t V_{\varphi(t,\cdot)}(u(t,\cdot))(x,\xi)+\left(i\xi\cdot\nabla_x-\frac{1}{2}|\xi|^2\right)V_{\varphi(t,\cdot)}(u(t,\cdot))(x,\xi)\\
&=V_{\varphi(t,\cdot)}\left(i\partial_t u(t,\cdot)+\frac{1}{2}\Delta u(t,\cdot)\right)(x,\xi)-V_{[i\partial_t\varphi(t,\cdot)+\frac{1}{2}\Delta\varphi(t,\cdot)]}(u(t,\cdot))(x,\xi)
\end{aligned}$$

に変換される．ここで, $u(t,x),\varphi(t,x)$ はシュレディンガー方程式の解であることを考慮すれば

$$i\partial_t V_{\varphi(t,\cdot)}(u(t,\cdot))(x,\xi)+\left(i\xi\cdot\nabla_x-\frac{1}{2}|\xi|^2\right)V_{\varphi(t,\cdot)}(u(t,\cdot))(x,\xi)=0$$

を得る．以上のことをまとめると, (3.1) は短時間フーリエ変換により

$$\begin{cases}(i\partial_t+i\xi\cdot\nabla_x-\frac{1}{2}|\xi|^2)V_{\varphi(t,\cdot)}(u(t,\cdot))(x,\xi)=0,\\ V_{\varphi(0,\cdot)}(u(0,\cdot))(x,\xi)=V_{\varphi_0}u_0(x,\xi).\end{cases}\tag{3.2}$$

に変換される．方程式 (3.2) は $V_{\varphi(t,\cdot)}(u(t,\cdot))(x,\xi)$ に関する 1 階の偏微分方程式なのでこれを解くと

$$V_{\varphi(t,\cdot)}(u(t,\cdot))(x,\xi)=e^{-\frac{1}{2}it|\xi|^2}V_{\varphi_0}u_0(x-\xi t,\xi)\tag{3.3}$$

を得る. 短時間フーリエ変換は可逆な変換なので両辺に $V^*_{\varphi(t,\cdot)}$ を作用させ $\langle \varphi(t,\cdot), \varphi(t,\cdot)\rangle$ で割れば

$$\begin{aligned} u(t,x) &= \frac{1}{\langle \varphi(t,\cdot), \varphi(t,\cdot)\rangle} V^*_{\varphi(t,\cdot)_{[(y,\xi)\to x]}}[e^{-\frac{1}{2}it|\xi|^2} V_{\varphi_0} u_0(y-\xi t, \xi)](x) \\ &= \frac{1}{\|\varphi_0\|^2_{L^2}} V^*_{\varphi(t,\cdot)_{[(y,\xi)\to x]}}[e^{-\frac{1}{2}it|\xi|^2} V_{\varphi_0} u_0(y-\xi t, \xi)](x). \end{aligned}$$

を得る. これが, 我々が目標とした短時間フーリエ変換を用いて得られる自由粒子のシュレディンガー方程式の新たな解表示である.

4. Applications

先ほど得られた (3.3) を応用してシュレディンガー方程式の解のモジュレーション空間における評価式を与える.

定義 5 (モジュレーション空間)**.** $1 \le p, q \le \infty$, $\phi \in \mathcal{S}(\mathbb{R}^n) \setminus \{0\}$ とする. このとき, モジュレーション空間 $M^{p,q}_\phi(\mathbb{R}^n)$ を

$$M^{p,q}_\phi(\mathbb{R}^n) = \left\{ f \in \mathcal{S}'(\mathbf{R}^n) \;\middle|\; \|f\|_{M^{p,q}_\phi} < \infty \right\}$$

で定める. ただし,

$$\begin{aligned} \|f\|_{M^{p,q}_\phi} &= \left(\int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} |V_\phi f(x,\xi)|^p dx \right)^{q/p} d\xi \right)^{1/q} \\ &= \left\| \|V_\phi f(x,\xi)\|_{L^p(\mathbb{R}^n_x)} \right\|_{L^q(\mathbb{R}^n_\xi)}. \end{aligned}$$

とする.

注意 6. モジュレーション空間は窓関数の取り方に依らないバナッハ空間である. より詳しく述べると任意の $\phi, \psi \in \mathcal{S}(\mathbb{R}^n) \setminus \{0\}$ に対して $M^{p,q}_\phi(\mathbb{R}^n) = M^{p,q}_\psi(\mathbb{R}^n) (= M^{p,q})$ であり $\|\cdot\|_{M^{p,q}_\phi}$ と $\|\cdot\|_{M^{p,q}_\psi}$ は互いに同値なノルムとなる. また $p = q = 2$ の場合 $M^{2,2}(\mathbb{R}^n) = L^2(\mathbb{R}^n)$ となる.

以上のことを踏まえると先ほど得られた (3.3) から次を得る.

命題 7. $1 \le p, q \le \infty$, $\varphi_0 \in \mathcal{S}(\mathbb{R}^n) \setminus \{0\}$, $\varphi(t,x) = (e^{\frac{1}{2}it\Delta}\varphi_0)(x)$ とする. このとき

$$\|u(t,\cdot)\|_{M^{p,q}_{\varphi(t,\cdot)}} = \|u_0\|_{M^{p,q}_{\varphi_0}}, \quad \forall u_0 \in M^{p,q}(\mathbb{R}^n).$$

また, (3.3) から次に述べるシュレディンガー作用素の $M^{p,q}$-評価および $M^{p,q}$-$M^{p',q}$ 評価も得ることができる.

命題 8. $1 \le p, q \le \infty$, $\varphi_0 \in \mathcal{S}(\mathbb{R}^n) \setminus \{0\}$ とする. このとき $\exists C > 0$ があって

$$\|u(t,\cdot)\|_{M^{p,q}_{\varphi(s,\cdot)}} \le C(1+|t|)^{n/2} \|u_0\|_{M^{p,q}_{\varphi(s,\cdot)}}, \quad \forall u_0 \in \mathcal{S}(\mathbb{R}^n), t, s \in \mathbf{R}.$$

注意 9. $s = 0$ の場合は Bényi-Gröchenig-Okoudjou-Rogers [1] の結果に対応する.

命題 10. $2 \le p \le \infty, 1 \le q \le \infty$, $\varphi_0 \in \mathcal{S}(\mathbb{R}^n) \setminus \{0\}$ とする．このとき $\exists C > 0$ があって

$$\|u(t,\cdot)\|_{M^{p,q}_{\varphi(s,\cdot)}} \le C(1+|t|)^{-n(1/2-1/p)}\|u_0\|_{M^{p',q}_{\varphi(s,\cdot)}}, \quad \forall u_0 \in \mathcal{S}(\mathbb{R}^n), t, s \in \mathbf{R}.$$

ただし, $1/p + 1/p' = 1$.

注意 11. $s = 0$ の場合は Wang-Hudzik [6] の結果に対応する.

命題 12. $2 \le p \le \infty, 1 \le q \le \infty$, $\varphi_0 \in \mathcal{S}(\mathbb{R}^n) \setminus \{0\}$ とする．このとき $\exists C > 0$ があって

$$\|u(t,\cdot)\|_{M^{p,q}_{\varphi(s,\cdot)}} \le C(1+|t|)^{n(1/2-1/p)}\|u_0\|_{M^{p,q}_{\varphi(s,\cdot)}}, \quad \forall u_0 \in \mathcal{S}(\mathbb{R}^n), t, s \in \mathbf{R}.$$

注意 13. $s = 0$ の場合は Wang-Hudzik [6] の結果に対応する.

5. Comment

この方面のさらなる発展に関しては Kato-Kobayashi-Ito [3–5] を参照してください.

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Hardy spaces with variable exponent

Yoshihiro Sawano and Eiichi Nakai

Abstract

The aim of this article is to explain Hardy spaces with variable exponent. The source is [7].

1 Introduction

Among function spaces, Lebesgue spaces with variable exponent are believed to be most difficult. Indeed, they are not rearrangement invariant and detailed research shows that the modular inequality is not available (see [4]). Therefore, Hardy spaces with variable exponent, whose underlying spaces are Lebesgue spaces with variable exponent, are difficult to analyse. In this note, we describe how to cope with this difficulty. With the technique developed in [7], we can analyse other function spaces.

The functions belonging to Hardy spaces on the cirle are considered in 1915 by Hardy [3] and F. Riesz named such function spaces Hardy spaces in [5, 6]. In 1960 attempts to define Hardy spaces on Euclidean spaces were made. Such attempts can be found in [1, 2, 8]. As in [2], the classical theory of $H^p(\mathbb{R}^d)$-spaces could be considered as a chapter of complex function theory.

In harmonic analysis, Hardy spaces are regarded as tools with which to state endpoint results. A typical result of such attempts is the boundedness of $H^1(\mathbb{R}^d)$-boundnedness of the Riesz transform.

Recall that Hardy spaces are given as follows:

1. Topologize $\mathcal{S}(\mathbb{R}^n)$ by norms $\{p_N\}_{N\in\mathbb{N}}$ given by

$$p_N(\varphi) \equiv \sum_{|\alpha|\le N} \sup_{x\in\mathbb{R}^n} (1+|x|)^N |\partial^\alpha \varphi(x)|$$

   for each $N \in \mathbb{N}$. Define $\mathcal{F}_N \equiv \{\varphi \in \mathcal{S}(\mathbb{R}^n) : p_N(\varphi) \le 1\}$.

2. Let $f \in \mathcal{S}'(\mathbb{R}^n)$. The grand maximal operator $\mathcal{M}f$ is given by

$$\mathcal{M}f(x) \equiv \sup\{|t^{-n}\psi(t^{-1}\cdot) * f(x)| \,:\, t>0,\, \psi \in \mathcal{F}_N\}\ (x \in \mathbb{R}^n),$$

   where we choose and fix a large integer N.

3. The Hardy space $H^p(\mathbb{R}^n)$ is the set of all $f \in \mathcal{S}'(\mathbb{R}^n)$ for which $\|f\|_{H^p} \equiv \|\mathcal{M}f\|_{L^p}$ is finite.

Hardy spaces have the following properties:

1. The Hardy space $H^p(\mathbb{R}^n)$ with $1 < p < \infty$ coincides with $L^p(\mathbb{R}^n)$.
2. The Hardy space $H^1(\mathbb{R}^n)$ is a proper subset of $L^1(\mathbb{R}^n)$.
3. The Hardy space $H^p(\mathbb{R}^n)$ with $0 < p < 1$ is not contained in $L^1_{\rm loc}(\mathbb{R}^n)$.

These three different properties make us fell that there is no need to consider $H^p(\mathbb{R}^n)$ with $1 < p < \infty$.

However, in this paper, we would like to say that it is still meaningful to consider $H^p(\mathbb{R}^n)$ with $1 < p < \infty$. Indeed, by considering Hardy spaces with variable exponent, we could obtain the atomic decomposition of Lebesgue spaces with variable exponent.

In Section 2, we recall Lebesgue spaces with variable exponent and in Section 3, we describe Hardy spaces with variable exponent. Finally in Section 4, we consider the atomic decomposition of Morrey spaces.

2 Lebesgue spaces with variable exponent

For a measurable function $f : \mathbb{R}^n \to \mathbb{C}$, define

$$\|f\|_{L^{p(\cdot)}} \equiv \inf \left\{ \lambda > 0 \, : \, \int_{\mathbb{R}^n} \left(\frac{|f(x)|}{\lambda} \right)^{p(x)} dx \le 1 \right\}.$$

The space $L^{p(\cdot)}(\mathbb{R}^n)$ is the set of all measurable functions $f : \mathbb{R}^n \to \mathbb{C}$ for which the norm $\|f\|_{L^{p(\cdot)}}$ is finite. The function $p(\cdot)$ in the definition is called the variable exponent. It is customary to denote $p_+ \equiv \sup_{x \in \mathbb{R}^n} p(x)$ and $p_- \equiv \inf_{x \in \mathbb{R}^n} p(x)$, which we shall do throughout this talk. As is often the case with many other cases, we postulate on $p(\cdot)$ the following conditions.

(log-Hölder continuity)

$$|p(x) - p(y)| \lesssim \frac{1}{\log(1/|x-y|)} \quad \text{for} \quad |x-y| \le \frac{1}{2}, \tag{1}$$

(decay condition)

$$|p(x) - p(y)| \lesssim \frac{1}{\log(e+|x|)} \quad \text{for} \quad |y| \ge |x|. \tag{2}$$

Denote by p_∞ the limit $\lim_{x \to \infty} p(x)$ ensured by the decay condition.

Variable Lebesgue spaces have a long history which starts from Nakano's observations. However, unlike other function spaces, it had been extremely difficult to analyze. One of the reasons is that the boundedness of Hardy-Littlewood maximal operators can not be proven in a satisfactory way.

Remark 2.1. We do not assume $p_+ \le 1$, nor $p_- > 1$. However, it seems to count that $L^{p(\cdot)}(\mathbb{R}^n)$ is isomorphic to the Hardy space $H^{p(\cdot)}(\mathbb{R}^n)$ with variable exponent.

3 Hardy spaces with variable exponent

The Hardy space $H^{p(\cdot)}(\mathbb{R}^n)$ with variable exponent is the set of all $f \in \mathcal{S}'(\mathbb{R}^n)$ for which $\|f\|_{H^{p(\cdot)} \equiv \|\mathcal{M}f\|_{L^{p(\cdot)}}}$ is finite.

We formulate here the definition of Hardy spaces with variable exponent.

Let $q \in [1, \infty]$. A function a is said to be a $(p(\cdot), q)$-atom if it is supported on a cube Q with the following properties.

1. $\|a\|_q \le \dfrac{|Q|^{1/q}}{\|\chi_Q\|_{L^{p(\cdot)}}}$. (Size condition)
2. $\displaystyle\int_Q a(x)x^\alpha\,dx = 0$ for all $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n)$ with $\sum_{j=1}^n \alpha_j \le n\left(\frac{1}{p_-} - 1\right)$. (Moment condition)

Suppose that we are given cubes $\{Q_j\}_{j=1}^\infty$ and complex constants $\{\lambda_j\}_{j=1}^\infty$. Then define

$$\|\{\lambda_j\}_{j=1}^\infty\|_{\mathcal{A}_1} := \left\| \sum_{j=1}^\infty \frac{|\lambda_j|\chi_{Q_j}}{\|\chi_{Q_j}\|_{L^{p(\cdot)}}} \right\|_{L^{p(\cdot)}}$$

$$\|\{\lambda_j\}_{j=1}^\infty\|_{\mathcal{A}_2} := \left\| \left(\sum_{j=1}^\infty \left(\frac{|\lambda_j|}{\|\chi_{Q_j}\|_{L^{p(\cdot)}}} \right)^{\min(1,p_-)} \chi_{Q_j} \right)^{1/\min(1,p_-)} \right\|_{L^{p(\cdot)}}.$$

The next theorem is one of the key theorems in [7].

Theorem 3.1. *Let $f \in \mathcal{S}'(\mathbb{R}^n)$. Then TFAE.*

1. *$f \in H^{p(\cdot)}(\mathbb{R}^n)$.*
2. *There exists a collection of $(p(\cdot), \infty)$-atoms such that $f = \sum_{j=1}^\infty \lambda_j a_j$ with*

$$\|\{\lambda_j\}_{j=1}^\infty\|_{\mathcal{A}_1} = \left\| \sum_{j=1}^\infty \frac{|\lambda_j|\chi_{Q_j}}{\|\chi_{Q_j}\|_{L^{p(\cdot)}}} \right\|_{L^{p(\cdot)}} < \infty.$$

3. There exists a collection of $(p(\cdot), q)$-atoms such that $f = \sum_{j=1}^{\infty} \lambda_j a_j$ with

$$\|\{\lambda_j\}_{j=1}^{\infty}\|_{\mathcal{A}_2} = \left\| \left(\sum_{j=1}^{\infty} \frac{|\lambda_j|^{\min(1,p_-)} \chi_{Q_j}}{\|\chi_{Q_j}\|_{L^{p(\cdot)}}^{\min(1,p_-)}} \right)^{1/\min(1,p_-)} \right\|_{L^{p(\cdot)}} < \infty.$$

Here q in the definition of atoms satisfies

$$q > \max(1, p_+) \tag{3}$$

.

Remark 3.2. In [7], the condition (3) was $q \gg 1$. Later it was improved.

4 Morrey spaces and atomic decomposition

Recall that the Morrey norm $\|f\|_{\mathcal{M}_q^p}$ is given by

$$\|f\|_{\mathcal{M}_q^p} = \sup_B |B|^{\frac{1}{p}-\frac{1}{q}} \|f\|_{L^q(B)},$$

where B runs over all balls. If we use the Hölder inequality, then we have

$$\mathcal{M}_q^p(\mathbb{R}^n) \subset \mathcal{M}_r^p(\mathbb{R}^n)$$

when $1 < r \le q \le p < \infty$. So we can measure the quality of the obtained result with respect to the parameter q in $\mathcal{M}_q^p(\mathbb{R}^n)$.

In [9], we obtained the following theorems:

Theorem 4.1. *Suppose that the parameters p, q, s, t satisfy*

$$1 < q \le p < \infty, \quad 1 < t \le s < \infty, \quad q < t, \quad p < s.$$

Assume that $\{Q_j\}_{j=1}^{\infty} \subset \mathcal{D}$, $\{a_j\}_{j=1}^{\infty} \subset \mathcal{M}_t^s(\mathbb{R}^n)$ and $\{\lambda_j\}_{j=1}^{\infty} \subset [0, \infty)$ fulfill

$$\|a_j\|_{\mathcal{M}_t^s} \le |Q_j|^{1/s}, \quad \mathrm{supp}(a_j) \subset Q_j, \quad \left\| \sum_{j=1}^{\infty} \lambda_j \chi_{Q_j} \right\|_{\mathcal{M}_q^p} < \infty.$$

Then $f = \sum_{j=1}^{\infty} \lambda_j a_j$ converges in $\mathcal{S}'(\mathbb{R}^n) \cap L^q_{\mathrm{loc}}(\mathbb{R}^n)$ and satisfies

$$\|f\|_{\mathcal{M}_q^p} \le C_{p,q,s,t} \left\| \sum_{j=1}^{\infty} \lambda_j \chi_{Q_j} \right\|_{\mathcal{M}_q^p}.$$

Theorem 4.2. *Suppose that the real parameters p, q, v, L satisfy*

$$v > 0, \quad 1 < q \le p < \infty, \quad L \in \mathbb{N} \cup \{0\}.$$

Let $f \in \mathcal{M}^p_q(\mathbb{R}^n)$. Then there exists a triplet $\{\lambda_j\}_{j=1}^\infty \subset [0,\infty)$, $\{Q_j\}_{j=1}^\infty \subset \mathcal{Q}(\mathbb{R}^n)$ and $\{a_j\}_{j=1}^\infty \subset L^\infty(\mathbb{R}^n)$ such that $f = \sum_{j=1}^\infty \lambda_j a_j$ in $\mathcal{S}'(\mathbb{R}^n)$ and that

$$|a_j| \le \chi_{Q_j}, \quad \int_{\mathbb{R}^n} x^\alpha a_j(x)\,dx = 0$$

for all α with $|\alpha| \le L$ and

$$\left\| \left(\sum_{j=1}^\infty (\lambda_j \chi_{Q_j})^v \right)^{1/v} \right\|_{\mathcal{M}^p_q} \le C_{p,q,v} \|f\|_{\mathcal{M}^p_q}.$$

Remark 4.3. Theorem 4.2 follows from a generality but Theorem 4.1 needs a delicate argument. Indeed, the condition about $\mathcal{M}^r_s$ is by no means trivial.

As applications, we reproved the following theorem:

Theorem 4.4. *Let $0 < \alpha < n$, $1 < p \le p_0 < \infty$, $1 < q \le q_0 < \infty$ and $1 < r \le r_0 < \infty$. Suppose that*

$$q > r, \ \frac{1}{p_0} > \frac{\alpha}{n}, \ \frac{1}{q_0} \le \frac{\alpha}{n}, \tag{4}$$

and that

$$\frac{1}{r_0} = \frac{1}{q_0} + \frac{1}{p_0} - \frac{\alpha}{n}, \ \frac{r}{r_0} = \frac{p}{p_0}. \tag{5}$$

Then

$$\|g \cdot I_\alpha f\|_{\mathcal{M}^{r_0}_r} \le C \|g\|_{\mathcal{M}^{q_0}_q} \|f\|_{\mathcal{M}^{p_0}_p},$$

where the constant C is independent of f and g.

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SEPARATION THEOREM FOR CHAIN COMPLETE PARTIALLY ORDERED VECTOR SPACES

TOSHIKAZU WATANABE

1. Introduction

A separation theorem for convex sets is one of the most fundamental theorems in the optimization theory and functional analysis theory. Let X be a vector space, $X^{'}$ its algebraic dual space and A a subset of X. We denote ${}^{l}A$ the linear span of A and ${}^{i}A$ denotes the *relatively algebraic interior* of A, where

$${}^{i}A = \left\{ y \in Y \;\middle|\; \begin{array}{l} \text{any } y^{'} \in {}^{l}A \text{ there exists } \varepsilon > 0 \text{ with} \\ y + \lambda(y^{'} - y) \in A \text{ for any } \lambda \in [0, \varepsilon) \end{array} \right\}.$$

If ${}^{l}A = X$, then it is called *algebraic interior*, core (A), of A. Then we can obtain the separation theorem in the vector space as follows, see [1, 15]:

Theorem 1.1. *Let X be a vector space, $X^{'}$ its algebraic dual space, A, B convex subsets of X such that relatively algebraic interior ${}^{i}A$ and ${}^{i}B$ are non-empty. Then there exists $u \in X^{'}$, $u \neq 0$, and $\lambda \in R$ such that $\langle u, x\rangle \leq \lambda \leq \langle u, y\rangle$ for any $x \in A$ and $y \in B$. Moreover, $\langle u, z\rangle \neq \lambda$ for at least one $z \in A \cup B$ if and only if ${}^{i}A \cap {}^{i}B = \emptyset$.*

In [10, 19], Theorem 1.1 is generalized in the Cartesian product space of a vector space and a Dedekind complete partially ordered vector space. Under certain assumptions, two nonvoid subsets of a product space can be separated by an affine manifold of that product space. Its proof is due to Hahn-Banach's theorem and Dedekind completeness, both assumptions are equivalent, see [3, 5]. The same direction, Zowe[23] also treated a separation theorem and a Fenchel duality in a Dedekind complete vector lattice. Moreover, in [10], the equivalence between a separation theorem and a Fenchel duality are discussed in an ordered vector space with Dedekind completeness.

On the other hand, in [11], Fel'dman consider the Hahn-Banach exact approximation property and conflict the subgradient of convex and sublinear mapping. In [4, 6], Borwein consider the subgradient of convex mapping taking value in an ordered vector space under chain complete, countable chain complete (monotone sequence property), Daniel property and Dini property. In [11, 6], they give examples of chain completeness. Particularly, any closed subcone of a DANIEILL cone (infima exist as topological limits) has is chain complete (chain property) and DANIEILL cone is an important example in optimization theory, see [4, 6].

Above results inspired us to consider the separation theorem for the partially ordered vector space under the chain completeness. When we consider the Cartesian product of a vector space and a chain complete partially ordered vector space, two subsets in that product space are not necessary separated by an affine manifold. So, we can not rely on the method in [10, 19].

In this paper, we give a separation theorem in the Cartesian product of a vector space and a partially ordered vector space with the chain completeness using a Gerstewitz (Tammer) scalarization method [13].

2. Preliminaries

Let R be the set of a real number, N the set of a natural number, I an indexed set. Let X and Y be real vector spaces. We denote by $\mathcal{L}(X,Y)$ a linear mapping from X into Y. In particular, $X^{'} = \mathcal{L}(X,R)$. We give a convex cone K and define its algebraic dual cone $K^{\sharp}$ by

$$K^{\sharp} := \{x^{'} \in X^{'} \mid \langle x^{'}, x\rangle \geq 0 \text{ for all } x \in K\},$$

where $\langle x^{'}, x\rangle$ denotes dual pair, and its quasi interior is defined by

$$K^{\sharp 0} := \{x^{'} \in X^{'} \mid \langle x^{'}, x\rangle > 0 \text{ for all } x \in K \setminus \{0\}\}.$$

A partial ordering on Y with respect to K is defined by $x \leqq_K y$ if $y - x \in K$ for all x, $y \in Y$. If $y - x \in K \setminus \{0\}$ for all x, $y \in Y$, we denote by $x \leq_K y$. We assume K is a proper convex cone (that is, $K \neq \emptyset$, $K \neq \{\theta\}$, where θ denotes the zero element in Y, $K \neq Y$, $\lambda K \subset K$ for all $\lambda \geq 0$, and $K + K \subset K$). It is well known that $\leqq_K$ is reflexive and transitive. Moreover, $\leqq_K$ has invariable properties to vector space structures as translation and scalar multiplication. In the sequel, we consider $(Y, \leqq_K)$ as a partially ordered vector space, where K is a proper convex cone. In particular, we assume that K is pointed, that is, $K \cap K = \{0\}$, then K is antisymmetric.

Let Z be a subset of Y. The set Z is called a *chain* if any two elements are comparable, that is, $x \leqq_K y$ or $y \leqq_K x$ for any x, $y \in Z$. An element $x \in Y$ is called a *lower bound* (resp., *upper bound*) of Z if $x \leqq_K y$ (resp., $y \leqq_K x$) for any $y \in Z$, *minimum* (resp., *maximum*) of Z if x is a lower bound (resp., *upper bound*) of Z and $x \in Z$. If there exists a lower bound (resp., an upper bound) of Z, then Z is said to be *bounded from below* (resp., *bounded from above*). If the set of all lower bounds of Z has the maximum, then the maximum is called an *infimum* of Z and denoted by $\inf Z$. If the set of all upper bounds of Z has the minimum, then the minimum is called a *supremum* of Z and denoted by $\sup Z$. A partially ordered vector space Y is said to be *chain complete* if every nonempty chain of Y which is bounded from below has an infimum; *Dedekind complete* if every nonempty subset of Y which is bounded from below has an infimum; *Dedekind σ-complete* if every nonempty countable subset of Y which is bounded from below has an infimum. A partially ordered vector space Y is (upward) *directed* if for any x,$y \in Y$ there exists $z \in Y$ such that $x \leqq_K z$ and $y \leqq_K z$. For the further information of a partially ordered vector space and a partially ordered set, see [7, 8, 18, 20, 21].

It is clear that if Y is Dedekind complete, then it is chain complete. However, the converse is not true in general. The following example shows this fact.

Example 2.1. The set of all continuous functions on the interval $[0,1]$, $C([0,1])$ is chain complete. In fact, when we consider an increasing sequence of continuous functions which is bounded from above, then it is a chain and has a supremum. Since the supremum is also a uniformly convergent limit of a sequence, it is continuous. However, $C([0,1])$ is not a Dedekind σ-complete space, (see [18, Example 23.3. (ii)]).

The following, we give elementary properties for the vector space and sacalarizing function.

Definition 2.2. A point $x \in X$ is *linearly accessible* from A if there exists $a \in A$ with $a \neq x$ such that $(a,x) \subset A$. We write lina (A) for the set of all such x and put lin $(A) = A \cup$ lina (A). A subset A of X is said to be *algebraically closed* if $A =$ lin (A).

Let $\overline{R} = R \cup \{\infty\}$ and $\varphi : Y \to \overline{R}$. Then we define the domain and epigraph of φ by

$$\text{dom}\,(\varphi) = \{y \in Y \mid \varphi(y) < \infty\},\ \text{epi}\,(\varphi) = \{(y,t) \in Y \times R \mid \varphi(y) \leq t\},$$

respectively. We say that φ is convex if epi (φ) is a convex set; proper if dom $(\varphi) \neq \emptyset$ and $\varphi(y) > -\infty$ for all $y \in Y$. If we take $k^0 \in K$, then we have

$$K + [0,\infty) \cdot k^0 \subset K. \tag{2.1}$$

A function f from X into R is said to be *sublinear* if the following conditions are satisfied.

(S1) For any $x, y \in X$, $f(x+y) \leq f(x) + f(y)$.
(S2) For any $x \in X$ and $\alpha \geq 0$, $f(\alpha x) = \alpha f(x)$.

Gerstewitz (Tammer) [12] considers the sublinear scalarizing function defined by

$$\varphi_{K,k^0}(y) = \inf\{t \in R \mid y \in tk^0 - K\}.$$

For this function, we have the following proposition, see [13, Theorem 2.3.1].

Proposition 2.3. Let K be a closed proper convex cone and $k^0 \in K$. Then we have dom $(\varphi_{K,k^0}) = R \cdot k^0 - K$,

$$\{y \in Y \mid \varphi_{K,k^0}(y) \leq \lambda\} = \lambda k^0 - K \tag{2.2}$$

and

$$\varphi_{K,k^0}(y + \lambda k^0) = \varphi_{K,k^0}(y) + \lambda. \tag{2.3}$$

Moreover, we have the following results:

(i): φ_{K,k^0} is convex.
(ii): K is cone if and only if φ_{K,k^0} satisfies $\varphi_{K,k^0}(\lambda y) = \lambda \varphi_{K,k^0}(y)$ for any $\lambda > 0$.
(iii): φ_{K,k^0} is proper if and only if K does not contain the lines parallel to $k^0 \in Y \setminus \{0\}$, that is,

$$\text{for any } y \in Y \text{ there exists } t \in R \text{ such that } y + tk^0 \notin K. \tag{2.4}$$

(iv): φ_{K,k^0} is finite-valued if and only if K does not contain the lines parallel to $k^0 \in Y \setminus \{0\}$ and

$$R \cdot k^0 - K = Y. \tag{2.5}$$

(v): φ_{K,k^0} is K-monotone, that is, $y_2 - y_1 \in K$ implies $\varphi_{K,k^0}(y_1) \leq \varphi_{K,k^0}(y_2)$.

Proof. By the definition of φ_{K,k^0}, $\varphi_{K,k^0}(y + \lambda k^0) = \varphi_{K,k^0}(y) + \lambda$ is clear. We only to prove the equation (2.2). The proofs of remainder are similar to that of [13, Theorem 2.3.1]. $\{y \in Y \mid \varphi_{K,k^0}(y) \leq \lambda\} \supset \lambda k^0 - K$ is obvious. For any $y \in \{y \in Y \mid \varphi_{K,k^0}(y) \leq \lambda\}$, we assume that $y \notin \lambda k^0 - K$. Since K is closed, $\lambda k^0 - K$ is also closed, and $y \notin$ lin $(\lambda k^0 - K)$. Then for any $d \in K$, there exists μ with $0 < \mu < 1$ such that $\mu(\lambda k^0 - d) + (1-\mu)y \notin \lambda k^0 - K$. Take $d = k^0$, we have

$$y \notin \lambda k^0 + \frac{\mu}{1-\mu} k^0 - K.$$

Thus $\varphi_{K,k^0}(y) \geq \lambda + \frac{\mu}{1-\mu} > \lambda$. Contradict the fact $y \in \{y \in Y \mid \varphi_{K,k^0}(y) \leq \lambda\}$. □

Let X be a vector space and $(Y, \leqq_K)$ a partially ordered vector space, where K is a convex cone. A mapping $f : X \to Y$ is called sublinear if for all x,$y \in X$ and all $\lambda \geq 0$, f satisfies (S1) and (S2). We denote $\mathcal{L}(X, Y)$ the real vector space of all linear mapping from X into Y. For a chain complete partially ordered vector space, Fel'dman [11] gives the following theorem.

Theorem 2.4. *Let X be a vector space, $(Y, \leqq_K)$ a chain complete partially ordered vector space, where K is a convex cone. Let f be a sublinear mapping from a X into Y and x_0 a point in X. Then there exists $g \in \mathcal{L}(X, Y)$ such that $g(x) \leqq_K f(x)$ for any $x \in X$ and $g(x_0) = f(x_0)$.*

3. Separation theorem

Let X be a vector space and $(Y, \leqq_K)$ a chain complete directed partially ordered vector space, where K is a proper closed convex cone. Let $f \in \mathcal{L}(X, Y)$, $g \in \mathcal{L}(Y, Y)$, t_0 a point in R, $k^0 \in \mathrm{core}\,(K)$ and φ_{K,k^0} a scalarizing function from Y into R. Then

$$H = \{(x,\, y) \in X \times Y \mid \varphi_{K,k^0}(f(x) + g(y)) = t_0\}$$

is a subset in $X \times Y$. Let A, B be nonempty subsets of $X \times Y$. It is said that H separates A and B if

$$H_- = \{(x,\, y) \in X \times Y \mid \varphi_{K,k^0}(f(x) + g(y)) \leq t_0\} \supset A$$

and

$$H_+ = \{(x,\, y) \in X \times Y \mid \varphi_{K,k^0}(f(x) + g(y)) \geq t_0\} \supset B$$

hold. The operator P_X defined by $P_X(x,\, y) = x$ for any $(x,\, y) \in X \times Y$ is called the projection of $X \times Y$ onto X. Similarly, we define the projection P_Y of $X \times Y$ onto Y by $P_Y(x,\, y) = y$. Then $P_X \in \mathcal{L}(X \times Y, X)$ and $P_Y \in \mathcal{L}(X \times Y, Y)$. We define

$$P_X(A) = \{x \in X \mid \text{ there exists } y \in Y \text{ such that } (x,\, y) \in A\}$$

and

$$P_Y(A) = \{y \in Y \mid \text{ there exists } x \in X \text{ such that } (x,\, y) \in A\}.$$

Then for each $* = X, Y$, we have $P_*(A + B) = P_*(A) + P_*(B)$. We take a chain $C \subset P_Y(A - B)$ and define

$$P_X^C(A - B) = \{x \in X \mid \text{ there exists } y \in C \text{ such that } (x,\, y) \in A - B\}.$$

The set

$$\mathrm{cone}\,(A) = \{\lambda z \in X \times Y \mid \lambda \geq 0,\, z \in A\}$$

is called a *cone span* of A. If A is convex, then cone (A) is convex. We called a subset Z in X is expansive if for at least one $a \in {}^iZ$ and for each $z \in Z$, it holds that $a + \lambda(z - a) \in {}^iZ$.

We obtain a separation theorem for the Cartesian product of a vector space and a chain complete directed partially ordered vector space, as follows:

Theorem 3.1. *Let X be a vector space, $(Y, \leqq_K)$ a chain complete directed partially ordered vector space, where K is a proper closed convex cone, and $k^0 \in \mathrm{core}\,(K)$. Let A, B be non-empty subsets of $X \times Y$ such that* cone $(A - B)$ *is a convex cone, and C a chain of $P_Y(A - B)$. Assume that the following* (i) *and* (ii) *hold* :
(i) $0 \in {}^iP_X^C(A - B)$ *and* ${}^lP_X^C(A - B) = X$.
(ii) *If $(x,\, y_1) \in A$ and $(x,\, y_2) \in B$, then we have $y_2 \leqq_K y_1$.*
Then there exist $f \in \mathcal{L}(X, Y)$ and $t_0 \in R$ such that $H = \{(x,\, y) \in X \times Y \mid \varphi_{K,k^0}(f(x) - y) = t_0\}$ separates A and B.

Proof. By assumption (i) and the definition of ${}^iP_X^C(A - B)$, for any $x \in X$, there exists $\varepsilon > 0$ and $y \in C$ such that $(\lambda x,\, y) \in A - B$ for any $\lambda \in [0, \varepsilon)$. For any $x \in X$, we define $C_x = \{y \in C \mid (x,\, y) \in \mathrm{cone}\,(A - B)\}$, where C is a chain in Y. Since $\lambda^{-1} y \in C_x$ for any $\lambda \in (0,\, \varepsilon)$, we have $C_x \neq \emptyset$ for all $x \in X$. Moreover, for any $y \in C_0$ with $y \neq 0$, there exist $\lambda > 0$, $(x_1,\, y_1) \in A$ and $(x_2,\, y_2) \in B$ such that $(0,\, y) = \lambda\{(x_1,\, y_1) - (x_2,\, y_2)\}$. Then we have $x_1 = x_2$ and $y = \lambda(y_1 - y_2)$. By assumption (ii), we obtain $0 \leqq_K \lambda(y_1 - y_2) = y$. Thus $y \in Y_+ = \{y \in Y \mid 0 \leqq_K y\}$. Since cone $(A - B)$ is a convex cone, we have $C_x + C_{x'} \subset C_{x+x'}$ for any $x,\, x' \in X$. For any $x \in X$, there exists $y' \in Y$ with $-y' \in C_{-x}$ by the definition. Then we have $y - y' \in C_x + C_{-x} \subset C_0 \subset Y_+$ for any $y \in C_x$. Thus we have $y' \leqq_K y$ for any $y \in C_x$. Put $p(x) = \inf\{y \mid y \in C_x\}$, then p is sublinear. Since Y is chain complete, by Theorem 2.4, there exists $f \in \mathcal{L}(X, Y)$ such that

$f(x) \leqq_K p(x)$ for all $x \in X$. For any $(x_1, y_1) \in A$ and $(x_2, y_2) \in B$, if we take $x = x_1 - x_2$, then we have

$$f(x_1 - x_2) \leqq_K p(x_1 - x_2) \leqq_K y_1 - y_2.$$

Therefore we have

$$f(x_1) - y_1 \leqq_K f(x_2) - y_2.$$

Since

$$(f(x_2) - y_2) - (f(x_1) - y_1) \in K,$$

there exists $t_0 \in R$ such that

$$\varphi_{K,k^0}(f(x_1) - y_1) \leq t_0 \leq \varphi_{K,k^0}(f(x_2) - y_2)$$

for any $(x_1, y_1) \in A$ and $(x_2, y_2) \in B$. □

Let X be a vector space and two linear subspaces A and B of X are called *algebraically complementary* to each other if each $x \in X$ can be represented in one and only one way as a sum $x = y + z$ with $y \in A$ and $z \in B$. Then by Theorem 3.1, we obtain the following theorem.

Corollary 3.2. *Let X be a vector space, $(Y, \leqq_K)$ a chain complete directed partially ordered vector space, where K is a proper closed convex cone, and $k^0 \in$ core (K). Let A, B be subsets of $X \times Y$ such that* cone $(A - B)$ *is a convex cone and C a chain of $P_Y(A - B)$. We assume that $P_X^C(A - B)$ is expansive. We also assume that the following* (i) *and* (ii) *hold* :
(i) $0 \in {}^i P_X^C(A - B)$.
(ii) *If $(x, y_1) \in A$ and $(x, y_2) \in B$, then we have $y_2 \leqq_K y_1$.*
Then there exist $f \in \mathcal{L}(X, Y)$ and $t_0 \in R$ such that $H = \{(x, y) \in X \times Y \mid \varphi_{K,k^0}(f(x) - y) = t_0\}$ separates A and B.

Proof. Since $P_X^C(A - B)$ is expansive, ${}^l P_X^C(A - B) = {}^{li} P_X^C(A - B)$ hold, see [2]. We put $X_1 = {}^l P_X^C(A - B) = {}^{li} P_X^C(A - B)$. Then X_1 is a subspace of X. The sets A, B, $A - B$ and cone $(A - B)$ are subsets of X_1. By Theorem 3.1, there exists $f_1 \in \mathcal{L}(X_1, Y)$ such that

$$f_1(x_1 - x_2) \leqq_K y_1 - y_2$$

for any $(x_1, y_1) \in A$ and $(x_2, y_2) \in B$. Let X_2 be an algebraical complementary space of X_1. Then an arbitrary $z \in X$ has a unique representation $z = x + y$ with $x \in X_1$ and $y \in X_2$, see [16, page 51 and 54]. We define $f \in \mathcal{L}(X, Y)$ by $f(z) = f_1(x)$ for all $z \in X$. Then f satisfies the assertion of Corollary. □

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QUADRANTS AND AN EXTENSION OF INTERVALS IN A VECTOR LATTICE

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1. Open sets in a vector lattice

Definition 1.1. Let X be a vector lattice with unit.

A subset $D \subset X$ is said to be open if for any $x \in D$ and for any $e \in \mathcal{K}_X$ there exists $\varepsilon \in \mathcal{K}_{\mathbf{R}}$ such that $[x - \varepsilon e, x + \varepsilon e] \subset D$. Let $\mathcal{O}_X$ be the class of open subsets of X.

2. Quadrants and an extension of intervals in a vector lattice

First some concepts required in the following arguments are defined.

Let X be a vector lattice with unit, $e \in \mathcal{K}_X$, $a, b \in D \subset X$ with $a \neq b$. Let $\mathbf{CSIP}_e(a, b)$ be the class of mappings φ from $[0, 1]$ into D satisfying the following conditions (P), (CP_e) and (SI), $\mathbf{CSDP}_e(a, b)$ the class of mappings φ from $[0, 1]$ into D satisfying the following conditions (P), (CP_e) and (SD) and $\mathbf{CSMP}_e(a, b) = \mathbf{CSIP}_e(a, b) \cup \mathbf{CSDP}_e(a, b)$:

(P) $\varphi(0) = a$ and $\varphi(1) = b$;

(CP_e) For any $t \in [0, 1]$ and for any $\varepsilon \in \mathcal{K}_{\mathbf{R}}$ there exists $\delta \in \mathcal{K}_{\mathbf{R}}$ such that for any $s \in [0, 1]$ if $|s - t| \leq \delta$, then $|\varphi(s) - \varphi(t)| \leq \varepsilon e$;

(SI) $\varphi(t_1) < \varphi(t_2)$ if $t_1 < t_2$;

(SD) $\varphi(t_1) > \varphi(t_2)$ if $t_1 < t_2$.

Remark 2.1. Let $\varphi^{rev}(t) = \varphi(1 - t)$. Then $\varphi \in \mathbf{CSIP}_e(a, b)$ is equivalent to $\varphi^{rev} \in \mathbf{CSDP}_e(b, a)$ and $\varphi \in \mathbf{CSDP}_e(a, b)$ is equivalent to $\varphi^{rev} \in \mathbf{CSIP}_e(b, a)$.

Let X be a vector lattice with unit. Let $|\mathcal{K}_X|$ be the class of x satisfying $|x| \in \mathcal{K}_X$. For any $x \in |\mathcal{K}_X|$ let $x_+^\perp = \{0 \vee x\}^\perp$, $x_-^\perp = \{0 \vee (-x)\}^\perp$,

$$Q(x) = \{x_1 \mid x_1 \in |\mathcal{K}_X|,\ (x_1)_+^\perp = x_+^\perp,\ (x_1)_-^\perp = x_-^\perp\}$$

and

$$\overline{Q}(x) = \left(\bigcup_{x_1, x_2 \in Q(x)} [0 \wedge x_1, 0 \vee x_2] \right) \setminus \{0\}.$$

Remark 2.2. The class of $Q(x)$'s is an equivalence class of $|\mathcal{K}_X|$. Therefore each $x \in |\mathcal{K}_X|$ belongs to a unique $Q(x)$.

Lemma 2.3. *Let X be a vector lattice with unit satisfying the principal projection property and $x \in |\mathcal{K}_X|$.*

Then $x_+^\perp \oplus x_-^\perp = X$.

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Proof. Since X satisfies the principal projection property, it holds that $B(\{x_1\}) \oplus \{x_1\}^{\perp} = X$ for any $x_1 \in X$, where $B(A)$ is the smallest band containing $A \subset X$. Let $x_1 = 0 \vee (-x)$. Then $x_-^{\perp} = \{x_1\}^{\perp}$. Since $x_1 \wedge (0 \vee x) = 0$ and $x_+^{\perp}$ is a projection band, it holds that $B(\{x_1\}) \subset x_+^{\perp}$. Let $x_2 \in x_+^{\perp} \cap x_-^{\perp}$. Then

$$(0 \vee x) \wedge |x_2| = 0, \ (0 \vee (-x)) \wedge |x_2| = 0$$

and

$$\begin{aligned} |x| \wedge |x_2| &= ((0 \vee x) + (0 \vee (-x))) \wedge |x_2| \\ &\leq (0 \vee x) \wedge |x_2| + (0 \vee (-x)) \wedge |x_2| = 0 \end{aligned}$$

and hence $x_2 = 0$. Therefore $x_+^{\perp} \oplus x_-^{\perp} = X$. □

Lemma 2.4. *Let X be a vector lattice with unit satisfying the principal projection property and $x \in |\mathcal{K}_X|$.*

If $(x_1)_+^{\perp} = x_+^{\perp}$ and $(x_1)_-^{\perp} = x_-^{\perp}$, then $x_1 \in |\mathcal{K}_X|$.

Proof. By Lemma 2.3 it holds that $(x_1)_+^{\perp} \oplus (x_1)_-^{\perp} = x_+^{\perp} \oplus x_-^{\perp} = X$. Therefore $(x_1)_+^{\perp} \cap (x_1)_-^{\perp} = \{0\}$. Assume that $x_1 \notin |\mathcal{K}_X|$. Then there exists $x_2 \in X$ with $x_2 > 0$ such that $|x_1| \wedge x_2 = 0$. Therefore

$$\begin{aligned} (0 \vee x_1) \wedge x_2 &\leq |x_1| \wedge x_2 = 0, \\ (0 \vee (-x_1)) \wedge x_2 &\leq |x_1| \wedge x_2 = 0 \end{aligned}$$

and hence $x_2 \in (x_1)_+^{\perp} \cap (x_1)_-^{\perp}$. It is a contradiction. Therefore $x_1 \in |\mathcal{K}_X|$. □

Remark 2.5. By Lemmas 2.3 and 2.4 if X satisfies the principal projection property, then

$$\begin{aligned} Q(x) &= \{x_1 \mid x_1 \in |\mathcal{K}_X|, \ (x_1)_+^{\perp} = x_+^{\perp}\} \\ &= \{x_1 \mid x_1 \in |\mathcal{K}_X|, \ (x_1)_-^{\perp} = x_-^{\perp}\} \\ &= \{x_1 \mid (x_1)_+^{\perp} = x_+^{\perp}, \ (x_1)_-^{\perp} = x_-^{\perp}\}. \end{aligned}$$

Lemma 2.6. *Let X be a vector lattice with unit satisfying the principal projection property and $x \in |\mathcal{K}_X|$.*

Then a mapping $|\cdot|_{Q(x)}$ from $Q(x)$ into $\mathcal{K}_X$ defined by $|x_1|_{Q(x)} = |x_1|$ is bijective.

Proof. By Lemma 2.3 for any $e \in \mathcal{K}_X$ and for any $x \in |\mathcal{K}_X|$ there exist $x_1 \in x_+^{\perp}$ and $x_2 \in x_-^{\perp}$ such that $x_1 + x_2 = e$. Since $x_1 \perp x_2$, it holds that $|x_1 - x_2| = |x_1 + x_2|$. Therefore $|x_2 - x_1| = e$. Note that $x_2 \perp x_3$ for any $x_3 \in x_+^{\perp}$. Since

$$\begin{aligned} (0 \vee (x_2 - x_1)) \wedge |x_3| &= (0 \vee (2x_2 - e)) \wedge |x_3| \\ &\leq (0 \vee (2x_2)) \wedge |x_3| = 0, \end{aligned}$$

it holds that $x_3 \in (x_2 - x_1)_+^{\perp}$ and hence $x_+^{\perp} \subset (x_2 - x_1)_+^{\perp}$. Note that $x_1 \perp x_3$ for any $x_3 \in x_-^{\perp}$. Since

$$\begin{aligned} (0 \vee (x_1 - x_2)) \wedge |x_3| &= (0 \vee (2x_1 - e)) \wedge |x_3| \\ &\leq (0 \vee (2x_1)) \wedge |x_3| = 0, \end{aligned}$$

it holds that $x_3 \in (x_2 - x_1)_-^{\perp}$ and hence $x_-^{\perp} \subset (x_2 - x_1)_-^{\perp}$. Since $x_2 - x_1 \in |\mathcal{K}_X|$, by Lemma 2.3 it holds that $(x_2 - x_1)_+^{\perp} \oplus (x_2 - x_1)_-^{\perp} = X$, $(x_2 - x_1)_+^{\perp} = x_+^{\perp}$ and $(x_2 - x_1)_-^{\perp} = x_-^{\perp}$. Therefore $x_2 - x_1 \in Q(x)$ and $|\cdot|_{Q(x)}$ is surjective.

To prove that $|\cdot|_{Q(x)}$ is injective it should be proved that if $|x_1| = |x_2| = e$ and $x_1 \neq x_2$, then $Q(x_1) \neq Q(x_2)$. Note that $0 \vee (-x_1) \in (x_1)_+^{\perp}$ and $0 \vee (-x_2) \in (x_2)_+^{\perp}$. In general

$$(0 \vee x_1) \wedge (0 \vee (-x_2)) + (0 \vee x_2) \wedge (0 \vee (-x_1)) = \frac{1}{2}(|x_1| + |x_2| - |x_1 + x_2|)$$

and $|x_1+x_2| \wedge |x_1-x_2| = ||x_1|-|x_2||$. Since $|x_1|=|x_2|=e$, it holds that $|x_1+x_2| \notin \mathcal{K}_X$ and it does never hold that $|x_1|+|x_2| = |x_1+x_2|$. Therefore

$$(0 \vee x_1) \wedge (0 \vee (-x_2)) + (0 \vee x_2) \wedge (0 \vee (-x_1)) > 0$$

and either $(0 \vee x_1) \wedge (0 \vee (-x_2)) > 0$ or $(0 \vee x_2) \wedge (0 \vee (-x_1)) > 0$, thus either $0 \vee (-x_2) \notin (x_1)^{\perp}_{+}$ or $0 \vee (-x_1) \notin (x_2)^{\perp}_{+}$. Therefore $(x_1)^{\perp}_{+} \neq (x_2)^{\perp}_{+}$ and hence $Q(x_1) \neq Q(x_2)$. □

Lemma 2.7. *Let X be a vector lattice with unit satisfying the principal projection property and $x \in |\mathcal{K}_X|$.*

If $x_1, x_2 \in Q(x)$, then $x_1 \wedge x_2, x_1 \vee x_2 \in Q(x)$.

Proof. Since

$$|x_1 \wedge x_2| = \frac{1}{2}|x_1 + x_2 - |x_1 - x_2|| \geq \frac{1}{2}||x_1+x_2| - |x_1-x_2|| = |x_1| \wedge |x_2|$$

and

$$|x_1 \vee x_2| = \frac{1}{2}|x_1 + x_2 + |x_1 - x_2|| \geq \frac{1}{2}||x_1+x_2| - |x_1-x_2|| = |x_1| \wedge |x_2|,$$

it holds that $x_1 \wedge x_2, x_1 \vee x_2 \in |\mathcal{K}_X|$. If $x_3 \in x^{\perp}_{-} = (x_1)^{\perp}_{-} = (x_2)^{\perp}_{-}$, then

$$\begin{aligned}(0 \vee (-(x_1 \wedge x_2))) \wedge |x_3| &\leq (0 \vee (-x_1) + 0 \vee (-x_2)) \wedge |x_3| \\ &\leq (0 \vee (-x_1)) \wedge |x_3| + (0 \vee (-x_2)) \wedge |x_3| = 0\end{aligned}$$

and hence $x_3 \in (x_1 \wedge x_2)^{\perp}_{-}$. Conversely if $x_3 \in (x_1 \wedge x_2)^{\perp}_{-}$, then

$$(0 \vee (-x_1)) \wedge |x_3| \leq (0 \vee (-(x_1 \wedge x_2))) \wedge |x_3| = 0$$

and hence $x_3 \in (x_1)^{\perp}_{-} = x^{\perp}_{-}$. Therefore $(x_1 \wedge x_2)^{\perp}_{-} = x^{\perp}_{-}$. By Remark 2.5 it holds that $x_1 \wedge x_2 \in Q(x)$. The rest can be proved similarly. □

Lemma 2.8. *Let X be a vector lattice with unit and $x \in |\mathcal{K}_X|$.*

If $x_1 \in \overline{Q}(x)$, $0 \wedge x_1 \leq x_2 \leq 0 \vee x_1$ and $x_2 \neq 0$, then $x_2 \in \overline{Q}(x)$.

Proof. Since $x_1 \in \overline{Q}(x)$, there exist $x_3, x_4 \in Q(x)$ such that $x_1 \in [0 \wedge x_3, 0 \vee x_4] \setminus \{0\}$. Since $0 \wedge x_1 \leq x_2 \leq 0 \vee x_1$ and $x_2 \neq 0$, it holds that $x_2 \in [0 \wedge x_3, 0 \vee x_4] \setminus \{0\}$. Therefore $x_2 \in \overline{Q}(x)$. □

Lemma 2.9. *Let X be a vector lattice with unit.*

(1) *Then $\alpha x_1 \in \overline{Q}(x)$ for any $x_1 \in \overline{Q}(x)$ and for any $\alpha \in \mathcal{K}_{\mathbf{R}}$.*

(2) *If X satisfies the principal projection property, then $x_1 + x_2 \in \overline{Q}(x)$ for any $x_1, x_2 \in \overline{Q}(x)$.*

Proof. (1) Since $x_1 \in \overline{Q}(x)$, there exist $x_3, x_4 \in Q(x)$ such that $x_1 \in [0 \wedge x_3, 0 \vee x_4] \setminus \{0\}$. Since $\alpha \in \mathcal{K}_{\mathbf{R}}$, it holds that $\alpha x_1 \in [0 \wedge (\alpha x_3), 0 \vee (\alpha x_4)] \setminus \{0\}$. Since

$$(0 \vee x) \wedge |\alpha x_3| \leq (1 \vee \alpha)((0 \vee x) \wedge |x_3|) = 0$$

and

$$(0 \vee x) \wedge |\alpha x_4| \leq (1 \vee \alpha)((0 \vee x) \wedge |x_4|) = 0,$$

it holds that $\alpha x_3, \alpha x_4 \in Q(x)$. Therefore $\alpha x_1 \in \overline{Q}(x)$.

(2) Since $x_1, x_2 \in \overline{Q}(x)$, there exist $x_3, x_4, x_5, x_6 \in Q(x)$ such that $x_1 \in [0 \wedge x_3, 0 \vee x_4] \setminus \{0\}$ and $x_2 \in [0 \wedge x_5, 0 \vee x_6] \setminus \{0\}$. Note that $0 \vee x_4, 0 \vee x_6 \in x^{\perp}_{-}$ and $0 \vee (-x_3), 0 \vee (-x_5) \in x^{\perp}_{+}$. Assume that $x_2 = -x_1$. Then

$$\begin{aligned}x_1 = x_1 \wedge (-x_2) &\leq (0 \vee x_4) \wedge (0 \vee (-x_5)) = 0, \\ x_2 = x_2 \wedge (-x_1) &\leq (0 \vee x_6) \wedge (0 \vee (-x_3)) = 0\end{aligned}$$

and hence $x_1 = x_2 = 0$. It is a contradiction. Therefore $x_2 \neq -x_1$ and $x_1 + x_2 \in [0 \wedge 2(x_3 \wedge x_5), 0 \vee 2(x_4 \vee x_6)] \setminus \{0\}$. By Lemma 2.7 and the proof of (1) it holds that $2(x_3 \wedge x_5), 2(x_4 \vee x_6) \in Q(x)$. Therefore $x_1 + x_2 \in \overline{Q}(x)$. □

Let X be a vector lattice with unit and $a, b \in D \subset X$ with $a \neq b$. Let $\mathbf{CSSMP}(a,b)$ be the class of mappings φ from $[0,1]$ into D satisfying the following conditions:

(CS1) There exists a natural number r_φ and $\{e^i_\varphi \mid e^i_\varphi \in \mathcal{K}_X \text{ for } i = 1, \ldots, r_\varphi\}$ such that a mapping φ^i from $[0,1]$ into D defined by $\varphi^i(s) = \varphi(\frac{s+i-1}{r_\varphi})$ belongs to $\mathbf{CSMP}_{e^i_\varphi}(\varphi(\frac{i-1}{r_\varphi}), \varphi(\frac{i}{r_\varphi}))$;

(CS2) There exists $x \in |\mathcal{K}_X|$ such that $\varphi^i(1) - \varphi^i(0) \in \overline{Q}(x)$ for any $i = 1, \ldots, r_\varphi$;

(CS3) $\varphi([0,1]) \subset [a \wedge b, a \vee b]$.

φ^i satisfies either (SI) or (SD). For convenience, φ^i is said to be **CSIP** if φ^i satisfies (SI) and φ^i is **CSDP** if φ^i satisfies (SD), respectively.

Remark 2.10. By Remark 2.1, $\varphi \in \mathbf{CSSMP}(a,b)$ is equivalent to $\varphi^{rev} \in \mathbf{CSSMP}(b,a)$. Since $(\varphi^{rev})^{rev} = \varphi$, a mapping φ corresponding to φ^{rev} is bijective.

Definition 2.11. Let X be a vector lattice with unit and $D \subset X$.

A subset D is said to be connected if $\mathbf{CSSMP}(a,b) \neq \emptyset$ for any $a, b \in D$ with $a \neq b$. Let $\mathcal{CO}_X$ be the class of connected open subsets of X.

Definition 2.12. Let X be a vector lattice with unit and $a, b \in D \in \mathcal{CO}_X$.

The following subset

$$\langle a|b \rangle = \begin{cases} \bigcup_{\varphi \in \mathbf{CSSMP}(a,b)} \varphi([0,1]) & \text{if } a \neq b, \\ \{a\} & \text{if } a = b \end{cases}$$

is said to be a stepwise interval from a to b.

Remark 2.13. By Remark 2.10 it holds that $\varphi([0,1]) = \varphi^{rev}([0,1])$. Therefore $\langle a|b \rangle$ and $\langle b|a \rangle$ coincide as a set. However the first one means an “interval from a to b”, the second one means another “interval from b to a” and both are distinguished.

Remark 2.14. By (CS1) and (CS3) it holds that $\langle a|b \rangle \subset [a \wedge b, a \vee b] \cap D$.

Definition 2.15. Let X be a vector lattice with unit, Y a complete vector lattice and $a, b \in D \in \mathcal{CO}_X$.

The subset $\langle c|d \rangle$ is said to be a subinterval of $\langle a|b \rangle$ if $c, d \in \langle a|b \rangle$ and there exists $x \in |\mathcal{K}_X|$ such that $c - a, d - c, b - d \in \overline{Q}(x)$.

Remark 2.16. By Lemma 2.9 and (CS2) if X satisfies the principal projection property, then $\langle c|d \rangle \subset \langle a|b \rangle$.

Let X be a vector lattice with unit, $e \in \mathcal{K}_X$ and $a, b \in X$ with $a \leq b$. For $[a,b] \in \mathcal{I}_X$ we consider the following subset:

$$[a,b]^e = \{x \mid \text{ there exists some } \varepsilon \in \mathcal{K}_{\mathbf{R}} \text{ such that } x - a \geq \varepsilon e \text{ and } b - x \geq \varepsilon e\}.$$

Lemma 2.17. *Let X be a vector lattice with unit, $e \in \mathcal{K}_X$ and $a, b \in X$ with $a \leq b$.*

Then $[a,b]^e \neq \emptyset$ if and only if there exists $\varepsilon \in \mathcal{K}_{\mathbf{R}}$ such that $b - a \geq \varepsilon e$.

Proof. Suppose that $[a,b]^e \neq \emptyset$. Let $x \in [a,b]^e$. By definition there exists $\varepsilon \in \mathcal{K}_{\mathbf{R}}$ such that $x - a \geq \frac{1}{2}\varepsilon e$ and $b - x \geq \frac{1}{2}\varepsilon e$. Therefore $b - a \geq \varepsilon e$.

Conversely suppose that there exists $\varepsilon \in \mathcal{K}_{\mathbf{R}}$ such that $b - a \geq \varepsilon e$. Let $x = \frac{1}{2}(a+b)$. Then $x - a = b - x = \frac{1}{2}(b-a) \geq \frac{1}{2}\varepsilon e$. Therefore $x \in [a,b]^e$. □

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On Lusin's theorem for non-additive ordered vector space-valued measures

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1 Introduction

Lusin's theorem can be discussed in a metric space. For the real-valued non-additive measure, in [8], Li and Mesiar show that Lusin's theorem remains valid for the measure has condition (E), which is equivalent to the Egoro condition, and pseudometric generating property. Further, Kawabe investigates the Li and Yasuda's work for the Riesz space-valued non-additive measure, see [5].

In this study, we consider the ordered vector space-valued non-additive measure. In [19], we prove that equivalence of Egoro 's theorem and Egoro condition for an ordered vector space-valued non-additive measure. By using this fact and method in [8], we show that Lusin's theorem remains valid for ordered vector space-valued non-additive Borel measures on a metric space that have pseudometric generating property and satisfy the Egoro condition by assuming that the vector space is order separable.

2 Preliminaries and Definitions

In what follows, let V be an ordered vector space. Let Θ be the set of all mappings from $\mathbb{N}$ into $\mathbb{N}$. Let S be a non-emptyset and $\mathcal{F}$ a - eld of S. we say that (u_n) in V converges to $u \in V$ (write $u_n \to u$) if there exists $\{p_k\}$ in V with $p_k \searrow 0$ such that $\quad p_k \quad u_n \quad u \quad p_k$.

Definition 2.1. (1) *A double sequence $\{r_{m,n}\}_{(m,n)\in\mathbb{N}^2}$ in V is called a regulator if it is order bounded and $r_{m,n} \searrow 0$ for each $m \in \mathbb{N}$, that is, $r_{m,n} \quad r_{m,n+1}$ for any $m, n \in \mathbb{N}$ and $\inf_n r_{m,n} = 0$ for any $m \in \mathbb{N}$.*
(2) *We say that V has the Egoroff property if for any regulator $\{r_{m,n}\}_{(m,n)\in\mathbb{N}^2}$ in V, there exists a sequence $\{p_k\}_{k\in\mathbb{N}}$ in V such that, for each $(k,m) \in \mathbb{N}^2$, one can find $n(k,m) \in \mathbb{N}$ satisfying that $r_{m,n(k,m)} \quad p_k$.*
(3) *We say that V is called order separable if every non-empty subset D possessing a supremum contains an at most countable subset possessing the same supremum in D.*

A set function $\mu : \mathcal{F} \to V$ is called a non-additive measure if it satis es the following two conditions.
(1) $\mu(\emptyset) = 0$.
(2) If $A, B \in \mathcal{F}$ and $A \subset B$, then $\mu(A) \quad \mu(B)$.

Definition 2.2. *Let $\mu : \mathcal{F} \to V$ be a non-additive measure.*
(1) *μ is called strongly order continuous if it is continuous from above at measurable sets of measure 0, that is, for any $\{A_n\}_{n\in N} \subset \mathcal{F}$ and $A \in \mathcal{F}$ with $A_n \searrow A$ and $\mu(A) = 0$, it*

holds that $\mu(A_n) \searrow 0$.
(2) μ *is called weakly null-additive if* $\mu(A \cup B) = 0$ *whenever* $A, B \in \mathcal{F}$ *and* $\mu(A) = \mu(B) = 0$.
(3) μ *is said to have pseudometric generating property if there exists* $\{p_k\}_{k\in N}$ *with* $p_k \searrow 0$, *and for any* $k \in N$, *there exists* n_k *such that for any* $A, B \in \mathcal{F}$, $\mu(A) \le p_{n_k}$ *and* $\mu(B) \le p_{n_k}$ *implies* $\mu(A \cup B) \le p_k$.

Definition 2.3. *Let* $\mu : \mathcal{F} \to V$ *be a non-additive measure. Let* $\{f_n\}_{n\in\mathbb{N}}$ *be a sequence of* $\mathcal{F}$*-measurable real-valued functions on* S *and* f *also such a function.*
(1) $\{f_n\}_{n\in\mathbb{N}}$ *is called convergent* μ*-a.e. to* f *if there exists an* $A \in \mathcal{F}$ *with* $\mu(A) = 0$ *such that* $\{f_n\}_{n\in\mathbb{N}}$ *converges to* f *on* $S \setminus A$.
(2) $\{f_n\}_{n\in\mathbb{N}}$ *is called* μ*-almost uniformly convergent to* f *if there exists a decreasing net* $\{B_\gamma : \gamma \in \Gamma\} \subset \mathcal{F}$ *such that there exists a* $\gamma \in \Gamma$ *such that* $\mu(B_\gamma) \to 0$ *and* $\{f_n\}_{n\in\mathbb{N}}$ *converges to* f *uniformly on each subset* $S \setminus B_\gamma$.

Definition 2.4. *Let* $\mu : \mathcal{F} \to V$ *be a non-additive measure.*
(1) *A double sequence* $\{A_{m,n}\}_{(m,n)\in\mathbb{N}^2} \subset \mathcal{F}$ *is called a* μ*-regulator if it satisfies the following two conditions.*
(D1) $A_{m,n} \supset A_{m,n'}$ *whenever* $n \le n'$.
(D2) $\mu(\bigcup_{m=1}^{\infty} \bigcap_{n=1}^{\infty} A_{m,n}) = 0$.
(2) μ *satisfies the Egoroff condition if for any* μ*-regulator* $\{A_{m,n}\}_{(m,n)\in\mathbb{N}^2}$,
$\inf_{\theta\in\Theta} \mu(\bigcup_{m=1}^{\infty} A_{m,\theta(m)}) = 0$ *holds.*

Theorem 2.1. *Let* S *be a metric space and* $\mathcal{B}(S)$ *a* σ*-field of all Borel subsets of* S. *Let* $\mu : \mathcal{B}(S) \to V$ *be a non-additive Borel measure on* S *that has pseudometric generating property and satisfies the Egoroff condition. We assume that* V *is order separable. Then* μ *is regular.*

3 Egoroff's Theorem

We show a version of Egoroff's theorem for an ordered vector space-valued case. For a real-valued monotone measure case, see [8]. For a real-valued fuzzy measure case, see [9], and the Riesz space-valued case, see [3]. We have obtained the following result in [3, 19].

Theorem 3.1. *Let* $\mu : \mathcal{F} \to V$ *be a non-additive measure which satisfies the Egoroff condition and* S *be a set. Let* $\{f_n\}_{n\in\mathbb{N}}$ *be a sequence of measurable real-valued functions on* S *and* f *also such a function. If* $\{f_n\}_{n\in\mathbb{N}}$ *converges* μ*-a.e. to* f, *then there exists an increasing sequence* $\{A_m\}_{m\in\mathbb{N}} \subset \mathcal{F}$ *such that* $\mu(S \setminus \bigcup_{m=1}^{\infty} A_m) = 0$ *and* $\{f_n\}_{n\in\mathbb{N}}$ *converges to* f *uniformly on* A_m *for each* $m \in \mathbb{N}$.

When μ is Borel and satisfies pseudometric generating property, we can obtain a slightly stronger conclusion. In order to point up the difference between an usual measure theory and a non-additive measure, we shall show the proof of the following theorem.

Theorem 3.2. *Let* S *be a metric space and* $\mu : \mathcal{B}(S) \to V$ *a non-additive Borel measure that has pseudometric generating property and satisfies the Egoroff condition. We assume that* V *is order separable. Let* $\{f_n\}_{n\in\mathbb{N}}$ *be a sequence of Borel measurable real-valued functions on* S *and* f *also sych a function. If* $\{f_n\}_{n\in\mathbb{N}}$ *converges* μ*-a.e. to* f, *then for any*

$m \in \mathbb{N}$, there exists an increasing sequence $\{F_m\}_{m\in\mathbb{N}}$ of closed sets such that $\mu(S\backslash F_m) \searrow 0$ as $m \to \infty$ and $\{f_n\}_{n\in\mathbb{N}}$ converges to f uniformly on each $\{F_m\}_{m\in\mathbb{N}}$.

Proof. Since $\{f_n\}_{n\in\mathbb{N}}$ converges μ-a.e. to f, by Theorem 3.1, there exists an increasing sequence $\{A_m\}_{m\in\mathbb{N}} \subset \mathcal{B}(S)$ such that $\{f_n\}_{n\in\mathbb{N}}$ converges to f uniformly on A_m for each $m \in \mathbb{N}$ and $\mu(S \setminus \cup_{m=1}^{\infty} A_m) = 0$. By Theorem 2.1, since μ is regular, there exists an increasing sequence $\{F_{m,n}\}_{(m,n)\in\mathbb{N}^2}$ of closed sets such that $F_{m,n} \subset A_m$ and $\mu(A_m \backslash F_{m,n}) \searrow 0$ for each $m \in \mathbb{N}$ as $n \to \infty$. Without loss of generality, we can assume that for each $m \in \mathbb{N}$, $\{A_m \setminus F_{m,n}\}_{(m,n)\in\mathbb{N}^2}$ is decreasing. Then we have

$$A_m \setminus F_{m,n} \searrow \bigcap_{n=1}^{\infty} (A_m \setminus F_{m,n}) \text{ as } n \to \infty.$$

Put $S_{m,n} = (S \setminus \cup_{m=1}^{\infty} A_m) \cup (A_m \setminus F_{m,n})$ and $D_m = \cap_{n=1}^{\infty} S_{m,n}$. Then for each $m \in \mathbb{N}$, $S_{m,n} \searrow D_m$ as $n \to \infty$. Since

$$\mu\left(\bigcap_{n=1}^{\infty} (A_m \setminus F_{m,n})\right) \quad \mu\left(A_m \setminus F_{m,n}\right),$$

and

$$\mu\left(A_m \setminus F_{m,n}\right) \to 0 \text{ as } n \to \infty,$$

we have

$$\mu\left(\bigcap_{n=1}^{\infty} (A_m \setminus F_{m,n})\right) = 0.$$

Since we can show easily that μ is weakly null additive, we have $\mu(D_m) = 0$ for each $m \in \mathbb{N}$. By the Egoro condition, we can show that there exists an increasing sequence $\{\theta_k\}_{k\in\mathbb{N}} \subset \Theta$ such that

$$\mu\left(\bigcup_{m=1}^{\infty} S_{m,\theta_k(m)}\right) \to 0 \text{ as } k \to \infty.$$

Since $S \setminus \cup_{m=1}^{\infty} F_{m,\theta_k(m)} \subset \cup_{m=1}^{\infty} S_{m,\theta_k(m)}$, we have

$$\mu\left(S \setminus \bigcup_{m=1}^{\infty} F_{m,\theta_k(m)}\right) \to 0 \text{ as } k \to \infty.$$

Put $q_k = \mu\left(S \setminus \bigcup_{m=1}^{\infty} F_{m,\theta_k(m)}\right)$. Since $\{F_{m,n}\}_{(m,n)\in\mathbb{N}^2}$ is increasing, we have $q_k \searrow 0$ as $k \to \infty$.

On the other hand, since

$$\bigcup_{m=1}^{\infty} F_{m,\theta_k(m)} \setminus \bigcup_{m=1}^{n} F_{m,\theta_k(m)} \searrow \emptyset$$

as $n \to \infty$ and μ is strongly order continuous,

$$\mu\left(\bigcup_{m=1}^{\infty} F_{m,\theta_k(m)} \setminus \bigcup_{m=1}^{n} F_{m,\theta_k(m)}\right) \searrow 0 \text{ as } n \to \infty.$$

For each $k \in \mathbb{N}$ we choose $n_k \in \mathbb{N}$ with $n_k \quad n_{k+1}$, then we have

$$\mu\left(\bigcup_{m=1}^{\infty} F_{m,\theta_k(m)} \setminus \bigcup_{m=1}^{n_k} F_{m,\theta_k(m)}\right) \to 0 \text{ as } k \to \infty.$$

Put $r_k = \mu\left(\cup_{m=1}^{\infty} F_{m,\theta_k(m)} \setminus \cup_{m=1}^{n_k} F_{m,\theta_k(m)}\right)$. Since $\{F_{m,n}\}_{(m,n)\in\mathbb{N}^2}$ is increasing, we have $r_k \searrow 0$ as $k \to \infty$. Put $p_k = r_k + q_k$. Since $\inf_k r_k + \inf_k q_k = \inf_k p_k$, $p_k \searrow 0$ as $k \to \infty$. Noting that $0 \quad r_k \quad p_k$, $0 \quad q_k \quad p_k$, since μ has pseudometric generating property, for any $k \in \mathbb{N}$, we have

$$\mu\left(\left(S \setminus \bigcup_{m=1}^{\infty} F_{m,\theta_k(m)}\right) \bigcup \left(\bigcup_{m=1}^{\infty} F_{m,\theta_k(m)} \setminus \bigcup_{m=1}^{n_k} F_{m,\theta_k(m)}\right)\right) \quad p_k.$$

Thus we have

$$\mu\left(\left(S \setminus \bigcup_{m=1}^{\infty} F_{m,\theta_k(m)}\right) \bigcup \left(\bigcup_{m=1}^{\infty} F_{m,\theta_k(m)} \setminus \bigcup_{m=1}^{n_k} F_{m,\theta_k(m)}\right)\right) \to 0$$

as $k \to \infty$. Since

$$S \setminus \bigcup_{m=1}^{n_k} F_{m,\theta_k(m)} \subset \left(S \setminus \bigcup_{m=1}^{\infty} F_{m,\theta_k(m)}\right) \bigcup \left(\bigcup_{m=1}^{\infty} F_{m,\theta_k(m)} \setminus \bigcup_{m=1}^{n_k} F_{m,\theta_k(m)}\right),$$

we have

$$\mu\left(S \setminus \bigcup_{m=1}^{n_k} F_{m,\theta_k(m)}\right) \to 0 \text{ as } k \to \infty.$$

Denote $F_k = \cup_{m=1}^{n_k} F_{m,\theta_k(m)}$, then F_k is a closed set, $\mu(S \setminus F_k) \to 0$ as $k \to \infty$ and $F_k \subset \cup_{m=1}^{n_k} A_m$. It is easy to see that $\{f_n\}_{n\in\mathbb{N}}$ converges to f uniformly on F_k. ∎

4 Lusin's Theorem

We shall further generalize Lusin's theorem. For the real-valued case, see [8, 9]. And for the Riesz space-valued case, see [5].

Theorem 4.1. *Let S be a metric space and $\mu : \mathcal{B}(S) \to V$ a non-additive Borel measure on S which has pseudometric generating property and satisfies the Egoroff condition. We assume that V is order separable. If f is a Borel measurable real-valued function on S, then there exists an increasing sequence $\{F_k\}_{k\in\mathbb{N}}$ of closed sets such that $\mu(S \setminus F_k) \searrow 0$ as $k \to \infty$ and f is continuous on each F_k.*

Further, we obtain the sufficient condition for the establishment of Lusin's theorem.

Definition 4.1. (1) *A regulator* $\{u_{m,n}\}_{(m,n)\in\mathbb{N}^2}$ *is said to be summative if any* $m, n \in \mathbb{N}$, *if holds that* $u_{m+1,n} + u_{m+1,n} \quad u_{m,n}$.
(2) *Let* $\mu : \mathcal{F} \to E$ *be a non-additive measure.* μ *is said to be multiple continuous from above if for any* μ*-regulator* $\{A_{m,n}\}_{(m,n)\in\mathbb{N}^2}$ *there exists a summative regulator* $\{u_{m,i}\}_{(m,i)\in\mathbb{N}^2}$ *such that for any* $i \in \mathbb{N}$ *there exists* $\theta_i \in \Theta$ *such that for any* $m \in \mathbb{N}$ *and* $(n_1, n_2, \ldots, n_m) \in \mathbb{N}^m$, $\mu(\bigcup_{j=1}^m A_{j,n_j} \cup A_{m+1,n} \cup A) \quad \mu(\bigcup_{j=1}^m A_{j,n_j} \cup A) \quad u_{m+2,i}$ *for any* $n \quad \theta_i(m+1)$ *whenever* $A \in \mathcal{F}$ *and satisfies* $\bigcup_{j=1}^m A_{j,n_j} \cup A_{m+1,n} \cup A \searrow \bigcup_{j=1}^m A_{j,n_j} \cup A$ *as* $n \to \infty$.

Theorem 4.2. *If* μ *is multiple continuous from above and continuous from below, then* μ *satisfies the Egoroff condition.*

Theorem 4.3. *Let* S *be a metric space, and* $\mu : \mathcal{B}(S) \to V$ *a non-additive Borel measure on* S *that is multiple continous from above, condinuous from below and has pseudometric generating property. We assume that* V *is order separable. If* f *is a Borel measurable real-valued function on* S, *then there exists an increasing sequence* $\{F_k\}_{k\in\mathbb{N}}$ *of closed sets such that* $\mu(S \setminus F_k) \searrow 0$ *and* f *is continuous on each* F_k.

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A notion of algebraic duals of commutative Banach algebras and its applications

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Abstract

Let A be a commutative semisimple Banach algebra with maximal ideal space Δ_A. The algebraic dual A_* of A is the norm closure of $\mathrm{span}(\Delta_A)$ in A^*, and the canonical embedding $\phi : A \to {A_*}^*$, $< a, \psi >=< \psi, \phi(a) >$ $(a \in A, \psi \in A_*)$ are defined. In 962, Birtel defined Arens type product in ${A_*}^*$ so that ${A_*}^*$ becomes a commutative semisimple Banach algebra and ϕ being a norm-decreasing isomorphism. Using these $A_*, {A_*}^*$ and ϕ, we give a characterization theorem of commutative semisimple convolution measure algebras.

§1. **Introduction**

Let A be a commutative semisimple Banach algebra with maximal ideal space Δ_A. For $a \in A$, $\hat{a}$ denotes the Gelfand transform of a. A_* denotes the norm closure of $\mathrm{span}(\Delta_A)$ in A^*, the dual space of A, and ${A_*}^*$ the dual space of A_*. For $a \in A, \psi \in A_*$, and $\xi \in {A_*}^*$ we use notations $\psi(a) =< a, \psi >$ and $\xi(\psi) =< \psi, \xi >$. The canonical embedding ϕ of A into ${A_*}^*$ is defined by $< \psi, \phi(a) >=< a, \psi >$ $(\psi \in A_*, a \in A)$.

It was in Birtel [1] that Arens type product in ${A_*}^*$ was first considered. The product makes ${A_*}^*$ a commutative Banach algebra, and the canonical embedding of A into ${A_*}^*$ is a norm decreasing Banach algebra isomorphism. Birtel called ${A_*}^*$ a commutative extension of A, and his definition of product given in [1] is:

$< p, \xi \quad \zeta >= \sum_{\varphi \in \Delta_A} \hat{p}(\varphi) < \varphi, \xi >< \varphi, \zeta >$ $\quad (\xi, \zeta \in {A_*}^*, p = \sum_{\varphi \in \Delta_A} \hat{p}(\varphi)\varphi \in \mathrm{span}(\Delta_A))$.
The above is enough to determine multiplication because $\mathrm{span}(\Delta_A)$ is dense in A_*.

Lemma 1.1 *Let A be a commutative semisimple Banach algebra. Then the canonical embedding ϕ of A into ${A_*}^*$ satisfies $\|\phi(a)\|_{{A_*}^*} \quad \|a\|_A$ $(a \in A)$ and has a weak*-dense image.*

Proof. That ϕ satisfies $\|\phi(a)\|_{{A_*}^*} \quad \|a\|_A$ $(a \in A)$ is evident. Since A_* forms a separating vector space of linear functionals on ${A_*}^*$, the A_*-topology, that is the weak*-topology of ${A_*}^*$ makes ${A_*}^*$ a locally convex space whose dual space is A_* ([3, Theorem 3.10]). Therefore, if $\phi(A)$ is not weak*-dense in ${A_*}^*$, there exists a nonzero $f \in A_*$ such that $< a, f >=< f, \phi(a) >= 0$ $(a \in A)$. But this is impossible because f is a non-zero linear functional on A. □

For $a \in A$, we define $\|\hat{a}\|_{BSE} = \sup\{|< a.p >|: p \in \mathrm{span}(\Delta_A), \|p\|_{A^*} \quad 1\}$. One can easily see that the canonical embedding ϕ of A into ${A_*}^*$ is isometric if and only if $\|a\|_A = \|\hat{a}\|_{BSE}$ $(a \in A)$.

In [5] J.L.Taylor introduced the notion of convolution measure algebras. Group algebras and measure algebras on locally compact Hausdorff groups are important examples of convolution measure algebras. Furthermore, measure algebras $M(S)$ and its L-subalgebras on locally compact Hausdorff semigroups S are convolution measure algebras. He proved in [5] an important representation theorem for commutative convolution measure algebras.

Definition 1. Let A be a Banach algebra. A is said to be a convolution measure algebra if the following conditions are satisfied:
(i) A is a complex L-space (c.f., for the definition, see [2]);

(ii) If A_+ is the positive cone of A and $a, b \in A_+$, then $ab \in A_+$ with $\|ab\|_A = \|a\|_A\|b\|_A$;

(iii) Suppose $a, b, c \in A_+$ such that $c \quad ab$. For any $\varepsilon > 0$, there exist $a_1, ..., a_m, b_1, ..., b_n \in A_+$ and $a_{ij} \in [0, 1]$ $(i = 1, ..., m, j = 1, ..., n)$ with $\sum_{i=1}^{m} a_i \quad a$ and $\sum_{j=1}^{n} b_j \quad b$ such that

$$\left\| \sum_{i,j=1}^{m,n} a_{ij} a_i b_j \quad c \right\|_A \quad \varepsilon. \tag{1}$$

From now on, we restrict ourselves to commutative Banach algebras. A convolution measure algebra means a commutative convolution measure algebra.

Let A be a commutative semisimple Banach algebra with maximal ideal space Δ_A. Hereafter, the following conditions for A are important:

(a) A multiplication   and an involution $*$ are defined in $\Delta_A \cup \{0\}$ so that $(\Delta_A \cup \{0\}, \ , *)$ becomes a commutative semigroup with involution with a unit element;

(b) For $p_i = \sum_{\varphi \in \Delta_A} \hat{p_i}(\varphi)\varphi \ \in \mathrm{span}(\Delta_A), i = 1, 2$, we define multiplication of p_1 and p_2 by $p_1 \quad p_2 = \sum_{\varphi, \psi \in \Delta_A} \hat{p_1}(\varphi)\hat{p_2}(\psi)\varphi \quad \psi \in \mathrm{span}(\Delta_A)$. Then $\|p \quad p^*\|_{A^*} = \|p\|_{A^*}^2$ $(p \in \mathrm{span}(\Delta_A))$;

(c) For $a \in A$ and $\psi \in \Delta_A$, there exists an $a^\psi \in A$ such that $\widehat{a^\psi}(\varphi) = \hat{a}(\psi \quad \varphi)$ $(\varphi \in \Delta_A)$;

(d) $\|a\|_A = \|\hat{a}\|_{BSE}$ $(a \in A)$.

In §2, we show that any semisimple convolution measure algebra A satisfies the conditions (a), (b), (c), and (d) above. In §3, we show that if A is a commutative semisimple Banach algebra satisfying the conditions (a), (b), (c), and (d) above, then there exists a natural order in A so that A becomes a convolution measure algebra. Finally in §4, using results in §2 and §3, we give a theorem which characterizes convolution measure algebras. Also we will show, as a corollary, that when A is a semisimple convolution measure algebra, Birtel's canonical embedding of A into the commutative extension ${A_*}^*$ is just equal to Taylor's representation of A.

§2 Necessary conditions for A to be a convolution measure algebra

In this section we will show that the four conditions (a), (b), (c) and (d) in §1 for A are sufficient to ensure A to be a convolution measure algebra.

If S is a compact Hausdorff topological space, $C(S)$ and $M(S)$ denote the usual Banach space of continuous functions with supremum norm and the usual Banach space of bounded regular complex Borel measures, respectively. Further, if S is a commutative locally compact Hausdorff topological semigroup, $\hat{S}$ denotes the set of bounded continuous semicharacters of S, and $M(S)$ the usual convolution measure algebra. A subspace N of $M(S)$ is called an L-subspace if N is a closed subspace and satisfies; "if $\mu \in N, \nu \in M(S)$ with $\nu << \mu$, then $\nu \in N$". When $\mathfrak{M}_i$ is an L-subspace of $M(S_i)$ of a locally compact Hausdorff space S_i $(i = 1, 2)$, a bounded linear map ϕ of $\mathfrak{M}_1$ to $\mathfrak{M}_2$ is called an L-homomorphism, if for any $0 \quad \mu \in \mathfrak{M}_1$, we have (i) $\phi(\mu) \quad 0$, (ii) $\|\phi(\mu)\| = \|\mu\|$, (iii) $\phi\{\nu \in \mathfrak{M}_1 : 0 \quad \nu \quad \mu\} = \{\omega \in \mathfrak{M}_2 : 0 \quad \omega \quad \phi(\mu)\}$.

Proposition 2.1 *If A is a semisimple convolution measure algebra, it satisfies the conditions (a), (b), (c), and (d) in §1.*

Proof. As is well known, A has a concrete representation (cf. [2, Theorem 1.7]) : "there exists a locally compact Hausdorff space Ω and a regular positive Borel measure, m, on Ω such that A is isometrically linear and order isomorphic to $L^1(\Omega, m)$".

Now we identify A with $L^1(\Omega, m)$. Then $A^* = L^\infty(\Omega, m)$, which is a C*-algebra, having an identity 1, with respect to pointwise multiplication and conjugation as involution. If X denotes the maximal ideal space of A^* the Gelfand transform $\psi \to \hat{\psi}$ is an order preserving *-isometric homomorphism of A^* onto $C(X)$, and there is an order preserving linear isometry $a \to a_X$ of A into $M(X)$ (this is called the standard representation of A which is characterized by $< a, \psi >= \int_X \hat{\psi} da_X$ $(\psi \in A^*, a \in A)$. We denote $A_X = \{a_X : a \in A\}$(cf. [6, §2.4]). The multiplication " " in A_X is defined, of course, so that the map $a \to a_X$ becomes an algebra isomorphism, and the maximal ideal space of A_X is

$\Delta_X := \{f \in C(X) : 0 \neq f, < f, \mu \quad \nu >=< f, \mu >< f, \nu > \quad (\mu, \nu \in A_X)\}$.

For $x, y \in X$, we define $x \sim y$ if and only if $f(x) = f(y)$ for all $f \in \Delta_X$. Then $\sim$ is an equivalent relation in X, and for $x \in X$ denote $[x] := \{y \in X : y \sim x\}$ and put $S := X/\sim$ with the quotient topology, which makes S a compact Hausdorff space. Since A_X is an L-subspace of $M(X)$, the map $\mu \to \mu_S := \mu \circ \alpha^{\ \ 1}$ is an L-homomorphism of A_X onto an L-subspace $A_S(:= A_X \circ \alpha^{\ \ 1})$ (cf. [6, Proposition 2.2.2]).

For $f \in \Delta_X$, put $\tilde{f}(s) = f(x),\ x \in s \in S$. Then $\tilde{f} \in C(S)$ with $\tilde{f} \circ \alpha = f$. If $\mu_S = \mu \circ \alpha^{\ \ 1} = 0$, we have $0 = \int_S \tilde{f} d\mu \circ \alpha^{\ \ 1} = \int_X \tilde{f} \circ \alpha d\mu = \int_X f d\mu$ $(f \in \Delta_X)$. This implies $\mu = 0$ since A_X is semisimple. Therefore the map $\mu \to \mu_S (= \mu \circ \alpha^{\ \ 1})$ is injective, and hence it is isometry (cf. [6, Corollary 2.4.5]).

We define multiplication " " in A_S so that $a \to (a_X)_S$ becomes an algebra homomorphism. Then the relation $\int_S \tilde{f} d\mu_S = \int_X f d\mu$ $(f \in \Delta_X, \mu \in A_X)$, where $\mu \to \mu_S (= \mu \circ \alpha^{\ \ 1})$ induces an onto isometric isomorphism of A_X to A_S, implies that $\Delta_S = \{\tilde{f} : f \in \Delta_X\}$ is the maximal ideal space of A_S. Obviously, Δ_S separates points of S.

By the above arguments, A is isomorphic to A_S as a convolution measure algebra with maximal ideal space Δ_S, and hence it is sufficient to show that (a), (b), (c), and (d) hold for A_S.

(a) By Definition 1 (ii), $1_S \in \Delta_S$ follows, and $\Delta_S = \overline{\Delta_S}$ is obvious. Let $f, g \in \Delta_S$. At first, we show that

$$f(\mu \quad \nu) = f\mu \quad f\nu \ (\mu, \nu \in A_{S+}) \text{ where } A_{S+} = \{\lambda \in A_S : 0 \quad \lambda\}. \tag{2}$$

Since $\|f\|_\infty \quad 1, \{f(t) : t \in S\}$ is contained in the closed unit disc of the complex plane. Let $\mu, \nu \in A_{S+}$. For any $\varepsilon > 0$, we can choose disjoint Borel subsets $S_\ell, \ell = 1, ..., N_\varepsilon$ of S such that

$$S = \cup_{\ell=1}^{N_\varepsilon} S_\ell, \ | f(t) \quad f(t') | \quad \varepsilon/4 \ \ (t, t' \in S_\ell, \ell = 1, ..., N_\varepsilon).$$

For each $\ell \in \{1, ..., N_\varepsilon\}$, choose an element $t_\ell \in S_\ell$ and fix. We will show that for any ℓ and ℓ' in $\{1, ..., N_\varepsilon\}$, if ξ_{S_ℓ} denotes the characteristic function of S_ℓ, we have

$$| f(t) \quad f(t_\ell) f(t_{\ell'}) | \quad \varepsilon/2 \quad (a.e.\ t/(\xi_{S_\ell}\mu) \quad (\xi_{S_{\ell'}}\nu)). \tag{3}$$

To prove (3), it suffices to show that, for any $0 \quad \omega \quad (\xi_{S_\ell}\mu) \quad (\xi_{S_{\ell'}}\nu)$, we have

$$|< f, \omega/\|\omega\| > \quad f(t_\ell) f(t_{\ell'}) | \quad \varepsilon/2. \tag{4}$$

By (1), we can assume that ω is an element of the form $\omega = \sum_{j=1}^m \sum_{k=1}^n a_{j,k}\mu_j \ \nu_k$, where $0 \quad a_{j,k} \quad 1$ and $\mu_j, \nu_k \in (A_S)_+ (j = 1, .., m, k = 1, ..., n)$ with $\sum_{j=1}^m \mu_j \quad \xi_{S_\ell}\mu,\ \sum_{k=1}^n \nu_k \quad \xi_{S_{\ell'}}\nu$. Since $\sum_{j,k=1}^n a_{j,k}\|\mu_j\|\|\nu_k\| = \|\omega\|$, (4) can be shown as follows:

$$\begin{aligned}
\left|< f, \frac{\omega}{\|\omega\|} > \quad f(t_\ell)f(t_{\ell'})\right| &= \left|\sum_{j,k} a_{j,k} < f, \mu_j \quad \nu_k > \quad f(t_\ell)f(t_{\ell'}) \sum_{j,k} a_{j,k}\|\mu_j\|\|\nu_k\|\right| / \|\omega\| \\
& \left|\sum_{j,k} a_{j,k} < f, \mu_j >< f, \nu_k > \quad \sum_{j,k} a_{j,k} f(t_\ell)\|\mu_j\| < \nu_k, f >\right| / \|\omega\| \\
+ & \left|\sum_{j,k} a_{j,k} f(t_\ell)\|\mu_j\| < f, \nu_k > \quad \sum_{j,k} a_{j,k} f(t_\ell) f(t_{\ell'})\|\mu_j\|\|\nu_k\|\right| / \|\omega\| \\
& \sum_{j,k} a_{j,k} \int_S | f \quad f(t_\ell) | \, d\left(\frac{\mu_j}{\|\mu_j\|}\right) \|\mu_j\| \, |< f, \nu_k >| \, / \|\omega\| \\
+ & \sum_{j,k} a_{j,k} | f(t_\ell) | \, \|\mu_j\| \int_S | f \quad f(t_{\ell'}) | \, d\left(\frac{\nu_k}{\|\nu_k\|}\right) \|\nu_k\| / \|\omega\| \quad \varepsilon/2.
\end{aligned}$$

Next we show

$$\|f(\mu \quad \nu) \quad f\mu \quad f\nu\| \quad \varepsilon\|\mu\|\|\nu\|. \tag{5}$$

Since $\mu = \sum_{\ell=1}^{N_\varepsilon} \xi_{S_\ell}\mu$ and $\nu = \sum_{\ell'=1}^{N_\varepsilon} \xi_{S_{\ell'}}\nu$, we have from (3)

$$\begin{aligned}
\|f(\mu \quad \nu) \qquad f\mu \quad f\nu\| \quad & \sum_{\ell,\ell'=1}^{N_\varepsilon} \left\| f((\xi_{S_\ell}\mu) \quad (\xi_{S_{\ell'}}\nu)) \quad (f(\xi_{S_\ell}\mu)) \quad (f(\xi_{S_{\ell'}}\nu)) \right\| \\
& \sum_{\ell,\ell'=1}^{N_\varepsilon} \left\| f((\xi_{S_\ell}\mu) \quad (\xi_{S_{\ell'}}\nu)) \quad f(t_\ell)f(t_{\ell'})((\xi_{S_\ell}\mu) \quad (\xi_{S_{\ell'}}\nu)) \right\| \\
+ \quad & \sum_{\ell,\ell'=1}^{N_\ell} \left\| (f(t_\ell)(\xi_{S_\ell}\mu)) \quad (f(t_{\ell'})(\xi_{S_{\ell'}}\nu)) \quad (f(\xi_{S_\ell}\mu)) \quad (f(t_{\ell'})(\xi_{S_{\ell'}}\nu)) \right\| \\
+ \quad & \sum_{\ell,\ell'=1}^{N_\varepsilon} \left\| (f(\xi_{s_\ell}\mu))) \quad (f(t_{\ell'})(\xi_{S_{\ell'}}\nu)) \quad (f(\xi_{S_\ell}\mu)) \quad (f(\xi_{S_{\ell'}}\nu)) \right\| \\
& \sum_{\ell,\ell'=1}^{N_\varepsilon} \int_S |\, f \quad f(t_\ell)f(t_{\ell'}) \,|\, d((\xi_{S_\ell}\mu) \quad (\xi_{S_{\ell'}}\nu)) \\
+ \quad & \sum_{\ell,\ell'=1}^{N_\varepsilon} \int_S |\, f(t_\ell) \quad f \,|\, d(\xi_{S_\ell}\mu)\|f(s_{\ell'})(\xi_{S_{\ell'}}\nu)\| \\
+ \quad & \sum_{\ell,\ell'=1}^{N_\varepsilon} \|f(\xi_{S_\ell}\mu)\| \int_S |\, f(t_{\ell'}) \quad f \,|\, d(\xi_{S_{\ell'}}\nu) \\
& \left(\frac{\varepsilon}{2} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4}\right) \sum_{\ell,\ell'=1}^{N_\varepsilon} \|\xi_{S_\ell}\mu\|\|\xi_{S_{\ell'}}\nu\| = \varepsilon\|\mu\|\|\nu\|.
\end{aligned}$$

Thus (5) holds. Since $\varepsilon > 0$ in (5) can be chosen arbitrarily small, we get (2).

Let $\mu_i = \mu_{i,1} \quad \mu_{i,2} + i\mu_{i,3} \quad i\mu_{i,4}$, $\mu_i, \mu_{i,j} \in (A_S)_+ (i = 1, 2\ j = 1, 2, 3, 4)$. Then by (2)

$$\begin{aligned}
< fg, \mu_1 \quad \mu_2 > \; &= \; < g, (f\mu_{1,1} \quad f\mu_{1,2} + if\mu_{1,3} \quad if\mu_{1,4}) \quad (f\mu_{2,1} \quad f\mu_{2,2} + if\mu_{2,3} \quad if\mu_{2,4}) > \\
&= \; < g, f\mu_1 >< g, f\mu_2 > = < fg, \mu_1 >< fg, \mu_2 > .
\end{aligned}$$

Hence $fg \in \Delta_S \cup \{0\}$. Thus (b) holds.

(b) If $p = \sum_{i=1}^n c_i f_i \in \mathrm{span}(\Delta_S)$, there exists a point $t_0 \in S$ such that $\|p\|_\infty = p(t_0)$. Then

$$\left\| \left(\sum_{i=1}^n c_i f_i\right) \left(\sum_{j=1}^n c_j f_j\right)^* \right\|_\infty = \|p\overline{p}\|_\infty = \mid p(t_0)\overline{p}(t_0) \mid = \mid p(t_0) \mid^2 = \left\| \sum_{i=1}^n c_i f_i \right\|_\infty^2 .$$

(c) For $\mu \in A_S$ and $g \in \Delta_S$, if we put $\mu^g = g\mu$, then $\widehat{\mu^g}(f) = < f, g\mu > = \hat{\mu}(fg)\ (f \in \Delta_S)$.

(d) If $\tilde{p} = \sum_k c_k \tilde{f}_k \in \mathrm{span}(\Delta_S)$ and $p = \tilde{p} \circ \alpha = \sum_k c_k f_k \in \mathrm{span}(\Delta_X)$, then

$$\begin{aligned}
\|\tilde{p}\|_{A_S^*} \;&=\; \sup\{|< \tilde{p}, \mu_S >| : \mu_S = \mu \circ \alpha^{\quad 1} \in A_S, \mu \in A_X, \|\mu\| \quad 1\} \\
&=\; \sup\{|< p, \mu >| : \mu \in A_X, \|\mu\| \quad 1\} = \|p\|_\infty = \|\tilde{p}\|_\infty.
\end{aligned}$$

Hence, for any $a_S \in A_S$, we have

$$\begin{aligned}
\|a_S\| \;&=\; \sup\{|< \tilde{p}, a_S >| : \tilde{p} \in \mathrm{span}(\Delta_S), \|\tilde{p}\|_\infty \quad 1\} \\
&=\; \sup\{|< \tilde{p}, a_S >| : \tilde{p} \in \mathrm{span}(\Delta_S), \|\tilde{p}\|_{A_S^*} \quad 1\} = \|\widehat{a_S}\|_{BSE},
\end{aligned}$$

since $\mathrm{span}(\Delta_S)$ is dense in $C(S)$. □

Remark 2.2 For later use in $ 4, we remark here that in Proposition 2.1, for an $a \in A$, $0 \quad a$ if and only if $\widehat{a_S}$ is positive definite on the *-semigroup $\Delta_S \cup \{0\}$.

§3. Sufficient conditions for A to be a convolution measure algebra

Lemma 3.1 *Let S be a commutative compact Hausdorff topological semigroup such that $\hat{S}$ separates points of S. Let $0 \quad \mu \in M(S)$ and $h \in L^1(\mu)$ be given. Then for any $\varepsilon > 0$, there exists a $p \in \mathrm{span}(\hat{S})$ such that $\int_S | h \quad p | d\mu < \varepsilon$.*

Proof. We can suppose without loss of generality that $h \in L^\infty(\mu)$. By Lusin's theorem (cf. [4, p.55]), we can choose a $g \in C(S)$ such that

$$\|g\|_\infty \quad \|h\|_\infty, \mu\left(\left\{s \in S : g(s) \neq h(s)\right\}\right)\|h\|_\infty \quad \frac{1}{4}\varepsilon.$$

Since $\mathrm{span}(\hat{S})$ is uniformly dense in $C(S)$, we can choose a $p \in \mathrm{span}(\hat{S})$ such that $\|g \quad p\|_\infty \|\mu\| < \frac{1}{2}\varepsilon$. Then we have

$$\begin{aligned} \int_S | h \quad p | d\mu & \quad \int_S | h \quad g | d\mu + \int_S \|g \quad p\|_\infty d\mu \\ & \quad \mu\left(\left\{s \in S : h(s) \neq g(s)\right\}\right) 2\|h\|_\infty + \|g \quad p\|_\infty \|\mu\| < \varepsilon. \ \Box \end{aligned}$$

Proposition 3.2 *Let A be a commutative semisimple Banach algebra with maximal ideal space Δ_A. Suppose that A satisfies the conditions (a), (b), (c), and (d) in §1. Then we have:*

(i) *Define an order in A by "for $a \in A$, $0 \quad a$ if and only if $\hat{a}$ is positive definite on Δ_A", then A becomes a partially ordered Banach algebra;*

(ii) *The multiplication and involution in $\Delta_A \cup \{0\}$ can be extended to A_* so that A_* is a commutative C^*-algebra;*

Here S denotes the maximal ideal space of A_*. Then A_* is isomorphic to $C(S)$ through the Gelfand transform. We identify A_* with $C(S)$, and hence $A_*{}^* = M(S)$, where

$$< a, \psi >= \int_S \psi(s) d\phi(a)(s) \quad (\psi \in A_*, a \in A).$$

(iii) *We can define a multiplication in S so that S becomes a commutative compact Hausdorff topological semigroup;*

(iv) *The Arens product in $M(S)$ coincides with convolution;*

(v) *The canonical embedding ϕ is a bipositive isometric homomorphism;*

Here, we put $\phi(A) = A_S$.

(vi) *A_S is a closed subalgebra of $M(S)$, and $\Delta_{A_S} = \Delta_A = \hat{S}$;*

(vii) *A_S is an L-subalgebra of $M(S)$;*

(viii) *A is a convolution measure algebra.*

Proof. (i) It is easy to see that A_+ is a closed convex algebraic cone, and hence A becomes an ordered Banach algebra with positive cone A_+.

(ii) By (a) and (b), multiplication " " and involution $*$ can be extended to $\mathrm{span}(\Delta_A)$ so that $\mathrm{span}(\Delta_A)$ becomes a normed *-algebra satisfying $\|p \quad p^*\|_{A^*} = \|p\|_{A^*}^2$ $(p \in \mathrm{span}(\Delta_A)$. From this it follows easily that $\|p\|_{A^*} = \|p^*\|_{A^*}$ $(p \in \mathrm{span}(\Delta_A))$. For $\psi \in A_*$, choose a sequence $\{p_n\}$ which converges to ψ, then $\{p_n^*\}$ is a Cauchy sequence in A^* and hence converges to an element, $\psi^* \in A_*$. It is easy to see that an involution $*$ in A_* is thus defined which makes A_* a C^*-algebra with identity.

(iii) For each $s \in S$ we denote by δ_s the unit point mass at s. Evaluation at s of elements of $\mathrm{span}(\Delta_A)$ induces a non-zero bounded complex homomorphism and vice versa. For $s, t \in S$, let $\delta_s \quad \delta_t$ denotes the Arens product of δ_s and δ_t in $M(S)$. Then

$$\mathrm{span}(\Delta_A) \to \mathbf{C} : \sum_{f \in \Delta_A} \hat{p}(f) f \to \sum_{f \in \Delta_A} \hat{p}(f) f(s) f(t) = \sum_{f \in \Delta_A} \hat{p}(f) < f, \delta_s \quad \delta_t >$$

is a bounded nonzero multiplicative linear functional, and hence exists a unique element, $st \in S$, such that $< f, \delta_{st} >= f(s)f(t) =< f, \delta_s \quad \delta_t >$ $(f \in \Delta_A)$, and hence $\delta_{st} = \delta_s \quad \delta_t$. One can easily see that this multiplication $S \quad S \to S : (s,t) \to st$ makes S a commutative compact topological semigroup.

(iv) From (iii), $\Delta_A \subseteq \hat{S}$ is evident. If $\mu \in M(S)$, we describes $\hat{\mu}(f) =< f, \mu > \ (f \in \Delta_A)$. For $\mu, \nu \in M(S)$, let $\mu \quad \nu$ and $\mu * \nu$ be the Arens product and convolution product in $M(S)$, respectively. Then

$$< f, \mu \quad \nu >= \hat{\mu}(f)\hat{\nu}(f) = \widehat{\mu * \nu}(f) \ (f \in \Delta_A).$$

Since $\mathrm{span}(\Delta_A)$ is dense in $C(S)$, we have $\mu \quad \nu = \mu * \nu$.

(v) It is easy to see that, for a $\mu \in M(S)$ $0 \quad \mu$ if and only if $\hat{\mu}$ is positive definite on $\hat{S}$, and hence the canonical embedding of A into $M(S)$ is, by (d), a bipositive isometric homomorphism.

(vi) That A_S is a closed subalgebra of $M(S)$ and $\Delta_{A_S} = \Delta_A \subseteq \hat{S}$ follows from (v). If $f \in \hat{S}$, $A_S \to \mathbf{C} : \mu \to \int_S f d\mu$ is a nonzero complex homomorphism, and hence there is a $g \in \Delta_{A_S}$ such that $\int_S g(s)d\mu(s) = \int_S f(s)d\mu(s) \ (\mu \in A_S)$. Since A_S is weak*-dense in $M(S)$ by Lemma 1.1, we must have $g(s) = f(s) \ (s \in S)$. Hence $f = g \in \Delta_{A_S}$.

(vii) Let $a \in A$ and $g \in \Delta_A$. By (c), there is an $a^g \in A$ such that $\widehat{a^g}(f) = \hat{a}(fg) \ (f \in \Delta_A)$. Hence

$$< f, \phi(a^g) >=< a^g, f >=< a, fg >=< fg, \phi(a) >=< f, g\phi(a) > \quad (f \in \Delta_A).$$

This implies $g\phi(a) = \phi(a^g) \in A_S$. Let $\mu \in A_S$ and $\nu \in M(S)$ be such that $\nu << \mu$, and choose an $h \in L^1(\mid \mu \mid)$ so that $\nu = h\mu$. By Lemma 3.1, for any $\varepsilon > 0$ there is a $p \in \mathrm{span}(\hat{S})$ such that $\int_S \mid h \quad p \mid d \mid \mu \mid < \varepsilon$. From above $p\mu \in A_S$ and $\|p\mu \quad \nu\| = \|p\mu \quad h\mu\| \quad \int_S \mid h \quad p \mid d \mid \mu \mid < \varepsilon$. Since A_S is a closed subspace of $M(S)$, it follows that $\nu \in A_S$.

(viii) A and A_S are isomorphic as ordered Banach algebras through the mapping $a \to \phi(a)$. Since A_S is a convolution measure algebra, it follows that A is also a convolution measure algebra. □

§4 Main results

Theorem 4.1 *Let A be a commutative semisimple Banach algebra with maximal ideal space Δ_A. Then the following statements (i) and (ii) are equivalent each other:*

(i) We can define a partial order in A so that A becomes a convolution measure algebra.

(ii) A satisfies the conditions (a), (b), (c), and (d) in §1.

Proof. (i) implies (ii) by Proposition 2.1, and (ii) implies (i) by Proposition 3.2. □

Corollary 4.2 (*Taylor's representation theorem***)** *Let A be a commutative semisimple convolution measure algebra with maximal ideal space Δ_A, and ϕ Birtel's canonical embedding of A into ${A_*}^*$. Then there is a commutative compact Hausdorff topological semigroup S such that:*

(i) *$A_* = C(S)$ and ${A_*}^* = M(S)$, where $\Delta_A = \hat{S}$, the Arens product in $M(S)$ coincides with the convolution product in $M(S)$, and $< a, \psi >= \int_S \psi(s)d\phi(a)(s) \ (\psi \in C(S), a \in A)$;*

(ii) *$A_S := \phi(A)$ is a weak*-dense L-subalgebra of $M(S)$ and $\hat{S}$ separates points in S;*

(iii) *ϕ is an order preserving isometric homomorphism of A onto A_S.*

Proof. By Proposition 2.1, A satisfies the conditions (a), (b), (c), and (d) in §1. Then Proposition 3.2 with Remark 2.2 provide the proof of this corollary. □

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CLASSIFICATION OF COMMUTATIVE BANACH ALGEBRAS AND SEGAL ALGEBRAS WHICH ARE NEITHER BSE NOR BED

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Abstract. 先ず分類の原理を論じる。次にこの原理の一つの実現として、 BSE と BED と言う 2 つの概念を導入して可換 Banch 環の世界を 4 つに分類する。次に各クラスに属する典型的な可換 Banach 環の例を与える。最後に BSE でも BED でもない環が最近 (群環 $L^1(G)$ の) Segal 環の中に発見された事を報告にする。

1. 分類の原理とその実現

人間は何か対象物に出会ったときそれを理解したがる癖があります。それでは

「人が何かを理解するとはどういうことか」

を考えますと、一つに原理を設定し、それを実現する事によって何か分かった気分になると言う立場があります。そのような原理の一つに「分類の原理」があります。これは何か対象物 X を知りたいとするとき、その対象物を含む大きな集団を考え、その大きな集団を「何かの基準」を設定する事によって分類し、 X がどのクラス (剰余類) に属するか、そしてそのクラスが全体のどの位置あるかを知り、X を分った気分になることです。

例えばここに一本の木があるとします。この木は一体何だろうと考えるとき、先ずこの木が立っている山を見てこの山にあるすべての木を考え、「木の高さ」で分類してみます。この分類によってその木の高さとその位置の高さは逆である事に気付くと思います。つまり山の低い所に立っている木は高く、高い所に行くに従って木は低くなることをその分類で知るでしょう。このときお目当ての木が高ければ、低地に属する木として何か分った気分になるでしょう。

前述の「何かの基準」について安達恒雄 (1941-) 著「 $\sqrt{2}$ の不思議」から興味ある文章を紹介しましょう。中略を除いて原文のままです。

”数と言うものはその抽象性に本質があるのであって、具体的なものの数があるていどわかることと、抽象的な数がわかることの間に大きなギャップがあるのである。その違いの意味を説明するためには先ず「類別」と言う手法を説明せねばならない。類別、正確には「同値類別」、という概念は、何らかの意味で同じ性質を持ったものを一つのものとみなすという考え方で、単純であるが、同時に人類の思考法の土台ともなっている手法である。...... 中略 ...... われわれは抽象の本質が同値類別、同じことだが同値関係であるということを知った。抽象とはまたうまくできた言葉で、「共通の象 (かたち) を抽 (ひ) き出す」のである。無数の図形の中から、三角形、五角形といった形を抽象するのはたしかに精緻・深遠な思考の始まりであると思われる。かって赤摂也氏が「人間であることの必要十分条件は数学することだ」という説を唱えておられたが、わたしも今まで述べてきたような意味に数学することを解釈するなら、そのとおりだと思う。「数学すること」とは、集合をいろいろな方法で抽象する能力と言い換えてもよい。”

結局「数学する」と言う事は、「人間が人として生きる」と言う事でしょうか？可換 Banach 環の分類もこの延長上に位置づけられると思います。

2. 可換 Banach 環について

早稲田大学名誉教授の和田淳藏先生によりますと、Banach 環の一般論は 1936 年

M. Nagumo (南雲道夫) (Jap. J. Math. 13)　K. Yosida (吉田耕作) (Jap. J. Math. 13)

等によって始められたとされています。その頃の阪大での陣容は

南雲道夫 (31 才教授)　吉田耕作 (27 才助教授)　角谷静夫 (25 才助手)

だったそうです。もっとも同じ頃外国でも Banach 環の一般論に関する論文が出ていたそうです。1945 年 Warren Ambrose が初めて Banach algbera と言う用語を用いたらしいと言われています。1932 年 Norbert Wiener は次の有名な Lemma を発表しました：

$$f(x) = \sum_{n=-\infty}^{\infty} a_n e^{-inx} \neq 0 \ (x \in \mathbf{R}) : \sum_{n=-\infty}^{\infty} |a_n| < \infty$$

$$\Downarrow$$

$$\frac{1}{f(x)} = \sum_{n=-\infty}^{\infty} b_n e^{-inx} \ (x \in \mathbf{R}) : \sum_{n=-\infty}^{\infty} |b_n| < \infty$$

1941 年 Izrail' Moiseevich Gelfand は所謂「 Gelfand Theory 」を用いて上の Wiener's lemma を証明し、世間をアッと言わせました。その証明は

astonishingly simple proof, elementary proof, elegant proof

等の賛辞が送られています。

さて可換代数 (多元環) が完備なノルムを持つときそれを可換 Banach 環と言いますが、以後断らない限り A を複素可換 Banach 環とします。A の零でない複素数値準同形写像の全体を Φ_A で表しますと、これは相対汎弱位相で局所コンパクト Hausdorff 空間となりまが、これを A の Gelfand 空間と言います。別名極大イデアル空間とも呼ばれています。またこの相対汎弱位相を Gelfand 位相と言います。

3. Gelfand の表現定理及び Helgason-Wang の表現定理

可換 Banach 環 A が与えられたとき、各 $x \in A$ に対して

$$\hat{x}(\varphi) = \varphi(x) \ (\varphi \in \Phi_A)$$

と定義しますと、 $\hat{x}$ は Φ_A 上の複素数値関数となりますが、この $\hat{x}$ を x の Gelfand 変換と呼んでいます。Gelfand 位相と Gelfand 変換の定義から、$\hat{x}$ は無限遠点でゼロとなる連続関数である事が分ります。また 1 次方程式の解を考察する事によって

$$\|\hat{x}\|_\infty \leq \|x\| \ (x \in A)$$

が成り立ちます。さて

$$\mathrm{Rad}(A) \equiv \cap_{\varphi \in \Phi_A} \mathrm{Ker}\,(\varphi)$$

を A のラジカルと言い、もし $\mathrm{Rad}(A) = \{0\}$ ならば A は半単純であると言います。以上から次の定理が成り立ちます。

定理 1. Gelfand 変換 : $x \to \hat{x}$ は A から可換 C^*-環 $C_0(\Phi_A)$ への $\|\hat{x}\|_\infty \leq \|x\| \ (x \in A)$ を満たす連続な準同型写像で、そのカーネルは A のラジカルに一致する。特に A が半単純ならばこれは単射である。

さて A からそれ自身への有界線形作用素の全体を $B(A)$ で表しますと、これは作用素ノルムのもとで Banach 環を作ります。各 $a \in A$ に対して、

$$L_a(x) = ax \ (x \in A)$$

で定義される作用素は $B(A)$ に属しますが、これを a による積作用素と言います。積作用素は

$$L_a(xy) = xL_a(y) \ (x, y \in A)$$

と言う著しい性質を持っていますので

$$M(A) = \{T \in B(A) : T(xy) = xTy \ (\forall x, y \in A)\}$$

と定義し、$M(A)$ に属する作用素を乗作用素 (multiplier) と呼ぶ事にします。勿論 A が単位的ならば積作用素と乗作用素は一致しますが、一般には違って、乗作用素の方が積作用素より広い概念です。$M(A)$ は $B(A)$ の単位元を含む閉部分環である事は直ぐ分りますので、 $M(A)$ を乗作用素環 (multiplier algebra) と呼んでいます。またもし A が条件：

$$x \in A, Ax = \{0\} \Rightarrow x = 0$$

を満たせば、A は順序を持たない (without order) と言いますが、このとき、 $M(A)$ は $B(A)$ の極大可換部分環となる事を容易に示す事ができます。殆どの有用な A は順序を持たないので、その乗作用素環は単位的可換 Banach 環です。

さて $T \in M(A)$ を任意にとって来ます。いま任意の $\varphi \in \Phi_A$ に対して、$\varphi(a_\varphi) = 1$ なる元 $a_\varphi \in A$ を選び、

$$\hat{T}(\varphi) = \varphi(Ta_\varphi)$$

と定義しますと、これは well-defined となり、$\widehat{Tx}(\varphi) = \hat{T}(\varphi)\hat{x}(\varphi)$ $(x \in A)$ である事が容易に分ります。従って $\hat{T}$ は Φ_A 上の複素数値連続関数となりますが、更に $\|\hat{T}\|_\infty \leq \|T\|$ が成り立つ事が分かり、$\hat{T}$ は T の Helgason-Wang 変換と呼ばれています。以上から次の定理が成り立ちます。

定理 2. Helgason-Wang 変換 $T \to \hat{T}$ は $M(A)$ から単位的可換 C^*-環 $C^b(\Phi_A)$ への $\|\hat{T}\|_\infty \leq \|T\|$ を満たす連続な準同形写像である。もし A が半単純ならば、これは単射である。

4. 抽象的問題

可換 Banach 環 A が与えられたとき、その Gelfand 変換像や、乗作用素環の Helgason-Wang 変換像を何かの言葉で特徴付けられないかと言う漠然とした問題があります。もう少し正確に述べますと、次のようになります。

問題 1. $\hat{A} \equiv \{\hat{x} : x \in A\}$ を $C(\Phi_A)$ の中で特徴付けよ。

問題 2. $\hat{M}(A) \equiv \{\hat{T} : T \in M(A)\}$ を $C(\Phi_A)$ の中で特徴付けよ。

ただし $C(\Phi_A)$ は Φ_A 上の複素数値連続関数全体の代数を表します。しかしながらこれらは余りにも抽象的過ぎてこのままではどうしようもありません。それで街に出て具体的な例を探し、それをもとに抽象化すれば解答の一部を得るのではないかと考えました。

5. 街へ出てみる, I

G を LCA-群とし、$\hat{G}$ をその双対群、$M(G)$ を G の測度環とします。このとき次の定理が成り立ちます。

定理 3. $\hat{G}$ 上の複素数値連続関数 σ が $M(G)$ に属する測度の Fourier-Stieltjes 変換になっている為の必要十分条件は、全ての三角多項式 $p = \sum_{\gamma \in \hat{G}} \hat{p}(\gamma)\gamma$ に対して

$$\left| \sum_{\gamma \in \hat{G}} \hat{p}(\gamma)\sigma(\gamma) \right| \leq C\|p\|_\infty$$

が成り立つような定数 C が存在する事である。

上の定理は $M(G)$ の Fourier-Stieltjes 変換像を特徴付けたもので、Bochner-Schoenberg-Eberlein theorem として知られています。
さて

「優れた定理はそれ自身定義となり得る」

を信じて、定理 3 を可換 Banach 環の世界に焼き直して、 BSE-環という新しい可換 Banach 環のクラスを定義し、問題 2 の解答を考えます。
先ず各 $\mu \in M(G)$ に対して

$$T_\mu(f) = f * \mu \ (f \in L^1(G))$$

と定義しますと、写像 $\mu \to T_\mu$ は測度環 $M(G)$ と群環 $L^1(G)$ の乗作用素環 $M(L^1(G))$ の間の等距離同型写像を与えます。また $L^1(G)$ の共役空間と $L^\infty(G)$ とは等距離同型です。また $\Phi_{L^1(G)} \cong \hat{G}$ と見る事ができるので、 Gelfand 変換と Fouier 変換は同じものです。同様に Helgason-Wang 変換と Fourier-Stieltjes 変換は同じものです。以上の考察から定理 3 は次のような集合を示唆します：

$$C_{BSE}(\Phi_A) = \{\sigma \in C(\Phi_A) : \exists\beta > 0 : |\sum_{\varphi \in \Phi_A} \hat{p}(\varphi)\sigma(\varphi)| \leq \beta\|p\|_{A^*} \ (\forall p \in \operatorname{span}\Phi_A)\}.$$

ここで $\operatorname{span}\Phi_A$ は Φ_A の A^* における線形包を表します。従ってその元は $p = \sum_{\varphi \in \Phi_A} \hat{p}(\varphi)\varphi$ と一意に書く事ができます。このとき $C_{BSE}(\Phi_A)$ は定義の中の β の下限を σ のノルムとして半単純可換 Banach 環となる

事が分っています。我々は今便宜上これを A の BSE-拡大と呼び $\hat{A}_{BSE}$ と書く事にします。そこで $M(A)$ の Helgason-Wang 変換像を $\hat{M}(A)$ で表しますと、A が群環のとき定理 3 は

$$\hat{A}_{BSE} = \hat{M}(A)$$

である事を主張しています。このとき上の条件を満たす可換 Banach 環を BSE-環と名付けますと、Bochner-Schoenberg-Eberlein theorem は

「群環は BSE-環である」

と一言で述べる事ができます。勿論 BSE-環は問題 2 の解答を与える可換 Banach 環ですが、問題はそのような環が群環しかなかったら解答として意味がありません。幸い可換 C*-環や円板環も BSE-環である事が分ります。従ってこれらの環は BSE 眼鏡で見る同じく見える事になります。しかし BSE-環の探索は問題 2 に関連して重要ですが、それだけでは分類の原理の実現としては不十分です。他の眼鏡も欲しい所です。その眼鏡を次節で話しましょう。

6. 街へ出てみる, II

次の様な定理があります。

定理 4. G を Haar 測度 dx を持つ LCA-群とするとき、$\hat{G}$ 上の複素数値連続関数 σ が $L^1(G)$ の Fourier 変換像に属する為の必要十分条件は、任意の正数 $\varepsilon > 0$ に対して、次の条件を満たす正数 $\delta > 0$ と G のコンパクト部分集合 K が存在する事である：任意の三角多項式 $p = \sum_{i=1}^n c_i \gamma_i$ [1] に対して

$$\|p\|_\infty \leq 1, \int_K |p(x)|dx < \delta \Rightarrow \left|\sum_{i=1}^n c_i \sigma(\gamma_i)\right| < \varepsilon$$

が成り立つ。

これは $L^1(G)$ の Fourier 変換像を特徴づけたもので、 R. Doss の定理として知られています。

さてこの定理を一般の可換 Banach 環に焼き直して見ましょう。

「真理は不完全さを好む」

でしょうか？このキャッチフレーズに従って思いついたのが「擬位相」という概念です。これは一般の線形空間のある部分集合族 $\mathcal{Q}$ の事で、2 条件：

(i) $0 \in U \ (\forall U \in \mathcal{Q})$,

(ii) $\forall U, V \in \mathcal{Q}, \exists W \in \mathcal{Q} : W \subseteq U \cap V$

を満たすものを言います。このとき擬位相のクラスは自然な半順序構造を持ちます。

さて Φ_A 上の任意の複素数値関数 σ は

$$\tilde{\sigma}(p) = \sum_{\varphi \in \Phi_A} \hat{p}(\varphi)\sigma(\varphi) \ (p \in \operatorname{span} \Phi_A)$$

によって $\operatorname{span} \Phi_A$ 上の線形汎関数 $\tilde{\sigma}$ を引き起こします。それ故 $\operatorname{span} \Phi_A$ 上に 1 つの擬位相 $\mathcal{Q}$ が与えられたとき、$\tilde{\sigma}$ が "$\mathcal{Q}$-連続" であるような Φ_A 上の複素数値連続関数 σ の全体を $C(\Phi_A; \mathcal{Q})$ で表しますと、これは自然に線形空間を作ります。

今 $A = L^1(G)$ とおき、$\operatorname{span} \Phi_A$ を G 上の三角多項式全体の作る線形空間 $P(G)$ と同一視しますと、

$$\mathcal{Q}_D = \{U_{\delta,K} : \delta > 0, K \in \mathcal{K}(G)\}$$

は $P(G)$ 上の擬位相となります。ただし、$\mathcal{K}(G)$ は G のコンパクト部分集合全体を表し、$U_{\delta,K}$ は $P(G)$ の部分集合：

$$\{p \in P(G) : \|p\|_\infty \leq 1, \int_K |p(x)|dx < \delta\}$$

を表します。このとき定理 4 は $\hat{A} = C(\Phi_A; \mathcal{Q}_D)$ を主張しています。この擬位相 $\mathcal{Q}_D$ は $P(G)$ 上の線形演算と両立するどんな位相の原点での近傍の基底にも成り得ない事が分ります。

さて定理 4 から一般の可換 Banach 環 A に対しても

「 $\hat{A} = C(\Phi_A; \mathcal{Q})$ となるような $\operatorname{span} \Phi_A$ 上の擬位相 $\mathcal{Q}$ が存在するか？」

と言う自然な問題が発生しますが、これに関しては次の補題がそれに答えてくれます。

補題 1. $\mathrm{span}\,\Phi_A$ 上の相対汎弱位相に関する原点での近傍の族を $\mathcal{Q}_0 = \mathcal{Q}_0(A)$ とするとき、常に $\hat{A} = C(\Phi_A; \mathcal{Q}_0)$ が成り立ち、更に $\hat{A} = C(\Phi_A; \mathcal{Q})$ を満たす擬位相 $\mathcal{Q}$ は常に $\mathcal{Q}_0 \leq \mathcal{Q}$ である。

従って $\hat{A} = C(\Phi_A; \mathcal{Q})$ 且つ $\mathcal{Q}_0 < \mathcal{Q}$ であるような擬位相 $\mathcal{Q}$ が見つかれば、それは $\hat{A}$ を特徴付けるより精密な結果であると言えます。これが基本理念ですが、擬位相の研究は沢山の問題を内蔵しているのに殆どその研究は進んでいません。

所で R. Doss は次のような定理も発表しています。

定理 5. G を Haar 測度 dx を持つ LCA-群とするとき、$\hat{G}$ 上の複素数値連続関数 σ が $L^1(G)$ の Fourier 変換像に属する為の必要十分条件は、次の 2 条件が満たされる事である：

(i) 正数 $M > 0$ が存在して、任意の三角多項式 $p = \sum_{i=1}^{n} c_i \gamma_i{}^{1}$ に対して、

$$\|p\|_\infty \leq 1 \Rightarrow \left| \sum_{i=1}^{n} c_i \sigma(\gamma_i) \right| < M$$

が成り立つ。

(ii) 任意の正数 $\varepsilon > 0$ に対して、$\hat{G}$ のコンパクト部分集合 K が存在して

$$p = \sum_{i=1}^{n} c_i \gamma_i{}^{1} \ (\gamma_1, \cdots, \gamma_n \notin K), \|p\|_\infty \leq 1 \Rightarrow \left| \sum_{i=1}^{n} c_i \sigma(\gamma_i) \right| < \varepsilon$$

が成り立つ。

定理 5 から次のような新しい擬位相が発見されました。それは $\mathcal{K}(\Phi_A)$ を Φ_A のコンパクト部分集合全体とするとき、各 $K \in \mathcal{K}(\Phi_A)$ 及び $\delta > 0$ に対して、

$$U_{K,\delta} = \{p \in \mathrm{span}\,\Phi_A : \|p\|_{A^*} \leq 1, \exists q \in \mathrm{span}\,\Phi_A \text{ s.t. } \|q\|_{A^*} \leq \delta, \hat{p} \mid K = \hat{q} \mid K\}$$

とおいたとき、そのような $U_{K,\delta}$ の全体 $\mathcal{Q}^0_{BSE} = \mathcal{Q}^0_{BSE}(A)$ の事であります。このとき可換 Banach 環 $C(\Phi_A; \mathcal{Q}^0_{BSE})$ を A の BED 拡大と呼び $\hat{A}_{BED}$ で表す事にします。もし A が群環ならば、定理 5 から

$$\hat{A} = \hat{A}_{BED}$$

である事が導かれます。それ故そのような性質を持つ可換 Banach 環を BED-環と名付けました。

さて BED-環の探索は問題 1 の解決に繋がる訳ですが、ここで気になるのは BSE 性と BED 性の関係です。一体どんな関係があるのでしょうか？次の定理はある種の可換 Banach 環のクラスにおいては、 BSE 性と BED 性は同値であることを主張しています。

定理 6. 有界近似単位元を持つ半単純正則 Tauber 型可換 Banach 環においては、BSE 性と BED 性は同値である。

上の定理は近似単位元の有界性が大きな役割を果たしていますが、有界性を外すとこの 2 つは途端に無関係になります。次節の具体例にそれが良く現れています。

7. 可換 Banach 環の分類

これまでの考察から可換 Banach 環は次の 4 つに分類される事が分ります。

(I) BSE and BED,
(II) BSE and not BED,
(III) BED and not BSE,
(IV) not BED and not BSE.

次にそれぞれに属する可換 Banach 環の例を挙げましょう。しかし詳しい説明は割愛します。

(I) に属する例：$L^1(G), L^1_w(\mathbf{R}^d)$, $L^1(G)$ のある種の商環, $L^1(G)$ のある種の閉イデアル, 可換 C*-環, disk 環 $A(D)$, Hardy 環 $H^\infty(D)$, 実数直線 $\mathbf{R}$ 上のある種の Lipschitz 環 $\mathrm{Lip}^0_1(\mathbf{R})$

(II) に属する例：noncompact LCA-群 G 上の Segal 環：$S_p(G)$ $(1 < p < \infty)$, $A_p(G)$ $(1 \leq p < \infty)$

(III) に属する例：無限集合 S 上の l^1-環 $l^1(S)$, L^p-環 $L^p(G)$ (G : compact, $\sharp G = \infty, 1 < p < \infty$), 無限次元可換 H*-環, $C_0(X;\tau)$, A_τ, $A_{\tau(n)}$ $(1 \leq n \leq \infty)$

(IV) に属する例：単位閉区間 [0,1] 上の微分環 $C^1([0,1])$, 実数直線 $\mathbf{R}$ 上の微分環 $C_0^1(\mathbf{R})$, 非離散 LCA-群 G 上の測度環 $M(G)$, 半群 $\mathbf{N}_k \equiv k-1+\mathbf{N}$ $(k \geq 1)$ 上の半群環 $L^1(\mathbf{N}_k)$

8. BSE でも BED でもない Segal 環

A は正則でその Gelfand 変換はコンパクトな台を持つ有界近似単位元を有するものとします。このとき A の稠密な Banach イデアル S が近似単位元を持てば、それを A における Segal 環と言います。これは H. Reiter の群環における Segal 環の一般化になっており、研究対象としては面白いクラスを作っています。ただし A が単位的であれば、その Segal 環は A 自身となり意味がありません。従って A は非単位的でなければなりません。また A が前節のクラス (I) に属しているとき、A における Segal 環でクラス (I) に属しているものは A 自身しかありません。しかしながらこれまで知られた Segal 環で非自明なものは皆クラス (II) に属するかクラス (III) に属するかのどちらかで、クラス (IV) に属するものは見つかっていませんでした。そこでそのような Segal 環が存在するかと言う自然な問題が起こります。所が最近それが解決されたので、大雑把ですが、以下それについて述べたいと思います。

$\mathcal{A}$ を LCA-群 G 上の Fourier 環とします。$\tau \in C(G)$ が条件：

$$f\tau \in \mathcal{A} \ (\forall f \in \mathcal{A}_c)$$

を満たすとき、局所 $\mathcal{A}$-関数と言いそのような関数の全体を $\mathcal{A}_{loc}$ で表します。ただし $\mathcal{A}_c$ はコンパクトな台を持つ $\mathcal{A}$ の関数全体を表します。また

$$\mathcal{A}_{\tau(1)} = \{f \in \mathcal{A} : f\tau \in \mathcal{A}\}$$

と定義しますと、これは $\mathcal{A}$ における Segal 環になる事は分ります。このとき次のような定理が成り立ちます。

定理 7. $\mathcal{A}_{loc}$ に属する関数 τ で、$\mathcal{A}_{\tau(1)}$ が $\mathcal{A}$ におけるどんな "proper isometrically translation invariant" Segal 環にも含まれないようなものが存在する。

更に次のような定理も成り立ちます。

定理 8. $\mathcal{S}_1, \mathcal{S}_2$ を $\mathcal{A}$ における Segal 環とする。もし $\mathcal{S}_1$ が not BSE で、$\mathcal{S}_2$ が BSE で且つ $S_1 \not\subseteq S_2$ であれば、$\mathcal{A}$ における Segal 環 $\mathcal{S}_1 \cap \mathcal{S}_2$ はクラス (IV) に属する。

上の定理 7, 8 を応用すると、我々のゴールである次の定理を得る事が出来ます。

定理 9. τ を定理 7 で構成したものとする。G を非コンパクト非離散 LCA-群、$\mathcal{S}$ を $\mathcal{S}_p(G)$ $(1 < p < \infty)$ または $\mathcal{S} = \mathcal{A}_p(G)$ $(1 \leq p < \infty)$ とする。このとき $\mathcal{A}$ における Segal 環 $\mathcal{S} \cap \mathcal{A}_{\tau(1)}$ はクラス (IV) に属する。

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リーマン面上の有界正則関数環 — やり残したこと、自戒を込めて

Problems on Bounded Analytic Functions on Riemann Surfaces

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1 記号

$\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$, 単位開円板
$\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\} = \{e^{i\theta} : \theta \in \mathbb{R}\}$, 単位円周
$\mathbb{C}^* := \mathbb{C} \cup \{\infty\}$, リーマン球面

また、D (in $\mathbb{C}^*$) を平面領域として、

$$\begin{cases} H^\infty(D) := \{f : D \text{ 上の有界正則関数 }\} \\ \|f\|_\infty := \|f\|_{H^\infty(D)} := \|f\|_D := \sup_{z\in D} |f(z)|, D \text{ 上の sup-norm} \end{cases}$$

$$\begin{cases} H^p(D) := \{f : D \text{ 上の正則関数で、調和関数 } \exists u, |f|^p \quad u \ \} \ (0 < p < \infty) \\ \|f\|_p := \|f\|_{p,a} := \inf\{u(a)^{1/p} : u \text{ は } |f|^p \quad u \text{ をみたす調和関数 }\} \ (a \in D) \end{cases}$$

とおく。$D = \mathbb{D}$ の場合、a を原点 $a = 0$ にとって、

$$\|f\|_p = \sup_{0\le r<1} \left(\int_0^{2\pi} |f(re^{i\theta})|^p \frac{d\theta}{2\pi}\right)^{1/p} = \left(\int_0^{2\pi} |f(e^{i\theta})|^p \frac{d\theta}{2\pi}\right)^{1/p} = \|f\|_{L^p(\frac{d\theta}{2\pi})}$$

である。ただし、$f(e^{i\theta})$ は $f(z)$ の境界値関数を表す。

リーマン面 R の場合も、$H^\infty(R)$, $\|f\|_\infty, H^p(R)$, $\|f\|_p$ は同様に定義される。リーマン面を知らなければ、R も平面領域と思ってお読み頂きたい — ただし、その場合、既知、もしくは、明らかな結果、問題となってしまうものもある。

尚、リーマン面は（互いに同値な）３通りの方法で定義できる。

1. 複素１次元 (連結) 多様体
2. R 上に有理型関数 $f : R \to D$ があって、R は f により D 上に広がった面 (より正確には、分岐点を許す被覆面、即ち、f の解析接続により得られる面)。R 上の関数は、f を通して、D **上の (分岐) 多価関数**と見なせる。
3. 普遍被覆 $\pi : \mathbb{D} \to R$; このとき、π を普遍被覆写像、$\mathbb{D}$ を (R の) 普遍被覆面という。R 上の関数は π を通して $\mathbb{D}$ **上の１価関数**と見なせる。（正確には、普遍被覆面として、$\mathbb{C}^*$ または $\mathbb{C}$ となる場合もあるが、４つの特別な場合 $R = \mathbb{C}^*, \mathbb{C}, \mathbb{C} \setminus \{0\}, \mathbb{T}^2$ を除き、普遍被覆面は $\mathbb{D}$ となることが知られている）

また、具体的なリーマン面の例を作る場合には、主につぎの３通りがある。

1.「はさみ」と「のり」で、領域を切った、貼ったを繰り返してで作る。
2. 具体的な多価関数を考えて、そのリーマン面として作る。
3. 高次元領域、例えば $\mathbb{C}^n$ の中で方程式の零点集合や、関数のグラフにより作る。

2 非定数有界正則関数の存在

Problem 1 非定数有界正則関数をもつリーマン面の特徴付け

2.1 D が有界平面領域なら、座標関数 $id(z) = z$ が $H^\infty(D)$ の元となるので、明らかに $H^\infty(D) \neq \mathbb{C}$. また、$H^\infty(\mathbb{C}) = \mathbb{C}$ *(Liouville の定理)*. $\mathbb{C}^* \setminus D$ の連結成分の中に、2点以上含むものがあれば、リーマンの写像定理により、$H^\infty(D) \neq \mathbb{C}$.

コンパクト集合 $E \subset \mathbb{C}$ は、任意の開円板 U に対して、$H^\infty(U \setminus E) = H^\infty(U)$ となるとき **Painlev'e 零集合**（**AB-除去可能集合**、**AB-零集合**）と呼ばれることがある *(cf. [SaNa65, Chap.II 2D])*。また、D の境界点が**本質的境界点** *(essential boundary point)* とは、この境界点まで解析接続できない D 上の有界正則関数あることである。従って、$H^\infty(D) = \mathbb{C}$ は D が本質的境界点を持たないこととも同値である。*Painlev'e* 零集合は全不連結であり、解析容量 *(analytic capacity)* 零の集合としても特徴付けられる。いずれの特徴付けも言葉の言い換えに過ぎない所が難点である。

一方、具体な領域についてまだ研究の余地があると思われる。たとえば、コンパクト集合が直線上にあれば、「長さ零」で特徴付けられる— しかし、具体例について実際に解析容量を評価することは難しい問題である。

2.2 リーマン面の場合でも、R が閉リーマン面の部分領域 *(*種数有限と同値*)* ならば、平面領域の場合の拡張が成り立つ。一般のリーマン面では、*Myrberg* の例などを見る限り、特徴付けにはほど遠い。たとえば、次節で説明する *Royden'resolution* を考えた上で、非定数有界正則関数をもつリーマン面の特徴を探るという方法はあり得る。

3 点分離

以下では、有界正則関数環 $H^\infty(R)$ の性質について考える。$H^\infty(R) = \mathbb{C}$ なら考えるまでもない。従って、非定数有界正則関数がある場合に限って考える。

3.1 つぎの問題は有界正則関数がリーマン面の点を区別できる程に沢山あるかどうかを問うている。

Problem 2 リーマン面 R 上で有界正則関数が**点分離**かどうかの判定；すなわち、すべての異なる2点 $p, q \in R$ について、$f(p) \neq f(g)$ となる有界正則関数 f があるか？

平面領域 D については、非定数有界関数 f が1つでもあれば、$H^\infty(D)$ は点分離である。これは、$a \in D$ について、$g_a(z) = \frac{f(z)-f(a)}{z-a}$ がまた有界正則関数となることから言える。

リーマン面の場合でも、R が閉リーマン面の部分領域 *(*種数有限と同値*)* ならば、前節 §2 と同様、平面領域の場合の拡張が成り立つ。一般のリーマン面では、つぎの一般的な結果がある。

3.2 **定理** *(Royden[Ro65])* $\mathcal{A}$ をリーマン面 R 上の正則関数からなる環とし、非定数関数を含むものとする。このとき、以下の性質を満たす三つ組み $(\tilde{R}, \tilde{\mathcal{A}}, \Phi)$ が存在する。

(1) $\tilde{R}$ はリーマン面、$\tilde{\mathcal{A}}$ は $\tilde{R}$ 上の正則関数の環、$\Phi : R \to \tilde{R}$ は正則写像。

(2) $\tilde{f} \to \tilde{f} \circ \Phi$ は $\tilde{\mathcal{A}}$ から $\mathcal{A}$ の上への環同型。

(3) $\tilde{\mathcal{A}}$ は $\tilde{R}$ を**弱点分離**する。すなわち、相異なる 2 点 $\tilde{p}, \tilde{q} \in \tilde{R}$ $(\tilde{p} \neq \tilde{q})$ に対し、関数対 $\tilde{f}, \tilde{g} \in \tilde{\mathcal{A}}$ が存在して、$\frac{\tilde{f}}{\tilde{g}}(p) \neq \frac{\tilde{f}}{\tilde{g}}(q)$

(4) $\tilde{R}$ は (2),(3) を満たすリーマン面の中で *maximal* である：すなわち、$\tilde{R} \subsetneq W$ となるリーマン面 W があれば、$\tilde{\mathcal{A}}$ は W まで解析接続されない関数を含む。

このようなリーマン面 $\tilde{R}$ は等角同値を除き一意に定まる。$(\tilde{R}, \tilde{\mathcal{A}}, \Phi)$ を対 $(R, \mathcal{A})$ の **Royden's resolution** という。このとき、$\mathcal{A} = H^\infty(R)$ ならば $\tilde{\mathcal{A}} = H^\infty(\tilde{R})$ となる。

3.3 非定数有界正則関数をもつ平面領域 (閉リーマン面の部分領域) D については、$\tilde{D}$ は有界関数の *non-essential* 境界点 (*i.e.*, 除去可能特異点) を D に付け加えて得られる。この場合は、$H^\infty(D)$ が (弱) 点分離は**理想境界の性質**である.。一般のリーマン面については (弱) 点分離は理想境界の性質ではないが、極の集合 $\mathscr{P}(R)$(記号は次節) に関しての変形なら理想境界の性質となっている。([H99])

3.4 与えられたリーマン面 R について $H^\infty(R)$ が (弱) 点分離性には有効な判定方法は知られていない。*Myrberg* の例などを見る限りそれは難しい。「2葉の分岐被覆面」において、分岐点を中心に小閉円板列を取り除いて得られるリーマン面については、分岐点の分布により円板半径列の大きさの”条件”で点分離かどうかが変わる。この”条件”についても試みの段階にあり、特徴付けにはほど遠い状況である。([HKa98, HKoNa00] など)

3.5 *Royden* は $\tilde{R}$ を $\mathcal{A}$ 上の環準同型 ϕ の芽 (*germ*) から構成しているが、つぎの方法でも構成できる (*unpublished*)：任意の非定数関数 $h \in \mathcal{A}$ をとり、$h'(a) \neq 0$ となる点 $a \in R$ をとる。a を中心とする座標円板 U で、h が U 上で単射となるものがある。そこで、複素平面上の円板 $h(U)$ 上の正則関数族 $\mathcal{A}_a = \{f \circ (h|U)^{-1} : f \in \mathcal{A}\}$ の同時解析接続から得られるリーマン面を $\tilde{R}$ とすればよい。

Royden's resolution の性質から、$\tilde{R}$ は正則関数族 $\mathcal{A}$ の”自然領域” と見なせる。このことから、つぎの問題を考えた。

Problem 3 (同型問題) $H^\infty(R_1)$ と $H^\infty(R_2)$ が環として同型ならば、それぞれの *Royden's resolution* は等角同値か？

残念ながら、この問題にも反例がある ([H91, H99])：複素2次元 *Polydisc* $\mathbb{D}^2$ の1次元解析部分集合 (リーマン面) $R = \tilde{R}$ で $H^\infty(R) \cong H^\infty(\mathbb{D}^2)$ となるもので、互いに等角同値でないものが無数にある。この例の中に、$H^\infty(R)$ が弱点分離だが点分離でない場合や、R のある点ですべての $f \in H^\infty(R)$ について、$f'(a) = 0$ となるものもある。

この例では、$H^\infty(R)$ の”自然領域”は $\mathbb{D}^2$ であって、2次元複素となっている。そこで、一般のリーマン面 R についての漠然とした問題：

Problem 4 $H^\infty(R)$ の高次元 *resolution* を考えことで”自然領域”が得られるか？

(Wermer[We58]-)Bishop([Bi62])-Royden([Ro65]) の理論によれば、$\tilde{R}$ が $\tilde{\mathcal{A}}$-*convex* となる ($\tilde{R}$ の定義よりは、ここが彼らの理論の主要定理)。高次元 *resolution* を $\mathcal{A}$ の元からなる座標系が存在する部分に限れば、解析接続の方法で簡単に定義できる。しかし、$\tilde{\mathcal{A}}$-*convex* を高次元 *resolution* で示すには、特異点を *resolution*

に取り込むた解析空間 *(analytic space)* を考える必要があり、その定義が既に自明でない — 少なくとも私には。この方向では、*Bishop* の仕事の中に参考になりそうな結果がある。

4　極大イデアル空間 $\mathscr{M}(R)$

$H^\infty(R)$ の極大イデアル空間を $\mathscr{M}(R)$ とする。$\mathscr{M}(R)$ は $H^\infty(R)$ から複素数 $\mathbb{C}$ への nonzero 環準同型の全体と同一視され、双対 Banach 空間 $H^\infty(R)^*$ の weak*コンパクト集合となる。$\mathscr{M}(R)$ は通常この weak* 位相 (**Gelfand 位相**と呼ばれる) を考えてコンパクト Hausdorff 空間と見る。

$a \in R$ について、point-evalution $\phi_a(f) = f(a)$ $(f \in H^\infty(R))$ は明らかに環準同型となるので、$a \to \phi_a$ により、R を連続に $\mathscr{M}(R)$ に埋め込んで、R を $\mathscr{M}(R)$ の部分集合と考えるが、$H^\infty(R)$ が点分離で無いときは、単射埋め込みでないことに注意する。

以下では $H^\infty(R)$ は弱点分離と仮定する — $H^\infty(R) \cong H^\infty(\tilde{R})$ であるから、必要なら R を $\tilde{R}$ で置き換えれば一般性を失わない。この場合、R の $\mathscr{M}(R)$ への埋め込みは、R から高々可算個の点からなる離散集合を除けば単射となる。(cf. [Nr98])

4.1　定理　非定数有界正則関数をもつ平面領域 *(*有限種数リーマン面*)* D については、D は $\mathscr{M}(D)$ の開集合に位相同型である。

リーマン面ではつぎの結果がある。

4.2　定理 *([H87])*

(1) $H^\infty(R)$ を弱点分離とする。このとき、点 $a \in R$ について以下は同値である。
   (a) a を含む座標近傍 U があって、U は $\mathscr{M}(R)$ の開集合に位相同型で埋め込まれる。
   (b) a に極をもつ R 上の有理型関数でコンパクト集合の外で有界なものが存在する。

(2) 性質 *(b)* を満たす R の点の全体 $\mathscr{P}(R)$ を**極の集合 (pole set)** という。$\mathscr{P}(R) = \emptyset$ となるリーマン面や、$\emptyset \neq \mathscr{P}(R) \subsetneq R$ となるリーマン面が存在する。

その後の研究で、極の集合 $\mathscr{P}(R)$ に関する限り、*Runge* 型の近似定理などほぼ平面領域の場合と同様の関数論が成立する。*([GaH87]* など*)*

$\mathscr{M}(R)$ に関しては、この他にも問題が残っている。

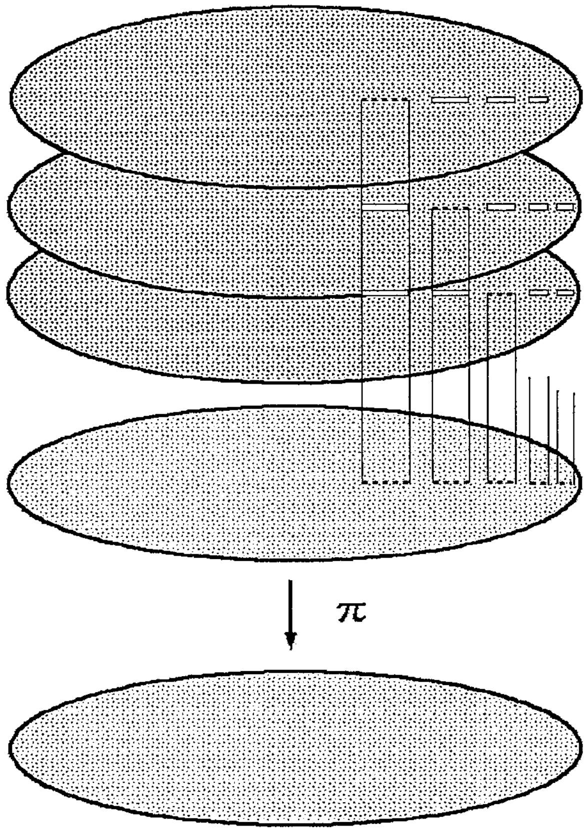
4.3　例えば、つぎの問題は *1993* 年に *Prof. Alex Izzo* から質問されて答えられなかった *(cf. [HNa**]).*

Problem 5 非定数有界正則関数をもつ平面領域 D で、極大イデアル $H_a^\infty(D) := \{f \in H^\infty(D) : f(a) = 0\}(a \in D)$ は主イデアルか？

リーマン面では $a \in \mathscr{P}(R)$ について $H_a^\infty(R)$ が主イデアルかを問うことになる— $H_a^\infty(R)$ が主イデアルなら $a \in \mathscr{P}(R)$ は明か。私の知っている例では、主イデアルとなることが明らかになる場合以外、リーマン面に範囲を広げても反例となる場合があるかどうか分かっていない。

また、つぎの問題は Gamelin 先生から1986 年に Oberwolfach へ行く車の中で聞かれたが未だ答えが分からない。

Problem 6 論文 *[H87]* の *Example 1*(右図) で射影 $\pi : R \to D$ について、D から分岐点に関わる *slits* をすべて除いた部分領域 D_0 を考える。点 $a \in D_0$ に対して、$\mathscr{M}(R)$ における *fiber* $\hat{\pi}^{-1}(a)$ は R に含まれるか? ただし、$\hat{\pi}$ は $\pi \in H^\infty(R)$ の *Gelfand* 変換を表す。



これを示すには、fiber $\pi^{-1}(a)$ が $\hat{\pi}^{-1}(a)$ で dense を示せばよく、つぎの corona 問題と同値である : $f \in H^\infty(R)$ が fiber $\pi^{-1}(a)$ 上で $|f| > 0$ ならば、$\pi^{-1}(a)$ 上で $gf = 1$ となる関数 $g \in H^\infty(R)$ が存在する。尚、同じ論文の Example 2 の場合は、fiber $\pi^{-1}(a)$ は可算無限個の点からなるのに対し、fiber $\hat{\pi}^{-1}(a)$ は Cantor 集合を含むので非可算濃度となる。

リーマン面の corona 定理については、B. Cole による最初の反例以来、Parreau-Widom 型での反例など色々与えられている。逆に、corona 定理の成り立つリーマン面については、Jones-Marshall により、Green 関数の特異点が補完点列になっている場合に得られている。また、Jones-Garnett により、平面の Denjoy 領域では corona 定理が示されている。私自身この貢献は $H^\infty(R) \cong H^\infty(\mathbb{D}^2)$ の場合しかない。

4.4 この節の最後に $H^\infty(R)$ の *Shilov* 境界 $\text{Ш}(R)$ についてふれる— **Shilov 境界**とは一般の極大イデアル空間のコンパクト部分集合で、最大値原理をみたす最小となるものである。

Problem 7 $\text{Ш}(R)$ は全不連結か?

4.5 定理 *(Gamelin[Ga72, Ga74])* 非定数有界正則関数をもつ平面領域 D において、$\text{Ш}(D)$ は全不連結である。

尚、$H^\infty(\mathbb{D})$ の場合、その Shilov 境界は $L^\infty(\frac{d\theta}{2\pi})$ の極大イデアル空間に同相となり、$\text{Ш}(\mathbb{D})$ はより不連結性の高い Stonean 空間となることが知られている。また、R が Parreau-Widom 型 (cf. 用語の記号の定義は次節参照) の場合でも、Shilov 境界は $L^\infty_\Gamma(\frac{d\theta}{2\pi}) \cong L^\infty(d\omega) \cong$ (R 上の複素値有界調和関数の境界関数族) の極大イデアル空間に同相となり、やはり Stonean 空間となる ([H79])。

小林保幸君と Gamelin のレクチャーノート [Ga72] を読んでいて気づいたことだが、Gamelin の証明を真似ると、つぎの形までは一般化できる。

4.6 定理 *(unpublished)* リーマン面 R 上の有界正則関数 f について、$\|f\|_R = \|f\|_{\mathscr{P}(R)}$ が成り立つならば、$\text{Ш}(R)$ は全不連結である。

[H87] の Example 1 はこの条件を満たしている。

5 Parreau-Widom 型リーマン面

A. Beuring(1949) は ℓ^2 上のシフト作用素の不変部分空間を Hardy 族 $H^2(\mathbb{D}) \cong H^2(\frac{d\theta}{2\pi})(\cong \ell^2)$ を用いて特徴付けた。現在、単位円上の不変部分空間定理はつぎの形で述べられる：

5.1 定理単位円周上で、$L^p(\frac{d\theta}{2\pi})$ $(1 \quad p \quad \infty)$ の *(weak* if* $p=\infty$*)* 閉部分空間 M が、$zM \subset M$ *(*z は座標関数 $\in H^\infty(\frac{d\theta}{2\pi})$ を満たすとき、M を**不変部分空間**という。不変部分空間 M はつぎのどちらかの形になる：

(a) $M = qH^p(\frac{d\theta}{2\pi})$*;* ただし、関数 q は $|q(e^{i\theta})| = 1\, a.e.\, d\theta$ を満たし、絶対値 1 の定数倍を除き *(*M によって*)* 一意に定まる。

(b) $M = \chi_E L^p(\frac{d\theta}{2\pi})$*;* ただし、$\chi_E$ は可測集合 $E(\subset \mathbb{T})$ の特性関数。

5.2 M が $L^p(\frac{d\theta}{2\pi})$ の不変部分空間ならば、$f \in H^\infty(\frac{d\theta}{2\pi})$ による乗法作用に関しても M は不変となる。すなわち、$fM \subset M$ $(\forall f \in H^\infty(\frac{d\theta}{2\pi}))$*;* これは、乗法作用素 f が z の多項式作用素で近似できることによる。従って、M は H^∞-加群でもある。

この観点から、不変部分空間定理を様々な *Hardy* 族に一般化する試みが行われた。リーマン面上の *Hardy* 族に関しては、初期には、有限リーマン面について、*D. Sarason, M. Voichick,* 荷見先生による研究があり、その後、*C.W. Neville,* 荷見先生 *(cf. [Hs10])* 等により、*Parreau-Widom* 型リーマン面にまで拡張され、最後の所で私も若干の貢献をさせて頂いた。

5.3 不変部分空間定理のリーマン面への拡張であるが、ここでは、すべて省略して不変被覆面 $\pi : \mathbb{D} \to R$ を用いて述べるに留める。本来ならリーマン面の境界の言葉を使った表現をする方がより好ましいと思われるが、そのためには、リーマン面の理想境界の理論が必要になるので割愛する。

*(*被覆写像 π の*)* **被覆変換**とは、$\mathbb{D}$ 上の *Möbius* 変換 γ で $\pi \circ \gamma = \pi$ を満たすものをいう。被覆変換全体 $\Gamma = \Gamma_R := \{\gamma\}$ は *(*合成写像の演算で*)* 群をなし、*(*被覆写像 π の*)* **被覆変換群**と呼ばれる。Γ_R は R の基本群 $\pi_1(R)$ と同型となることが知られている。$\mathbb{D}$ *(*$\mathbb{T}$*)* 上の関数 $F(z)$ $(F(e^{i\theta}))$ が、すべての $\gamma \in \Gamma$ について $F \circ \gamma = F$ *(* a.e. *)* を満たすとき **Γ-不変**であるといい、$H^p(\mathbb{D}), H^p(\frac{d\theta}{2\pi}), L^p(\frac{d\theta}{2\pi})$ の中で Γ-不変な関数全体をそれぞれ、$H^p_\Gamma(\mathbb{D}), H^p_\Gamma(\frac{d\theta}{2\pi}), L^p_\Gamma(\frac{d\theta}{2\pi})$ で表すこととする。明らかに、

$$H^p_\Gamma(\mathbb{D}) = H^p(R) \circ \pi$$

となっている。また、$L^p_\Gamma(\frac{d\theta}{2\pi})$ $(1 \quad p \quad \infty)$ の *(weak*)* 閉部分空間が $H^\infty_\Gamma(\frac{d\theta}{2\pi})$-加群となるとき、**不変部分空間**ということにする。

群準同型 $\xi : \Gamma_R \to \mathbb{T}$ を *(*群 Γ_R の*)* の**指標 (character)** という。指標全体 $\Gamma^* = \Gamma^*_R$ はコンパクト *Abel* 群をなし、**指標群**と呼ばれる。各指標 $\xi \in \Gamma^*$ に対して、

$$H^p(\mathbb{D}, \xi) := \{F \in H^p(\mathbb{D}) : F \circ \gamma = \xi(\gamma)F\}$$
$$H^p(\frac{d\theta}{2\pi}, \xi) := \{F \in H^p(\frac{d\theta}{2\pi}) : F \circ \gamma = \xi(\gamma)F\ a.e.\}$$

と定める。

(双曲型) リーマン面 R が「$\forall\xi \in \Gamma_R^* : H^\infty(\mathbb{D},\xi) \neq \{0\}$」を満たすとき **Parreau-Widom 型**であるという。

$\mathbb{T}$ 上の可測関数 Q が「$|Q| = 1,\ Q \circ \gamma = \xi(\gamma)Q$ (a.e.) $(\forall\gamma \in \Gamma)$」 を満たすとき、指標 ξ をもつ **i-関数**という。リーマン面 R について **(弱) 不変部分空間定理が成り立つ**とは、$L^p_\Gamma(\frac{d\theta}{2\pi})$ $(1 \quad p \quad \infty)$ の不変部分空間 M がつぎのどちらかの形になることをいう[*1]：

(a) $M = (\subset)QH^p(\frac{d\theta}{2\pi},\xi)$; ただし、関数 Q は指標 ξ^{-1} をもつ *i*-関数

(b) $M = \chi_E L^p_\Gamma(\frac{d\theta}{2\pi})$; ただし、$E(\subset \mathbb{T})$ は Γ_R-不変な可測集合。

5.4 定理 双曲型リーマン面 R で弱不変部分空間定理が成り立つための必要十分条件は、つぎのどちらかであること：

(a) $\dim HB(R) = 1$ *or* 2.

(b) R 上に非定数有界正則関数が存在し、*Royden's resolution* $\tilde{R}$ が *Parreau-Widom* 型である。

更に、不変部分空間定理が成り立つための必要十分条件は、*(b)* において、$a \in \tilde{R}$ を *(*任意の*)* 定点として、$\tilde{\xi} \in \Gamma^*_{\tilde{R}}$ の関数 $m(\tilde{\xi}) := \sup\{|\tilde{F}(a)| : \tilde{F} \in H^\infty(\mathbb{D},\tilde{\xi}), |\tilde{F}| \quad 1\}$ が連続となること。

5.5 最後に、*Parreau-Widom* 型リーマン面に関しての問題を二つ。一つ目は、不変部分空間定理が成り立てば正しいのだが、弱不変部分空間定理しか成り立たない場合だと分からない。

Problem 8 R が *Parreau-Widom* 型ならば $H^\infty(R)$ は $H^1(R)$ で *dense* か？

二つ目は、大野氏との共著論文 *([OhH86])* でリーマン面が種数有限かつ境界成分有限である場合までは一般化ができるが、更なる一般化は未解決：

Problem 9 *(Douglas* 問題の一般化*)* *Parreau-Widom* 型リーマン面 について、$H^\infty_\Gamma(\frac{d\theta}{2\pi}) \subset A \subset L^\infty_\Gamma(\frac{d\theta}{2\pi})$ を満たすノルム閉部分環 A を特徴付けよ。

尚、「$H^\infty_\Gamma(\frac{d\theta}{2\pi}) \subsetneq A$ ならば $(M^\infty(R) \circ \pi$ の境界値関数族$) \subset A$」*(unpublished)* までは示せる。ここで、$M^\infty(R)$ は R 上の有理型関数でコンパクト集合の外では有界となるもの全体。これは、$1/z \in M^\infty(\mathbb{D})$ に注意すれば、$R = \mathbb{D}$ のときの *Sarason* の定理「$H^\infty(\frac{d\theta}{2\pi}) \subsetneq A$ ならば $H^\infty(\frac{d\theta}{2\pi}) + C(\mathbb{T}) \subset A$」の一般化に対応している。

[*1] **[補足]** $L^p_\Gamma(\frac{d\theta}{2\pi})$ の不変部分空間を R 上の言葉で述べるには、更に用語が必要。その概略だけを説明する。まず、R 上の多価有理型関数 f は、$|f|$ が一価関数となっているとき、**乗法的**であるという。乗法的有理型関数の周期は、基本群 $\pi_1(R)$ の指標 ξ により、$f_C = \xi(C)f_{\boldsymbol{o}}$ で表せる；ただし、$f_{\boldsymbol{o}}$ は定点 $\boldsymbol{o} = \pi(0)$ における f の分枝、f_C は $\boldsymbol{o}$ を始点とする閉曲線 C に沿った $f_{\boldsymbol{o}}$ の解析接続を表す。また、基本群 $\pi_1(R)$ は 普遍被覆変換群 Γ_R と同型となることから、ξ を Γ_R の指標と同一視される。そこで、R 上で周期 ξ をもつ乗法的正則関数 f で、

$|f|^p \quad u$ を満たす R 上の調和関数 u をもつ ($p = \infty$ のときは、$|f|$ が有界)

を満たすもの全体を $H^p(R,\xi)$ とおく。このとき、$H^p(\mathbb{D},\xi) = H^p(R,\xi) \circ \pi$; すなわち、関数 $f \in H^p(R,\xi)$ を普遍被覆写像 $\pi : \mathbb{D} \to R$ により普遍被覆面 $\mathbb{D}$ に持ち上げ関数 $F = f \circ \pi$ 全体となる。乗法的正則関数 q で $Q = q \circ \pi$ が内関数となるものを**乗法的内関数**と呼べば、q の周期を $\xi \quad 1$ として、$qH^p(R,\xi)$ は $H^p(R)$ の不変部分空間となる。

更に、R の境界上で表現するには、リーマン面の理想境界の理論を使う。たとえば、Wierner コンパクト化における調和測度 $d\omega$, または、Maritin コンパクト化における調和測度 $d\chi$ を用いると、$L^\infty(d\omega)$, $L^\infty(d\chi)$ はどちらも R 上の複素値有界調和関数の境界値関数族と同一になる。また、$L^p(d\omega)$, $L^p(d\chi)$ は $L^p_\Gamma(\frac{d\theta}{2\pi})$ と等距離線型同型となる。この同型により、$L^p_\Gamma(\frac{d\theta}{2\pi})$ の不変部分空間を $L^p(d\omega)$, $L^p(d\chi)$ の部分空間として表現できる。R が Parreau-Widom 型ならば、荷見先生による Green-star 領域上の理論 (cf. [Hs10]) を用いて、$H^p(R,\xi)$ の境界値関数族 $H^p(d\chi,\xi)$ を直接定義することも可能である。

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Multiplicative isometries on the class M^p

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Abstract. In this paper, *multiplicative* (but not necessarily linear) isometries of $M^p(X)$ onto $M^p(X)$ will be described, where $M^p(X)\,(p \in \mathbb{N})$ are F-algebras included in the Smirnov class $N_*(X)$.

1. 準備

$n \geq 1$ とする。$\mathbf{C}^n$ を複素 n 次元 Euclid 空間とし、その点を表す座標を $z = (z_1, \ldots, z_n)$ と書くことにする。unit polydisk を $U^n = \{z \in \mathbf{C}^n : |z_j| < 1,\ \ 1 \leq j \leq n\}$、unit ball を $B_n = \{z \in \mathbf{C}^n : \sum_{j=1}^n |z_j|^2 < 1\}$ とし、$\mathbb{T}^n = \{z \in \mathbf{C}^n : |z_j| = 1,\ \ 1 \leqq j \leqq n\}$、$S_n = \{z \in \mathbf{C}^n : \sum_{j=1}^n |z_j|^2 = 1\}$ とする。以下、X は U^n か B_n を表し、∂X は $\mathbb{T}^n$ か S_n を表すものとする。また ∂X 上の normalized Lebesgue measure を $d\sigma$ で表す。

X 上の正則関数 f が $\displaystyle\sup_{0 \leq r < 1} \int_{\partial X} \log(1 + |f(rz)|)\, d\sigma(z) < \infty$ を満たすとき f は Nevanlinna class $N(X)$ に属するという。$f \in N(X)$ には有限な nontangential limit が a.e. $z \in \partial X$ で存在することが知られており、これを改めて $f(z)$ で表すものとする。また $f \in N(X)$ が以下の条件を満たすとき f は Smirnov class $N_*(X)$ に属するという：

$$\sup_{0 \leq r < 1} \int_{\partial X} \log(1 + |f(rz)|)\, d\sigma(z) = \int_{\partial X} \log(1 + |f(z)|)\, d\sigma(z).$$

$1 < p < \infty$ とする。X 上の正則関数 f が $\displaystyle\sup_{0 \leq r < 1} \int_{\partial X} (\log(1 + |f(rz)|))^p\, d\sigma(z) < \infty$ を満たすとき f は Privalov class $N^p(X)$ に属するという。以下、簡便のために $N^1(X) = N_*(X)$ と表すことにする。$N^p(X)$ $(p \geq 1)$ 上の距離を

$$d_{N^p(X)}(f, g) = \left(\int_{\partial X} (\log(1 + |f(z) - g(z)|))^p\, d\sigma(z) \right)^{\frac{1}{p}} \quad (f, g \in N^p(X))$$

で定義すると、$N^p(X)$ はこの距離に関して F-algebra（積に関して連続である、線形完備距離空間）であることが知られている。

次に関数空間 $M^p(X)$ を定義しよう。$0 < p < \infty$ に対し、以下を満たす X 上の正則関数 f の全体を $M^p(X)$ で表すことにする：

$$\int_{\partial X} \left(\log \left(1 + \sup_{0 \leq r < 1} |f(rz)| \right) \right)^p d\sigma(z) < \infty.$$

$M^p(X)$ 上の距離を

$$d_{M^p(X)}(f,g) = \left\{ \int_{\partial X} \left(\log \left(1 + \sup_{0 \le r < 1} |f(rz) - g(rz)| \right) \right)^p d\sigma(z) \right\}^{\frac{\alpha_p}{p}} \quad (f, g \in M^p(X))$$

とする（ただし $\alpha_p = \min(1,p)$ とおく）と、$M^p(X)$ はこの距離に関して F-algebra であることがわかっている。

これらのクラスに対し、以下の包含関係が成り立つことが知られている：

$$H^q(X) \subsetneq N^p(X) = M^p(X) \subsetneq M^1(X) \subsetneq N_*(X) \ \ (0 < q \le +\infty\,,\, p > 1).$$

ここで Hardy space を $H^p(X)$ で表し、そのノルムは $\|\cdot\|_p$ と表記することにする。特に Hardy algebra $H^\infty(X)$ は $N_*(X)$ や $N^p(X)\,,\, M^p(X)$ において稠密である。

2. $N_*(X)\,,\, N^p(X)$ における等長写像のこれまでの結果について

Smirnov class $N_*(X)$ における線形等長写像の結果は Stephenson [7] によって得られており、また Privalov class $N^p(X)$ における線形等長写像については、1 変数の場合は Iida-Mochizuki [5] による結果があり、多変数の場合は Subbotin [8] の結果が知られている。以上をまとめたものが次の定理である：

定理 2-1

> Let $p \ge 1$. $T : N^p(X) \to N^p(X)$ is a surjective linear isometry. Then there exists a holomorphic automorphism Φ on X with $\Phi(0) = 0$ such that $T(f) = \alpha f \circ \Phi$ for all $f \in N^p(X)$ where $\alpha \in \mathbb{C}, |\alpha| = 1$.

さて、$p \ge 1$ に対し $T : N^p(X) \to N^p(X)$ が $T(fg) = T(f)T(g)$ $(f\,,\, g \in N^p(X))$ を満たすとき、T は**乗法的（multiplicative）**であると呼ぶ。Smirnov class $N_*(X)$ における（必ずしも線形ではない）乗法的等長写像の結果は Hatori-Iida [2] によって得られており、また Privalov class $N^p(X)$ における（必ずしも線形ではない）乗法的等長写像については Hatori-Iida- Stević-Ueki [3] によって得られている。以下がその内容である：

命題 2-2

> Let n be a positive integer and let X be either B_n or U^n. Let $p \ge 1$ and suppose that $T : N^p(X) \to N^p(X)$ is a surjective isometry. If T is 2-homogeneous in the sense that $T(2f) = 2T(f)$ holds for every $f \in N^p(X)$, then either
>
> $$T(f) = \alpha f \circ \Phi \text{ for every } f \in N^p(X) \quad \text{or} \quad T(f) = \alpha \overline{f \circ \overline{\Phi}} \text{ for every } f \in N^p(X),$$
>
> where α is a complex number with the unit modulus and for $X = B_n$, Φ is unitary transformation; for $X = U^n$, $\Phi(z_1, \ldots, z_n) = (\lambda_1 z_{i_1}, \ldots, \lambda_n z_{i_n})$, where $|\lambda_j| = 1$, $1 \le j \le n$ and $(i_1, \ldots, i_n)$ is some permutation of the integers from 1 through n.

定理 2-3

Let n be a positive integer and let X be either B_n or U^n. Let T be a multiplicative (not necessarily linear) isometry from $N^p(X)$ $(p \geq 1)$ onto itself. Then there exists a holomorphic automorphism Φ on X such that either of the following holds:

$$T(f) = f \circ \Phi \text{ for every } f \in N^p(X) \quad \text{or} \quad T(f) = \overline{f \circ \overline{\Phi}} \text{ for every } f \in N^p(X),$$

where Φ is unitary transformation for $X = B_n$; $\Phi(z_1, \ldots, z_n) = (\lambda_1 z_{i_1}, \ldots, \lambda_n z_{i_n})$ for $X = U^n$, where $|\lambda_j| = 1$ for every $1 \leq j \leq n$ and $(i_1, \ldots, i_n)$ is some permutation of the integers from 1 through n.

以上の命題 2-2 と定理 2-3 の証明において大きな役割を果たしたのが、次の「Mazur-Ulam の定理」と呼ばれるものである（[6]）。

補題 2-4

X , Y を normed vector space とし、$U : X \to Y$ は $U(0) = 0$ を満たす onto isometry とする。このとき U は real-linear である。

３．$M^p(X)$ における（乗法的）等長写像について

$p > 0$ とする。関数空間 $M^p(X)$ における線形等長写像の結果は Subbotin [9, 10] によって得られている。その結果は $N_*(X)$, $N^p(X)$ のケースと全く同じである。

定理 3-1

Let $p > 0$. $T : M^p(X) \to M^p(X)$ is a surjective linear isometry. Then there exists a holomorphic automorphism Φ on X with $\Phi(0) = 0$ such that $T(f) = \alpha f \circ \Phi$ for all $f \in M^p(X)$ where $\alpha \in \mathbb{C}, |\alpha| = 1$.

今回、我々は $p \in \mathbb{N}$ に限定した場合について $M^p(X)$ における乗法的等長写像の結果を得た。

命題 3-2([4])

Let n be a positive integer and let X be either B_n or U^n. Let $p \in \mathbb{N}$ and suppose that $T : M^p(X) \to M^p(X)$ is a surjective isometry. If T is 2-homogeneous in the sense that $T(2f) = 2T(f)$ holds for every $f \in M^p(X)$, then either

$$T(f) = \alpha f \circ \Phi \text{ for every } f \in M^p(X) \quad \text{or} \quad T(f) = \alpha \overline{f \circ \overline{\Phi}} \text{ for every } f \in M^p(X),$$

where α is a complex number with the unit modulus and for $X = B_n$, Φ is unitary transformation; for $X = U^n$, $\Phi(z_1, \ldots, z_n) = (\lambda_1 z_{i_1}, \ldots, \lambda_n z_{i_n})$, where $|\lambda_j| = 1$, $1 \leq j \leq n$ and $(i_1, \ldots, i_n)$ is some permutation of the integers from 1 through n.

（証明の概略）

$f, g \in H^p(X)$ とする。T が 2-homogenuity であることから

$$\int_{\partial X}\left(\log\left(1+\sup_{0\le r<1}\left|\frac{f(rz)}{2^n}-\frac{g(rz)}{2^n}\right|\right)\right)^p d\sigma(z)$$
$$=\int_{\partial X}\left(\log\left(1+\sup_{0\le r<1}\left|\frac{(Tf)(rz)}{2^n}-\frac{(Tg)(rz)}{2^n}\right|\right)\right)^p d\sigma(z)$$

が成り立つ。[7] の Theorem 2.1 と同様にして、以下の等式が得られる：

$$(1)\qquad \int_{\partial X}\sup_{0\le r<1}|f(rz)-g(rz)|^p\,d\sigma(z)=\int_{\partial X}\sup_{0\le r<1}|(Tf)(rz)-(Tg)(rz)|^p\,d\sigma(z).$$

これより T は $H^p(X)$ 上のノルム $||f||_{H^p_m} := \left\{\int_{\partial X}\sup_{0\le r<1}|f(rz)|^p\,d\sigma(z)\right\}^{\frac{1}{p}}$ に関して等長であることがわかる（このノルムは $H^p(X)$ 上の通常のノルムと同値である）。(1) で $g=0$ とおき、さらに $T(2f)=2T(f)$ から $T(0)=0$ であるので $T(H^p(X))\subseteq H^p(X)$ が得られる。さらに $T|_{H^p(X)}$ は H^p-ノルム $\|\cdot\|_p$ によって導入される距離に関して等長である。同様の方法を T^{-1} に適用することによって $T^{-1}(H^p(X))\subseteq H^p(X)$ がわかる。よって $T(H^p(X))=H^p(X)$ が得られる。$T(0)=0$ なので、Mazur-Ulam の定理より $T|_{H^p(X)}$ が real-linear であることが示される。

さて、極限

$$\lim_{\varepsilon\to 0^+}\frac{1}{\varepsilon^{p+1}}\left\{(\varepsilon t)^p-(\log(1+\varepsilon t))^p\right\}=\frac{p}{2}t^{p+1}\quad (t\ge 0)$$

より T はノルム $||f||_{H^{p+1}_m}$ に関して等長であることが分かる ([8])。$p>0$ とすると、任意の $k\in\mathbb{N}$ に対して次の極限も存在する：

$$(2)\qquad \lim_{\varepsilon\to 0^+}\frac{1}{\varepsilon^{p+k}}\left\{(\log(1+\varepsilon t))^p-\sum_{n=0}^{k-1}c_n(\varepsilon t)^{n+p}\right\}=c_k t^{p+k},\quad (t\ge 0).$$

ここで c_n は関数 $\left(\frac{\log(1+t)}{t}\right)^p$ を Taylor 展開した時の係数を表す。(2) を用いることで T は $p\in\mathbb{N}$ とすべての $k\in\mathbb{N}$ に対して H^{p+k}_m 上で等長であることが帰納法より証明される。

$d\sigma$ は有限測度なので $\lim_{p\to\infty}||f||_{H^p_m}=||f||_{H^\infty_m}$ がすべての $f\in H^\infty(X)$ に対して成り立ち、また $||f||_{H^\infty_m}=||f||_\infty$ は明らかである。さらに $\|f\|_p=\|T(f)\|_p$ がすべての $f\in H^\infty(X)$ に対して成り立ち、$\lim_{p\to\infty}\|T(f)\|_p=\|T(f)\|_\infty$ なので $T(f)\in H^\infty(X)$ かつ $\|f\|_\infty=\|T(f)\|_\infty$ がすべての $f\in H^\infty(X)$ に対して成立する。同様にして $T(f)\in H^\infty(X)$ ならば $f\in H^\infty(X)$ が成り立つ。よって $T|_{H^\infty(X)}$ は $\|\cdot\|_\infty$ に関して $H^\infty(X)$ から $H^\infty(X)$ への全射等長写像である。$H^\infty(X)$ はその極大イデアル空間上の uniform algebra であり、その極大イデアル空間は「Šilov idempotent theorem」によって連結であるとしてよい。よって [1] より $T|_{H^\infty(X)}$ は complex-linear または conjugate linear であることがいえる。

$T|_{H^\infty(X)}$ が complex-linear であるとすると、$H^\infty(X)$ は $M^p(X)$ で稠密であり、その上の距離による収束は X のコンパクト部分集合上の一様収束より強いことを利用して、T が $M^p(X)$ 上 complex-linear であることが示される。よって命題中の最初の式が [7] の Corollary 2.3 から成立する。

一方 $T|_{H^\infty(X)}$ が conjugate linear であるとすると、T は前述のように $M^p(X)$ 上 conjugate linear である。ここで $\widetilde{T}: M^p(X) \to M^p(X)$ を $\widetilde{T}(f) = T(\widetilde{f})$ $(f \in M^p(X))$ のように定義する。ただし $\widetilde{f}(z_1, \ldots, z_n) = \overline{f(\overline{z_1}, \ldots, \overline{z_n})}$ $(f \in M^p(X))$ であるものとする。このとき $\widetilde{T}$ は $M^p(X)$ から $M^p(X)$ への complex-linear な等長写像である。[7] の Corollary 2.3 を再度用いることで、命題中の２番目の式も成り立つことが示される。

（証明終）

定理 3-3([4])

Let n be a positive integer and let X be either B_n or U^n. Let T be a multiplicative (not necessarily linear) isometry from $M^p(X)$ $(p \in \mathbb{N})$ onto itself. Then there exists a holomorphic automorphism Φ on X such that either of the following holds:

$$T(f) = f \circ \Phi \text{ for every } f \in M^p(X) \quad \text{or} \quad T(f) = \overline{f \circ \overline{\Phi}} \text{ for every } f \in M^p(X),$$

where Φ is unitary transformation for $X = B_n$; $\Phi(z_1, \ldots, z_n) = (\lambda_1 z_{i_1}, \ldots, \lambda_n z_{i_n})$ for $X = U^n$, where $|\lambda_j| = 1$ for every $1 \leq j \leq n$ and $(i_1, \ldots, i_n)$ is some permutation of the integers from 1 through n.

（証明の概略）

T は乗法的なので、[2] の Theorem 2.2 の証明と同様にして $T(1) = 1$, $T(2) = 2$, $T(\frac{1}{2}) = \frac{1}{2}$ であることが示される。よって再び T が乗法的であることを利用して、T は $T(2f) = 2T(f)$ を満たす全射等長写像であることがわかる。命題3-2から、ある複素数 α と、その命題と同様の holomorphic automorphism Φ に対して

$$T(f) = \alpha f \circ \Phi, \quad f \in M^p(X) \quad \text{または} \quad T(f) = \alpha \overline{f \circ \overline{\Phi}}, \quad f \in M^p(X)$$

が成り立つ。このとき $T(1) = 1$ から $\alpha = 1$ がわかり、本定理が証明される。

（証明終）

【注意】 $M^p(X)$ $(p \in \mathbb{N})$ における全射乗法的等長写像は Smirnov class や Privalov class における全射乗法的等長写像の構造と全く同じであることがわかる。今回の $M^p(X)$ の結果が自然数以外で成り立つかどうかはまだ分かっていない。

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On ψ-direct sums of Banach spaces with a strictly monotone norm

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We shall discuss the uniform non-squareness and uniform non-ℓ_1^n-ness of ψ-direct sums $(X_1 \oplus \cdots \oplus X_N)_\psi$ with a strict monotone norm for Banach spaces $X_1, \ldots, X_N$.

1. Absolute norms on $\mathbb{C}^N$ and convex functions

A norm $\|\cdot\|$ on $\mathbb{C}^N$ is called *absolute* if

$$\|(z_1, \cdots, z_N)\| = \|(|z_1|, \cdots, |z_N|)\|$$

for all $(z_1, \cdots, z_N) \in \mathbb{C}^N$ and *normalized* if

$$\|(1, 0, \cdots, 0)\| = \cdots = \|(0, \cdots, 0, 1)\| = 1.$$

The collection of all such norms on $\mathbb{C}^N$ is denoted by AN_N. In K.-S. Saito et al. [10] it was shown that there is a one-to-one correspondence between AN_N and the class Ψ_N of convex functions on the standard N-simplex Δ_N stated below (the case N=2 was shown by Bonsall and Duncan[1]): Let $\|\cdot\| \in AN_N$. Let

$$\psi(s) = \|(1 - \sum_{i=1}^{N-1} s_i, s_1, \cdots, s_{N-1})\| \quad \text{for } s = (s_1, \cdots, s_{N-1}) \in \Delta_N, \tag{1}$$

where $\Delta_N = \{s = (s_1, \cdots, s_{N-1}) \in \mathbb{R}^{N-1} : \sum_{i=1}^{N-1} s_i \leq 1,\ s_i \geq 0\}$. Then ψ is convex (continuous) on the convex set Δ_N and satisfies the following:

$$
\begin{array}{lrcll}
(A_0) & \psi(0, \cdots, 0) & = & \psi(1, 0, \cdots, 0) = \cdots = \psi(0, \cdots, 0, 1) = 1, & \\
(A_1) & \psi(s_1, \cdots, s_{N-1}) & \geq & \Big(\sum_{i=1}^{N-1} s_i\Big)\psi\Big(\frac{s_1}{\sum_{i=1}^{N-1} s_i}, \cdots, \frac{s_{N-1}}{\sum_{i=1}^{N-1} s_i}\Big) & \text{if } 0 < \sum_{i=1}^{N-1} s_i \leq 1, \\
(A_2) & \psi(s_1, \cdots, s_{N-1}) & \geq & (1 - s_1)\psi\Big(0, \frac{s_2}{1 - s_1}, \cdots, \frac{s_{N-1}}{1 - s_1}\Big) & \text{if } 0 \leq s_1 < 1, \\
 & \cdots\cdots\cdots & & & \\
(A_N) & \psi(s_1, \cdots, s_{N-1}) & \geq & (1 - s_{N-1})\psi\Big(\frac{s_1}{1 - s_{N-1}}, \cdots, \frac{s_{N-2}}{1 - s_{N-1}}, 0\Big) & \text{if } 0 \leq s_{N-1} < 1
\end{array}
$$

The converse holds true: Denote by Ψ_N the family of all convex functions ψ on Δ_N satisfying $(A_0),(A_1),\cdots,(A_N)$. For any $\psi \in \Psi_N$ define

$$\|(z_1,\cdots,z_N)\|_\psi = \begin{cases} (|z_1|+\cdots+|z_N|)\psi\Big(\dfrac{|z_2|}{|z_1|+\cdots+|z_N|},\cdots,\dfrac{|z_N|}{|z_1|+\cdots+|z_N|}\Big) \\ \qquad\qquad\qquad\qquad \text{if } (z_1,\cdots,z_N)\neq(0,\cdots,0), \\ 0 \qquad\qquad\qquad\qquad \text{if } (z_1,\cdots,z_N)=(0,\cdots,0) \end{cases} \tag{2}$$

Then $\|\cdot\|_\psi \in AN_N$ and $\|\cdot\|_\psi$ satisfies (1). Thus we have a one-to-one correspondence between AN_N and Ψ_N with the equation (1).

By (1) the conditions $(A_0),(A_1),(A_2),\ldots,(A_N)$ are equivalent to:

$$\begin{array}{llcl}
(A_0) & \|(1,0,\cdots,0)\| & = & \|(0,1,0\cdots,0)\| = \cdots = \|(0,\cdots,0,1)\| = 1, \\
(A_1) & \|(1-\sum_{i=1}^{N-1} s_i, s_1,\cdots,s_{N-1})\| & \geq & \|(0,s_1,\cdots,s_{N-1})\| \quad \text{if } 0<\sum_{i=1}^{N-1} s_i \leq 1, \\
(A_2) & \|(1-\sum_{i=1}^{N-1} s_i, s_1,\cdots,s_{N-1})\| & \geq & \|(1-\sum_{i=1}^{N-1} s_i, 0, s_2\cdots,s_{N-1})\| \qquad \text{if } 0\leq s_1<1, \\
 & \cdots\cdots\cdots & & \\
(A_N) & \|(1-\sum_{i=1}^{N-1} s_i, s_1,\cdots,s_{N-1})\| & \geq & \|(1-\sum_{i=1}^{N-1} s_i, s_1,\cdots,s_{N-2},0)\| \quad \text{if } 0\leq s_{N-1}<1
\end{array}$$

The ℓ_p-norms are typical examples of such a norm. The function corresponding to the ℓ_p-norm, which is denoted by ψ_p, is given by

$$\psi_p(s_1,\cdots,s_{N-1}) = \begin{cases} \Big\{\Big(1-\sum_{i=1}^{N-1} s_i\Big)^p + s_1^p+\cdots+s_{N-1}^p\Big\}^{1/p} & \text{if } \ 1\leq p<\infty, \\ \max\{1-\sum_{i=1}^{N-1} s_i, s_1,\cdots,s_{N-1}\} & \text{if } \ p=\infty \end{cases}$$

for $(s_1,\cdots,s_{N-1}) \in \Delta_N$. For any $\|\cdot\| \in AN_N$ we have $\|\cdot\|_\infty \leq \|\cdot\| \leq \|\cdot\|_1$.

Let $X_1,...,X_N$ be Banach spaces and let $\psi \in \Psi_N$. The ψ-direct sum $(X_1\oplus\cdots\oplus X_N)_\psi$ is the direct sum of $X_1,...,X_N$ equipped with the norm

$$\|(x_1,\cdots,x_N)\|_\psi := \|(\|x_1\|,\cdots,\|x_N\|)\|_\psi \quad \text{for } (x_1,\ldots,x_N) \in X_1\oplus\cdots\oplus X_N.$$

2. Strict monotonicity

A norm $\|\cdot\|$ on $\mathbb{C}^N$ is called *monotone* provided that $\|(z_1,\ldots,z_N)\|_\psi \leq \|(w_1,\ldots,w_N)\|_\psi$ if $|z_j| \leq |w_j|$ for all $1\leq j\leq N$ and *strictly monotone* provided that $\|(z_1,\ldots,z_N)\|_\psi < \|(w_1,\ldots,w_N)\|_\psi$ if $|z_j| \leq |w_j|$ for all $1\leq j\leq N$ and $|z_{j_0}| < |w_{j_0}|$ for some $1\leq j_0\leq N$.

Lemma 1 *Let F be a convex function on $\mathbb{R}_+^N$. Let $(p_1, \ldots, p_N) \in \mathbb{R}_+^N$ and $p_{j_0} \leq q_{j_0}$ with $q_{j_0} \in \mathbb{R}_+$ for some $1 \leq j_0 \leq N$. If $F(p_1, \ldots, \underbrace{0}_{j_0}, \ldots, p_N) \leq F(p_1, \ldots, q_{j_0}, \ldots, p_N)$, then $F(p_1, \ldots, p_{j_0}, \ldots, p_N) \leq F(p_1, \ldots, q_{j_0}, \ldots, p_N)$.*

Proposition 2 ([10]) *Let $\psi \in \Psi_N$. Then ψ-norm $\|\cdot\|_\psi$ is monotone.*

For the strict monotonicity of the ψ-norm we have the following.

Proposition 3 *Let $\psi \in \Psi_N$. Let $(z_1, \ldots, z_N) \in \mathbb{C}^N$ and $0 < |z_{j_0}| < |w_{j_0}|$ with $w_{j_0} \in \mathbb{C}^N$ for some $1 \leq j_0 \leq N$. Then the following are equivalent.*

(i) $\|(z_1, \ldots, z_{j_0}, \ldots, z_N)\|_\psi < \|(z_1, \ldots, w_{j_0}, \ldots, z_N)\|_\psi$

(ii) $\|(z_1, \ldots, \underbrace{0}_{j_0}, \ldots, z_N)\|_\psi < \|(z_1, \ldots, w_{j_0}, \ldots, z_N)\|_\psi$

Corollary 4 ([4]) *Let $\psi \in \Psi_N$. Then $\|\cdot\|_\psi$ is strictly monotone if and only if ψ satisfies the following conditions:*

$$
\begin{aligned}
(sA_1) \quad \psi(s_1, \cdots, s_{N-1}) &> (s_1 + \cdots + s_{N-1})\psi\Big(\frac{s_1}{s_1 + \cdots + s_{N-1}}, \cdots, \frac{s_{N-1}}{s_1 + \cdots + s_{N-1}}\Big) \\
&\qquad \textit{if } 0 < s_1 + \cdots + s_{N-1} < 1, \\
(sA_2) \quad \psi(s_1, \cdots, s_{N-1}) &> (1 - s_1)\psi\Big(0, \frac{s_2}{1 - s_1}, \cdots, \frac{s_{N-1}}{1 - s_1}\Big) \qquad \textit{if } 0 < s_1 < 1, \\
&\cdots\cdots\cdots \\
(sA_N) \quad \psi(s_1, \cdots, s_{N-1}) &> (1 - s_{N-1})\psi\Big(\frac{s_1}{1 - s_{N-1}}, \cdots, \frac{s_{N-2}}{1 - s_{N-1}}, 0\Big) \\
&\qquad \textit{if } 0 < s_{N-1} < 1
\end{aligned}
$$

3. A class of convex functions $\Psi_N^{(1)}$

In the present authors [9] a new class of convex functions $\Psi_N^{(1)}$ is introduced as follows: Let $\psi \in \Psi_N$. We say $\psi \in \Psi_N^{(1)}$ if there is an element $(s_1, \ldots, s_{N-1}) \in \Delta_N$ such that for some nonempty subset S of $\{1, \ldots, N-1\}$ with $0 < M := \sum_{i=1}^{N-1} \chi_S(i) s_i < 1$,

$$
\begin{aligned}
\psi(s_1, \ldots, s_{N-1}) &= M\psi\left(\frac{\chi_S(1)s_1}{M}, \ldots, \frac{\chi_S(N-1)s_{N-1}}{M}\right) \\
&\quad + (1 - M)\,\psi\left(\frac{\chi_{S^c}(1)s_1}{1 - M}, \ldots, \frac{\chi_{S^c}(N-1)s_{N-1}}{1 - M}\right).
\end{aligned}
$$

In other words,

$$\|(1-\sum_{i=1}^{N-1}s_i, s_1,\ldots,s_{N-1})\|_\psi = \|(0,\chi_S(1)s_1,\ldots,\chi_S(N-1)s_{N-1})\|_\psi$$
$$+\|(1-\sum_{i=1}^{N-1}s_i,\chi_{S^c}(1)s_1,\ldots,\chi_{S^c}(N-1)s_{N-1})\|_\psi$$

In the case $N=2$, $\Psi_N^{(1)}=\Psi_2^{(1)}=\{\psi_1\}$.

Theorem 5 ([9]) *Let $\psi\in\Psi_N$. Then the following are equivalent.*
(i) $\psi\in\Psi_N^{(1)}$
(ii) *There exists $(a_1,\ldots,a_N)\in\mathbb{R}_+^N$ such that with some nonempty proper subset T of $\{1,\ldots,N\}$*

$$\|(a_1,\ldots,a_N)\|_\psi=\|(\chi_T(1)a_1,\ldots,\chi_T(N)a_N)\|_\psi+\|(\chi_{T^c}(1)a_1,\ldots,\chi_{T^c}(N)a_N)\|_\psi,$$

where $(\chi_T(1)a_1,\ldots,\chi_T(N)a_N)$ and $(\chi_{T^c}(1)a_1,\ldots,\chi_{T^c}(N)a_N)$ are nonzero.

4. Uniform non-squareness and uniform non-ℓ_1^n-ness

A Banach space X is called *uniformly non-square* if there exists a constant $\varepsilon>0$ such that $\min\{\|x+y\|,\|x-y\|\}\le 2(1-\varepsilon)$ for all $x,y\in B_X$. More generally, X is called *uniformly non-ℓ_1^n* if there exists a constant $\varepsilon>0$ such that $\min\{\|\sum_{j=1}^n\theta_jx_j\|:\theta_j=\pm1\}\le N(1-\varepsilon)$ for all $x_1,\ldots,x_n\in B_X$.

Theorem 6 *For $(X_1\oplus\cdots\oplus X_N)_\psi$ with a strictly monotone norm the following are equivalent.*
(i) *$(X_1\oplus\cdots\oplus X_N)_\psi$ is uniformly non-square.*
(ii) *All $X_1,\ldots,X_N$ are uniformly non-square and $\psi\notin\Psi_N^{(1)}$*

According to Theorem 6, under the condition that the norm of $(X_1\oplus\cdots\oplus X_N)_\psi$ is strictly monotone, we have the following: For uniformly non-square spaces $X_1,\ldots,X_N$ the uniform non-squareness of $(X_1\oplus\cdots\oplus X_N)_\psi$ is equivalent to $\psi\notin\Psi_N^{(1)}$. For the uniform non ℓ_1^N-ness the condition $\psi\notin\Psi_N^{(1)}$ is weakened as follows.

Theorem 7 *Assume that $X_1,\ldots,X_N$ are uniformly non-square and the norm of $(X_1\oplus\cdots\oplus X_N)_\psi$ is strictly monotone. Then the following are equivalent.*
(i) *$(X_1\oplus\cdots\oplus X_N)_\psi$ is uniformly non-ℓ_1^N.*
(ii) $\psi\neq\psi_1$

Corollary 8 *Assume that the ψ-norm $\|\cdot\|$ on $\mathbb{C}^N$ is strictly monotone. Then the following are equivalent.*

(i) $(\mathbb{C}^N, \|\cdot\|_\psi)$ *is uniformly non-*ℓ_1^N.

(ii) $\psi \neq \psi_1$

The next result by the present authors [8] should be compared with Theorem 7.

Theorem 9 ([8]) *Let $X_1, \ldots, X_N$ be uniformly non-square. Then $(X_1 \oplus \cdots \oplus X_N)_{\psi_1}$ is uniformly non-*ℓ_1^{N+1}.

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A remark on Taylor coefficients of complete Pick kernels on the unit disk

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Abstract

A condition on complete Pick kernels on the unit disk which implies convergence of certain operator power series is given.

Let $\mathbb{D}$ denote the open unit disk in the complex plane, that is, we set $\mathbb{D} = \{\lambda \in \mathbb{C} : |\lambda| < 1\}$. and k_λ will denote the reproducing kernel of a reproducing kernel Hilbert space consisting of holomorphic functions on $\mathbb{D}$. Throughout this article, we assume that monomials are mutually orthogonal in those Hilbert function spaces and $k_\lambda(0) = 1$. Then, setting $k_\lambda(z) = \sum_{n=0}^\infty a_n \overline{\lambda}^n z^n$, we have $a_n > 0$ for any $n \geq 0$. The following result is well known (see Theorem 7.33 in [1]):

Theorem 1 *Let d_n be the nth Taylor coefficient of $1/\sum_{n=0}^\infty a_n w^n$ at the origin, that is, we set $1/\sum_{n=0}^\infty a_n w^n = \sum_{n=0}^\infty d_n w^n$. Then k_λ is a complete Pick kernel if and only if $d_n \leq 0$ for any $n \geq 1$.*

Let T be a bounded linear operator acting on a Hilbert space $\mathcal{H}$. Our main interest is the following operator power series in the case where k_λ is a complete Pick kernel:

$$\Delta_T = \sum_{n=0}^\infty d_n T^n T^{*n},$$

which is Agler's hereditary functional calculus in our setting (see Chapter 14 in [1] for details).

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Lemma 1 Let k_λ be a complete Pick kernel. If $\Delta_T \geq 0$ and $\sum_{n\geq 0} a_n \|T^{*n}x\|^2$ is finite for some x in $\mathcal{H}$. Then $\sum_{n\geq 0, k\geq 1} a_n |d_k| \|T^{*n+k}x\|^2$ is finite.

Proof Since we have

$$\begin{aligned}\sum_{n=0}^{\infty} a_n \sum_{k=1}^{\infty} |d_k| \|T^{*n+k}x\|^2 &= \sum_{n=0}^{\infty} a_n (\|T^{*n}x\|^2 - \|\sqrt{\Delta_T} T^{*n}x\|^2) \\ &= \sum_{n=0}^{\infty} a_n \|T^{*n}x\|^2 - \sum_{n=0}^{\infty} a_n \|\sqrt{\Delta_T} T^{*n}x\|^2 \\ &< +\infty,\end{aligned}$$

we have the conclusion by Fubini's theorem.

Theorem 2 Let k_λ be a complete Pick kernel. If $\Delta_T \geq 0$ and there exists some dense subspace $\mathcal{V}$ of $\mathcal{H}$ such that $\sum_{n\geq 0} a_n \|T^{*n}x\|^2$ is finite for any x in $\mathcal{V}$ then $\sum_{n=0}^{\infty} a_n T^n \Delta_T T^{*n} = I$.

Proof We have the following identities by the definition of $\{d_n\}_n$:

$$\begin{cases} a_0 d_0 = 1 \\ a_0 d_l + a_1 d_{l-1} + \cdots + a_l d_0 = 0 \quad (l \geq 1). \end{cases} \tag{0.1}$$

Then, for any x in $\mathcal{V}$, we have

$$\begin{aligned}\langle x, x\rangle &= \|x\|^2 \\ &= \sum_{n\geq 0} a_n \|T^{*n}x\|^2 - \sum_{l\geq 1} a_l \|T^{*l}x\|^2 \\ &= \sum_{n\geq 0} a_n \|T^{*n}x\|^2 + \sum_{l\geq 1} \sum_{n+k=l, n\geq 0, k\geq 1} a_n d_k \|T^{*l}x\|^2 \\ &= \sum_{n\geq 0} a_n \|T^{*n}x\|^2 + \sum_{n\geq 0, k\geq 1} a_n d_k \|T^{*n+k}x\|^2 \\ &= \sum_{n=0}^{\infty} a_n (\|T^{*n}x\|^2 + \sum_{k=1}^{\infty} d_k \|T^{*n+k}x\|^2) \\ &= \sum_{n=0}^{\infty} a_n \langle \Delta_T T^{*n}x, T^{*n}x\rangle \\ &= \sum_{n=0}^{\infty} \|\sqrt{a_n \Delta_T} T^{*n}x\|^2 \\ &= \sum_{n=0}^{\infty} \langle a_n T^n \Delta_T T^{*n}x, x\rangle\end{aligned}$$

by Lemma 1, Fubini's theorem and (0.1). Since $\mathcal{V}$ is dense in $\mathcal{H}$, we have the conclusion.

Remark 1 The condition $\sum_{n=0}^{\infty} a_n T^n \Delta_T T^{*n} = I$ can be seen in many articles. For example, in p. 80 of [2], this is called "convergence condition".

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On the number of periodic orbits with prescribed energies of a Hamiltonian system near an equilibrium point

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1　ベクトル場の定める自励系常微分方程式と周期解の存在

X を Banach 空間，$f\colon X \to X$ は X 上の局所 Lipschitz 連続なベクトル場とする．このとき X 上の自励系方程式（力学系）

$$\frac{d}{dt}u(t) = f(u(t)), \quad u(0) = u_0 \tag{1}$$

は任意の初期値 u_0 に対して局所解 $u(t) =: \varphi(u_0, t)$ を持つが，f がさらにある程度の条件（e.g., 有界）を満たせば，X 上の連続な flow $\varphi\colon \mathbb{R} \times X \to X$ が定まる．

u_0 が f の平衡点，すなわち $f(u_0) = 0$ となっている場合は，$u(t) \equiv u_0$ が (1) の解（定常解）となる．平衡点の近傍での flow（解曲線）の様子は様々であるが，力学系に関する基本的な問題として詳しく研究されている．本稿では特に Hamilton 系について，エネルギー値を指定した場合の，平衡点付近の周期軌道の個数について取り扱うが，証明を述べるゆとりがないため，結果の報告のみであることをご容赦頂きたい．なお，ここで「周期軌道の個数」と言っているのは，周期解自体は，周期軌道上の各点を初期値と考えると連続無限個存在するため，幾何学的に異なる周期軌道の個数にこそ意味があるからである．

Lyapunov Center Theorem　個数について考える前に，有限次元空間上の力学系について，平衡点付近の周期軌道の存在を示した Lyapunov の基本的な結果を振り返っておこう．

$f \in C^1(\mathbb{R}^n, \mathbb{R}^n)$, $f(0) = 0$ として自励系方程式 $\dot{x} = f(x)$ を考える．ここではさらに $\dot{x} = f(x)$ が C^2 級の第一積分 G を持つとする．

定理 1 (Lyapunov Center Theorem) 以上の条件の下で，$A := D_x f(0)$ は非退化で，$\lambda_1 = \mathrm{i}\omega$, $\lambda_2 = -\mathrm{i}\omega$ を固有値に持つものとする．また，E でこれらの固有値に対する固有ベクトルの実部と虚部の生成する 2 次元空間を表す．

もしも A の他の固有値 λ_k $(k = 3, \ldots, n)$ が非共鳴条件

$$\frac{\lambda_k}{\lambda_1} \notin \mathbb{Z}$$

を満たし，$(D^2G)|_E(0) > 0$ であれば，$\dot{x} = f(x)$ は十分小な $\varepsilon > 0$ に対して，$G = G(0) + \varepsilon^2$ 上に値を取る周期 $T(\varepsilon)$ の周期解で次の性質を持つものをただ一つ持つ：$\varepsilon \to 0$ のとき，この周期解は L^∞ で 0 に収束し，$T(\varepsilon) \to 2\pi/\omega$ を満たす．

Lyapunov Center Theorem は，$f(x)$ の 0 における線形近似の固有値についてのあからさまな仮定の下で周期解の存在を示しているが，非共鳴条件のため，個数については最小限の存在を保証しているのみである．これに対して，Hamilton 力学系に考察を限定すると，非共鳴条件にとらわれず，周期解の個数について optimal な結果が得られている．

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2 Hamilton 力学系の周期解

2.1 Hamiltonian, symplectic form and Hamiltonian System

$\mathbb{R}^{2n}$ 上の Hamilton 力学系は，Hamiltonian $H \in C^2(\mathbb{R}^{2n}, \mathbb{R})$ と $\mathbb{R}^{2n}$ の symplectic 構造 ω から定まるベクトル場に関する自励系方程式である．ここで symplectic 構造 ω とは，$\mathbb{R}^{2n}$ 上の非退化双線形交代形式であり，正準座標系 $(p, q) = (p_1, p_2, \ldots, p_n; q_1, q_2, \ldots, q_n)$ では，

$$J = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$$

と表現される線形作用素 J によって $\omega(x, y) = \langle Jx, y\rangle$ と表される（$\langle\cdot, \cdot\rangle$ は標準内積）．

Hamilton 方程式（力学系） Hamiltonian H の gradient ∇H と J（ω から定まる）の合成で得られるベクトル場が定める自励系方程式が Hamilton 力学系である：

$$\text{(HS)} \qquad \dot{z}(t) = J\nabla H(z(t)), \quad z = (p, q) \in \mathbb{R}^n \times \mathbb{R}^n$$

周知のように，Hamilton 方程式は成分で表すと次のようになる：

$$\begin{cases} \dot{p}_i(t) = -H_{q_i}, \\ \dot{q}_i(t) = H_{p_i} \end{cases} \qquad (i = 1, 2, \ldots, n)$$

また，きわめてよく知られたことであるが，H は (HS) の第一積分であり，(HS) の解 $z(\cdot)$ に対して $H(z(\cdot))$ は定数となる．これが周期解を H の値を指定した中で探す理由である．

以下では $H \in C^2(\mathbb{R}^{2n}, \mathbb{R})$ で，$H(0) = 0, \nabla H(0) = 0$ と仮定する．

線形化 Hamilton 方程式 (HS) を原点で線形化した方程式を (LHS) で表す:

$$\text{(LHS)} \qquad \dot{z}(t) = JH''(0)z(t)$$

ここで $H''(0)$ は Fréchet 微分の意味での H の原点における 2 階微分係数であるが，座標成分で表せば，原点における H の Hesse 行列である．

(LHS) については，$JH''(0)$ を生成作用素とする群 $\mathcal{S} := \{e^{tJH''(0)}\}_{t\in\mathbb{R}}$ によって，初期値 z_0 に対する解が $u(t) = e^{tJH''(0)}u_0$ と表示される．

2.2 Weinstein–Moser's Theorem

前小節に述べた H に対する仮定の下で，周期軌道の個数について次のことが証明されている．

定理 2 (Weinstein [5], 1966) $H''(0)$ が positive definite であれば，十分小な $\varepsilon > 0$ に対して，(HS) はエネルギー曲面 $H = \varepsilon$ 上に少なくとも n 個の幾何学的に相異なる周期解を持つ．

Lyapunov Center Theorem のように，ベクトル場を線形化したときの固有値を表に出した形では次のようになる．

定理 3 (Moser [4], 1970) $\mathbb{R}^{2n} = E \oplus F$ が $\mathcal{S}$ 不変な部分空間への分解で，$\mathcal{S}$ は E では周期 τ_0 であり，F 内には $\{0\}$ 以外に $\mathcal{S}$ の周期 τ_0 の周期軌道は存在しないとする．さらに $H''(0)$ が E 上では positive definite であるとすると，十分小な $\varepsilon > 0$ に対して，(HS) はエネルギー曲面 $H = \varepsilon$ 上に少なくとも $\dim E/2$ 個の幾何学的に相異なる周期解を持つ．

Remark. (i) Weinstein の定理は Moser の定理から容易に導かれる．

(ii) 定理 2, 3 における個数評価の n や dim $E/2$ が optimal であることを示す例は容易に構成できる．

証明のキーポイント Moser の定理の証明にはいくつかの方法があるが，基本的には Lyapunov–Schmidt の分解を用いる．ここでは個数を導くところで変分法 (min-max 理論）を使う証明法の道筋を紹介するが，個数の評価には positive definite という仮定が本質的に関わっている．

1. symplectic 変換による $\mathbb{R}^{2n}$ 上の正値 2 次形式 q の対角化 (known to K. Weierstrass).
   $\exists\varphi \in \mathcal{L}(\mathbb{R}^{2n}, \mathbb{R}^{2n})$ satisfying $\omega \circ \varphi = \omega$ s.t.
   $$q \circ \varphi(x) = \frac{1}{2}\sum_{j=1}^{n} \lambda_j (x_j^2 + x_{n+j}^2)$$
2. 周期パラメータ τ の導入により周期 1 の解の存在問題に帰着させる：
   $$(\mathrm{HS})_\tau \qquad \dot{z}(t) = \tau J \nabla H(z(t))$$
3. Ljapunov–Schmidt procedure により，関数空間 $H^1(S^1, E)$ 上の制約条件 $\mathcal{H}(v) = \varepsilon$（$H = \varepsilon$ に対応）下での汎関数 $\mathcal{A}(v)$（作用積分に対応）の極値問題に帰着させる．
4. $H^1(S^1, E)$ には $[T(\theta)v](\,\cdot\,) := v(\,\cdot\, + \theta)$ ($v \in H^1(S^1, E)$, $\theta \in S^1$) によって群 $S^1 = [0, 1]/\sim$ が連続に作用する．このことから，S^1-index を用いた min-max 定理により上の制約条件付き極値問題が少なくとも $\dim E/2$ 個の幾何学的に異なる軌道を持つことが結論される．

2.3 正定値性を外した場合の結果

定理 3 において $H''(0)$ を E に制限したものが positive definite あるいは negative definite とならない場合についても，周期解の個数の評価が研究されている．

定理 4 (Fadell–Rabinowitz [2],1978) $H \in C^2(\mathbb{R}^{2n}, \mathbb{R})$ で，$H(0) = 0$, $\nabla H(0) = 0$ とし，$\mathbb{R}^{2n} = E \oplus F$ が $\mathcal{S}$ 不変な部分空間への分解で，$\mathcal{S}$ は E では周期 τ_0 であり，F 内には $\{0\}$ 以外に $\mathcal{S}$ の周期 τ_0 の周期軌道は存在しないとする．さらに $E \setminus \{0\}$ には平衡点が存在しないとする．このとき，$H''(0)$ が E 上で定める 2 次形式の符号数 2ν が 0 でないとすると，次のどちらかが成り立つ：(i) 0 は (HS) の τ_0 周期解の極限；(ii) k, $m \in \mathbb{Z}_+$ で $k + m \geq |\nu|$ をみたすものと，τ_0 のある左側近傍 $\mathcal{I}_\ell$ とある右側近傍 $\mathcal{I}_r$ が存在して，任意の $\lambda \in \mathcal{I}_\ell$ (resp. $\lambda \in \mathcal{I}_r$) に対して (HS) は少なくとも k (resp. m) 個の相異なる λ-周期解を持つ．

Fadell–Rabinowitz の定理については，次のことに注意する必要がある:

- Fadell–Rabinowitz は S^1 が作用する空間に対して “cohomological index” を導入し，それによる min-max 理論で変分法的に解の存在を示している．
- Fadell–Rabinowitz の結果は，Weinstein–Moser とは異なり， 周期をパラメータとしており，エネルギーを指定した場合の個数をカウントしていない．

Fadell–Rabinowitz の結果に対して，エネルギー値を指定した場合の個数について，Thomas Bartsch は 1997 年に次のことを定理として主張した:

Claim (Bartsch [1], 1997) Fadell–Rabinowitz の定理と同じ仮定のもとで，符号数 $2\nu \neq 0$ とする．このとき次のどちらかの主張が成り立つ．

(i) $H^{-1}(0) \setminus \{0\}$ の中に，各 k に対して周期が τ_k である (HS) の周期解の列 $\{x_k\}_k$ で，$k \to \infty$ のとき $x_k \to 0$ かつ $\tau_k \to \tau_0$ となるものが存在する．

(ii) ある $\lambda_0 > 0$ があって，$\lambda \cdot \nu > 0$ かつ $0 < |\lambda| \leq \lambda_0$ を満たす任意の λ に対して $H^{-1}(\lambda)$ 上に (HS) は，周期が τ_0 に近い，少なくとも ν 個の異なる周期解が存在する．これらの解は $\lambda \to 0$ のとき 0 に収束する．（Claim 終わり）

この主張の Bartsch による証明は，Conley index, length という cohomological index, Morse decomposition 等を駆使した変分法的方法であるが，誤った主張や直ちに正当化できない言明を多数含んでいる．

我々は Bartsch による証明の欠陥をすべて正して，上記の主張を定理として確立することができた ([3])が，その詳細は非常に多岐にわたるため，ここでは述べることができない．

2.4 Bartsch による主張の optimality

Bartsch の主張の optimality を示す例が次のように構成できた．これは Moser による例にヒントを得て一般次元まで拡張したものである．

$$H := \frac{1}{2}\sum_{i=1}^{n}(p_i^2+q_i^2) + \frac{1}{4}\sum_{j=2}^{n}\left(\sum_{i=2}^{j}(p_i^2+q_i^2)\right)^2 + \frac{1}{2}\sum_{i=1}^{m}(\tilde{p}_i^2+\tilde{q}_i^2) - \frac{1}{2}\sum_{i=1}^{m}(\hat{p}_i^2+\hat{q}_i^2) + \sum_{i=1}^{m}(\tilde{p}_i^2+\tilde{q}_i^2+\hat{p}_i^2+\hat{q}_i^2)(\tilde{p}_i\hat{p}_i - \tilde{q}_i\hat{q}_i) \tag{2}$$

この Hamiltonian のヒントになったのは，$n = 0, m = 1$ の場合に Moser が与えた次の Hamiltonian である：

$$H = \frac{1}{2}(p_1^2+q_1^2) - \frac{1}{2}(p_2^2+q_2^2) + (p_1^2+q_1^2+p_2^2+q_2^2)(p_1p_2 - q_1q_2) \tag{3}$$

(3) の Hamiltonian の場合，Hamilton 方程式の解に対して

$$\frac{d}{dt}(p_1q_2+p_2q_1) = 4(p_1p_2-q_1q_2)^2 + (p_1^2+q_1^2+p_2^2+q_2^2) \geq 0$$

なので，周期解が存在するとすればそれは恒等的に 0 となる．従って，符号数 0 の場合には非自明な周期解は，エネルギーを指定しなくても，存在しない場合がある．

(2) で与えられる，一般の m, n での H の場合は，$E = \mathbb{R}^{2(n+2m)}$ で，$H''(0)$ の符号は $(2(m+n), 2m)$ となるので，その符号数は $2n$．そして，(3) の場合と同様にして $H = 0$ を満たす周期解は 0 しかないことが分かり，Bartsch の主張のケース (i) は起きない．さらに，$H = \varepsilon > 0$ における周期解はちょうど n 個であることが分かって，Bartsch の主張による最低個数である n 個の周期解しか存在しない．

2.5 J–不変周期軌道の存在

Hamiltonian $H \in C^2(\mathbb{R}^{2n}, \mathbb{R})$ が対称性を備えていれば，Hamilton 方程式が対応する対称性を備えた解を持つことを期待するのは自然である．Hamiltonian が symplectic form に由来する J に関して不変 ($H(Jz) = H(z)$, $\forall z \in \mathbb{R}^{2n}$) なときに，幾何学的に J–不変な軌道を持つ周期解の存在について次の部分的な結果が得られた．

定理 5 Hamiltonian $H \in C^2(\mathbb{R}^2, \mathbb{R})$ が J–不変で $H(0) = 0$, $\nabla H(0) = 0$ かつ $H''(0)$ は正定値とすると，十分小な任意の $\varepsilon > 0$ に対して，Hamilton 方程式には J–不変な軌道を持つ周期解で $H = \varepsilon$ を満たすものが存在する．

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